Integration by substitution Method :-

Q.1 Evaluate $\int (x^3+x)^5 (3x^2+1)\,dx$

let

$x^3+x = u$

$\Rightarrow du = \frac{du}{dx}\cdot dx$

$\Rightarrow du = (3x^2+1)\,dx$

$= \int u^5 \cdot du$

$= \frac{u^6}{6} + C$

$= \frac{(x^3+x)^6}{6} + C.$

Q.2 Evaluate $\int \sqrt{2x+1}\,dx$

let

$u = 2x+1$

$\Rightarrow du = \frac{du}{dx}\cdot dx$

$\Rightarrow du = 2\,dx$

$\Rightarrow dx = \frac{du}{2}$

$= \int u^{1/2} \cdot \frac{du}{2}$

$= \frac{1}{2}\int u^{1/2}\,du$

$= \frac{1}{2}\,\frac{u^{3/2}}{3/2} + C$

$= \frac{1}{3}\,u^{3/2} + C$

$= \frac{1}{3}(2x+1)^{3/2} + C.$

Q.3 find $\int \sec^2(5x+1) \cdot 5\, dx$

let

$5x+1 = u$

$\Rightarrow du = \frac{du}{dx} \cdot dx$

$\Rightarrow du = 5\, dx$

$\Rightarrow dx = \frac{du}{5}$

$= \int \sec^2 u \cdot du$

$= \tan u + c$

$= \tan(5x+1) + c.$

Q.4 Find $\int \cos(7\theta+3)\, d\theta.$

let

$7\theta + 3 = u$

$\Rightarrow du = \frac{du}{d\theta} \cdot d\theta.$

$du = 7\, d\theta.$

$= \int \cos u \cdot \frac{du}{7}$

$= \frac{1}{7} \int \cos u\, du$

$= \frac{1}{7} \sin u + c$

$= \frac{1}{7} \sin(7\theta+3) + c$

Q.5 $\int x^2 e^{x^3} dx$

let

$x^3 = u$

$3x^2 dx = du$

$x^2 dx = \frac{du}{3}$

$= \int e^u \cdot \frac{du}{3}$

$= \frac{1}{3} e^u + c$

$= \frac{1}{3} e^{x^3} + c$

Q.6

$$\int x\sqrt{2x+1}\,dx$$

let

$$\boxed{2x+1 = u}$$

As $du = \frac{du}{dx}\cdot dx$

$du = 2\,dx$

$$\boxed{dx = \frac{du}{2}}$$

Also $2x = u-1$

$$\boxed{x = \frac{u-1}{2}}$$

$$= \int \frac{u-1}{2}\cdot u^{1/2}\cdot\frac{du}{2}$$

$$= \frac{1}{4}\int \left(u^{3/2} - u^{1/2}\right)du$$

$$= \frac{1}{4}\int u^{3/2}\,du - \frac{1}{4}\int u^{1/2}\,du$$

$$= \frac{1}{4}\,\frac{u^{5/2}}{5/2} - \frac{1}{4}\,\frac{u^{3/2}}{3/2} + C$$

$$= \frac{1}{10}u^{5/2} - \frac{1}{6}u^{3/2} + C$$

$$= \frac{1}{10}(2x+1)^{5/2} - \frac{1}{6}(2x+1)^{3/2} + C$$

Q.7

$$\int \sin^2 x\,dx$$

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$= \int\left(\frac{1-\cos 2x}{2}\right)dx$$

$$= \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x\,dx$$

$$= \frac{1}{2}x - \frac{1}{2}\,\frac{\sin 2x}{2} + C.$$

$$= \frac{x}{2} - \frac{1}{4}\sin 2x + C$$

Q.8

$$\int \cos^2 x \, dx$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$= \int \frac{1+\cos 2x}{2} \, dx$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \cos 2x \, dx$$

$$= \frac{1}{2}x + \frac{1}{2}\,\frac{\sin 2x}{2} + c$$

$$= \frac{x}{2} + \frac{1}{4}\sin 2x + c.$$

Q.9

$$\int \tan x \, dx$$

$$= \int \frac{\sin x}{\cos x} \, dx$$

let

$$u = \cos x$$

As $du = \frac{du}{dx} \cdot dx$

$$du = -\sin x \cdot dx$$

$$\Rightarrow \sin x \, dx = -du$$

$$= \int \frac{-du}{u}$$

$$= -\int \frac{du}{u}$$

$$= -\ln|u| + c$$

$$= -\ln|\cos x| + c.$$

$$= \ln|\cos x|^{-1} + c$$

$$= \ln\left|\frac{1}{\cos x}\right| + c$$

$$= \ln|\sec x| + c.$$

Q.10

$$\int \frac{dx}{e^{x}+e^{-x}}$$

xing & dividing by e^{x}.

$$= \int \frac{e^{x}\,dx}{e^{2x}+1}$$

let

$e^{x} = u$

$e^{x}\,dx = du$

$$= \int \frac{du}{u^{2}+1}$$

$$= \tan^{-1} u + c$$

$$= \tan^{-1}(e^{x}) + c.$$

Q.11

$$\int \frac{2z\,dz}{\sqrt[3]{z^{2}+1}}$$

let

$(z^{2}+1)^{1/3} = u$

$(z^{2}+1) = u^{3}$

$2z\,dz = 3u^{2}\,du$

$$= \int \frac{3u^{2}\,du}{u}$$

$$= 3\int u\,du$$

$$= 3\,\frac{u^{2}}{2} + c$$

$$= \frac{3}{2}(z^{2}+1)^{2/3} + c$$

Q.12

$$\int \frac{dx}{1+3x^2}$$

let

$\sqrt{3}\,x = u$

$\sqrt{3}\,dx = du$

$dx = \frac{du}{\sqrt{3}}$

$$= \int \frac{du/\sqrt{3}}{1+u^2}$$

$$= \frac{1}{\sqrt{3}} \int \frac{du}{1+u^2}$$

$$= \frac{1}{\sqrt{3}} \tan^{-1} u + c$$

$$= \frac{1}{\sqrt{3}} \tan^{-1} (\sqrt{3}\,x) + c.$$

Q.13

$$\int \left(\frac{1}{x} + \sec^2 \pi x\right) dx$$

$$= \int \frac{1}{x}\,dx + \int \sec^2 \pi x\,dx$$

$$= \ln|x| + \int \sec^2 \pi x\,dx$$

let

$\pi x = u$

$\pi\,dx = du$

$dx = \frac{du}{\pi}$

$$= \ln|x| + \int \sec^2 u \cdot \frac{du}{\pi}$$

$$= \ln|x| + \frac{1}{\pi} \int \sec^2 u\,du$$

$$= \ln|x| + \frac{1}{\pi} \tan u + c$$

$$= \ln|x| + \frac{1}{\pi} \tan(\pi x) + c$$

Q.14

$$\int \sin^2 x \cdot \cos x \, dx$$

let

$\sin x = u$

$\cos x \, dx = du$

$$= \int u^2 \, du$$

$$= \frac{u^3}{3} + c$$

$$= \frac{1}{3} \sin^3 x + c.$$

Q.15

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx$$

let

$\sqrt{x} = u$

$\frac{1}{2} x^{-1/2} \, dx = du$

$\frac{1}{2\sqrt{x}} \, dx = du$

$\frac{dx}{\sqrt{x}} = 2 \, du$

$$= \int e^u \cdot 2 \, du$$

$$= 2 \int e^u \, du$$

$$= 2 e^u + c$$

$$= 2 e^{\sqrt{x}} + c.$$

Q.16

$$\int t^4 \cdot \sqrt{3 - 5t^5} \, dt$$

solve the question by subsituting $3 - 5t^5 = u$

Q.17

$$\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$$

let

$e^x = u$

$e^x\,dx = du$

$$= \int \frac{du}{\sqrt{1-u^2}}$$

$$= \sin^{-1} u + c$$

$$= \sin^{-1}(e^x) + c.$$

Q.18

$$\int x^2 \sqrt{x-1}\,dx$$

let

$x-1 = u \longrightarrow (1)$

$dx = du$

squaring (1)

$(x-1)^2 = u^2$

$x^2 - 2x + 1 = u^2$

$x^2 = u^2 + 2x - 1$

$= u^2 + 2(u+1) - 1$

$= u^2 + 2u + 1$

$$= \int (u^2 + 2u + 1)\sqrt{u}\,.\,du$$

$$= \int u^{5/2}\,du + 2\int u^{3/2}\,du + \int u^{1/2}\,du$$

$$= \frac{2}{7}u^{7/2} + \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2} + c$$

$$= \frac{2}{7}(x-1)^{7/2} + \frac{4}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + c.$$

Q.19

$$\int \cos^3 x\,dx$$

$$= \int \cos^2 x \cdot \cos x\,dx$$

$$= \int (1-\sin^2 x)\cos x\,dx$$

$$= \int \cos x\,dx - \int \sin^2 x \cdot \cos x\,dx$$

$$= \sin x - \frac{\sin^3 x}{3} + c$$

Q. 20

$$\int \frac{dx}{a^2+x^2}$$

$$= \frac{1}{a^2}\int \frac{dx}{1+\left(\frac{x}{a}\right)^2}$$

$\frac{x}{a} = u$

$\frac{1}{a}dx = du$

$$= \frac{1}{a^2}\int \frac{a\,du}{1+u^2}$$

$$= \frac{1}{a}\int \frac{du}{1+u^2}$$

$$= \frac{1}{a}\tan^{-1} u + c$$

$$= \frac{1}{a}\tan^{-1}\frac{x}{a} + c.$$

Points to Remember

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c.$$

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + c.$$