Chapter 4: 4.4 Projectile Motion

Let's consider the case for two-dimensional motion: a particle moves in a vertical plane with initial velocity v_o and its acceleration is always directed downward. This particle is called **projectile** and its motion is called **projectile motion**. The projectile is launched with an initial velocity $\mathbf{v_o}$ that can be written as

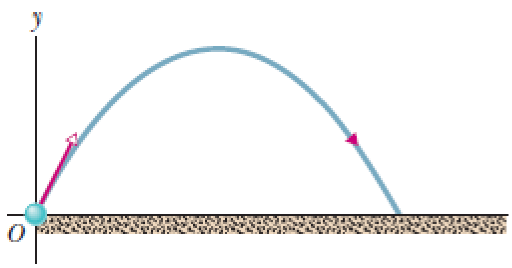
$$\vec{v}_0 = v_{0x}\hat{\mathrm{i}} + v_{0y}\hat{\mathrm{j}}.$$

$$v_{0x} = v_0 \cos\theta_0 \quad \text{and} \quad v_{0y} = v_0 \sin\theta_0.$$

During its two-dimensional motion, the projectile's position vector $\vec{r}$ and velocity vector $\vec{v}$ change continuously, but its acceleration vector $\vec{a}$ is constant and *always* directed vertically downward. The projectile has *no* horizontal acceleration.

> In projectile motion, the horizontal motion and the vertical motion are independent of each other; that is, neither motion affects the other.

Consider the experiment



Lets derive the expression for two types of motion involved in it vertical and the horizontal motion.

The Horizontal Motion

Now we are ready to analyze projectile motion, horizontally and vertically. We start with the horizontal motion. Because there is *no acceleration* in the horizontal direction, the horizontal component v_x of the projectile's velocity remains unchanged from its initial value v_{0x} throughout the motion, as demonstrated in Fig. 4-12. At any time t, the projectile's horizontal displacement $x - x_0$ from an initial position x_0 is given by Eq. 2-15 with $a = 0$, which we write as

$S = V_0 t$ x-component

$$x - x_0 = v_{0x}t.$$

Because $v_{0x} = v_0 \cos\theta_0$, this becomes

$$x - x_0 = (v_0 \cos\theta_0)t.$$

The Vertical Motion

The vertical motion is the motion we discussed in Module 2-5 for a particle in free fall. Most important is that the acceleration is constant. Thus, the equations of Table 2-1 apply, provided we substitute $-g$ for a and switch to y notation. Then, for example, Eq. 2-15 becomes

Table 2-1 Equations for Motion with Constant Acceleration[a]

Equation Number	Equation	Missing Quantity
2-11	$v = v_0 + at$	$x - x_0$
2-15	$x - x_0 = v_0 t + \frac{1}{2}at^2$	v ←
2-16	$v^2 = v_0^2 + 2a(x - x_0)$	t
2-17	$x - x_0 = \frac{1}{2}(v_0 + v)t$	a
2-18	$x - x_0 = vt - \frac{1}{2}at^2$	v_0

[a]Make sure that the acceleration is indeed constant before using the equations in this table.

$$y - y_0 = v_{0y}t - \tfrac{1}{2}gt^2$$
$$= (v_0 \sin \theta_0)t - \tfrac{1}{2}gt^2,$$

where the initial vertical velocity component v_{0y} is replaced with the equivalent $v_0 \sin \theta_0$. Similarly, Eqs. 2-11 and 2-16 become

$v = V_0 + at$

$$v_y = v_0 \sin \theta_0 - gt \tag{4-23}$$

$$v_y^2 = (v_0 \sin \theta_0)^2 - 2g(y - y_0). \tag{4-24}$$

$v^2 - v_0^2 = 2gS$

Eq. 4-23, the vertical velocity component behaves just as for a ball thrown vertically upward. It is directed upward initially, and its magnitude steadily decreases to zero, which marks the maximum height of the path. The vertical velocity component then reverses direction, and its magnitude becomes larger with time.

The Equation of the Path

We can find the equation of the projectile's path (its **trajectory**) by eliminating time t between Eqs. 4-21 and 4-22. Solving Eq. 4-21 for t and substituting into Eq. 4-22, we obtain, after a little rearrangement,

$$x - x_0 = (v_0 \cos \theta_0)t. \qquad (4\text{-}21)$$

$$\begin{aligned} y - y_0 &= v_{0y}t - \tfrac{1}{2}gt^2 \\ &= (v_0 \sin \theta_0)t - \tfrac{1}{2}gt^2, \end{aligned} \qquad (4\text{-}22)$$

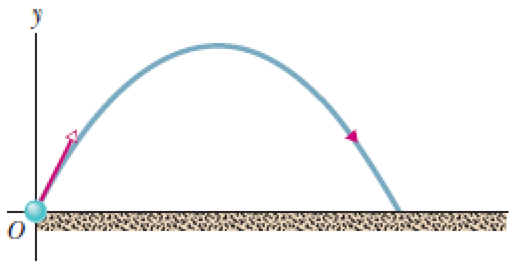
$$t = \frac{x - x_0}{v_0 \cos \theta_0}$$

$$y - y_0 = \cancel{v_0} \sin \theta_0 \frac{x - x_0}{\cancel{v_0} \cos \theta_0} - \frac{1}{2} g \frac{(x - x_0)^2}{(v_0 \cos \theta)^2}$$

$$= \cancel{v_0}(x - x_0)\tan \theta - \frac{1}{2} g \frac{(x - x_0)^2}{(V_0 \cos \theta_0)^2}$$

$$y = (\tan \theta_0)x - \frac{gx^2}{2(v_0 \cos \theta_0)^2} \qquad \text{(trajectory).} \qquad (4\text{-}25)$$

This is the equation of the path shown in Fig. 4-9. In deriving it, for simplicity we let $x_0 = 0$ and $y_0 = 0$ in Eqs. 4-21 and 4-22, respectively. Because g, θ_0, and v_0 are constants, Eq. 4-25 is of the form $y = ax + bx^2$, in which a and b are constants. This is the equation of a parabola, so the path is *parabolic*.



The Horizontal Range

The *horizontal range* R of the projectile is the *horizontal* distance the projectile has traveled when it returns to its initial height (the height at which it is launched). To find range R, let us put $x - x_0 = R$ in Eq. 4-21 and $y - y_0 = 0$ in Eq. 4-22, obtaining

R

$$x - x_0 = (v_0 \cos \theta_0)t. \tag{4-21}$$

$$y - y_0 = v_{0y}t - \tfrac{1}{2}gt^2$$

$$0 \qquad = (v_0 \sin \theta_0)t - \tfrac{1}{2}gt^2, \tag{4-22}$$

$$y - y_0 = v_{0y}t - \tfrac{1}{2}gt^2$$

$$0 \qquad = (v_0 \sin \theta_0)t - \tfrac{1}{2}gt^2, \qquad (4\text{-}22)$$

$$V_0 \sin\theta_0 = \frac{1}{2} g t \;,\; t = \frac{2 v_0 \sin\theta_0}{g}$$

Eliminating t between these two equations yields

$$R = \frac{2v_0^2}{g} \sin \theta_0 \cos \theta_0.$$

This equation does *not* give the horizontal distance traveled by a projectile when the final height is not the launch height. Note that R in Eq. 4-26 has its maximum value when $\sin 2\theta_0 = 1$, which corresponds to $2\theta_0 = 90°$ or $\theta_0 = 45°$.

Using the identity $\sin 2\theta_0 = 2 \sin \theta_0 \cos \theta_0$ (see Appendix E), we obtain

$$R = \frac{v_0^2}{g} \sin 2\theta_0.$$