Lyapunov Optimization Proof

I. Derivation of Lyapunov dift function

Consider we have the following queues.

$$Q_n^m(t+1) = \max\{Q_n^m(t) - \delta_n^m(t)\mu_n(t), 0\} + \lambda_n^m(t)$$

$$Z^m(t+1) = \max\{Z^m(t) - \sum_n \delta_n^m(t)\mu_n(t) + u^m(t), 0\}$$

$$G_n(t+1) = \max\{G_n(t) - \sum_m \lambda_n^m(t) + \gamma_n(t), 0\}$$

We define $\boldsymbol{\Theta}(t) = [\boldsymbol{Q}(t), \boldsymbol{Z}(t), \boldsymbol{G}(t)]$. Then we have the following quadratic Lyapunov function

$$L(\boldsymbol{\Theta}(t)) = \frac{1}{2}[\sum_n \sum_m (Q_n^m(t))^2 + \sum_m (Z^m(t))^2 + \sum_n (G_n(t))^2]$$

Denote $\Delta(\boldsymbol{\Theta}(t)) = L(\boldsymbol{\Theta}(t+1)) - L(\boldsymbol{\Theta}(t))$ as the one-step drift. Thus, we have

$$\begin{aligned}
\Delta(\boldsymbol{\Theta}(t)) &= L(\boldsymbol{\Theta}(t+1)) - L(\boldsymbol{\Theta}(t)) \\
&= \frac{1}{2}[\sum_n \sum_m ((Q_n^m(t+1))^2 - (Q_n^m(t)^2)) + \\
&\qquad \sum_m ((Z^m(t+1))^2 - (Z^m(t)^2)) + \\
&\qquad \sum_n ((G_n(t+1))^2 - (G_n(t)^2))]
\end{aligned}$$

Squaring both sides of queues, we have

$$\begin{aligned}
(Q_n^m(t+1))^2 - (Q_n^m(t))^2 \le{}& (\lambda_n^m(t))^2 + (\delta_n^m(t)\mu_n(t))^2 \\
&+ 2Q_n^m(t)(\lambda_n^m(t) - \delta_n^m(t)\mu_n(t))
\end{aligned}$$

$$\begin{aligned}
(Z^m(t+1))^2 - (Z^m(t))^2 \le{}& (u^m(t))^2 + (\sum_n \delta_n^m(t)\mu_n(t))^2 \\
&+ 2Z^m(t)(u^m(t) - \sum_n \delta_n^m(t)\mu_n(t))
\end{aligned}$$

$$\begin{aligned}
(G_n(t+1))^2 - (G_n(t))^2 \le{}& (\gamma_n(t))^2 + (\sum_m \lambda_n^m(t))^2 \\
&+ 2G_n(t)(\gamma_n(t) - \sum_m \lambda_n^m(t))
\end{aligned}$$

where the inequality holds by that, for any non-negative x, y, z, $(max\{x-y, 0\} + z)^2 \le y^2 + z^2 + 2x(z-y)$, and with $(max\{x, 0\})^2 \le x^2$.

Then, by summing over the above three inequalities for all m and n, we have

$$\begin{aligned}
\Delta(\boldsymbol{\Theta}(t)) \le \frac{1}{2}(&\sum_n \sum_m ((\lambda_n^m(t))^2 + (\delta_n^m(t)\mu_n(t))^2) \\
&+ \sum_m ((u^m(t))^2 + (\sum_n \delta_n^m(t)\mu_n(t))^2) \\
&+ \sum_n (\gamma_n(t))^2 + (\sum_m \lambda_n^m(t))^2) \\
&+ \sum_n \sum_m Q_n^m(t)(\{\lambda_n^m(t) - \delta_n^m(t)\mu_n(t)) \\
&+ \sum_m Z^m(t)(u^m(t) - \sum_n \delta_n^m(t)\mu_n(t)) \\
&+ \sum_n G_n(t)(\{\gamma_n(t) - \sum_m \lambda_n^m(t))
\end{aligned}$$

Let B be the upper bound of the first term in the above equation, and take conditional expectations. Hence, by adding $V(\overline{\Phi(\boldsymbol{\gamma})} - R(\overline{\boldsymbol{\delta}}))$ on both sides, we derive Lyapunov dift function, where B is a constant only related to the upper bound of $\lambda_n^m(t)$, $\mu_n(t)$, $\gamma_n(t)$.

II. Proof of Performance

Theorem : If the control variable V is fixed and the Lyapunov algorithm is used for all t, then we have:

a) Any of $Q_n^m(t)$ is stable and for all t and we have:

$$\limsup_{t\to\infty} \frac{1}{t} \sum_{\tau=0}^{t} \sum_n \sum_m \mathbb{E}\{Q_n^m(\tau)\} \le \frac{B}{\xi} + \frac{V(|WU|K + |R|)}{\xi}$$

where ξ is a positive constant, K is the Lipschiz constant, and $|WU| = \sum_n \omega_n U_n$, and $|R| = \sum_m \sum_n R_n^m$.

b) Let $\boldsymbol{obj_1^*} = \overline{\Phi(\boldsymbol{\gamma}^*)} - R(\overline{\boldsymbol{\delta}^*})$ be the objective achieved by our algorithm and $\boldsymbol{obj_1^{opt}} = \overline{\Phi(\boldsymbol{\gamma}^{opt})} - R(\overline{\boldsymbol{\delta}^{opt}})$ be the optimal objective. Then, we have:

$$\limsup_{t\to\infty} (\boldsymbol{obj_1^{opt}} - \boldsymbol{obj_1^*}) \le \frac{B}{V}$$

Proof: To prove the performance of Lyapunov-based strategies, let $\widetilde{\gamma_n}(t)$, $\widetilde{\lambda_n^m}(t)$, and $\widetilde{\delta_n^m}(t)$ be a random policy. According to the feasibility of the original problem, there exists a positive constant ξ, which yields

$$\mathbb{E}\{\widetilde{\lambda_n^m}(t) - \widetilde{\delta_n^m}(t)\mu_n(t)\} \le -\xi$$

$$\mathbb{E}\{u^m(t) - \sum_n \widetilde{\delta_n^m}(t)\mu_n(t)\} = 0$$

$$\mathbb{E}\{\gamma_n(t) - \sum_m \widetilde{\lambda_n^m}(t)\} = 0$$

Intuitively, such a policy can be easily found, if we always choose $\widetilde{\gamma_n}(t) = \sum_m \widetilde{\lambda_n^m}(t)$ and $\widetilde{u^m}(t) = \sum_n \widetilde{\delta_n^m}(t)\mu_n(t)$.

By replacing the decision variables in Lyapunov drift function, the following inequality can be derived.

$$\Delta(\boldsymbol{\Theta}(t)) - V\mathbb{E}\{\Phi(\boldsymbol{\gamma}(t)) - \boldsymbol{R}(\boldsymbol{\delta}(t))|\boldsymbol{\Theta}(t)\} \leq B$$
$$-V\mathbb{E}\{\sum_n \omega_n \phi_n(\widetilde{\gamma_n}(t)) - \sum_n\sum_m \widetilde{\Delta\delta_n^m}(t)R_n^m\}$$
$$+\sum_n\sum_m Q_n^m(t)\mathbb{E}\{\widetilde{\lambda_n^m}(t) - \widetilde{\delta_n^m}(t)\mu_n(t)\}$$

Rearranging the terms and summing over for all t, we have

$$\xi\sum_{t=0}^{t}\sum_n\sum_m Q_n^m(t) \leq Bt - V\sum_{t=0}^{t}((\Phi(\widetilde{\boldsymbol{\gamma}}(t)) - \boldsymbol{R}(\widetilde{\boldsymbol{\delta}}(t)))$$
$$-\mathbb{E}\{\Phi(\boldsymbol{\gamma}(t)) - \boldsymbol{R}(\boldsymbol{\delta}(t))|\boldsymbol{\Theta}(t)\})$$

Recall that $\phi_n(\cdot)$ satisfies Lipschiz condition and $\delta_n^m(t)$ is a binary variable.

$$\Phi(\boldsymbol{\gamma}(t)) - \Phi(\widetilde{\boldsymbol{\gamma}}(t)) = \sum_n \omega_n(\phi_n(\gamma_n(t)) - \phi_n(\widetilde{\gamma_n}(t)))$$
$$\leq K\sum_n \omega_n\,|\gamma_n(t) - \widetilde{\gamma_n}(t))| \leq |WU|K$$
$$\boldsymbol{R}(\boldsymbol{\delta}(t)) - \boldsymbol{R}(\widetilde{\boldsymbol{\delta}}(t)) = \sum_n\sum_m(\Delta\delta_n^m(t) - \widetilde{\Delta\delta_n^m}(t))R_n^m \leq |R|$$

By using the above inequalities, the average queue length satisfies

$$\xi\sum_{t=0}^{t}\sum_n\sum_m Q_n^m(t) \leq Bt + V\sum_{t=0}^{t}(|WU|K + |R|)$$

Divide both sides by ξt and take $\limsup_{t\to\infty}$. Finally, we have finished the proof.

Part b) is easy to be proved, similarly.

III. Proof of Performance with Delay Queue

We introduce another virtual queue $D_n^m(t+1)$ and modify $Q_n^m(t)$ as follows.

$$Q_n^m(t+1) = \max\{Q_n^m(t) - \delta_n^m(t)\mu_n(t) - d_n^m(t), 0\} + \lambda_n^m(t)$$
$$D_n^m(t+1) = \eta_n^m(t)\max\{D_n^m(t) - \delta_n^m(t)\mu_n(t) - d_n^m(t) + \sigma_n^m, 0\}$$

Combining this queue with the queues in Section I, we can have another control policies, which have the following performance.

Theorem 3: If the decision variables are determined by the Lyapunov algorithm for all t, the queue lengths of $Q_n^m(t)$ and $D_n^m(t)$ are bounded for all t as follows:

$$Q_n^m(t) \leq (Q_n^m)^{max}, \quad D_n^m(t) \leq (D_n^m)^{max}$$

provided that the above inequalities hold at $t = 0$.

Proof: Here, we leverage the mathematical induction to derive the bound of $(Q_n^m)^{max}$.

As all queues are initially set to be empty, we have $Q_n^m(0) \leq (Q_n^m)^{max}$ for all n and m. Suppose $Q_n^m(t) \leq (Q_n^m)^{max}$ holds at time t, and we have two conditions.

1) $Q_n^m(t) \leq V\beta_m$. Then, $Q_n^m(t+1) \leq V\beta_m + A^m$. This is because $Q_n^m(t)$ can increase by at most A^m at any slot.

2) $Q_n^m(t) \geq V\beta_m$. Then, according to (40), the rejecting decision is chosen by $d_n^m(t) = (d_n^m(t))^{max}$. Under this situation, if $Q_n^m(t) - \delta_n^m(t)\mu_n(t) \leq d_{max}$, the rejecting decision choose to clear $Q_n^m(t)$ by dropping all the remaining requests, and so $Q_n^m(t+1) = \lambda_n^m(t) \leq (Q_n^m)^{max}$. Otherwise, if $Q_n^m(t) - \delta_n^m(t)\mu_n(t) \geq d_{max}$, we have $d_n^m(t) = d_{max}$. By the boundedness assumption of d_{max}, the queue $Q_n^m(t)$ cannot increase in the next slot. Namely, $Q_n^m(t+1) \leq Q_n^m(t) \leq (Q_n^m)^{max}$

Overall, we prove $Q_n^m(t) \leq (Q_n^m)^{max}$.

The proof of the $(D_n^m)^{max}$ is almost the same.

IV. Proof of Performance with Delay Queue 2

For the variable $d_n^m(t)$, we have the following theorem.

Theorem 5: For a fixed $V > 0$, the average value, achieved by the Lyapunov algorithm, satisfies the following:

$$\limsup_{t\to\infty}\frac{1}{t}\sum_{n,m}\beta_m d_n^m(t) \leq \frac{B' + \sum_{n,m}\eta(V\beta_m + \sigma_n^m)}{V}$$
$$+(|WU|K + |R|)$$

where W, U, K and R are constants.

Proof: Suppose that there exists a randomized stationary control policy $\widehat{\gamma_n}(t)$, $\widehat{\lambda_n^m}(t)$, $\widehat{\delta_n^m}(t)$, and $\widehat{d_n^m}(t) = 0$, which satisfies the following.

$$\mathbb{E}\{\widehat{\lambda_n^m}(t) - \widehat{\delta_n^m}(t)\mu_n(t) - \widehat{d_n^m}(t)\} \leq \epsilon$$
$$\mathbb{E}\{u^m(t) - \sum_n \widehat{\delta_n^m}(t)\mu_n(t)\} \leq \epsilon$$
$$\mathbb{E}\{\gamma_n(t) - \sum_m \widehat{\lambda_n^m}(t)\} \leq \epsilon$$

where ϵ is a non-negative constant. Intuitively,such a policy can be easily obtained by choosing $d_n^m(t) = 0$ for all t.

Besides, as σ_n^m is a pre-defined constant, constrained by $0 \leq \sigma_n^m \leq A^m$, there exists a non-negative constant η yields

$$\mathbb{E}\{\sigma_n^m - \widehat{\lambda^m}(t)\} \leq \eta$$

Then, by plugging the above inequalities into Lyapunov drift function and taking $\epsilon \to 0$, we can derive

$$\Delta(\boldsymbol{\Theta}(t)) - V\mathbb{E}\{\Phi(\boldsymbol{\gamma}(t)) - \boldsymbol{R}(\boldsymbol{\delta}(t)) - \boldsymbol{\beta d}(t)|\boldsymbol{\Theta}(t)\} \leq B'$$
$$-V\mathbb{E}\{\sum_n \omega_n\phi_n(\widehat{\gamma_n}(t)) - \sum_n\sum_m \widehat{\Delta\delta_n^m}(t)R_n^m\} + \sum_n\sum_m D_n^m(t)\eta$$

Rearranging the terms and summing over for all t yields

$$V\sum_t\sum_n\sum_m \beta_m d_n^m(t) \leq B't + \sum_t\sum_n\sum_m D_n^m(t)\eta$$
$$+V\sum_t(\sum_n \omega_n(\phi_n(\gamma_n(t)) - \phi_n(\widehat{\gamma_n}(t)))$$
$$-\sum_n\sum_m R_n^m(\Delta\delta_n^m(t) - \widehat{\Delta\delta_n^m}(t)))$$

According to the previous analysis, we can obtain

$$V\sum_t\sum_n\sum_m \beta_m d_n^m(t) \leq B't + \sum_t\sum_n\sum_m \eta(V\beta_m + \sigma_n^m)$$
$$+V\sum_t(|WU|K + |R|)$$

Dividing both sides by Vt and taking $\limsup_{t\to\infty}$ yields

$$\limsup_{t\to\infty}\frac{1}{t}\sum_{n,m}\beta_m d_n^m(t) \leq \frac{B' + \sum_{n,m}\eta(V\beta_m + \sigma_n^m)}{V}$$
$$+(|WU|K + |R|)$$

Hence, we prove the theorem.