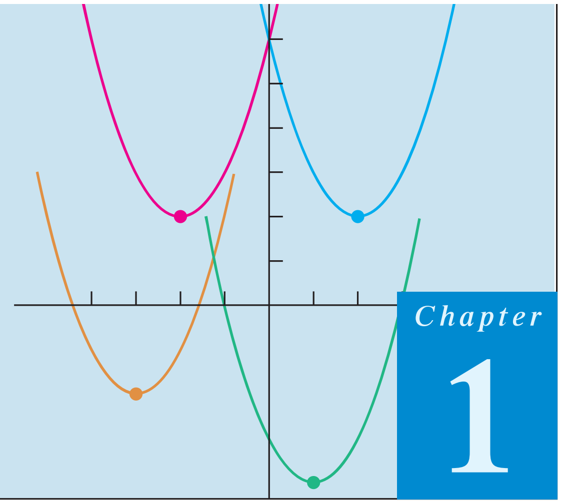


Chapter

1 PRELIMINARIES

OVERVIEW This chapter reviews the basic ideas you need to start calculus. The topics include the real number system, Cartesian coordinates in the plane, straight lines, parabolas, circles, functions, and trigonometry. We also discuss the use of graphing calculators and computer graphing software.

1.1 Real Numbers and the Real Line

This section reviews real numbers, inequalities, intervals, and absolute values.

Real Numbers

Much of calculus is based on properties of the real number system. **Real numbers** are numbers that can be expressed as decimals, such as

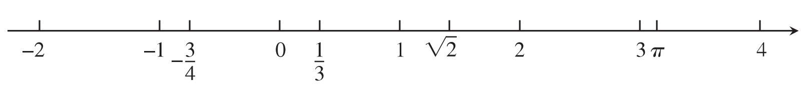
$$-\frac{3}{4} = -0.75000\ldots$$

$$\frac{1}{3} = 0.33333\ldots$$

$$\sqrt{2} = 1.4142\ldots$$

The dots ... in each case indicate that the sequence of decimal digits goes on forever. Every conceivable decimal expansion represents a real number, although some numbers have two representations. For instance, the infinite decimals .999 ... and 1.000 ... represent the same real number 1. A similar statement holds for any number with an infinite tail of 9's.

The real numbers can be represented geometrically as points on a number line called the **real line**.



The symbol $\mathbb{R}$ denotes either the real number system or, equivalently, the real line.

The properties of the real number system fall into three categories: algebraic properties, order properties, and completeness. The **algebraic properties** say that the real numbers can be added, subtracted, multiplied, and divided (except by 0) to produce more real numbers under the usual rules of arithmetic. *You can never divide by* 0.

The **order properties** of real numbers are given in Appendix 4. The following useful rules can be derived from them, where the symbol $\Rightarrow$ means "implies."

Rules for Inequalities

If a, b, and c are real numbers, then:

1. $a < b \Rightarrow a + c < b + c$
2. $a < b \Rightarrow a - c < b - c$
3. $a < b$ and $c > 0 \Rightarrow ac < bc$
4. $a < b$ and $c < 0 \Rightarrow bc < ac$
   Special case: $a < b \Rightarrow -b < -a$
5. $a > 0 \Rightarrow \dfrac{1}{a} > 0$
6. If a and b are both positive or both negative, then $a < b \Rightarrow \dfrac{1}{b} < \dfrac{1}{a}$

Notice the rules for multiplying an inequality by a number. Multiplying by a positive number preserves the inequality; multiplying by a negative number reverses the inequality. Also, reciprocation reverses the inequality for numbers of the same sign. For example, $2 < 5$ but $-2 > -5$ and $1/2 > 1/5$.

The **completeness property** of the real number system is deeper and harder to define precisely. However, the property is essential to the idea of a limit (Chapter 2). Roughly speaking, it says that there are enough real numbers to "complete" the real number line, in the sense that there are no "holes" or "gaps" in it. Many theorems of calculus would fail if the real number system were not complete. The topic is best saved for a more advanced course, but Appendix 4 hints about what is involved and how the real numbers are constructed.

We distinguish three special subsets of real numbers.

1. The **natural numbers**, namely $1, 2, 3, 4, \ldots$
2. The **integers**, namely $0, \pm 1, \pm 2, \pm 3, \ldots$
3. The **rational numbers**, namely the numbers that can be expressed in the form of a fraction m/n, where m and n are integers and $n \neq 0$. Examples are

$$\frac{1}{3}, \quad -\frac{4}{9} = \frac{-4}{9} = \frac{4}{-9}, \quad \frac{200}{13}, \quad \text{and} \quad 57 = \frac{57}{1}.$$

The rational numbers are precisely the real numbers with decimal expansions that are either

(a) terminating (ending in an infinite string of zeros), for example,

$$\frac{3}{4} = 0.75000\ldots = 0.75 \qquad \text{or}$$

(b) eventually repeating (ending with a block of digits that repeats over and over), for example

$$\frac{23}{11} = 2.090909\ldots = 2.\overline{09}$$

The bar indicates the block of repeating digits.

A terminating decimal expansion is a special type of repeating decimal since the ending zeros repeat.

The set of rational numbers has all the algebraic and order properties of the real numbers but lacks the completeness property. For example, there is no rational number whose square is 2; there is a "hole" in the rational line where $\sqrt{2}$ should be.

Real numbers that are not rational are called **irrational numbers**. They are characterized by having nonterminating and nonrepeating decimal expansions. Examples are π, $\sqrt{2}$, $\sqrt[3]{5}$, and $\log_{10} 3$. Since every decimal expansion represents a real number, it should be clear that there are infinitely many irrational numbers. Both rational and irrational numbers are found arbitrarily close to any point on the real line.

Set notation is very useful for specifying a particular subset of real numbers. A **set** is a collection of objects, and these objects are the **elements** of the set. If S is a set, the notation $a \in S$ means that a is an element of S, and $a \notin S$ means that a is not an element of S. If S and T are sets, then $S \cup T$ is their **union** and consists of all elements belonging either to S or T (or to both S and T). The **intersection** $S \cap T$ consists of all elements belonging to both S and T. The **empty set** $\varnothing$ is the set that contains no elements. For example, the intersection of the rational numbers and the irrational numbers is the empty set.

Some sets can be described by *listing* their elements in braces. For instance, the set A consisting of the natural numbers (or positive integers) less than 6 can be expressed as

$$A = \{1, 2, 3, 4, 5\}.$$

The entire set of integers is written as

$$\{0, \pm 1, \pm 2, \pm 3, \dots\}.$$

Another way to describe a set is to enclose in braces a rule that generates all the elements of the set. For instance, the set

$$A = \{x \mid x \text{ is an integer and } 0 < x < 6\}$$

is the set of positive integers less than 6.

Intervals

A subset of the real line is called an **interval** if it contains at least two numbers and contains all the real numbers lying between any two of its elements. For example, the set of all real numbers x such that $x > 6$ is an interval, as is the set of all x such that $-2 \leq x \leq 5$. The set of all nonzero real numbers is not an interval; since 0 is absent, the set fails to contain every real number between -1 and 1 (for example).

Geometrically, intervals correspond to rays and line segments on the real line, along with the real line itself. Intervals of numbers corresponding to line segments are **finite intervals**; intervals corresponding to rays and the real line are **infinite intervals**.

A finite interval is said to be **closed** if it contains both of its endpoints, **half-open** if it contains one endpoint but not the other, and **open** if it contains neither endpoint. The endpoints are also called **boundary points**; they make up the interval's **boundary**. The remaining points of the interval are **interior points** and together comprise the interval's **interior**. Infinite intervals are closed if they contain a finite endpoint, and open otherwise. The entire real line $\mathbb{R}$ is an infinite interval that is both open and closed.

Solving Inequalities

The process of finding the interval or intervals of numbers that satisfy an inequality in x is called **solving** the inequality.

TABLE 1.1 Types of intervals

	Notation	Set description	Type	Picture
Finite:	(a, b)	$\{x \mid a < x < b\}$	Open	
	$[a, b]$	$\{x \mid a \le x \le b\}$	Closed	
	$[a, b)$	$\{x \mid a \le x < b\}$	Half-open	
	$(a, b]$	$\{x \mid a < x \le b\}$	Half-open	
Infinite:	(a, ∞)	$\{x \mid x > a\}$	Open	
	$[a, \infty)$	$\{x \mid x \ge a\}$	Closed	
	$(-\infty, b)$	$\{x \mid x < b\}$	Open	
	$(-\infty, b]$	$\{x \mid x \le b\}$	Closed	
	$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	Both open and closed	

EXAMPLE 1 Solve the following inequalities and show their solution sets on the real line.

(a) $2x - 1 < x + 3$ **(b)** $-\frac{x}{3} < 2x + 1$ **(c)** $\frac{6}{x - 1} \ge 5$

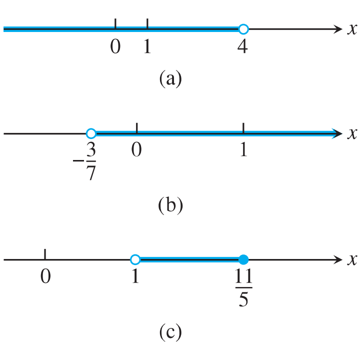


FIGURE 1.1 Solution sets for the inequalities in Example 1.

Solution

(a)

$$
\begin{aligned}
2x - 1 &< x + 3 \\
2x &< x + 4 && \text{Add 1 to both sides.} \\
x &< 4 && \text{Subtract } x \text{ from both sides.}
\end{aligned}
$$

The solution set is the open interval $(-\infty, 4)$ (Figure 1.1a).

(b)

$$
\begin{aligned}
-\frac{x}{3} &< 2x + 1 \\
-x &< 6x + 3 && \text{Multiply both sides by 3.} \\
0 &< 7x + 3 && \text{Add } x \text{ to both sides.} \\
-3 &< 7x && \text{Subtract 3 from both sides.} \\
-\frac{3}{7} &< x && \text{Divide by 7.}
\end{aligned}
$$

The solution set is the open interval $(-3/7, \infty)$ (Figure 1.1b).

(c) The inequality $6/(x-1) \geq 5$ can hold only if $x > 1$, because otherwise $6/(x-1)$ is undefined or negative. Therefore, $(x-1)$ is positive and the inequality will be preserved if we multiply both sides by $(x-1)$, and we have

$$\begin{aligned} \frac{6}{x-1} &\geq 5 & \\ 6 &\geq 5x - 5 & \text{Multiply both sides by } (x-1). \\ 11 &\geq 5x & \text{Add 5 to both sides.} \\ \frac{11}{5} &\geq x. & \text{Or } x \leq \frac{11}{5}. \end{aligned}$$

The solution set is the half-open interval $(1, 11/5]$ (Figure 1.1c). ■

Absolute Value

The **absolute value** of a number x, denoted by $|x|$, is defined by the formula

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases}$$

EXAMPLE 2 Finding Absolute Values

$$|3| = 3, \quad |0| = 0, \quad |-5| = -(-5) = 5, \quad |-|a|| = |a|$$ ■

Geometrically, the absolute value of x is the distance from x to 0 on the real number line. Since distances are always positive or 0, we see that $|x| \geq 0$ for every real number x, and $|x| = 0$ if and only if $x = 0$. Also,

$$|x - y| = \text{the distance between } x \text{ and } y$$

on the real line (Figure 1.2).

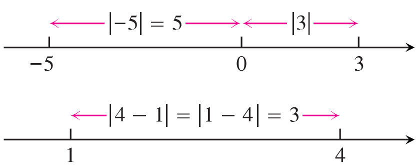


FIGURE 1.2 Absolute values give distances between points on the number line.

Since the symbol $\sqrt{a}$ always denotes the *nonnegative* square root of a, an alternate definition of $|x|$ is

$$|x| = \sqrt{x^2}.$$

It is important to remember that $\sqrt{a^2} = |a|$. Do not write $\sqrt{a^2} = a$ unless you already know that $a \geq 0$.

The absolute value has the following properties. (You are asked to prove these properties in the exercises.)

Absolute Value Properties

1. $|-a| = |a|$ — A number and its additive inverse or negative have the same absolute value.
2. $|ab| = |a||b|$ — The absolute value of a product is the product of the absolute values.
3. $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ — The absolute value of a quotient is the quotient of the absolute values.
4. $|a + b| \leq |a| + |b|$ — The **triangle inequality**. The absolute value of the sum of two numbers is less than or equal to the sum of their absolute values.

Note that $|-a| \neq -|a|$. For example, $|-3| = 3$, whereas $-|3| = -3$. If a and b differ in sign, then $|a + b|$ is less than $|a| + |b|$. In all other cases, $|a + b|$ equals $|a| + |b|$. Absolute value bars in expressions like $|-3 + 5|$ work like parentheses: We do the arithmetic inside *before* taking the absolute value.

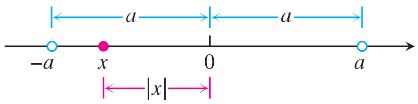


FIGURE 1.3 $|x| < a$ means x lies between $-a$ and a.

EXAMPLE 3 Illustrating the Triangle Inequality

$$
\begin{aligned}
|-3 + 5| &= |2| = 2 < |-3| + |5| = 8 \\
|3 + 5| &= |8| = |3| + |5| \\
|-3 - 5| &= |-8| = 8 = |-3| + |-5|
\end{aligned}
$$

■

The inequality $|x| < a$ says that the distance from x to 0 is less than the positive number a. This means that x must lie between $-a$ and a, as we can see from Figure 1.3.

The following statements are all consequences of the definition of absolute value and are often helpful when solving equations or inequalities involving absolute values.

Absolute Values and Intervals

If a is any positive number, then

5. $|x| = a$ if and only if $x = \pm a$

6. $|x| < a$ if and only if $-a < x < a$

7. $|x| > a$ if and only if $x > a$ or $x < -a$

8. $|x| \leq a$ if and only if $-a \leq x \leq a$

9. $|x| \geq a$ if and only if $x \geq a$ or $x \leq -a$

The symbol $\Leftrightarrow$ is often used by mathematicians to denote the "if and only if" logical relationship. It also means "implies and is implied by."

EXAMPLE 4 Solving an Equation with Absolute Values

You Try It

Solve the equation $|2x - 3| = 7$.

Solution By Property 5, $2x - 3 = \pm 7$, so there are two possibilities:

$$
\begin{aligned}
2x - 3 &= 7 & 2x - 3 &= -7 && \text{Equivalent equations without absolute values} \\
2x &= 10 & 2x &= -4 && \text{Solve as usual.} \\
x &= 5 & x &= -2
\end{aligned}
$$

The solutions of $|2x - 3| = 7$ are $x = 5$ and $x = -2$. ■

EXAMPLE 5 Solving an Inequality Involving Absolute Values

Solve the inequality $\left|5 - \dfrac{2}{x}\right| < 1$.

Solution We have

$$
\begin{aligned}
\left|5-\frac{2}{x}\right|<1 &\Leftrightarrow -1<5-\frac{2}{x}<1 && \text{Property 6}\\
&\Leftrightarrow -6<-\frac{2}{x}<-4 && \text{Subtract 5.}\\
&\Leftrightarrow 3>\frac{1}{x}>2 && \text{Multiply by } -\tfrac{1}{2}.\\
&\Leftrightarrow \frac{1}{3}<x<\frac{1}{2}. && \text{Take reciprocals.}
\end{aligned}
$$

Notice how the various rules for inequalities were used here. Multiplying by a negative number reverses the inequality. So does taking reciprocals in an inequality in which both sides are positive. The original inequality holds if and only if $(1/3) < x < (1/2)$. The solution set is the open interval $(1/3, 1/2)$. ■

EXAMPLE 6 Solve the inequality and show the solution set on the real line:

(a) $|2x - 3| \le 1$ **(b)** $|2x - 3| \ge 1$

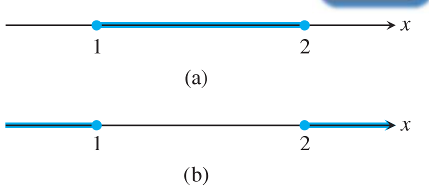


FIGURE 1.4 The solution sets (a) $[1, 2]$ and (b) $(-\infty, 1] \cup [2, \infty)$ in Example 6.

Solution

(a)

$$
\begin{aligned}
|2x-3| &\le 1 &&\\
-1 \le 2x-3 &\le 1 && \text{Property 8}\\
2 \le 2x &\le 4 && \text{Add 3.}\\
1 \le x &\le 2 && \text{Divide by 2.}
\end{aligned}
$$

The solution set is the closed interval $[1, 2]$ (Figure 1.4a).

(b)

$$
\begin{aligned}
&|2x-3| \ge 1 &&\\
2x-3 \ge 1 \quad &\text{or} \quad 2x-3 \le -1 && \text{Property 9}\\
x-\frac{3}{2} \ge \frac{1}{2} \quad &\text{or} \quad x-\frac{3}{2} \le -\frac{1}{2} && \text{Divide by 2.}\\
x \ge 2 \quad &\text{or} \quad x \le 1 && \text{Add } \tfrac{3}{2}.
\end{aligned}
$$

The solution set is $(-\infty, 1] \cup [2, \infty)$ (Figure 1.4b). ■

EXERCISES 1.1

Exercises

Decimal Representations

1. Express 1/9 as a repeating decimal, using a bar to indicate the repeating digits. What are the decimal representations of 2/9? 3/9? 8/9? 9/9?

2. Express 1/11 as a repeating decimal, using a bar to indicate the repeating digits. What are the decimal representations of 2/11? 3/11? 9/11? 11/11?

Inequalities

3. If $2 < x < 6$, which of the following statements about x are necessarily true, and which are not necessarily true?

a. $0 < x < 4$ **b.** $0 < x - 2 < 4$

c. $1 < \frac{x}{2} < 3$ **d.** $\frac{1}{6} < \frac{1}{x} < \frac{1}{2}$

e. $1 < \frac{6}{x} < 3$ **f.** $|x - 4| < 2$

g. $-6 < -x < 2$ **h.** $-6 < -x < -2$

4. If $-1 < y - 5 < 1$, which of the following statements about y are necessarily true, and which are not necessarily true?

a. $4 < y < 6$ **b.** $-6 < y < -4$

c. $y > 4$ **d.** $y < 6$

e. $0 < y - 4 < 2$ **f.** $2 < \frac{y}{2} < 3$

g. $\frac{1}{6} < \frac{1}{y} < \frac{1}{4}$ **h.** $|y - 5| < 1$

In Exercises 5–12, solve the inequalities and show the solution sets on the real line.

5. $-2x > 4$ **6.** $8 - 3x \geq 5$

7. $5x - 3 \leq 7 - 3x$ **8.** $3(2 - x) > 2(3 + x)$

9. $2x - \frac{1}{2} \geq 7x + \frac{7}{6}$ **10.** $\frac{6 - x}{4} < \frac{3x - 4}{2}$

11. $\frac{4}{5}(x - 2) < \frac{1}{3}(x - 6)$ **12.** $-\frac{x + 5}{2} \leq \frac{12 + 3x}{4}$

Absolute Value

Solve the equations in Exercises 13–18.

13. $|y| = 3$ **14.** $|y - 3| = 7$ **15.** $|2t + 5| = 4$

16. $|1 - t| = 1$ **17.** $|8 - 3s| = \frac{9}{2}$ **18.** $\left|\frac{s}{2} - 1\right| = 1$

Solve the inequalities in Exercises 19–34, expressing the solution sets as intervals or unions of intervals. Also, show each solution set on the real line.

19. $|x| < 2$ **20.** $|x| \leq 2$ **21.** $|t - 1| \leq 3$

22. $|t + 2| < 1$ **23.** $|3y - 7| < 4$ **24.** $|2y + 5| < 1$

25. $\left|\frac{z}{5} - 1\right| \leq 1$ **26.** $\left|\frac{3}{2}z - 1\right| \leq 2$ **27.** $\left|3 - \frac{1}{x}\right| < \frac{1}{2}$

28. $\left|\frac{2}{x} - 4\right| < 3$ **29.** $|2s| \geq 4$ **30.** $|s + 3| \geq \frac{1}{2}$

31. $|1 - x| > 1$ **32.** $|2 - 3x| > 5$ **33.** $\left|\frac{r + 1}{2}\right| \geq 1$

34. $\left|\frac{3r}{5} - 1\right| > \frac{2}{5}$

Quadratic Inequalities

Solve the inequalities in Exercises 35–42. Express the solution sets as intervals or unions of intervals and show them on the real line. Use the result $\sqrt{a^2} = |a|$ as appropriate.

35. $x^2 < 2$ **36.** $4 \leq x^2$ **37.** $4 < x^2 < 9$

38. $\frac{1}{9} < x^2 < \frac{1}{4}$ **39.** $(x - 1)^2 < 4$ **40.** $(x + 3)^2 < 2$

41. $x^2 - x < 0$ **42.** $x^2 - x - 2 \geq 0$

Theory and Examples

43. Do not fall into the trap $|-a| = a$. For what real numbers a is this equation true? For what real numbers is it false?

44. Solve the equation $|x - 1| = 1 - x$.

45. A proof of the triangle inequality Give the reason justifying each of the numbered steps in the following proof of the triangle inequality.

$$\begin{aligned} |a + b|^2 &= (a + b)^2 && (1)\\ &= a^2 + 2ab + b^2 &&\\ &\leq a^2 + 2|a||b| + b^2 && (2)\\ &= |a|^2 + 2|a||b| + |b|^2 && (3)\\ &= (|a| + |b|)^2 &&\\ |a + b| &\leq |a| + |b| && (4) \end{aligned}$$

46. Prove that $|ab| = |a||b|$ for any numbers a and b.

47. If $|x| \leq 3$ and $x > -1/2$, what can you say about x?

48. Graph the inequality $|x| + |y| \leq 1$.

49. Let $f(x) = 2x + 1$ and let $\delta > 0$ be any positive number. Prove that $|x - 1| < \delta$ implies $|f(x) - f(1)| < 2\delta$. Here the notation $f(a)$ means the value of the expression $2x + 1$ when $x = a$. This *function notation* is explained in Section 1.3.

50. Let $f(x) = 2x + 3$ and let $\epsilon > 0$ be any positive number. Prove that $|f(x) - f(0)| < \epsilon$ whenever $|x - 0| < \frac{\epsilon}{2}$. Here the notation $f(a)$ means the value of the expression $2x + 3$ when $x = a$. (See Section 1.3.)

51. For any number a, prove that $|-a| = |a|$.

52. Let a be any positive number. Prove that $|x| > a$ if and only if $x > a$ or $x < -a$.

53. a. If b is any nonzero real number, prove that $|1/b| = 1/|b|$.

b. Prove that $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ for any numbers a and $b \neq 0$.

54. Using mathematical induction (see Appendix 1), prove that $|a^n| = |a|^n$ for any number a and positive integer n.