FINM 32000: Homework 2

Due Thursday April 13, 2023 at 11:59pm

Problem 1

The Gold Dragon Coin (GDC) is a unit of currency in Westeros. Let S denote the GDC/USD exchange rate (the USD value of 1 GDC). Assume S has dynamics

$$\mathrm{d}S_t = (r-q)S_t\mathrm{d}t + \sigma(S_t,t)S_t\mathrm{d}W_t,$$

where W is Brownian motion under the [USD] risk-neutral probability measure. The USD interest rate is $r = 0.06$, the GDC interest rate is $q = 0.01$, today's time-0 spot is $S_0 = 100$, and

$$\sigma(S,t) := \min[0.2 + 5(\log(S/100))^2 + 0.1e^{-t},\ 0.6].$$

(a) Find the time-0 price of an American-style put on the GDC. The put is struck at 95 and expires at time 0.75.

(b) Find the time-0 price of a European-style call, with strike 10 and expiry 0.25, on an American put on the GDC, to be issued at time 0.25 if the European call is exercised (therefore the put will not already have been exercised prior to time 0.25). The American put has strike 95 and expiry 0.75.

This call is an example of a *compound option*. At time 0.25 it gives the call holder the right to buy the underlying put for 10. The underlying put will have the usual exercise privilege on the time interval $[0.25, 0.75]$, at strike 95.

Example of usage: A company whose expenses are in USD but anticipates receiving a GDC revenue stream may want to have the put (from part (a)) to hedge the FX risk. But suppose that the revenue stream may or may not happen, depending on whether or not the ruling house of Westeros selects this company as a vendor at time 0.25, so the company does not want to pay full price for a put that may or may not turn out to be needed. This motivates the company to ask the Iron Bank what it would cost, to buy a call on the put. You are to help the Iron Bank quote a price.

All prices are, as usual, in USD unless stated otherwise.

Complete the coding of the function `price_compound_localvol` in the provided `ipynb` file. Use a trinomial tree. Your code may reject N for which the call expiry fails to be represented in the tree. In choosing Δx, follow L2.10 and choose the "representative" volatility σ_{avg} to be $\sigma(S_0, 0)$.

The amount of work done by your algorithm in this problem should grow like N^2 as N grows (no proof required). If it grows like N^3 in this problem, then your algorithm has some major inefficiency.

Problem 2

(a) In the Black-Scholes model with interest rate r, no dividends, and volatility σ, approximate the time-0 *delta* of an at-the-money ($K = S_0$) vanilla call with expiry T, by applying a first-order Taylor expansion to the exact formula, and obtaining an explicit approximation formula in terms of the given parameters.

Then evaluate this approximation to two decimal places, assuming $\sigma = 0.2$ and $T = 0.25$ and $r = 0.01$.

(b) Suppose that some option, or combination of options – let's call it the "combination" – has a time-0 pricing function $C(S)$ with respect to an underlying stock S.

Define the combination's time-0 *dollar delta* to be its delta multiplied by S_0:

$$S_0 \frac{\partial C}{\partial S},$$

which equals the dollar value of the stock position in the delta hedge (quantity $\frac{\partial C}{\partial S}$ shares $\times$ price S_0 dollars per share).

Define the combination's time-0 *dollar gamma* to be its gamma multiplied by $S_0^2/100$:

$$\frac{S_0^2}{100} \times \frac{\partial^2 C}{\partial S^2},$$

which equals the dollar value of stock shares that need to be purchased/sold in order to rebalance a delta hedge, per 1 *percent* movement in the stock price.

(because: the *shares* of stock to be purchased/sold is $\frac{\partial^2 C}{\partial S^2}$ per *dollar* movement in S.

So the *dollars* of stock to be purchased/sold is $S_0 \frac{\partial^2 C}{\partial S^2}$ per *dollar* movement in S.

So the *dollars* of stock to be purchased/sold is $S_0 \frac{\partial^2 C}{\partial S^2} \times \frac{S_0}{100}$ per 1 *percent* change in S.)

Suppose that at time 0, the underlying stock has price $S_0 = 4$, and the option combination has price 5, *dollar delta* 3, and *dollar gamma* 0.02.

Use a second-order Taylor expansion (with zeroth, first, and second order terms) to approximate the time-0 value of the contract, given an underlying stock price 3.6.

Hint: convert the dollar delta and dollar gamma into delta and gamma, and apply Taylor expansion to C.