CS 461
Artificial Intelligence

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Constraint Satisfaction Problems

Constraint Satisfaction Problems

- A CSP consists of **variables with constraints** on them. It contains
  - *Finite set of variables $X_1, X_2, \ldots, X_n$*
  - *Nonempty domain of possible values for each variable $D_1, D_2, \ldots D_d$*
  - *Finite set of constraints $C_1, C_2, \ldots, C_m$*
  - *Each constraint C_i limits the values that variables can take, e.g., $X_1 \neq X_2$*
- **A state is defined as an assignment of values to some or all variables.**

Constraint Satisfaction Problems

Assignment:

- A state of the problem is defined by assigning values to some or all of the variables, $\{X_i = v_{i,} X_j = v_j\}$.

Consistent assignment:

- If the assignment *does not violate the constraints*.

Complete assignment:

- An assignment is ***complete*** *when every variable is assigned a value.*

Commutative:

- Variable assignments are ***commutative***
  - **e.g.** ***[step 1: WA = red; step 2: NT = green] equivalent to [step 1: NT = green; step 2: WA = red]***

Constraint Satisfaction Problems

Solution to CSP:

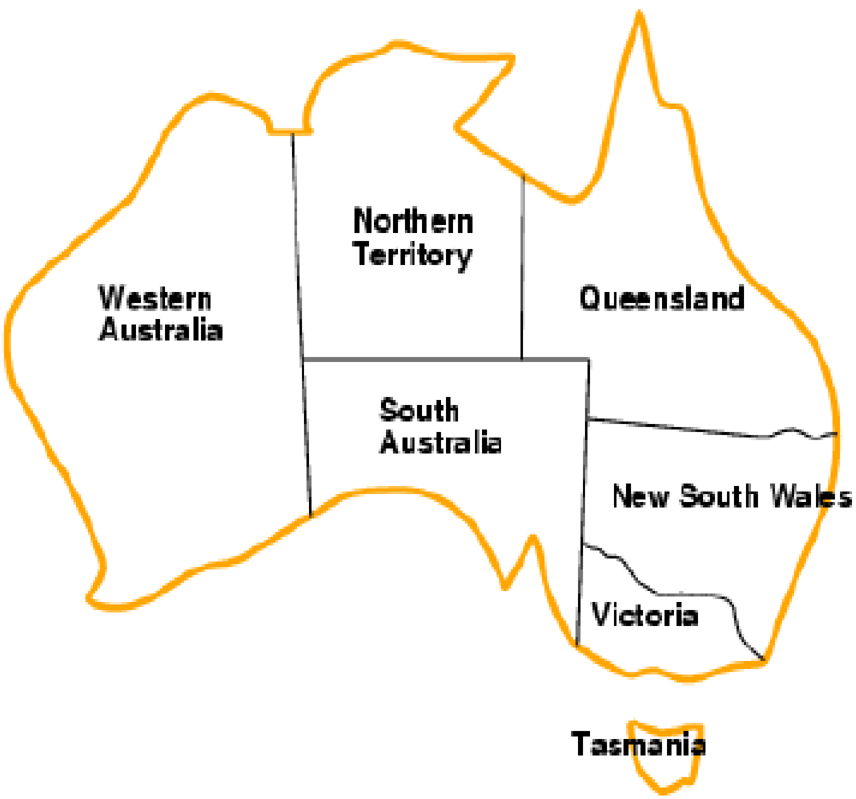
- A *solution to a CSP is a* ***complete assignment*** *that* ***satisfies all constraints****.*
- Some CSPs require a solution that maximizes an *objective function.*

Domain:

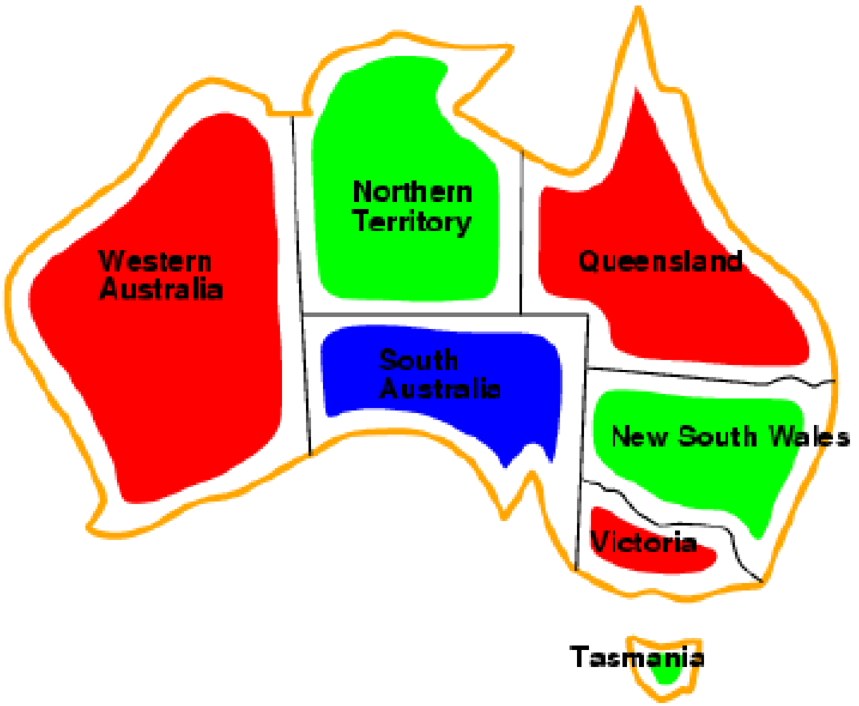
- Each variable X_i has a nonempty domain D_i of possible values.
  - e.g. Color is assigned to a variable X_i. Domain D_i may be set of possible colors like {R, G, B}.

Map-Colouring

- **Variables:**
  WA, NT, Q, NSW, V, SA, T
- **Domains:**
  D_i = red; green; blue
- **Constraints:**
  adjacent regions must have different colours
  - e.g., $WA \neq NT$
  - $(WA; NT) \in [(red;\ green);$
    $(red;\ blue);$
    $(green;\ red);$
    $(green;\ blue)\ \ldots\ ]$



Map-Colouring



Solutions are **complete** and **consistent** **assignments,**

- e.g., WA = red, NT = green,Q = red,NSW = green,
  V = red,SA = blue,T = green

Sudoku

	1	2	3	4	5	6	7	8	9
A			3		2		6		
B	9			3		5			1
C			1	8		6	4		
D			8	1		2	9		
E	7								8
F			6	7		8	2		
G			2	6		9	5		
H	8			2		3			9
I			5		1		3		

	1	2	3	4	5	6	7	8	9
A	4	8	3	9	2	1	6	5	7
B	9	6	7	3	4	5	8	2	1
C	2	5	1	8	7	6	4	9	3
D	5	4	8	1	3	2	9	7	6
E	7	2	9	5	6	4	1	3	8
F	1	3	6	7	9	8	2	4	5
G	3	7	2	6	8	9	5	1	4
H	8	1	4	2	5	3	7	6	9
I	6	9	5	4	1	7	3	8	2

- **Variables:** empty cells
- **Domains:** numbers between 1 to 9
- **Constraints:** rows, columns, boxes contain all different numbers

N-Queens

- **Variables:**

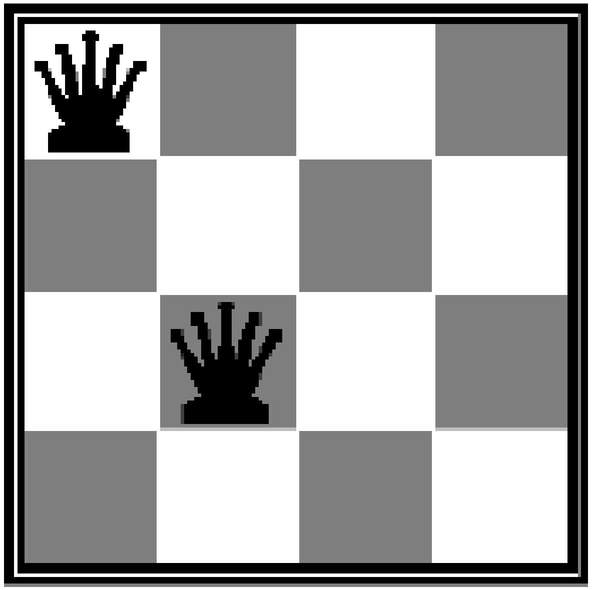
  Q_i

- **Domains:**

  $D_i = \{1, 2, 3, 4\}$

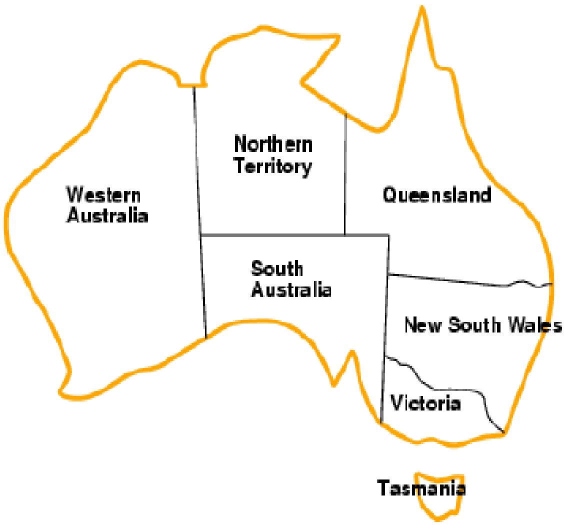
- **Constraints:**
  - Queen can NOT be in same row
  - Queen can NOT be in same column
  - Queen can NOT be in same diagonal

- **Valid values** for (Q_1, Q_2) are:
  - (1,3) (1,4) (2,4) (3,1) (4,1) (4,2)



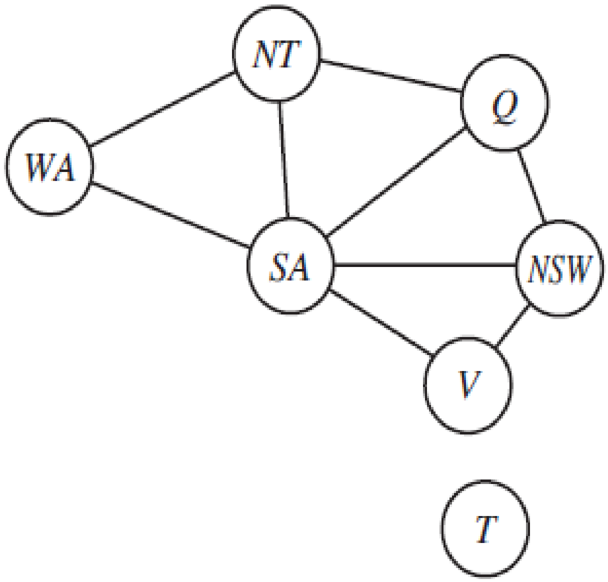
$Q_1 = 1 \quad Q_2 = 3$

Constraint Satisfaction Problems



<u>Constraint Graph</u>:

- Constraint Satisfaction Problem (CSP) can be visualized as a *constrained graph*.
  - The **nodes** of the graph correspond to *variables* of the problem
  - The **arcs** correspond to the *constraints*.



Constraint Satisfaction Problems

Finite Domain:

- The simplest kind of CSP involves variables that have **domains that are limited or restricted**.
  - Map coloring problems are of this kind.

Boolean CSP:

- Finite-domain CSPs include Boolean CSPs, *whose variables can either be true or false*.

Continuous Domain:

- Domain in which there is a sequence of assignment to the variables.
  - The scheduling experiments via telescope requires very precise timings of observation

Constraint Satisfaction Problems

Constraint Language:

- With **infinite domains**, it is *NO longer possible to describe constraints* by enumerating all combinations of values.
- Instead, a Constraint Language is used in which **set of rules are specified**.
  - If $job_{1,}$ which takes 5 days, must precede job_3 ,then a constraint language of algebraic inequalities such as $start\ job_1 + 5 <= start\ job_3$ will be required.

Constraint Satisfaction Problems

Types of Constraints

Unary Constraint:

- The simplest type of constraint, which **restricts the value of a single variable**, is called Unary Constraint.
  - e.g. SA $\neq$ Green

Binary Constraint:

- It relates **two variables** or **involves pair of variables**.
  - e.g. SA $\neq$ NSW

Constraint Hypergraph:

- Higher order constraints involve **three** or **more variables**. A Constraint Hypergraph represents these constraints.
  - e.g., crypt arithmetic column constraints

Crypt-arithmetic

- **Variables:**
  - D, E, M, N, O, R, S, Y
- **Domains:**
  - $D_i = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
- **Constraints:**
  - $M \neq 0, S \neq 0$
  - $D \neq E, D \neq M, D \neq N$
  - $Y = D + E$ OR $Y = D + E - 10$

$$
\begin{array}{r}
S\ E\ N\ D \\
+\ M\ O\ R\ E \\
\hline
M\ O\ N\ E\ Y
\end{array}
$$

Crypt-arithmetic

- **Variables:**
  - D, E, M, N, O, R, S, Y
- **Domains:**
  - $D_i = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
- **Constraints:**
  - $M \neq 0, S \neq 0$
  - $D \neq E, D \neq M, D \neq N$
  - $Y = D + E$ OR $Y = D + E - 10$

$$
\begin{array}{cccccc}
 & & S(9) & E(5) & N(6) & D(7) \\
+ & & M(1) & O(0) & R(8) & E(5) \\
\hline
 & M(1) & O(0) & N(6) & E(5) & Y(2)
\end{array}
$$

Constraint Satisfaction Problems

Linear Constraint:

- Constraint in which variable appears only in ***linear form*** is called **Linear Constraint**.
- Linear Constraints are solvable.

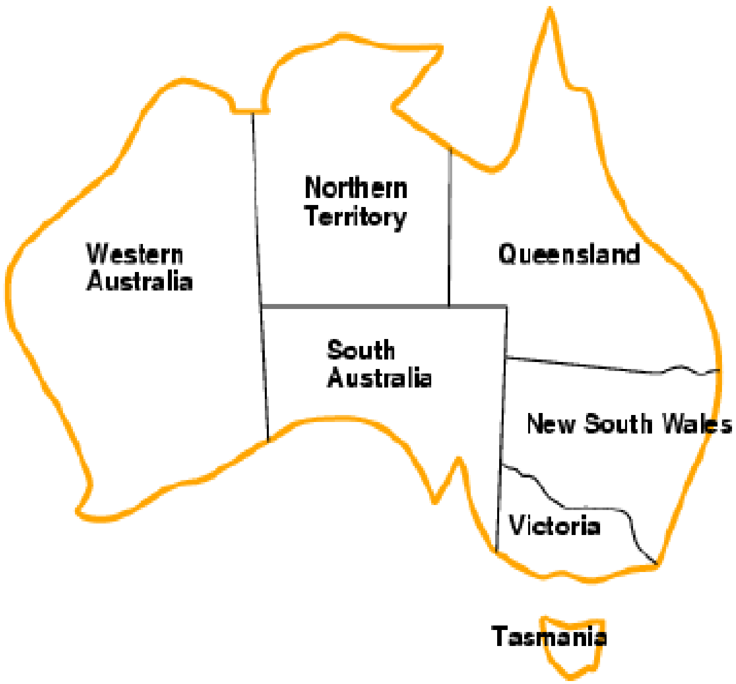
Non-Linear Constraint:

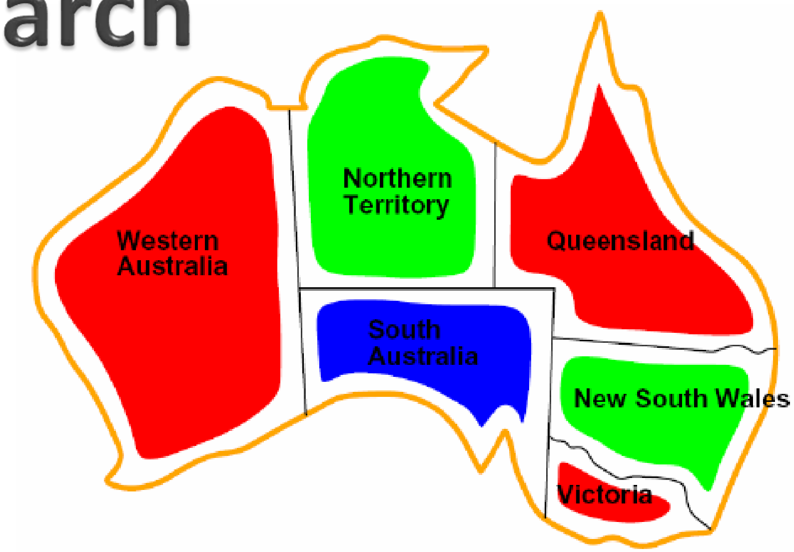
- Constraint in which variables appear in ***non-linear form*** is called **Non-linear Constraint**.
- Non-linear Constraints are undecidable.

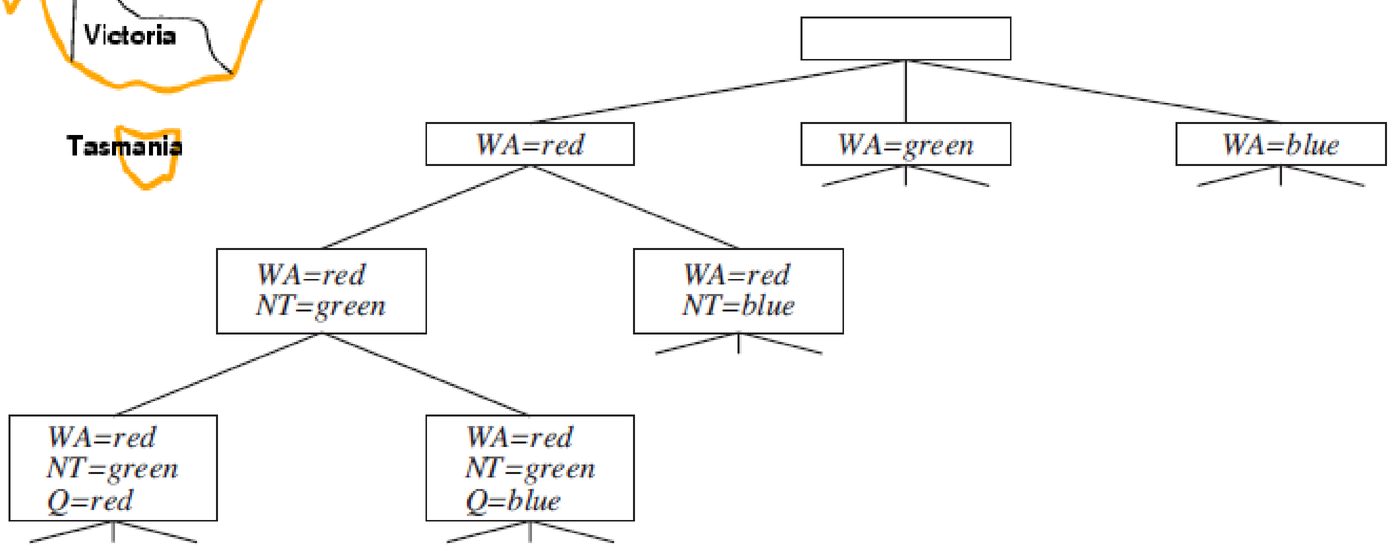
CSP as a Standard Search

- **A CSP can easily expressed as a <u>standard search</u> problem,**
  - ***Initial State:*** *the empty assignment {}*
  - ***Successor function:*** *Assign value to unassigned variable provided that there is no conflict*
  - ***Goal test:*** *the current assignment is complete*
  - ***Path cost:*** *a constant cost for every step*
- **Solution is found at depth n, for n variables**
  - Hence ***depth first search*** can be used
  - Only need to consider assignments to a single variable at each node

CSP as a Standard Search

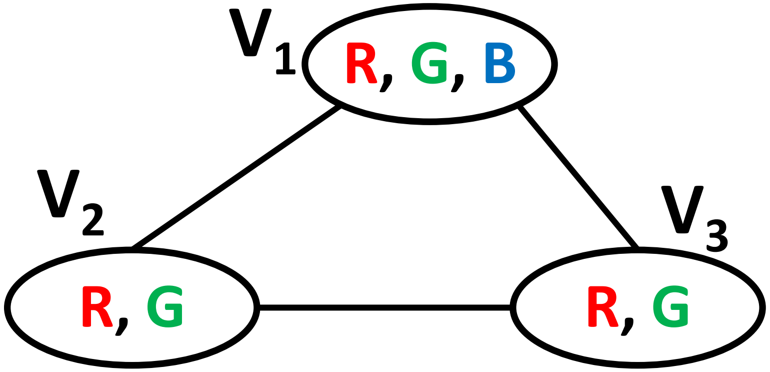




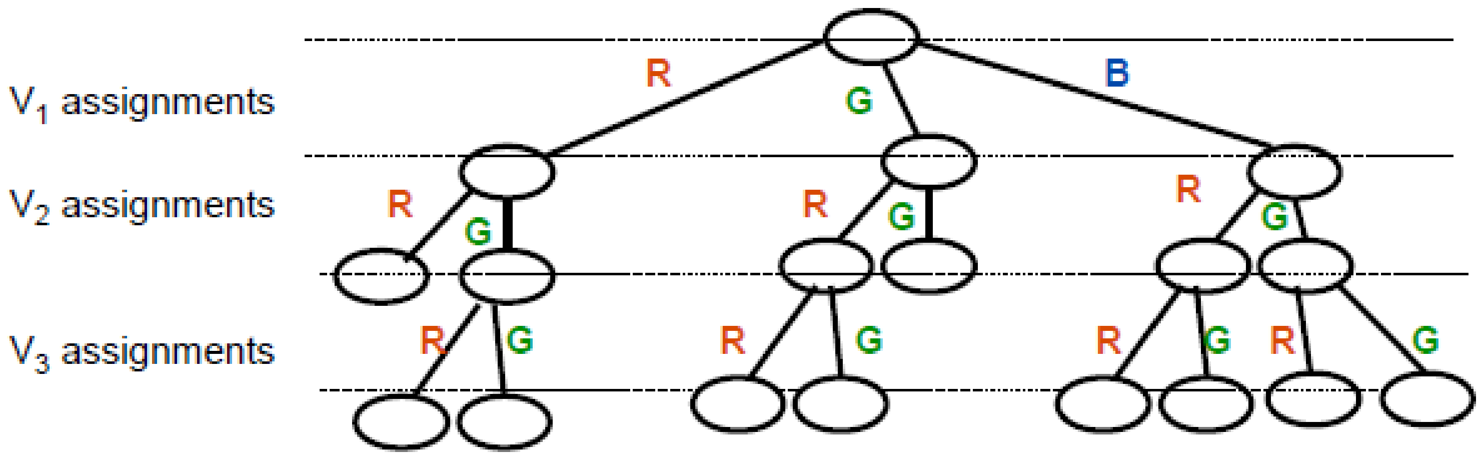


CSP as a Standard Search

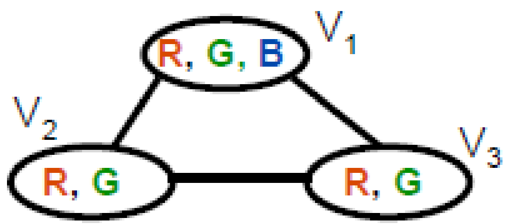
- **State:** assignment to k variables.
- ***Successor:*** The successor of a state is obtained by assigning a value to variable, keeping others Unchanged
- ***Start state:*** (V_1 = **R,G,B**, V_2= **R,G**, V_3 = **R,G**)
- ***Goal state:*** All variables **assigned colours (R,G,B)** with *constraints satisfied*.



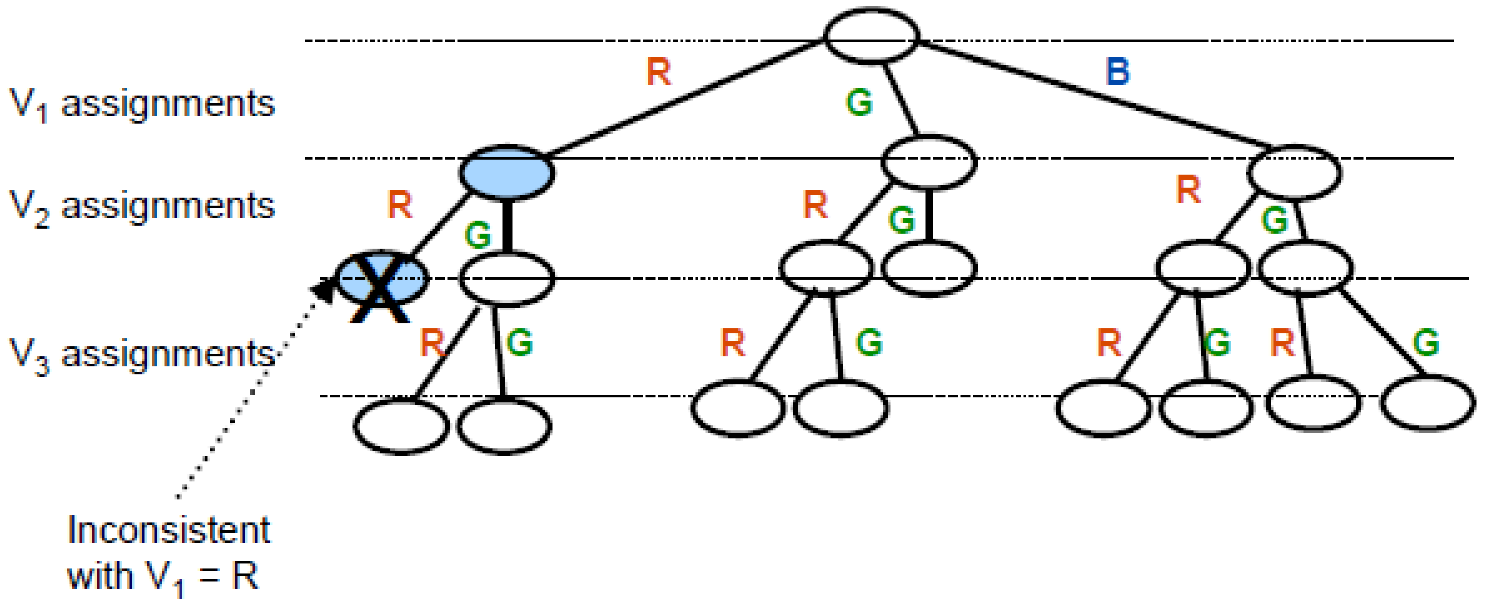
CSP as a Standard Search



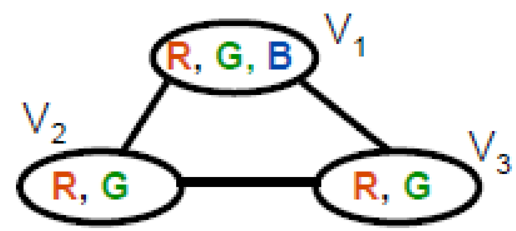
Depth First Search
can be performed



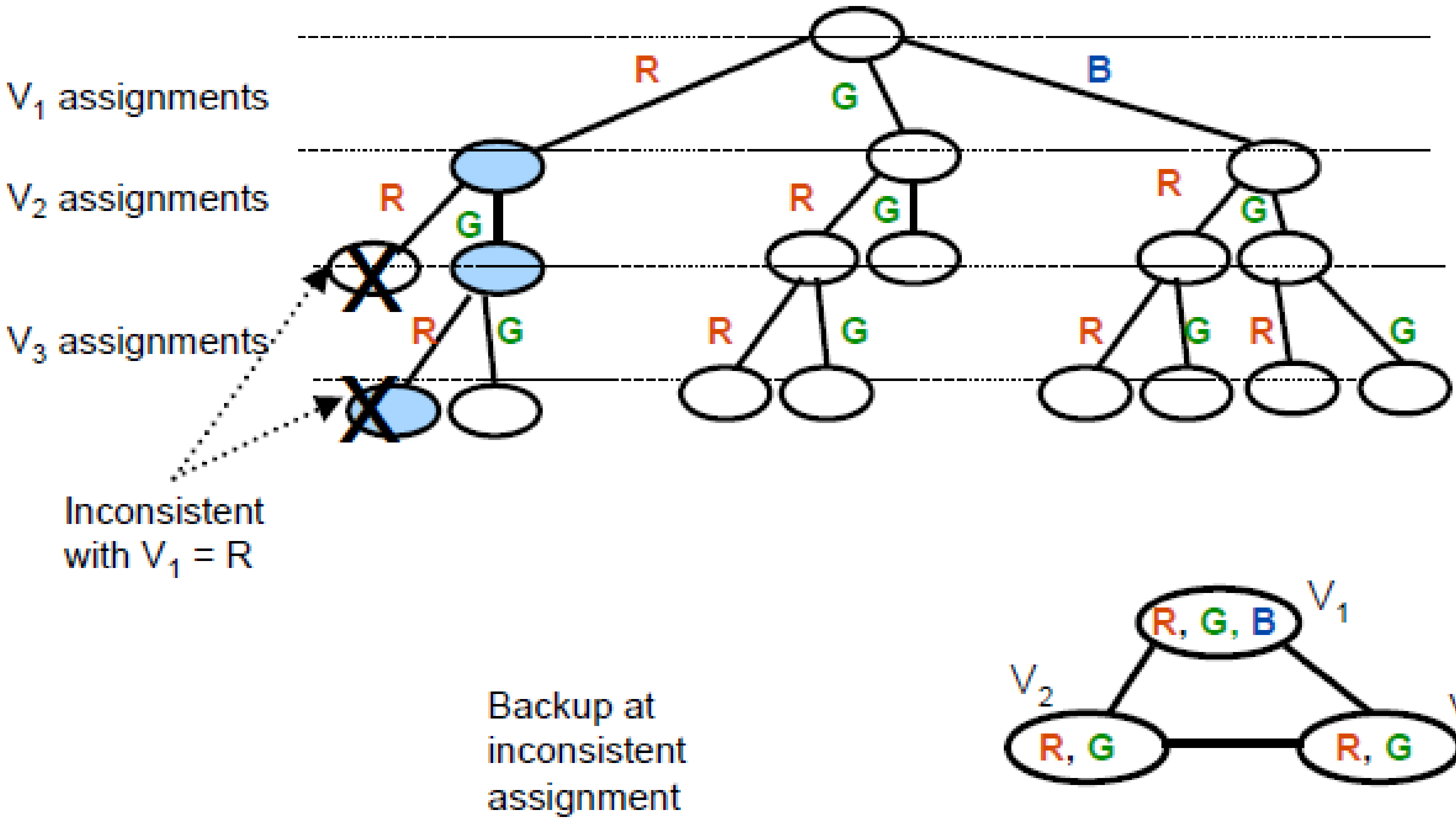
CSP as a Standard Search



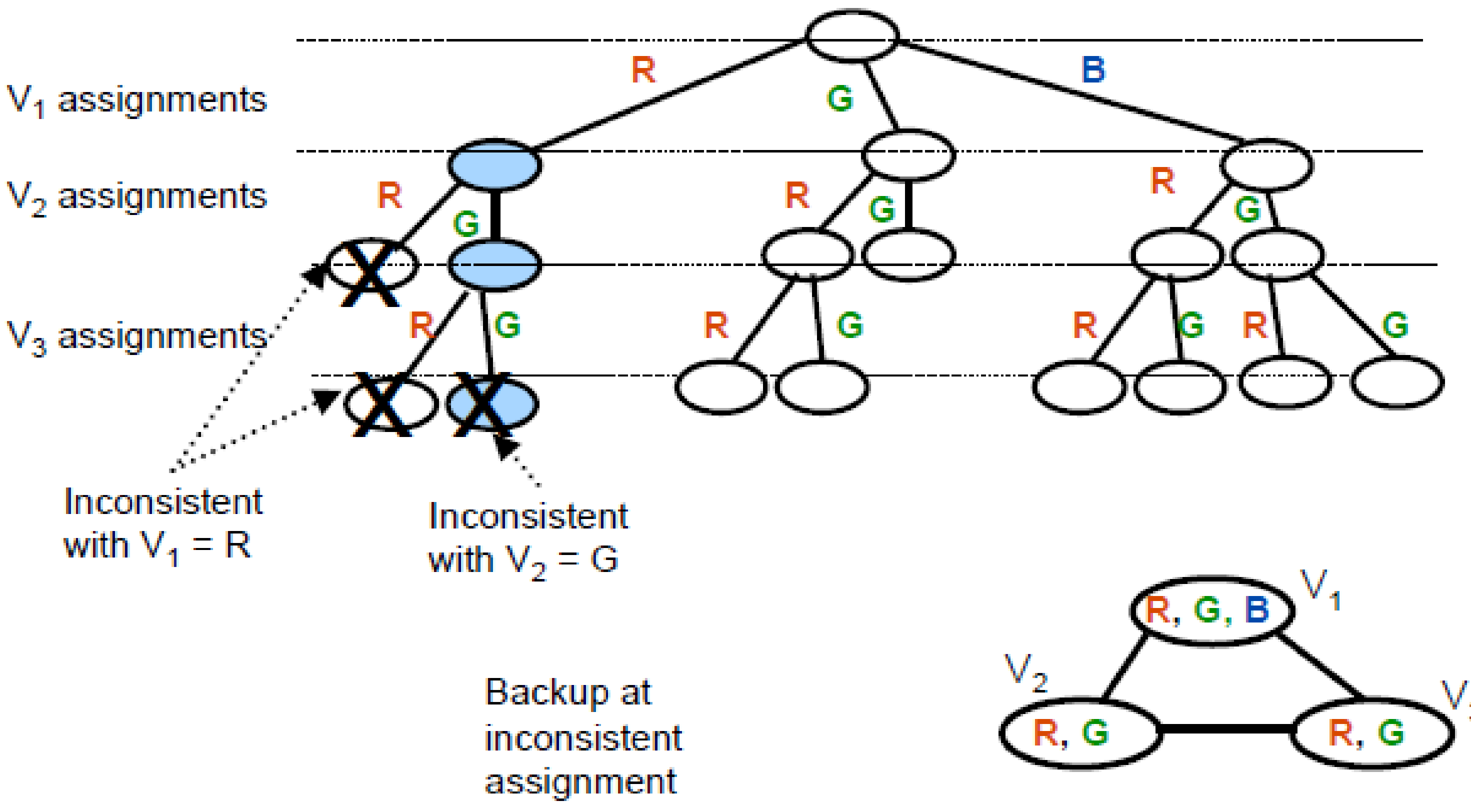
Backup at inconsistent assignment



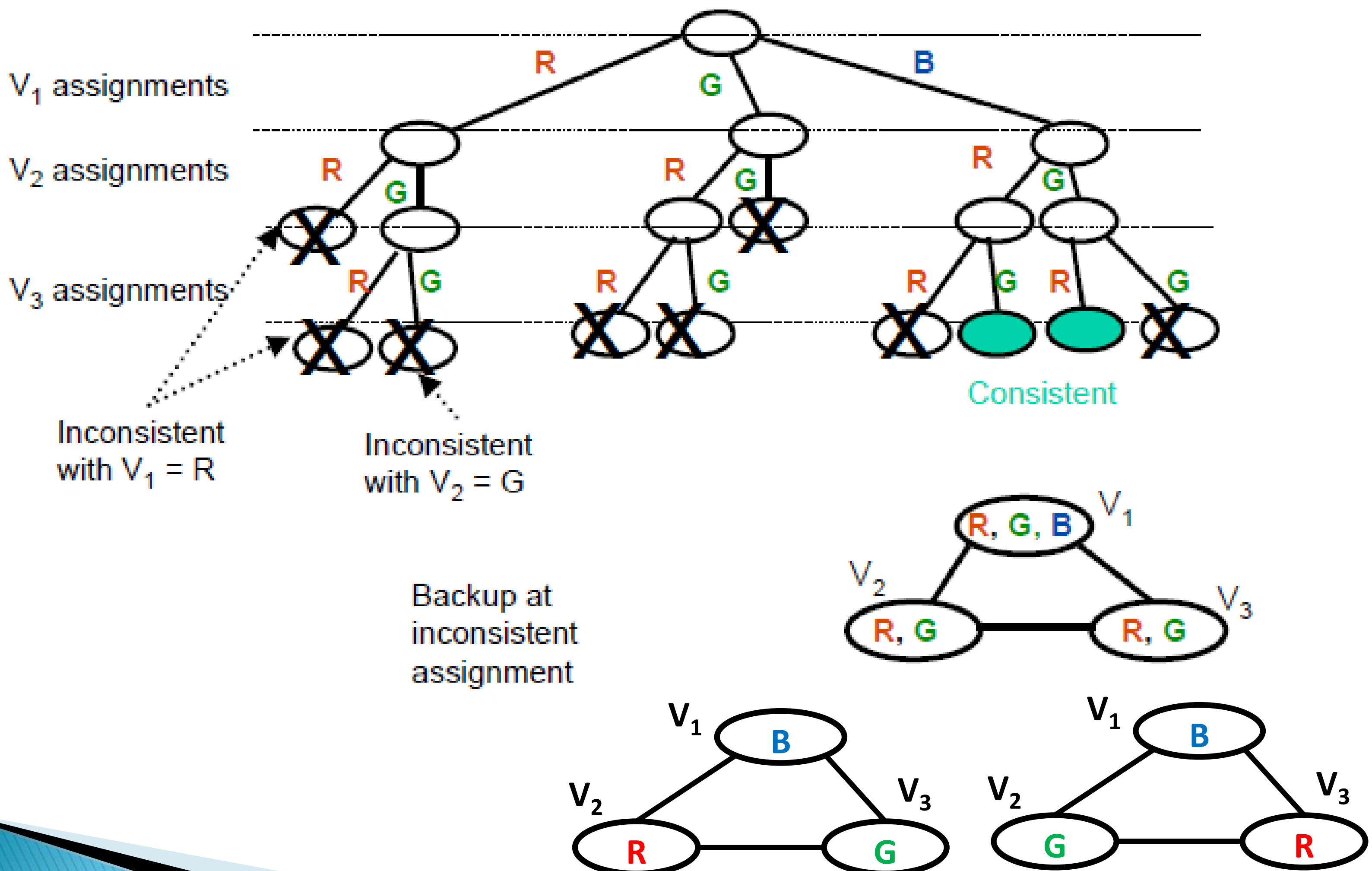
CSP as a Standard Search



CSP as a Standard Search

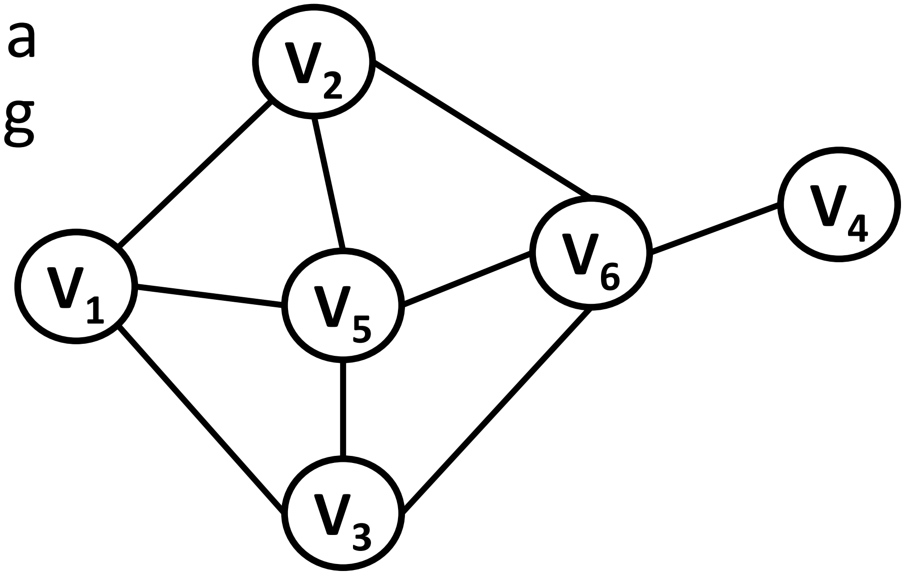


CSP as a Standard Search



CSP as a Standard Search

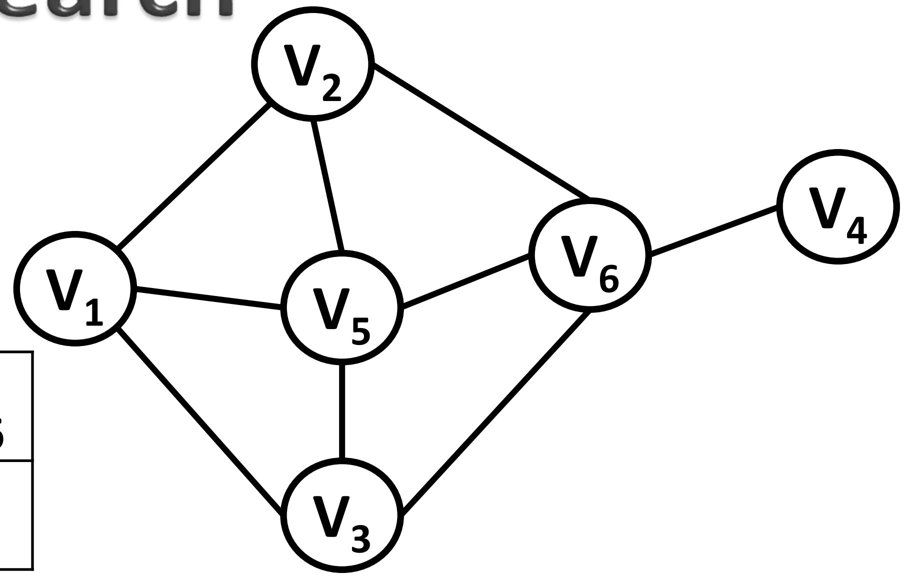
- **State:** assignment to k variables.
- ***Successor:*** The successor of a state is obtained by assigning a value to variable, keeping others unchanged.
- ***Start state:*** $(V_1 = ?, V_2 = ?, V_3 = ?, V_4 = ?, V_5 = ?, V_6 = ?)$
- ***Goal state:*** All variables **assigned colours (R,G,B)** with constraints satisfied.



CSP as a Standard Search

Depth First Search

can be performed



V_1	V_2	V_3	V_4	V_5	V_6
?	?	?	?	?	?

V_1	V_2	V_3	V_4	V_5	V_6
B	?	?	?	?	?

V_1	V_2	V_3	V_4	V_5	V_6
R	?	?	?	?	?

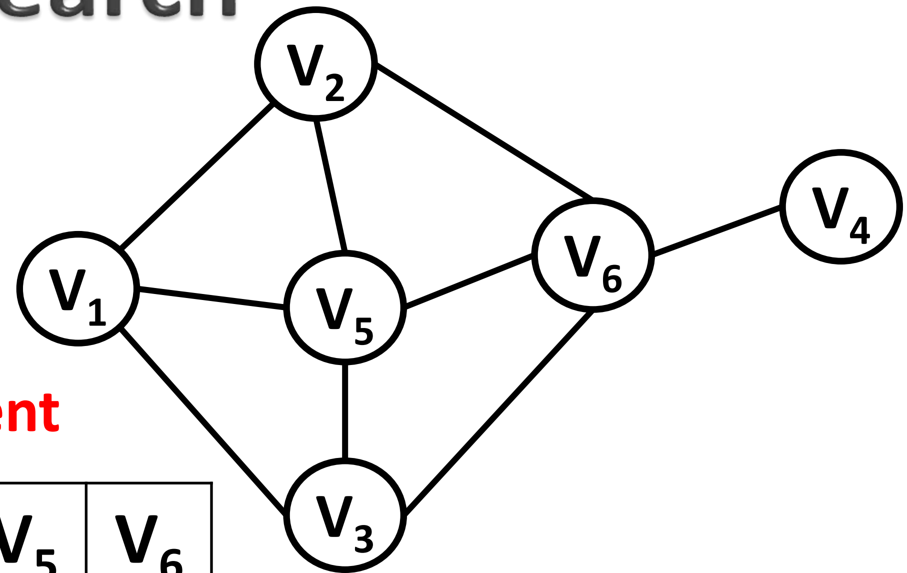
V_1	V_2	V_3	V_4	V_5	V_6
G	?	?	?	?	?

CSP as a Standard Search

V_1	V_2	V_3	V_4	V_5	V_6
B	?	?	?	?	?

Dumb Assignment

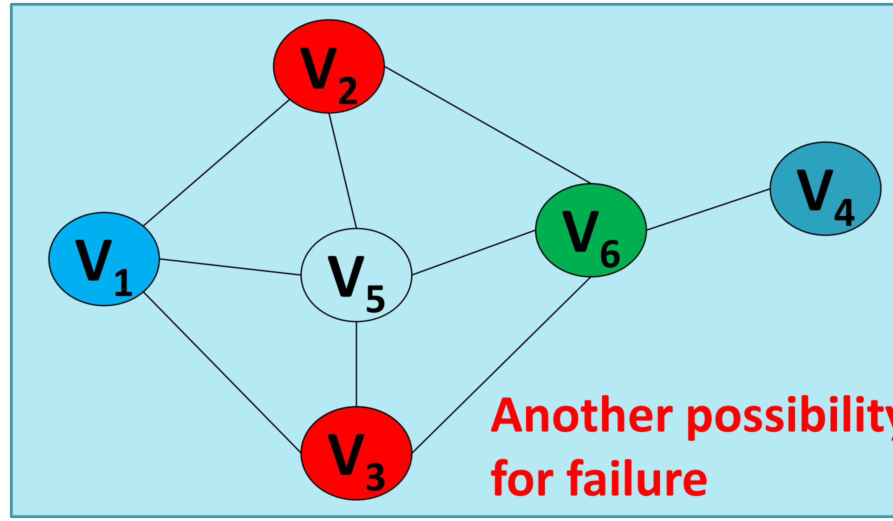
V_1	V_2	V_3	V_4	V_5	V_6
B	B	?	?	?	?



Recursively:

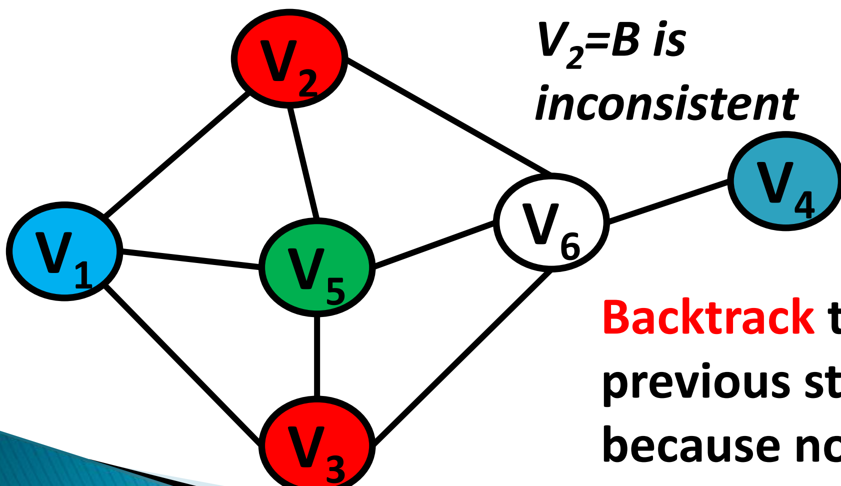
- For every possible value in ***D:***
  - Set the next unassigned variable in the successor to that value
- Evaluate the successor of current state with this variable assignment
- Stop as soon as a solution is found

V_1	V_2	V_3	V_4	V_5	V_6
B	?	?	?	?	?



V_1	V_2	V_3	V_4	V_5	V_6
B	B	?	?	?	?

V_2=B is inconsistent



V_1	V_2	V_3	V_4	V_5	V_6
B	R	?	?	?	?

V_1	V_2	V_3	V_4	V_5	V_6
B	R	R	?	?	?

V_1	V_2	V_3	V_4	V_5	V_6
B	R	R	B	G	?

Backtrack to the previous state because no valid assignment is for V_6

CSP as a Standard Search

- For every possible value for $\boldsymbol{x}$ *in* $\boldsymbol{D}$*:*
  - If assigning $\boldsymbol{x}$ *to the next unassigned variable* $\boldsymbol{V_{k+1}}$ ***does not violate*** *any constraint with the* $\boldsymbol{k}$ already assigned variables:
    - Set the value of the variable $\boldsymbol{V_{k+1}}$ to x
    - Evaluate the successors of the current state with this variable assignment
- If **no valid assignment** is found:
  - Backtrack to previous state
- Stop as soon as a solution is found

CSP as a Standard Search

- **Additional computation:** At each step, we *need to evaluate the constraints associated* with the current candidate assignment (**variable, value**).
- Uninformed search, **we can improve by predicting**:
  - What is the **effect of assigning a variable** on all of the other variables?
  - Which **variable should be assigned next** and in **which order** should the values be evaluated?
  - When a branch fails, how can we **avoid repeating the same mistake**?

Consistency

Node Consistency

- A single variable (corresponding to a node in the CSP network) is **node-consistent** if all the values in the variable's domain satisfy the variable's ***unary constraints***.

Example:

- In map-coloring problem where
  - SA dislike green,
  - the variable SA starts with domain $\{red\ , green, blue\}$,
  - we can make it node consistent by eliminating green,
  - SA with the reduced domain $\{red\ , blue\}$

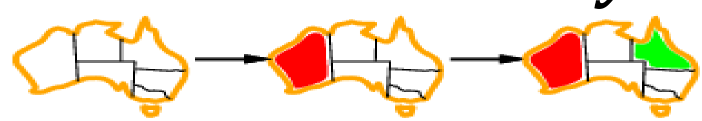
Arc Consistency

- A variable in a CSP is **arc-consistent** if every value in its domain satisfies the variable's ***binary constraints***.
- **Arc consistency** eliminates values from domain of variable that can never be part of a consistent solution.
- Directed arc $(\boldsymbol{V_i}, \boldsymbol{V_j})$ is arc consistent if

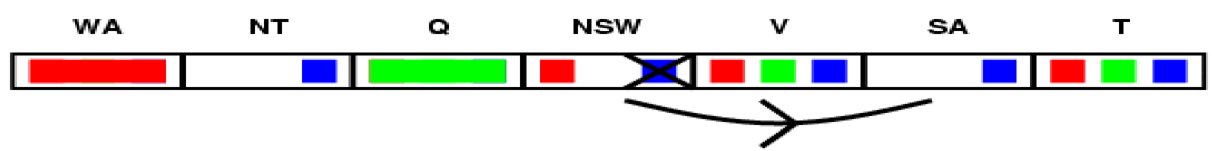
  $\forall \boldsymbol{x} \in \boldsymbol{D_i} \qquad \exists \boldsymbol{y} \in \boldsymbol{D_j}$

  such that $(\boldsymbol{x}, \boldsymbol{y})$ is allowed by constraint

- For every value $\boldsymbol{x}$ there is some allowed $\boldsymbol{y}$.



Inconsistant arc



Arc Consistency

Example:

- Consider the constraint $Y = X^2$
- The domain of both X and Y is the set of digits. We can write this constraint explicitly as

$$(X, Y), \{(0, 0), (1, 1), (2, 4), (3, 9))\}.$$

- To make X arc-consistent with respect to Y, we reduce X's domain to {0, 1, 2, 3}.
- If we also make Y arc-consistent with respect to X, then Y's domain becomes {0, 1, 4, 9}
- The whole CSP is arc-consistent.

Path Consistency

- **Path consistency** tightens the binary constraints by using implicit constraints that are inferred by looking at triples of variables.
- A two-variable set $\{X_i, X_j\}$ is path-consistent with respect to a third variable X_m if,
  - for every assignment $\{X_i = a, X_j = b\}$ consistent with the constraints on $\{X_i, X_j\}$,
  - there is an assignment to X_m that satisfies the constraints on $\{X_i, X_m\}$ and $\{X_m, X_j\}$.

k-consistency

- Stronger form of propagation
- A **CSP is k-consistent** if,
  - for any set of $k-1$ variables and
  - for any consistent assignment to those variables,
    - a consistent value can always be assigned to any k^{th} variable

1-consistency:

- given the empty set, we can make any set of one variable consistent: ***node consistency***.

k-consistency

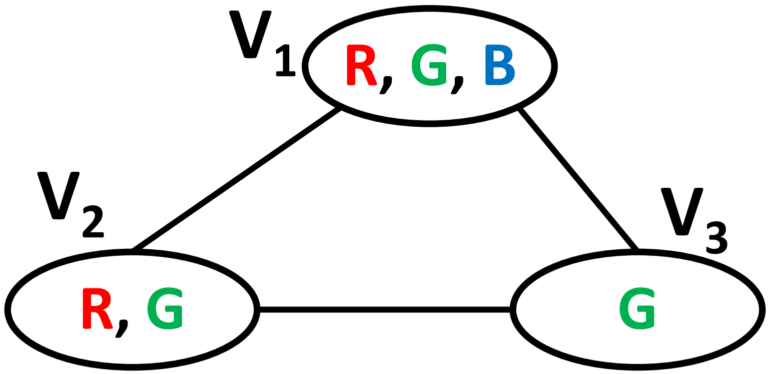
2-consistency:

- is the same as ***arc consistency***.
- Suppose a CSP with n nodes and make it strongly n-consistent (i.e., *strongly* k-consistent for $k = n$).

***k*-consistency:**

- A CSP is **strongly** k**-consistent** if it is k-consistent and is also (k − 1)-consistent, (k − 2)-consistent, . . . all the way down to 1-consistent

Arc Consistency



Different Colour Constraints

- Each undirected constraint arc is really **two directed constraint arcs**, the effects must be then from examining BOTH arcs.

Reading Material

- **Artificial Intelligence,** A Modern Approach

  Stuart J. Russell and Peter Norvig
  - **Chapter 6.**