

CS 461

Artificial Intelligence

Dr. Hashim Yasin

Logic

$$\alpha \models \beta \text{ if and only if } M(\alpha) \subseteq M(\beta)$$

Logic --- Valid

- ▶ A sentence is **valid** if it is **true** in all models (interpretations).
- ▶ **Valid Sentences:**

true, \neg false, $P \vee \neg P$

- ▶ Valid sentences are also known as **tautologies**—they are *necessarily true*.
- ▶ From the definition of entailment, we can derive the **deduction theorem**, for example

For any sentences α and β , $\alpha \models \beta$ if and only if the sentence $(\alpha \Rightarrow \beta)$ is valid.

Logic --- Satisfiability

- ▶ A sentence is **satisfiable** if it is **true** in, or satisfied by, some model.
- ▶ A sentence is **satisfiable** if and only if it's **true** in at least one interpretation.
- ▶ Satisfiable Sentence:
 - true, P , $\neg P$
- ▶ A sentence is **unsatisfiable** if and only if its truth value is **false** in all interpretation.
 - \neg true, false, $P \wedge \neg P$

Examples

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \rightarrow Q$	$Q \rightarrow P$	$P \leftrightarrow Q$
f	f	t	f	f	t	t	t
f	t	t	f	t	t	f	f
t	f	f	f	t	f	t	f
t	t	f	t	t	t	t	t

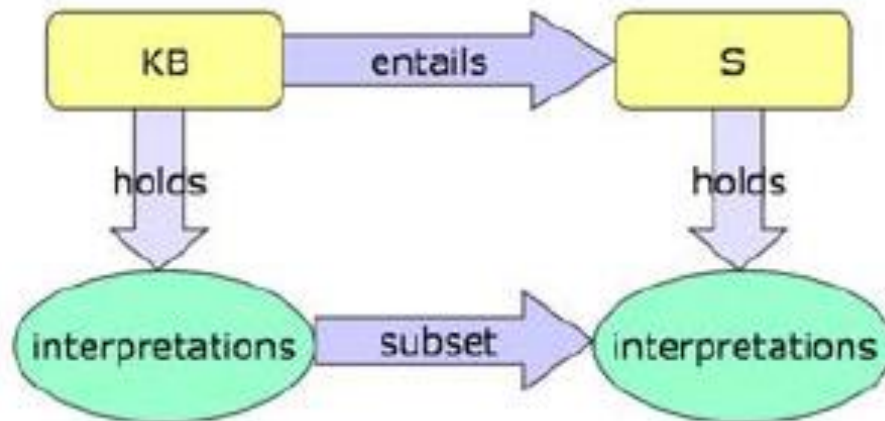
Sentences	Valid/ Satisfiability?	Interpretation
Smoke \rightarrow Smoke	Valid	
Smoke $\vee \neg$ Smoke	Valid	
Smoke \rightarrow Fire	Satisfiable	Smoke = t Fire = f, "smoke implies fire"
$(s \rightarrow f) \rightarrow (\neg s \rightarrow \neg f)$	Satisfiable	$s = f, f = t$ $(s \rightarrow f) = t, (\neg s \rightarrow \neg f) = f$ "Smoke implies fire implies not smoke implies not fire."
$(s \rightarrow f) \rightarrow (\neg f \rightarrow \neg s)$	Valid	Contrapositive "smoke implies fire implies not fire implies not smoke"

Logic --- Satisfiability

- ▶ The problem of satisfiability of a sentence is very **related** to **constraint satisfaction problem**.
- ▶ Have to **find a legal assignment** which can satisfy the constraint,
- ▶ Similarly, we have to *find interpretation in such as way that sentence holds that interpretation*.
- ▶ Can be solved the problem using **brute-force method** of **enumerating all possible interpretations**.

Entailment

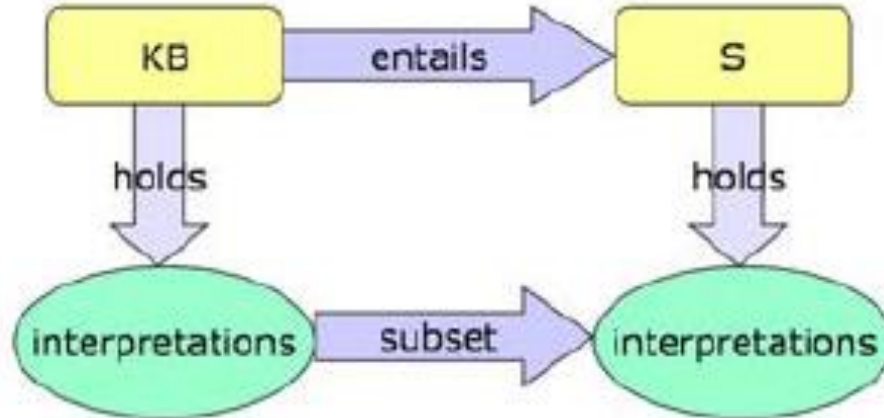
- ▶ Entailment that signifies a **relationship** between a **knowledge base** and **another sentence**.
- ▶ If every interpretation that **satisfies the *KB*** also **satisfies the conclusion**, we'll say that the ***KB* "entails"** the conclusion.



$$\alpha \models \beta \text{ if and only if } M(\alpha) \subseteq M(\beta)$$

Entailment

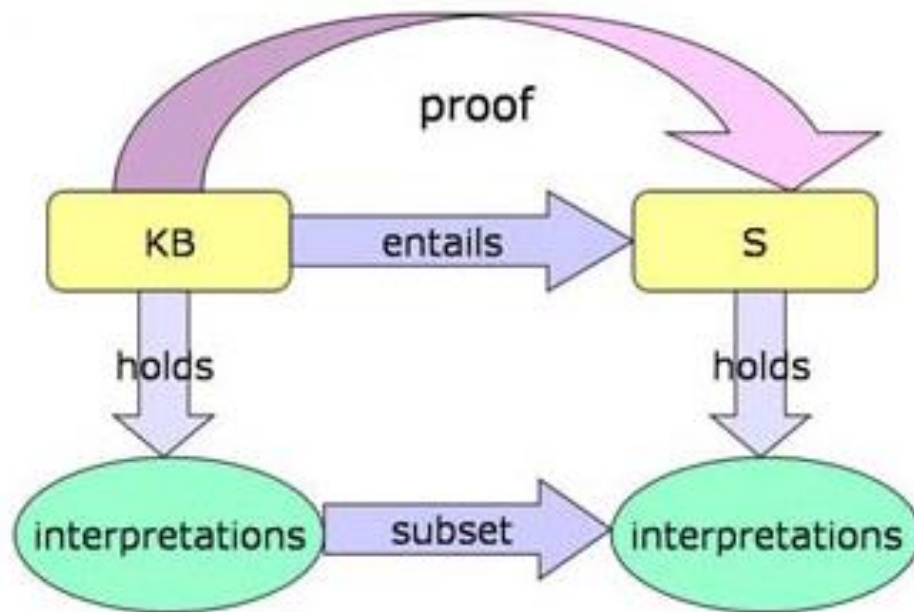
- ▶ Enumerate all interpretations
- ▶ Those interpretations are selected where all elements of ***KB*** are true.
- ▶ Check if ***S*** is true for all those interpretations.



Too many interpretations in general!

Entailment and **Proof**

- ▶ Proof is a way to test wheather a *KB* entails a sentence without enamurating too many interpretations.
- ▶ Proof is basically a sequences of sentences.
- ▶ Proof is a chain of conclusions that leads to the desired goal.



Inference

- ▶ If an inference algorithm ***i*** can derive ***α*** from ***KB***, we write

$$KB \vdash_i \alpha$$

- ▶ which is pronounced “***α* is derived from *KB* by *i***” or “***i* derives *α* from *KB*.**”

Inference Rules and Proofs

Some of Inference rules can be described as,

Modus Ponens

- ▶ If the sentences $\alpha \rightarrow \beta$ and α are known to be true, then *modus ponens* lets us infer β .

$$\frac{\alpha \rightarrow \beta \quad \alpha}{\beta}$$

Modus Tollens

- ▶ Under the inference rule *modus tollens*, if $\alpha \rightarrow \beta$ is known to be true and β is known to be false, we can infer $\neg \alpha$.

$$\frac{\alpha \rightarrow \beta \quad \neg \beta}{\neg \alpha}$$

Inference Rules and Proofs

And Elimination:

- ▶ And elimination allows us to infer the truth of either of the conjuncts from the truth of a conjunctive sentence. For instance, $\alpha \wedge \beta$ lets us conclude α and β are true.

And Introduction:

- ▶ And introduction lets us infer the truth of a conjunction from the truth of its conjuncts. For instance, if α and β are true, then $\alpha \wedge \beta$ is true.

$$\frac{\alpha \wedge \beta}{\alpha}$$

$$\frac{\alpha \quad \beta}{\alpha \wedge \beta}$$

Inference Rules and Proofs

- ▶ Inference rules can be applied to derive a proof.

$$\frac{\alpha \rightarrow \beta \quad \alpha}{\beta}$$

Modus
ponens

$$\frac{\alpha \rightarrow \beta \quad \neg \beta}{\neg \alpha}$$

Modus
tolens

$$\frac{\alpha \quad \beta}{\alpha \wedge \beta}$$

And-
introduction

$$\frac{\alpha \wedge \beta}{\alpha}$$

And-
elimination

Example

- Consider there are 3 sentences in our knowledge base, $P \wedge Q$, $P \rightarrow R$ and $(Q \wedge R) \rightarrow S$. We have to prove S .

Step	Formula	Derivations
1	$P \wedge Q$	Given
2	$P \rightarrow R$	Given
3	$(Q \wedge R) \rightarrow S$	Given

Example

- Consider there are 3 sentences in our knowledge base, $P \wedge Q$, $P \rightarrow R$ and $(Q \wedge R) \rightarrow S$. **We have to prove S.**

Step	Formula	Derivations
1	$P \wedge Q$	Given
2	$P \rightarrow R$	Given
3	$(Q \wedge R) \rightarrow S$	Given
4	P	1 And-Elimination
5	R	2,4 Modes Ponens
6	Q	1 And-Elimination
7	$Q \wedge R$	6,5 And-Introduction
8	S	3,7 Modes Ponens

Propositional Resolution



Propositional Resolution

- ▶ Resolution is one method for **automated theorem proving**.
- ▶ It helps logical agents **to reason** about the world.
- ▶ It helps logical agents **to prove new theorems**, and therefore helps them **to add to their knowledge**.
- ▶ Resolution requires all sentences to be first written in a special form - **conjunctive normal form**.

Propositional Resolution

- ▶ Input a knowledge base and an expression.
- ▶ It **negates the expression**, adds that to the knowledge base, and finds a **contradiction** if one exists
- ▶ If it **finds** a contradiction, then the negated statement is **false**.
- ▶ Therefore, the original statement **must be true**.

Conjunctive Normal Form

- ▶ A series of “conjunctions” (clauses joined together by “ \wedge ”).

$$\underbrace{(A \vee B \vee \neg C)}_{\text{clause}} \wedge \underbrace{(B \vee D)}_{\text{clause}} \wedge (\neg A) \wedge (B \vee C)$$

clauses

- ▶ A, B, $\neg C$ are **literals**.
- ▶ Inside the brackets, we only have “ \vee (**OR**)”, “ \neg (**NOT**)” symbols.
- ▶ *There must be no “implies” (\rightarrow) symbols anywhere.*

Conjunctive Normal Form

- ▶ Resolution algorithm 'resolves' clauses
- ▶ Each pair of clauses that contains complementary literals is resolved. Complementary literals have the property that one negates the other
 - $P, \neg P$
 - the clause $B_{1,1} \vee \neg B_{1,1} \vee P_{1,2}$ is equivalent to $True \vee P_{1,2}$ which is equivalent to $True$. Deducing that $True$ is true is not helpful.
- ▶ Therefore, *any clause in which two complementary literals appear can be discarded.*
- ▶ Each clause is a requirement that must be satisfied.
- ▶ It can be satisfied in multiple ways

Converting into CNF

Converting into CNF

- ▶ The first step is **to eliminate single and double arrows** using their definitions.

$$(\alpha \Leftrightarrow \beta) \equiv ((\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)) \quad \text{biconditional elimination}$$

$$(\alpha \Rightarrow \beta) \equiv (\neg\alpha \vee \beta) \quad \text{implication elimination}$$

- ▶ The next step is to **drive in negation**, De Morgan Laws

$$\neg(\alpha \wedge \beta) \equiv (\neg\alpha \vee \neg\beta) \quad \text{De Morgan}$$

$$\neg(\alpha \vee \beta) \equiv (\neg\alpha \wedge \neg\beta) \quad \text{De Morgan}$$

- ▶ The third step is to **distribute OR over AND**;

$$(\alpha \vee (\beta \wedge \gamma)) \equiv ((\alpha \vee \beta) \wedge (\alpha \vee \gamma)) \quad \text{distributivity of } \vee \text{ over } \wedge$$

Converting into CNF ... Example

$$(A \vee B) \rightarrow (C \rightarrow D)$$

1. Eliminate arrows

$$\neg(A \vee B) \vee (\neg C \vee D)$$

2. Drive in negations

$$(\neg A \wedge \neg B) \vee (\neg C \vee D)$$

3. Distribute

$$(\neg A \vee \neg C \vee D) \wedge (\neg B \vee \neg C \vee D)$$

Resolution

- ▶ Resolution Rule:

Resolution rule:

$$\frac{\alpha \vee \beta \quad \neg\beta \vee \gamma}{\alpha \vee \gamma}$$

- ▶ Wumpus-world Example

$$\frac{P_{1,1} \vee P_{3,1}, \quad \neg P_{1,1} \vee \neg P_{2,2}}{P_{3,1} \vee \neg P_{2,2}}$$

Resolution

Propositional Resolution:

- ▶ Convert all sentences into CNF
- ▶ Negate the desired conclusion (converted to CNF)
- ▶ Apply resolution rule until either
 1. Derive false (contradiction)
 - The conclusion follows from the axioms
 2. Can not apply anymore,
 - The conclusion can not be proved from axioms

Propositional Resolution is sound and complete.

Resolution Example

- Consider we are given "P or Q", "P implies R" and "Q implies R". We would like to conclude R from these three axioms.

1	$P \vee Q$
2	$P \rightarrow R$
3	$Q \rightarrow R$

Steps	Formula	Derivations
1	$P \vee Q$	Given
2	$\neg P \vee R$	Given
3	$\neg Q \vee R$	Given

Resolution Example

- Consider we are given "P or Q", "P implies R" and "Q implies R". We would like to conclude R from these three axioms

1	$P \vee Q$
2	$P \rightarrow R$
3	$Q \rightarrow R$

$$\begin{array}{l}
 \text{false} \vee R \\
 \neg R \vee \text{false} \\
 \hline
 \text{false} \vee \text{false}
 \end{array}$$

Steps	Formula	Derivations
1	$P \vee Q$	Given
2	$\neg P \vee R$	Given
3	$\neg Q \vee R$	Given
4	$\neg R$	Negated Conclusion
5	$Q \vee R$	1,2 Resolution Rule
6	$\neg P$	2,4
7	$\neg Q$	3,4
8	R	5,7
9	.	4,8

Resolution Example

- Consider we are given "P or Q", "P implies R" and "Q implies R". We would like to conclude R from these three axioms

1	$P \vee Q$
2	$P \rightarrow R$
3	$Q \rightarrow R$

$$\begin{array}{l}
 \text{false} \vee R \\
 \neg R \vee \text{false} \\
 \hline
 \text{false} \vee \text{false}
 \end{array}$$

Steps	Formula	Derivations
1	$P \vee Q$	Given
2	$\neg P \vee R$	Given
3	$\neg Q \vee R$	Given
4	$\neg R$	Negated Conclusion
5	$Q \vee R$	1,2 Resolution Rule
6	$\neg P$	2,4
7	$\neg Q$	3,4
8	R	5,7
9	.	4,8

Resolution Example 2

- ▶ Consider we are given following table. We would like to conclude R from these three axioms

1	$(P \rightarrow Q) \rightarrow Q$
2	$(P \rightarrow P) \rightarrow R$
3	$(R \rightarrow S) \rightarrow \neg(S \rightarrow Q)$

Resolution Example 2

- Consider we are given following table. We would like to conclude R from these three axioms

1	$(P \rightarrow Q) \rightarrow Q$
2	$(P \rightarrow P) \rightarrow R$
3	$(R \rightarrow S) \rightarrow \neg(S \rightarrow Q)$

Step	Formula & Derivations
1	$(P \rightarrow Q) \rightarrow Q$
2	$(\neg P \vee Q) \rightarrow Q$
3	$\neg(\neg P \vee Q) \vee Q$
4	$(P \wedge \neg Q) \vee Q$
5	$(P \vee Q) \wedge (\neg Q \vee Q)$
6	$(P \vee Q)$

Resolution Example 2

- Consider we are given following table. We would like to conclude R from these three axioms

1	$(P \rightarrow Q) \rightarrow Q$
2	$(P \rightarrow P) \rightarrow R$
3	$(R \rightarrow S) \rightarrow \neg(S \rightarrow Q)$

Step	Formula & Derivations
1	$(P \rightarrow P) \rightarrow R$
2	$(\neg P \vee P) \rightarrow R$
3	$\neg(\neg P \vee P) \vee R$
4	$(P \wedge \neg P) \vee R$
5	$(P \vee R) \wedge (\neg P \vee R)$

Resolution Example 2

- Consider we are given following table. We would like to conclude R from these three axioms

1	$(P \rightarrow Q) \rightarrow Q$
2	$(P \rightarrow P) \rightarrow R$
3	$(R \rightarrow S) \rightarrow \neg(S \rightarrow Q)$

Step	Formula & Derivations
1	$(R \rightarrow S) \rightarrow \neg(S \rightarrow Q)$
2	$(\neg R \vee S) \rightarrow \neg(\neg S \vee Q)$
3	$\neg(\neg R \vee S) \vee \neg(\neg S \vee Q)$
4	$(R \wedge \neg S) \vee (S \wedge \neg Q)$
5	$(R \vee S) \wedge (R \vee \neg Q) \wedge (\neg S \vee S) \wedge (\neg S \vee \neg Q)$
6	$(R \vee S) \wedge (R \vee \neg Q) \wedge (\neg S \vee \neg Q)$

Resolution Example 2

1	$(P \rightarrow Q) \rightarrow Q$
2	$(P \rightarrow P) \rightarrow R$
3	$(R \rightarrow S) \rightarrow \neg(S \rightarrow Q)$

Step	Formula	Derivations
1	$(P \vee Q)$	
2	$(P \vee R)$	
3	$(\neg P \vee R)$	
4	$(R \vee S)$	
5	$(R \vee \neg Q)$	
6	$(\neg S \vee \neg Q)$	
7	$\neg R$	Negation
8	S	4,7
9	$\neg Q$	6,8
10	P	1,9
11	R	3,10
12	.	7,11

Proof Strategies

- ▶ **Unit Preference:** Prefer a resolution step involving an unit clause (*clause with one literal*)
 - ▶ **Produce a shorter clause** which is good since we are trying to produce a **zero-length clause**, that is, a contradiction.
- ▶ **Set of support:** Choose *a resolution involving the negated goal* or any clause derived from negated goal
 - We're trying to produce a contradiction that follows from the negated goal, so these are “relevant” clauses.
 - If a contradiction exists one can find it using the set-of-support strategy.

Reading Material

- ▶ **Artificial Intelligence, A Modern Approach**
Stuart J. Russell and Peter Norvig
 - Chapter 7.

