$\boxed{Ex}$ if $y = (3x^2+1)^2$

find $\frac{dy}{dx} = ?$

$\boxed{Sol}$ $\frac{dy}{dx} = 2(3x^2+1)^{2-1} \frac{d}{dx}(3x^2+1)$

$= 2(3x^2+1) \cdot (6x)$

$= 12x(3x^2+1)$

$\boxed{Ex}$ if $x(t) = Cos(t^2+1)$ represents ~~velocity~~ position at any time "t".
find velocity at any time "t".

$\frac{dx}{dt} = -Sin(t^2+1) \frac{d}{dt}(t^2+1)$

$= -Sin(t^2+1) \cdot 2t$

$= -2t \, Sin(t^2+1)$

Here in this exp we have applied a formula

$\frac{d}{dx}(Cos u) = -Sin u \cdot \frac{du}{dx}$

$\boxed{Ex}$ Differentiate $Sin(x^2+e^x)$ w.r.t x.

$\frac{d}{dx}(Sin(x^2+e^x)) = \cancel{Cosx} \cdot Cos(x^2+e^x) \cdot \frac{d}{dx}(x^2+e^x)$

$= (2x+e^x)\, Cos(x^2+e^x).$

Explicit & Implicit Differentiation

An equation of the form $y = f(x)$ is said to be in explicit form because in this equation "y" appears in the form of "x" explicitly.

However, sometimes ftns are defined by equations in which we cannot express y as a ftn of x separatly i.e.

for exp $\quad xy + y + 1 = x \longrightarrow ①$

is not in the form of $y = f(x)$, but still it can be written as

$$y = \frac{x-1}{x+1} \longrightarrow ②$$

So ① which is in implicit form can be defined explicitly by ②.

Ex Consider $x^2 + y^2 = 25$

find ⓐ $\frac{dy}{dx}$

ⓑ find equation of tangent line to the given circle $x^2 + y^2 = 25$ at $(3,4)$

Sol $x^2 + y^2 = 25$

Diff w.r.t "x"

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Equation of tangent line at $(3,4)$ is

$$y - y_1 = m(x - x_1) \longrightarrow ①$$

$x_1 = 3$ $\quad$ $m = \left.\frac{dy}{dx}\right|_{(3,4)} = -\frac{3}{4}$

$y_1 = 4$

① $\Rightarrow$

$$y - 4 = -\frac{3}{4}(x - 3)$$

$$y - 4 = -\frac{3x}{4} + \frac{9}{4}$$

$$y \cancel{-4} = -\frac{3x}{4} + \frac{25}{4}$$

$$4y = -3x + 25$$

$$3x + 4y = 25$$

Question

find slope of tangent line to the curve

$$y^2 - x + 1 = 0$$

at $(2,-1)$ & $(2,1)$.

Ans. $m_1 = \left.\frac{dy}{dx}\right|_{(2,-1)} = -1/2$ $\qquad$ $m_2 = \left.\frac{dy}{dx}\right|_{(2,1)} = \frac{1}{2}$

(Ex)

(a) Use implicit differentiation to find $\frac{dy}{dx}$ if $x^3 + y^3 = 3xy$

(b) At what point in the 1st quadrant is the tangent line horizontal.

S.1 (a)

As $x^3 + y^3 = 3xy$

$$3x^2 + 3y^2 \frac{dy}{dx} = 3\left[x \frac{dy}{dx} + y(1)\right]$$

$$3x^2 + 3y^2 \frac{dy}{dx} = 3x \frac{dy}{dx} + 3y$$

$$(3y^2 - 3x)\frac{dy}{dx} = 3y - 3x^2$$

$$\frac{dy}{dx} = \frac{3y - 3x^2}{3y^2 - 3x}$$

$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$$

Ex $\quad xy = 1 \longrightarrow (1)$

Method 1

$y = \frac{1}{x}$

$\frac{dy}{dx} = -\frac{1}{x^2}$

Method 2

Diff (1) w.r.t x

$x\frac{dy}{dx} + y(1) = 0$

$x\frac{dy}{dx} + y = 0$

$x\frac{dy}{dx} = -y$

$\frac{dy}{dx} = -\frac{y}{x}$

$\frac{dy}{dx} = -\frac{1}{x}\left(\frac{1}{x}\right) = -1/x^2$

Question

Use implicit differentiation to find $\frac{dy}{dx}$ if

$5y^2 + \sin y = x^2$

Ans.

$10y\frac{dy}{dx} + \cos y\frac{dy}{dx} = 2x.$

$\frac{dy}{dx} = \frac{2x}{10y + \cos y}$

Question

Use implicit differentiation to find $\frac{d^2y}{dx^2}$ if $4x^2 - 2y^2 = 9.$

Ans.

$\frac{d^2y}{dx^2} = -\frac{9}{y^3}$

(b) Tangent line is horizontal, when

$$\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{y - x^2}{y^2 - x} = 0$$

$$\Rightarrow y - x^2 = 0$$

$$\Rightarrow y = x^2$$

As $x^3 + y^3 = 3xy$

The points where tangent line is horizontal, we have $y = x^2$

$$\Rightarrow x^3 + x^6 = 3x(x^2)$$

$$\cancel{x^3\left(1+x^3\right) = 3x^3}$$

$$\Rightarrow x^3 + x^6 = 3x^3$$

$$\Rightarrow x^6 - 2x^3 = 0$$

$$\Rightarrow x^3(x^3 - 2) = 0$$

Either $x^3 = 0$ or $x^3 - 2 = 0$

$\Rightarrow x = 0$ or $x^3 = 2$

$x = 2^{1/3}$

when $x = 0$, $y = 0$ $\longrightarrow (0, 0)$

when $x = 2^{1/3}$, $y = 2^{2/3}$ $\longrightarrow (2^{1/3}, 2^{2/3})$

The point in the 1st quadrant, where $\frac{dy}{dx} = 0$

$\Rightarrow (2^{1/3}, 2^{2/3})$.

Derivative of Logrithmic function

$\frac{d}{dx}(\ln x) = \frac{1}{x} \Rightarrow \frac{d}{dx}\left[\log_e x\right] = \frac{1}{x \ln e}$

$\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$: $x > 0$

Ex Consider $f(x) = \ln x$

(a) find domain of $f(x) = \ln x$.

(b) find Rang of $f(x) = \ln x$.

(c) Plot the graph of $f(x) = \ln x$.

(d) find equation of tangent line at $x = 1$

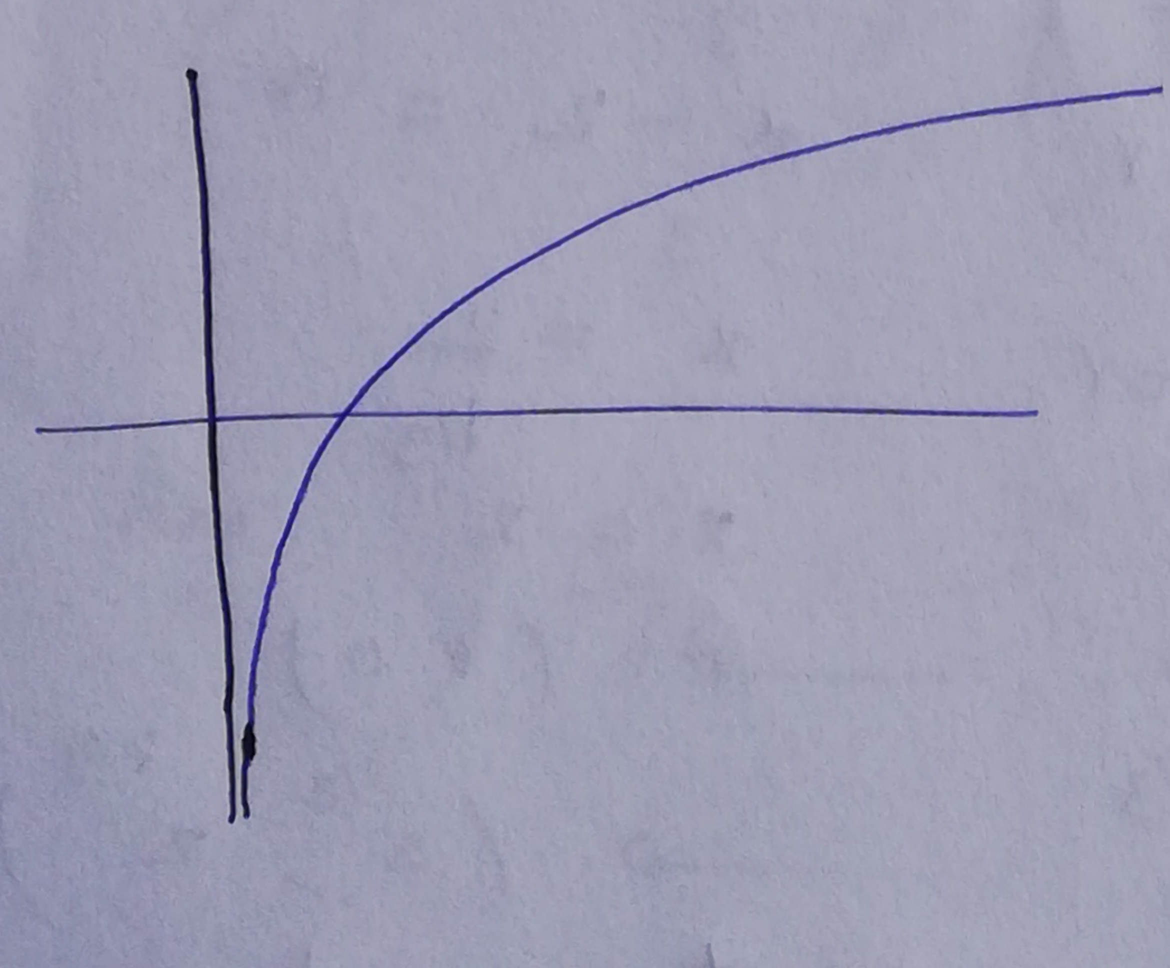
(e) show tangent line on the graph of $f(x)$ at $x = 1$.

(f) Does the graph of $f(x)$ have any horizontal tangent ?

Sol (a) $D(f(x)) = (0, \infty)$

(b) $R(f(x)) = (-\infty, \infty)$.

(c)



(d) $y - y_1 = m(x - x_1)$

$x_1 = 1$

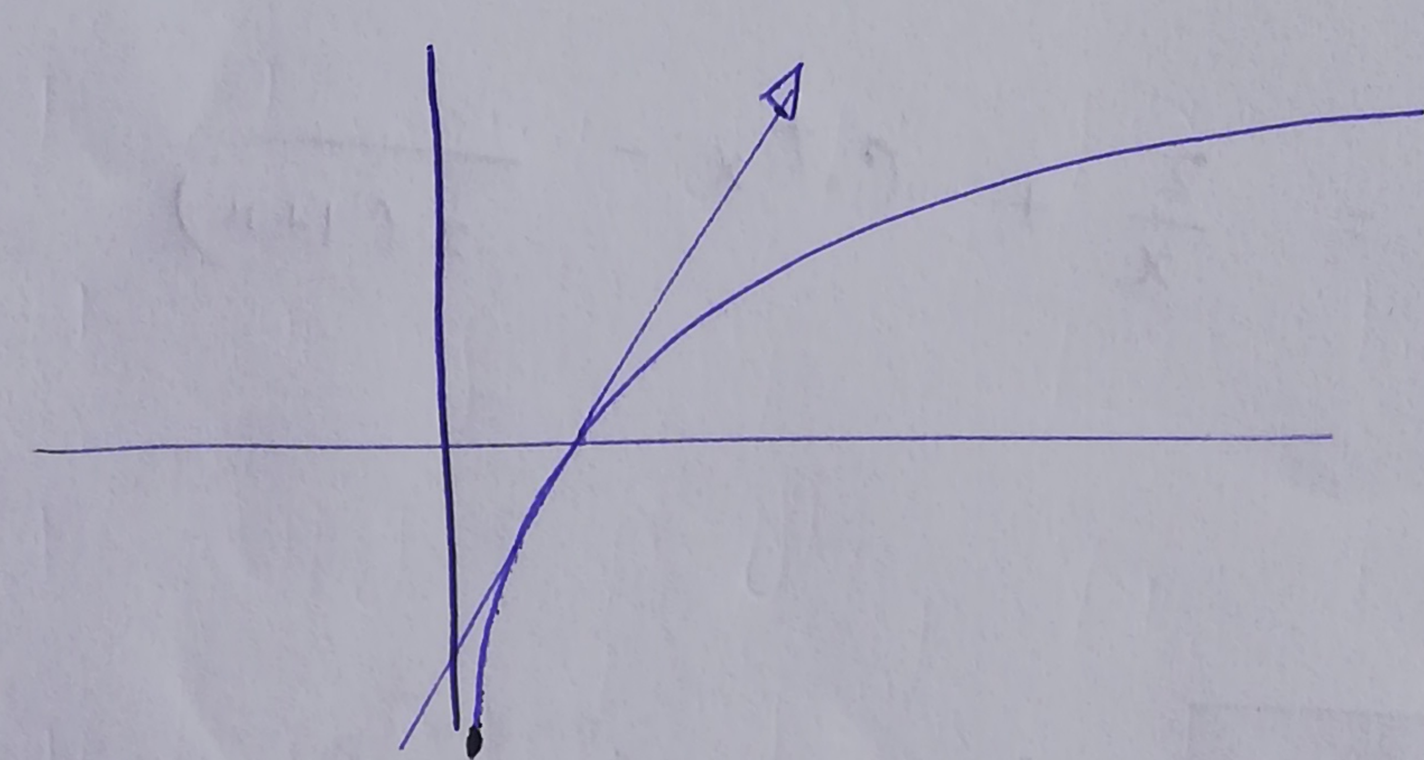
$y_1 = 0$

$m = \left.\frac{dy}{dx}\right|_{x=1} = \left.\frac{1}{x}\right|_{(1,0)} = 1$

$y - 0 = 1(x - 1)$

$y = x - 1$

(e)



(f) At horizontal tangent, $\frac{dy}{dx} = 0$

$\Rightarrow \frac{1}{x} = 0$

Not possible

So $f(x) = \ln x$ have no horizontal tangent.

Question find $\frac{d}{dx}\left(\ln(x^2+1)\right)$

Ans. $\frac{dy}{dx} = \frac{2x}{x^2+1}$

Ex $\quad \frac{d}{dx}\left(\ln\left(\frac{x^2 \sin x}{\sqrt{1+x}}\right)\right) = ?$

$$\ln\left(\frac{x^2 \sin x}{\sqrt{1+x}}\right) = \ln(x^2) + \ln(\sin x) - \ln\left(\sqrt{1+x}\right)$$

$$= 2\ln x + \ln(\sin x) - \frac{1}{2}\ln(1+x)$$

$$\frac{d}{dx}\left(\ln\left(\frac{x^2 \sin x}{\sqrt{1+x}}\right)\right) = 2\frac{1}{x} + \frac{1}{\sin x}\cos x - \frac{1}{2}\frac{1}{1+x}$$

$$= \frac{2}{x} + \cot x - \frac{1}{2(1+x)}$$

Ex $\quad \frac{d}{dx}\left(\frac{x^2 \sqrt[3]{7x-14}}{(1+x^2)^4}\right) = ?$