FINM 32000: Homework 6

Due Friday May 12, 2023 at 11:59pm

Problem 1

Let **S** be the column vector with components $S^{[1]}, S^{[2]}$, where the stock prices $S^{[j]}$ have risk-neutral dynamics

$$dS_t^{[j]} = rS_t^{[j]}dt + \sigma_{[j]}S_t^{[j]}dW_t^{[j]}$$
 $j = 1, 2$

with risk-free interest rate r = 0.05, and constant volatilities $\sigma_{[1]} = 0.3$, $\sigma_{[2]} = 0.2$.

The time-0 prices are $S_0^{[1]}=100,\ S_0^{[2]}=110.$ The \mathbb{P} -Brownian motions $W^{[i]}$ and $W^{[j]}$ have correlation $\rho=0.8$.

(a) Let **X** be the column vector with components $X^{[1]}, X^{[2]}$ where $X^{[j]} := \log S^{[j]}$. Find the covariance matrix of \mathbf{X}_T .

Hint: One possible approach is to write \mathbf{X}_T as a nonrandom vector plus $\Sigma \mathbf{W}_T$ where Σ is the nonrandom diagonal matrix with diagonal elements $\sigma_{[1]}, \sigma_{[2]}$, and \mathbf{W} is the random column vector with components $W^{[1]}, W^{[2]}$.

Then
$$\operatorname{Cov}(\mathbf{X}_T) = \mathbb{E}(\mathbf{X}_T \mathbf{X}_T^{\top}) = \mathbb{E}(\mathbf{\Sigma} \mathbf{W}_T \mathbf{W}_T^{\top} \mathbf{\Sigma}^{\top}) = \mathbf{\Sigma} \operatorname{Cov}(\mathbf{W}_T) \mathbf{\Sigma}^{\top}.$$

Consider a basket $H := \frac{1}{2}S^{[1]} + \frac{1}{2}S^{[2]}$ of one-half of a share of each stock.

(b) Using 10000 standard Monte Carlo simulations, estimate the time-0 price C of an option that pays $(H_T - 110)^+$ at time T = 1.0. Also give the standard error [the sample standard deviation, divided by the square root of the number of simulations] of your Monte Carlo estimate.

You may either use a random number generator that produces normals with a given covariance matrix (which you found in (a)), or alternatively use a random number generator that produces independent normals which you then transform to introduce correlation.

In either approach, each of the 10000 simulations should use just one \mathbb{R}^2 -valued random vector \mathbf{Z} of simulated normal zero-mean random variables.

- (c) Use 10000 antithetic pairs $(\mathbf{Z}, -\mathbf{Z})$ to estimate C, together with a standard error (L5.30). Consider the "geometric basket" $G := \left(S^{[1]}S^{[2]}\right)^{1/2}$.
- (d) The random variable $\log G_T$ is normally distributed (because it's a linear transformation of a multivariate normal vector). Show that $\log G_T$ has expectation

$$\frac{1}{2}\log(S_0^{[1]}S_0^{[2]}) + \left(r - \frac{\sigma_{[1]}^2 + \sigma_{[2]}^2}{4}\right)T$$

and variance

$$\frac{\sigma_{[1]}^2 + 2\rho\sigma_{[1]}\sigma_{[2]} + \sigma_{[2]}^2}{4}T.$$

(e) Let C^G be the time-0 price of a geometric basket option paying $(G_T - K)^+$ at time T. Express C^G in terms of the function C^{BS} defined in FINM 33000 L6.16. Specifically, fill in the blanks:

$$C^G = C^{BS}(\underline{\hspace{1cm}}, 0, K, T, \underline{\hspace{1cm}}, r, \underline{\hspace{1cm}}).$$

Your answer should be a general formula, in which you have not substituted 0.8 for ρ , etc. (You may *also* do the substitutions, but don't neglect the general formula).

(f) Using a geometric basket option as a control variate, run M=10000 Monte Carlo simulations to estimate C, together with a standard error. Use the control variate estimate $\hat{C}_M^{\text{cv},\hat{\beta}}$ from L6.6 or L6.7. Use the (asymptotically valid) standard error $\hat{\sigma}_M^{\text{cv},\hat{\beta}}/\sqrt{M}$.

See the ipynb file.

Problem 2

Let the bank account and non-dividend paying stock have risk-neutral dynamics

$$dB_t = rB_t dt$$

$$B_0 = 1$$

$$dS_t = rS_t dt + \sigma S_t dW_t$$

$$S_0 > 0$$

where $\sigma > 0$ and W is a P-Brownian motion.

Consider a K-strike T-expiry vanilla call option, and let C denote its time-0 price.

(a) Let $S_0 = 100$, $\sigma = 0.2$, r = 0.02, K = 150, T = 1.

Run 100000 ordinary Monte Carlo simulations to estimate C, together with a standard error.

(b) Suppose that we sample from a new probability measure \mathbb{P}^* , under which W now has constant drift λ instead of drift 0. Thus $W_t = W_t^* + \lambda t$ where W^* is a standard \mathbb{P}^* -BM.

Find the \mathbb{P}^* -expectation \mathbb{E}^*S_T in terms of S_0 , r, σ , λ , and T.

Calculate λ such that $\mathbb{E}^* S_T = 165$.

(Why did we choose 165? The picture in L6.16 shows that the optimal distribution from which to sample will have a mean that is greater than the strike K. So let's choose 10% higher than K. This will not be optimal, but we expect that it will be an improvement over ordinary Monte Carlo. There are more systematic ways to determine a reasonable drift adjustment, not utilized here.)

(c) Run 100000 importance sampling simulations, using the specific drift adjustment calculated in (b), to estimate C, together with a standard error. Be aware that your zero-mean normal random draws, here, simulate increments of W^* not W.

Each simulation should require only one number to be generated by rng.normal.

See the ipynb file.

Comment: of course, we do not need Monte Carlo to price a call under GBM. However, suppose you wanted to price a deep OTM option under an intractable *stochastic volatility* model, using importance sampling. You could still use a similar approach.