CS 461
Artificial Intelligence

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A* Search

A* Search

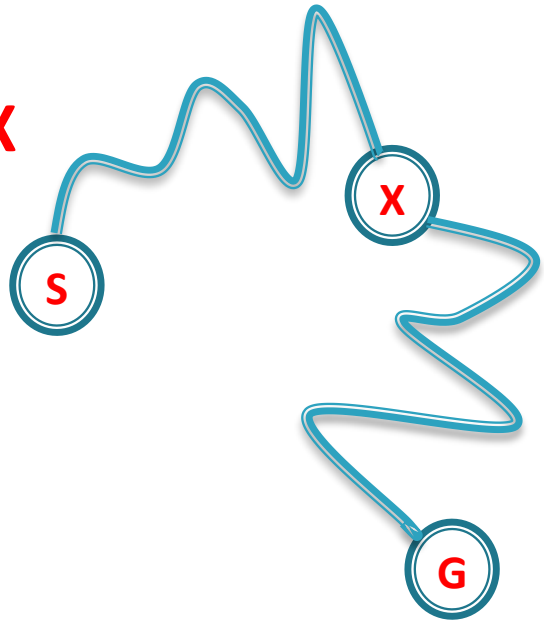
- We **can bias Uniform-cost search** to find the shortest path to the goal.
- In fact, we are interested in by using a **heuristic function $h(n)$** which is an estimate of the distance from a state to the goal.
- It evaluates nodes by combining $g(n)$, the cost to reach the node, and $h(n)$, the cost to get from the node to the goal:

$$f(n) = g(n) + h(n)$$

Optimality Principle

Dynamic Programming Optimality Principle

- *Given that **path length is additive**, the shortest path from **S** to **G** via a state **X** is made up of the shortest path from **S** to **X** and the shortest path from **X** to **G**.*
- **Only need to keep the shortest path**, discard all other paths.
- Once we expand path to state **X**, we do not need to expand any other path to state **X**.



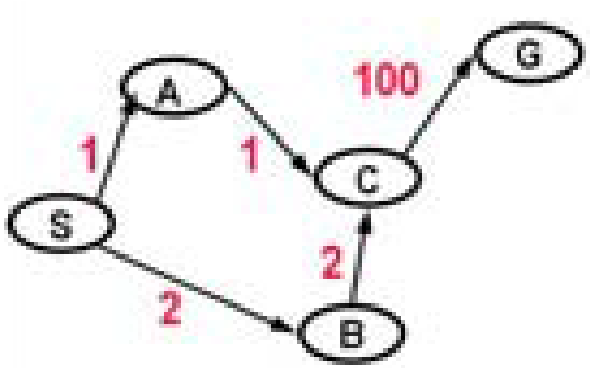
A* Search without Expanded List

A* Search

(without Expanded List)

- Pick best (by **path length + heuristic value**) element of Q
- Add path extensions to Q

	Q
1	(90 S)
2	**(101 A S)** **(3 B S)**
3	**(94 C B S)** (101 A S)
4	(101 A S) **(104 G C B S)**
5	**(92 C A S)** (104 G C B S)
6	**(102 G C A S)** (104 G C B S)



- **Blue Color represents added paths**
- Underline paths are selected for extension

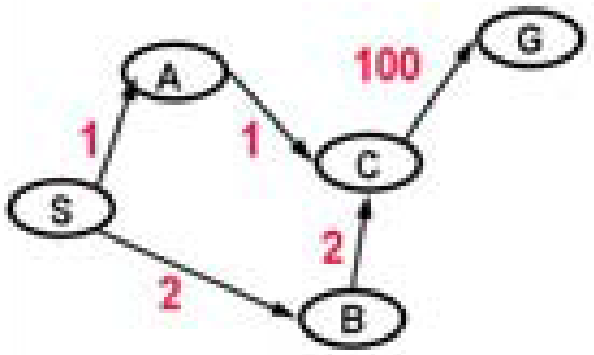
A* Search with Expanded List

A* Search

(with strict Expanded List)

- Pick best (by **path length + heuristic value**) element of Q
- Add path extensions to Q

	Q	Expanded
1	(90 S)	S
2	**(101 A S) (3 B S)**	S, B
3	**(94 C B S)** (101 A S)	S, B, C
4	(101 A S) **(104 G C B S)**	S, B, C, A
5	**(104 G C B S)** ~~(92 C A S)~~	S, B, C, A, **G**



- **Blue Color represents added paths**
- Underline paths are selected for extension.

But the shortest path to goal is **G,C,A,S** and the cost is 102.

A* Search

$$h(n) - h(n') \leq c(n, a, n')$$
$$h(n) \leq c(n, a, n') + h(n')$$

(with strict Expanded List)

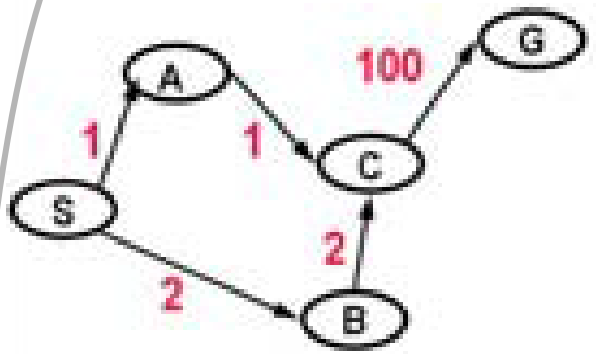
- Pick best (by **path length + heuristic value**) element of Q
- Add path extensions to Q

	Q	Expanded
1	(90 S)	S
2	**(101 A S)** **(3 B S)**	S, B
3	**(94 C B S)** (101 A S)	S, B, C
4	(101 A S) **(104 G C B S)**	S, B, C, A
5	**(104 G C B S)** ~~(92 C A S)~~	G, S, B, C, A

- **Blue Color represents added paths**
- Underline paths are selected for extension.

$$h(S) - h(B) \leq c(S, B)$$
$$90 - 1 \leq 2$$
$$89 \leq 1$$

$$h(S) - h(B) \leq c(S, B)$$
$$90 - 88 \leq 2$$



Heuristic Values

A=100 C=90 S=90

B=88 B=1 C=99 G=0

$$h(A) - h(C) \leq c(A, C)$$
$$100 - 90 \leq 1$$
$$10 \leq 1$$

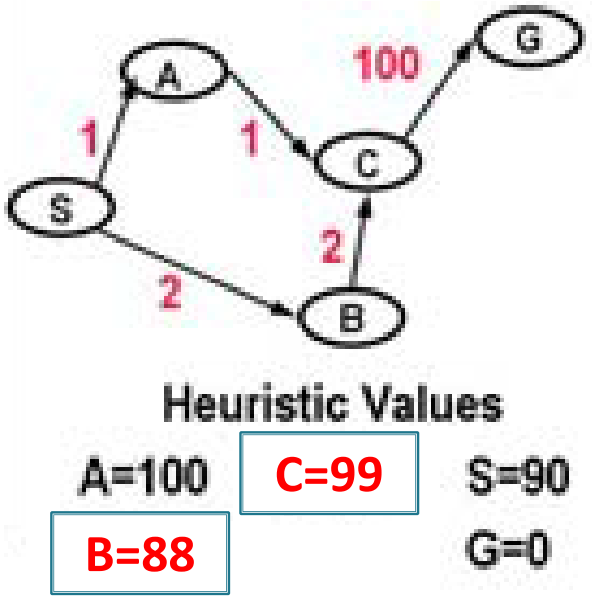
$$h(A) - h(C) \leq c(A, C)$$
$$100 - 99 \leq 1$$

A* Search

(with strict Expanded List)

- Pick best (by **path length + heuristic value**) element of Q
- Add path extensions to Q
- **Heuristic is admissible and consistent**

	Q	Expanded
1	(90 S)	S
2	(101 A S) (90 B S)	S, B
3	(103 C B S) (101 A S)	S, B, A
4	(101 C A S) ~~(103 C B S)~~	S, B, A, C,
5	(102 G C A S)	S, B, A, C, G



- **Blue Color represents added paths**
- Underline paths are selected for extension

A* Search with Pathmax

A* Search

(with pathmax and Expanded List)

$$h(n) - h(n') \leq c(n, a, n')$$
$$h(n) \leq c(n, a, n') + h(n')$$

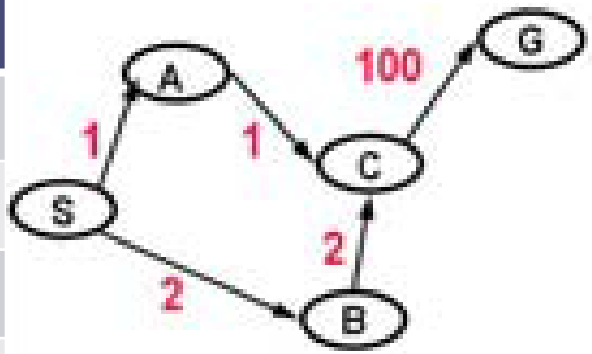
$$h(S) - h(A) \leq c(S, a, A)$$
$$90 - 100 \leq 1$$
$$-10 \leq 1$$

$$h(S) - h(B) \leq c(S, a, B)$$
$$90 - 1 \leq 2$$
$$89 \leq 1$$

- Pick best (**path length + heuristic value**) element of Q
- Add path extensions to Q
- **Heuristic is admissible but not consistent**

	Q	Expanded
1	(90 S)	[S, 90]
2	(101 A S) (90 B S)	[B, 90], S

Pathmax changes f value from 3 to 90



Heuristic Values

A=100 C=90 S=90

B=1 G=0

B=88

A* Search

$h(n) - h(n') \leq c(n, a, n')$

$h(n) \leq c(n, a, n') + h(n')$

$h(B) - h(C) \leq c(B, a, C)$

$88 - 90 \leq 2$

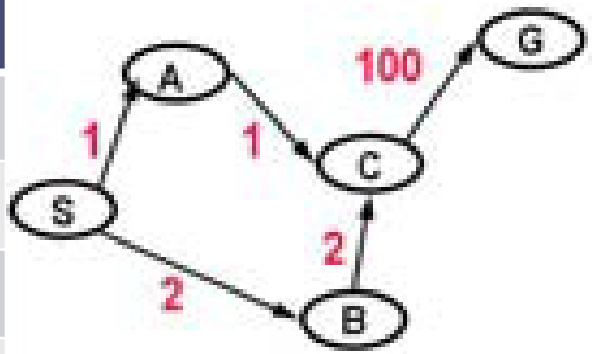
$-2 \leq 2$

(with pathmax and Expanded List)

- Pick best (**path length + heuristic value**) element of Q
- Add path extensions to Q
- **Heuristic is admissible but not consistent**

	Q	Expanded
1	(90 S)	[S, 90]
2	(101 A S) (90 B S)	[B, 90], S
3	(94 C B S) (101 A S)	[C, 94], B, S

Pathmax changes f value from 3 to 90



Heuristic Values

A=100 C=90 S=90

B=1 G=0

B=88

A* Search

$$h(n) - h(n') \leq c(n, a, n')$$
$$h(n) \leq c(n, a, n') + h(n')$$

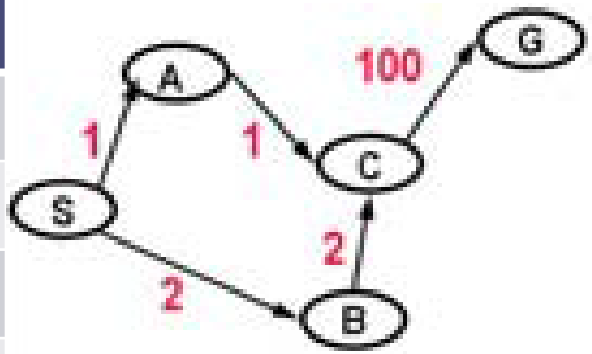
$$h(C) - h(G) \leq c(B, a, C)$$
$$90 - 0 \leq 100$$
$$90 \leq 100$$

(with pathmax and Expanded List)

- Pick best (by **path length + heuristic value**) element of Q
- Add path extensions to Q
- **Heuristic is admissible but not consistent**

	Q	Expanded
1	(90 S)	[S, 90]
2	**(101 A S)** **(90 B S)**	[B, 90], S
3	**(94 C B S)** (101 A S)	[C, 94], B, S
4	(101 A S) **(104 G C B S)**	[A, 101], B, S

Pathmax changes f value from 3 to 90



Heuristic Values

A=100 C=90 S=90

B=1 G=0

B=88

A* Search

$$h(n) - h(n') \leq c(n, a, n')$$
$$h(n) \leq c(n, a, n') + h(n')$$

$$h(A) - h(C) \leq c(B, a, C)$$
$$100 - 90 \leq 1$$
$$10 \leq 1$$

(with pathmax and Expanded List)

- Pick best (**path length + heuristic value**) element of Q
- Add path extensions to Q
- **Heuristic is admissible but not consistent**

	Q	Expanded
1	(90 S)	[S, 90]
2	**(101 A S)** **(90 B S)**	[B, 90], S
3	**(94 C B S)** (101 A S)	[C, 94], B, S
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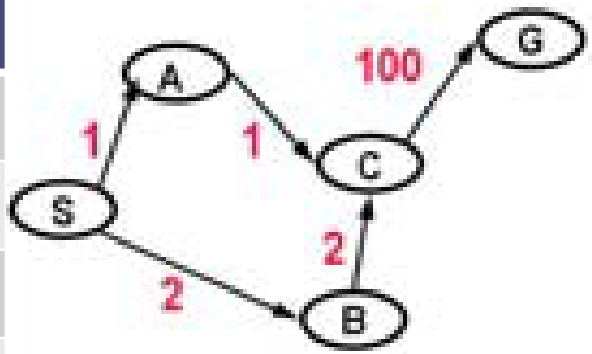


Heuristic Values

A=100 C=90 S=90

B=1 **C=99** G=0

B=88

Pathmax changes f value from 3 to 90

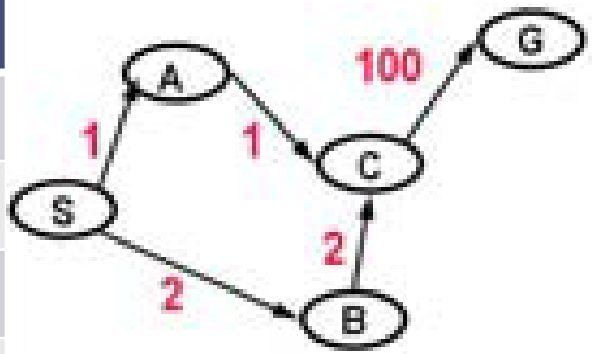
Pathmax changes f value from 92 to 101, node is added to Q even though C is on expanded list (and C is removed from expanded).

A* Search

(with pathmax and Expanded List)

- Pick best (by **path length + heuristic value**) element of Q
- Add path extensions to Q
- **Heuristic is admissible but not consistent**

	Q	Expanded
1	(90 S)	[S, 90]
2	**(101 A S)** **(90 B S)**	[B, 90], S
3	**(94 C B S)** (101 A S)	[C, 94], B, S
4	(101 A S) **(104 G C B S)**	[A, 101], B, S
5	**(101 C A S)** (104 G C B S)	[C, 101], A, B, S
6	**(102 G C A S)** ~~(104 G C B S)~~	[G, 0], A, B, S, C



Heuristic Values

A=100 C=90 S=90

B=1 **C=99** G=0

B=88

Pathmax changes f value from 3 to 90

Pathmax changes f value from 92 to 101, node is added to Q even though C is on expanded list (and C is removed from expanded).

A* Search

(with pathmax and expanded List)

- **In step 2**, when we generate a path to B, we need to modify the value of $\boldsymbol{h(B)}$ drastically,
  - The estimate at S is 90 and the edge length to B is 2, then the estimate at B is:

$$\boldsymbol{h(S) - h(B) \leq c(S,B)}$$
$$\boldsymbol{90 - ? \leq 2}$$

So, the lowest consistent value for $\boldsymbol{h(B)}$ is 88, and $\boldsymbol{f(B)}$ becomes,

$$\begin{aligned} \boldsymbol{f(B)} &= \boldsymbol{h(B) + c(S,B)} \\ &= \boldsymbol{88 + 2} \\ &= \boldsymbol{90\ (not\ 3).} \end{aligned}$$

A* Search

- **Complete:** Yes
  - A* is complete and optimal, provided that $h(n)$ is admissible (for TREE-SEARCH) or consistent (for GRAPH-SEARCH).
- **Optimal:** Yes, A* is **optimally efficient** for any given consistent heuristic.
- **Time:** Exponential
  - The complexity results depend very strongly on the **assumptions** made about the state space.
  - The complexity of A∗ often makes it **impractical** to insist on finding an optimal solution.
- **Space:** Keeps all nodes in memory,
  - A* usually runs out of space long before it runs out of time.

A* Search

- For problems with constant step costs, the growth in run time as a function of the optimal solution depth d is analyzed in terms of the **absolute error** or the **relative error** of the heuristic.
- The **absolute error** is defined as

$$\Delta \equiv h^* - h$$

  where h^* is the actual cost of getting from the root to the goal
- The **relative error** is defined as

$$\varepsilon \equiv (h^* - h)/h^*$$

A* Search

The 8-puzzle problem with single goal:

- The **time complexity of A*** is exponential in the maximum absolute error, that is,

$$O(b^{\Delta})$$

- For constant step costs, we can write this as

$$O(b^{\varepsilon d})$$

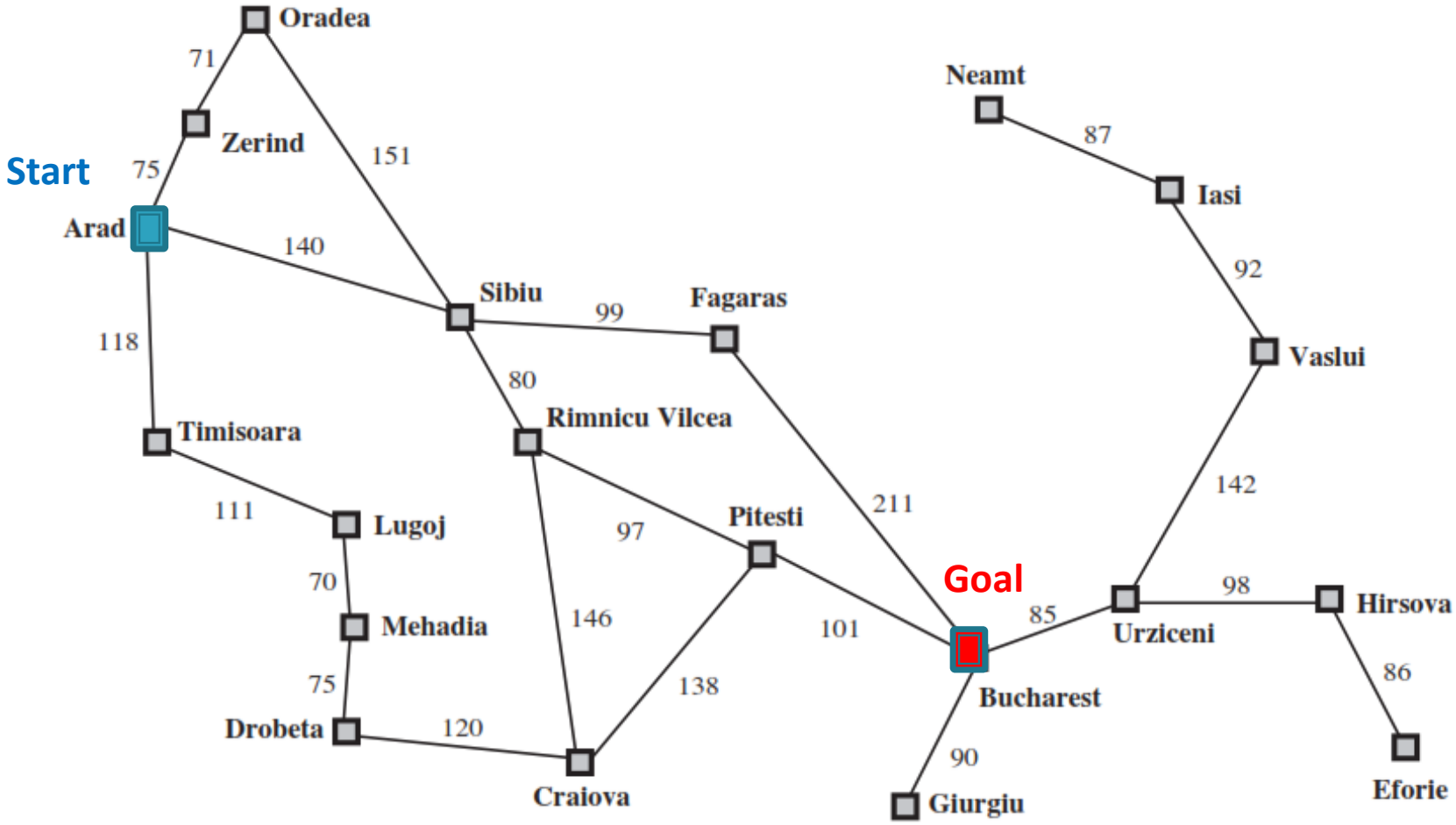
where d is the solution depth.

Memory-bounded Heuristic Search

Recursive Best-first Search (RBFS)

- **Recursive best-first search** (RBFS) is a **simple recursive algorithm** that attempts to mimic the operation of standard best-first search and A* search,
  - but using only linear space
- It uses the f_limit variable to keep track of the f_value of the best *alternative* path.
- If the current node exceeds this limit, the recursion unwinds back to the alternative path.
  - As the recursion unwinds, RBFS replaces the f_value of each node along the path with the best f_value of its children.

RBFS Example



A simplified road map of part of Romania.

RBFS Example

Values of h_{SLD}—straight-line distances to Bucharest

Arad	366	**Mehadia**	241
Bucharest	0	**Neamt**	234
Craiova	160	**Oradea**	380
Drobeta	242	**Pitesti**	100
Eforie	161	**Rimnicu Vilcea**	193
Fagaras	176	**Sibiu**	253
Giurgiu	77	**Timisoara**	329
Hirsova	151	**Urziceni**	80
Iasi	226	**Vaslui**	199
Lugoj	244	**Zerind**	374

RBFS Example

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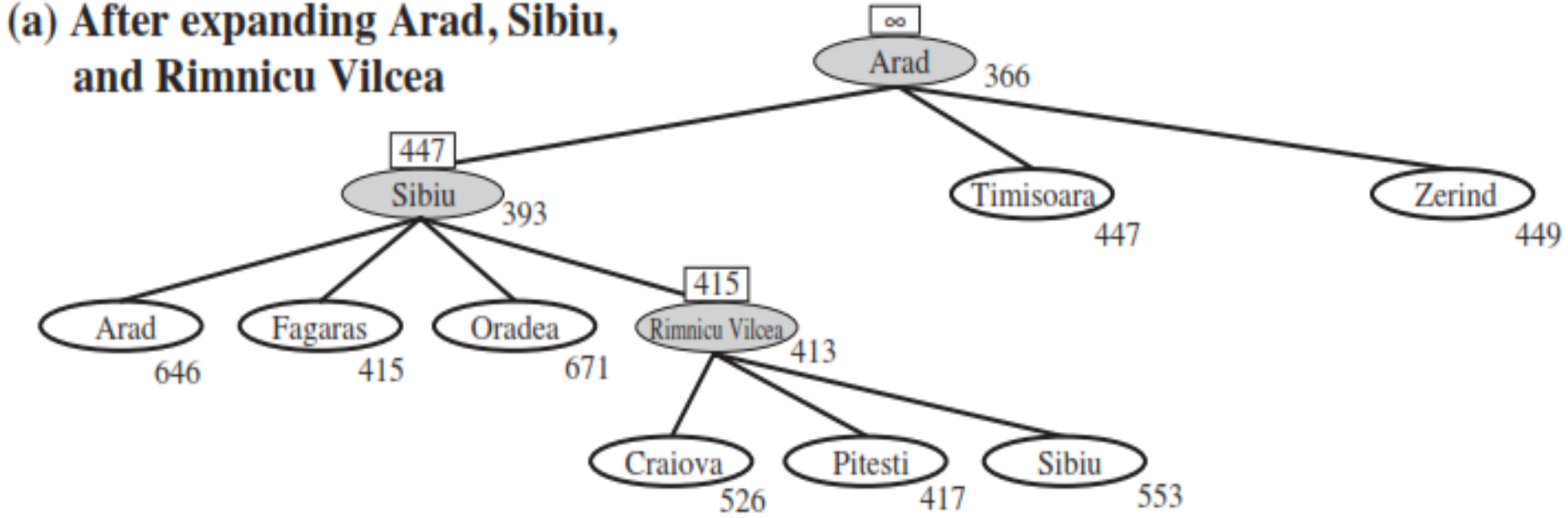
The f_limit value for each recursive call is shown on top of each current node, and every node is labeled with its f_cost.



RBFS Example

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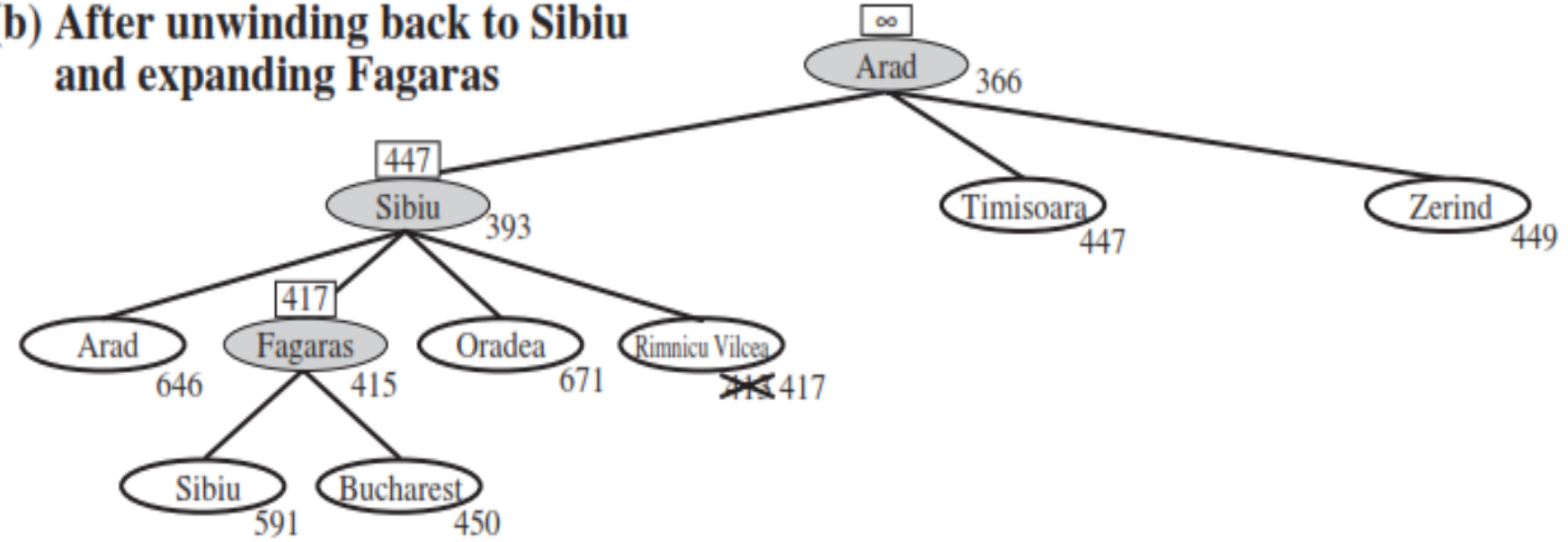
(b) After unwinding back to Sibiu and expanding Fagaras



RBFS Example

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Bucharest	0	**Neamt**	234
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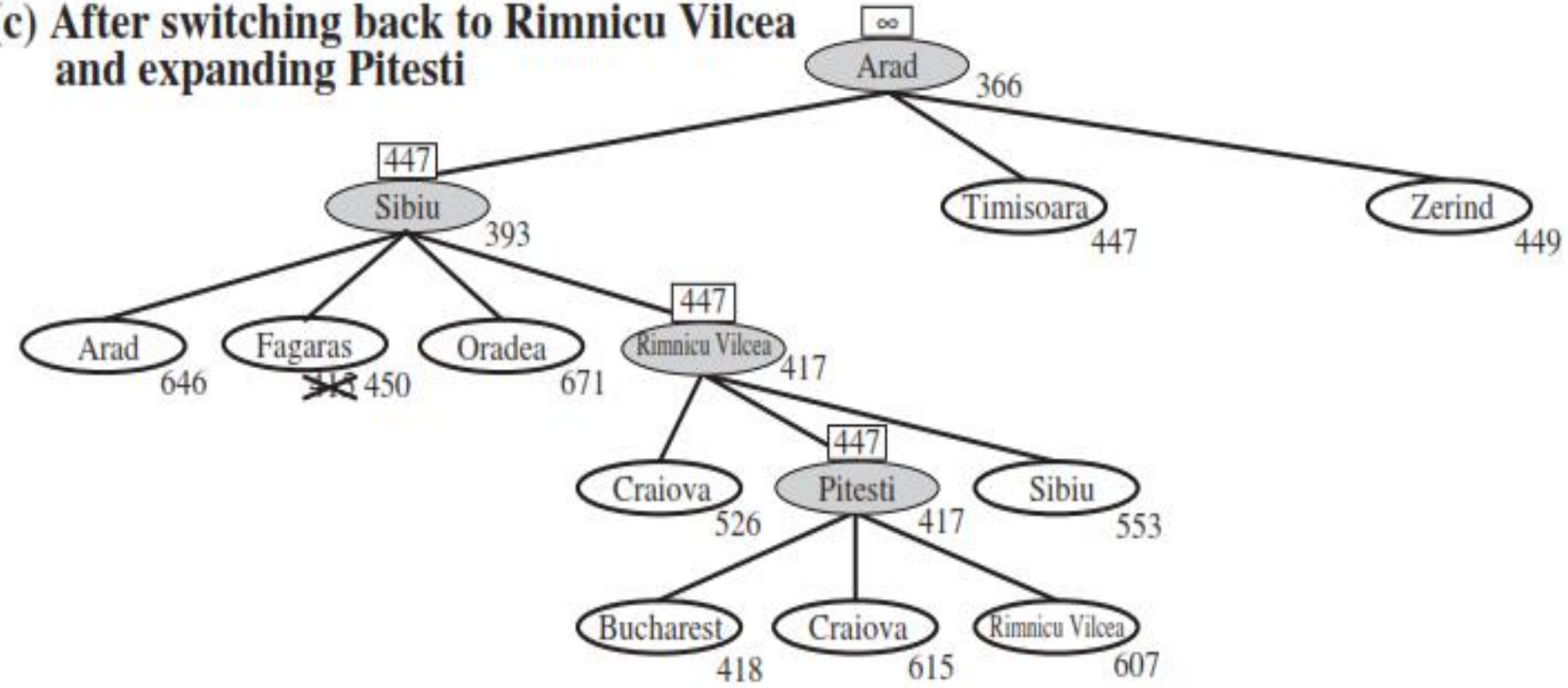
Iasi	226	**Vaslui**	199
Lugoj	244	**Zerind**	374

(c) After switching back to Rimnicu Vilcea and expanding Pitesti



Recursive Best-first Search (RBFS)

- Like A∗ tree search, RBFS is an **optimal algorithm** if the *heuristic function $h(n)$ is admissible*.
- Its **space complexity** is *linear* in the depth of the deepest optimal solution,
- Its **time complexity** is rather difficult to characterize: it depends both on
  - The *accuracy of the heuristic function* and
  - *How often the best path changes* as nodes are expanded.

Reading Material

- **Artificial Intelligence,** A Modern Approach

  Stuart J. Russell and Peter Norvig

  - **Chapter 3.**