19-10-20

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Derivatives

Ordinary Derivatives

e.g $\frac{dy}{dx}$

dy → dependent variable

dx → Independent variable.

partial Derivatives

eg $\frac{\partial y}{\partial x}$

→ "The equation in which ordinary derivatives are involved is known as ordinary differiential equations."

Differential Equation:

An equation containing the derivatives of one or more dependent variables w.r.t one or more independent variables is called Differential equation.

Ordinary D. Equations:

If an equation containing derivatives of one or more dependent variables w.r.t one independent variable is called ODE.

partial D. Equation:

If an equation containing derivatives of one or more dependent variables w.r.t two or more independent variables is called PDE.

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ODE

(i) $\frac{dy}{dx} + 2xy = 0$

Dependent

Independent

(ii) $\frac{d^2y}{dx^2} - \frac{dy}{dx} + 6y = 0 \rightarrow$ 2nd order ODE

(iii) $\frac{dx}{dt} + \frac{dy}{dt} = 2x + y$

Two dependent variables "x" & "y". One independent variable "t".

(iv) $\frac{d^3y}{dx^3} + 3xy = 0 \rightarrow$ 3rd order differential equation.

or

$y''' + 3xy = 0$

$x''' + 3xy = 0$ Here "x" is dependent variable.

PDE :

→ $\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0$ one dependent variable "u" & 2 independent variables "x" & "y".

→ $\frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} = 0$ 2 dependent variables & 2 independent variable.

Order of D. Equation:

The order of highest derivative involved in a D. Equation is called order of D. Equation.

Degree of D. Equation:

The power of highest derivative involved in a D. Equation.

e.g $\left(\frac{dy}{dx}\right)^{1} + 2xy = 0$ 1 = Degree

Linear / Non-Linear D. Equation:

An nth order ODE of the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + a_{n-2}(x)\frac{d^{n-2}y}{dx^{n-2}} +$$

$$------ + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

is called Linear D. Equation if it satisfy the two conditions:

(i) The dependent variable y and all its derivatives $y', y'', --- y^n$ are of the 1st degree.

(ii) The co-efficients $a_0, a_1, a_2, ---- a_n$ of $y, y', y'', ----, y^n$ depends at most

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independent variable x.

or Co-efficients must be either constant or function of x (independent var).

e.g

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$$

(1) power of y & its derivatives are one.

(2) Co-efficients are constants.

So, it is a 2nd order Liner ODE.

Questions: 1.1 Q: 1—10

$$(1)\ \frac{d^3y}{dx^3} + x\frac{dy}{dx} - 5y = e^x$$

$$a_3(x)\frac{d^3y}{dx^3} + a_1(x)\frac{d^2y}{dx^2} + a_0(x)y = f(x)$$

$$a_2(x) = 0$$

Liner 3rd order ODE.

Non-Linear D. Equations:

$$(1-y)\frac{dy}{dx} + 2y = e^x$$

(function of dependent variable.)

$$\frac{d^2y}{dx^2} + y^2 = 0$$

(The power of y & its derivatives is 2).

$$\frac{d^2y}{dx^2} + \sin y = 0 \quad \text{or} \quad \frac{d^2y}{dx} - \sin y$$

$(y^2-1)dx + xdy = 0 \Rightarrow (y^2-1)dx = -xdy$

$y^2-1 = \frac{-xdy}{dx}$.

1 (a) If y is dependent:

$$y^2-1 + x\frac{dy}{dx} = 0$$

$$x\frac{dy}{dx} + y^2 = 1$$

→ It is a non Linear 1st order ODE in dependent variable y.

(b) If x is dependent:

$$(y^2-1)\frac{dx}{dy} + x = 0$$

(function of independent variable)

1st order Linear ODE with dependent variable x.

Solution of diffriential equation:

"Any function y, defined on some interval (I) and possesing at least n derivatives that are continous on (I), which when substituted into an nth order ODE, reduces the equation to identity."

e.g $y = \frac{1}{16}x^4$ is a solution of $\frac{dy}{dx} = xy^{1/2}$ on the interval. $(\infty, -\infty)$

$$\frac{dy}{dx} = xy^{1/2}$$

L.H.S:

$$\frac{d}{dx}\left(\frac{1}{16}x^4\right) = \frac{1}{4}x^3$$

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$R.H.S = xy^{1/2} \Rightarrow x\left(\frac{1}{16}x^4\right)^{1/2} = \frac{1}{4}x^4$

So, $L.H.S = R.H.S.$

Hence, verified that $y = \frac{1}{16}x^4$ is a solution of $\frac{dy}{dx} = xy^{1/2}$ on interval $(-\infty, \infty)$.

Ex. 1-5

e.g $y'' - 2y' + y = 0$; $y = xe^x$

$\frac{dy}{dx} = xe^x + e^x$

$\frac{d^2y}{dx^2} = xe^x + e^x + e^x$

$\frac{d^2y}{dx^2} = xe^x + 2e^x$

$\Rightarrow \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$

$L.H.S = \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y$

$= \cancel{xe^x} + \cancel{2e^x} - \cancel{2xe^x} - \cancel{2e^x} + \cancel{xe^x}$

$= 0$

$= R.H.S.$

Explicit functions / Solutions of D.E :

A solution of the form $y = f(x)$ is called an explicit solution if:

"The value of dependent variables in terms of independent variable."

e.g $y = \frac{1}{16}x^4$ y interms of x

$y = xe^x$ y interms of x.

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Implicit Solution / function of D.E:

An implicit Solution defines one or more explicit Solutions.

Ex 1.1
Q: 1
11–18
19–24

e.g $x^2+y^2=25 \longrightarrow$ Implicit sol.

$y=\pm\sqrt{25-x^2}$

$\left.\begin{array}{l} y_1=+\sqrt{25-x^2} \\ y_2=-\sqrt{25-x^2} \end{array}\right\}$ two explicit solutions of DE

So, both are two explicit solution of $\frac{dy}{dx}+\frac{x}{y}=0$ on interval $(-5,5)$

for y_1: $\frac{dy}{dx}=\frac{1}{\cancel{2}\sqrt{25-x^2}}(-\cancel{2}x)=\frac{-x}{\sqrt{25-x^2}}=\frac{-x}{y}$

$L.H.S=\frac{-x}{y}+\frac{x}{y}=0$

$=R.H.S.$

$\frac{dy_2}{dx}=\frac{-1}{\cancel{2}\sqrt{25-x^2}}(-\cancel{2}x)=\frac{x}{y}$

$\sqrt{25-x^2}$

$\Rightarrow -\frac{x}{y}+\frac{x}{y}=0$

$\rightarrow$

Verify that $x^2+y^2=25$ is an implicit Solution of $\frac{dy}{dx}+\frac{x}{y}=0$ on interval $(-5,5)$

$$x^2+y^2=25$$

$$\frac{d}{dx}\left(x^2+y^2\right)=0$$

$$2x+2y\frac{dy}{dx}=0$$

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$$x + y\frac{dy}{dx} = 0$$

$$x = -y\frac{dy}{dx}$$

$$\boxed{\frac{dy}{dx} = -\frac{x}{y}} \Rightarrow \frac{dy}{dx} + xy = 0$$

Hence, $x^2+y^2=25$ is an implicit Solution of $\frac{dy}{dx} + \frac{x}{y} = 0$

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Q:19 $\ln\left(\frac{2x-1}{x-1}\right) = t$ is an implicit solution of $\frac{dx}{dt} = (x-1)(1-2x)$

Sol:-

$\ln\left(\frac{2x-1}{x-1}\right) = t$ is not of the form $x = f(t)$.

$$\ln(2x-1) - \ln(x-1) = t$$

$$\frac{d}{dt}\left[\ln(2x-1)\right] - \frac{d}{dt}\left[\ln(x-1)\right] = 1$$

$$\frac{1}{2x-1} \cdot 2 \cdot \frac{dx}{dt} - \frac{1}{x-1}\frac{dx}{dt} = 1$$

$$\frac{dx}{dt}\left(\frac{2}{2x-1} - \frac{1}{x-1}\right) = 1$$

$$\frac{dx}{dt}\left(\frac{2(x-1) - 2x + 1}{(2x-1)(x-1)}\right) = 1$$

$$\frac{dx}{dt}\left(\frac{\cancel{2x} - 2 - \cancel{2x} + 1}{-(1-2x)(x-1)}\right) = 1$$

$$\frac{dx}{dt}\left(\frac{\cancel{-}1}{\cancel{-}(1-2x)(x-1)}\right) = 1$$

$$\boxed{\frac{dx}{dt} = (x-1)(1-2x)}$$

Hence, Verified

Explicit form:

$$\frac{2x-1}{x-1} = e^t$$

$$2x - 1 = xe^t - e^t$$

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$$2x - xe^t = 1 - e^t$$

$$x(2 - e^t) = 1 - e^t$$

$$\boxed{x = \frac{1-e^t}{2-e^t}} \quad \text{or} \quad x = f(t)$$

Q:

$$2xy\,dx + (x^2 - y)dy = 0 \; ;$$

$$-2x^2y + y^2 = 1$$

implicit form of Solution

$-\cancel{2x}x^2$

$$\frac{d}{dy}\left[-2x^2y + y^2\right] = 0$$

$$-2x^2 - 4xy\frac{dx}{dy} + 2y = 0$$

$$-4xy\frac{dx}{dy} = 2x^2 - 2y$$

$$-2xy\frac{dx}{dy} = x^2 - y$$

$$xy\,dx = (y - x^2)dy$$

$$\boxed{xy\,dx + (x^2 - y)dy = 0}$$

$$-2x^2y = 1 - y^2$$

$$-2x^2 = \frac{1-y^2}{y}$$

$$x^2 = \frac{1-y^2}{-2y}$$

$$x^2 = \frac{y^2-1}{2y}$$

$$\boxed{x = \pm\sqrt{\frac{y^2-1}{2y}}}$$

Q: $y = \frac{1}{16}x^4$ is an explicit Solution of

$$\frac{dy}{dx} = xy^2$$

Note: $y=0$ is a trivial Solution of

$$\frac{dy}{dx} = xy^2.$$

Q: $y = xe^x$ is an explicit Solution of

$$y'' - 2y' + y = 0$$

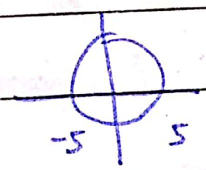
Note: $y=0$ is a trivial Solution of

$$y'' - 2y' + y = 0.$$

Solution curve:

The graph of the solution is called solution curve.

e.g $x^2 + y^2 = 25$



Linear D. Equation:

A first order differential equation of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x). \quad \text{---(i)}$$

is called 1st order Linear D.Eq.

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Note:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \rightarrow \text{General form of 1st order Eq.}$$

Pst

$$\frac{\cancel{a_1(x)}}{\cancel{a_1(x)}}\frac{dy}{dx} + \frac{a_0(x)}{a_1(x)}y = \frac{g(x)}{a_1(x)}$$

$$\frac{dy}{dx} + \frac{a_0(x)}{a_1(x)}y = \frac{g(x)}{a_1(x)}$$

Say:

$$\frac{a_0(x)}{a_1(x)} = p(x) \quad \&$$

$$\frac{g(x)}{a_1(x)} = f(x)$$

$$\boxed{\frac{dy}{dx} + p(x)y = f(x)}$$

This form of the equation is called standard form of 1st order Linear D. Eq.

Method of Solution

Given $a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$ —(i)

$\frac{dy}{dx}$ y'' $\frac{d^2y}{dx^2}$

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1. Convert (i) into standard form

$$\frac{dy}{dx} + P(x)y = f(x) \quad \text{—— (i)}$$

2. Calculate integerating factor:

$$IF = e^{\int p(x)\,dx}$$

3. Multiply eq. (ii) by I.F.

$$e^{\int p(x)dx}\frac{dy}{dx} + P(x)e^{\int p(x)dx}y = e^{\int p(x)dx}f(x)$$

$$\frac{d}{dx}\Big[\underbrace{e^{\int p(x)dx}}_{\text{I.F.}} \cdot \underbrace{y}_{\text{Dependent Variable.}}\Big] = e^{\int p(x)dx} \cdot f(x)$$

Apply integeration on b.S.

$$e^{\int p(x)dx} \cdot y = \int e^{\int p(x)dx} \cdot f(x)\,dx.$$

or

$$y = \frac{\int e^{\int p(x)dx} \cdot f(x)\,dx}{e^{\int p(x)dx}}.$$

2.3
Q.17 Example

$$\cos x\frac{dy}{dx} + \sin y = 1$$ → General form of Linear D.Eq.

$$\frac{dy}{dx} + \frac{\sin x}{\cos x}y = \frac{1}{\cos x} \quad \text{—— (i)}$$ → Standard form of D.Eq.

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where $p(x) = \frac{\sin x}{\cos x}$

$\int p(x)\,dx = -\int \frac{-\sin x}{\cos x}\,dx$ $\quad \because \int \frac{1}{x} = \ln x$

$\int p(x)\,dx = -\ln|\cos x|$

$I.F = e^{\int p(x)}$ $\quad e^{-\ln|\cos x|} = e^{\ln(\cos x)^{-1}}$

$\boxed{I.F = \cos x^{-1}}$

$\frac{dy}{dx} + \frac{\sin x}{\cos x} y = \frac{1}{\cos x}$ $\quad$ (i)

xing by I.F.

$\frac{1}{\cos x}\frac{dy}{dx} + \frac{\sin x}{\cos^2 x} y = \frac{1}{\cos^2 x}$.

$\sec x \frac{dy}{dx} + \sec x \tan x\, y = \sec^2 x$

$\frac{d}{dx}(u \cdot v) = u\frac{dv}{dx} + v\frac{du}{dx}$

Dependant var.

$\int \frac{d}{dx}\left[\sec x \cdot y\right] = \int \sec^2 x$.

Applying integration:

$\sec x \cdot y = \int \sec^2 x\,dx$.

$\sec x \cdot y = \tan x + c$ (implicit Solution)

or $y = \cancel{\cos x} \cdot \frac{\sin x}{\cancel{\cos x}} + c \cdot \cos x$

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$$\boxed{y = \sin x + c\cos x}$$ explicit function.

Q: 34 $\quad x(x+1)\frac{dy}{dx} + xy = 1 \; ; \; y(e) = 1$

initial condition.

Sol:

$$\frac{dy}{dx} + \frac{\cancel{x}}{\cancel{x}(x+1)}y = \frac{1}{x(x+1)}$$

$$\frac{dy}{dx} + \frac{1}{(x+1)}y = \frac{1}{x(x+1)} \quad \text{(i)}$$

$$\Rightarrow p(x) = \frac{1}{x+1}$$

$$\int \frac{1}{x+1} = \ln|x+1|$$

$$I.F = e^{\ln|x+1|}$$

$$\boxed{I.F = x+1}$$

xing (i) by I.F

$$(x+1)\frac{dy}{dx} + \frac{\cancel{x+1}}{\cancel{x+1}}y = \frac{\cancel{x+1}}{x(\cancel{x+1})}$$

$$(x+1)\frac{dy}{dx} + y(1) = \frac{1}{x}$$

$$\frac{d}{dx}\left[(x+1)\cdot y\right] = \frac{1}{x}$$

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$$(x+1)y = \int \frac{1}{x}\, dx$$

$$\boxed{(x+1)y = \ln(x) + c}$$ General Solution.

$$e+1 = \ln(e) + c$$

$$e+1 = 1 + c$$

$$\boxed{c = e}$$

$$\boxed{(x+1)y = \ln(x) + e}$$ perticular Solution.

Exercise: 2.3

39, 40, 30, 36

Q: 39 $\frac{dy}{dx} + 2xy = f(x)$; $y(0) = 2$

$$F(x) = \begin{cases} x & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$$

When: $0 \le x \le 1$

$$\frac{dy}{dx} + 2xy = x$$

When: $x > 1$

$$\frac{dy}{dx} + 2xy = 0$$

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Seperable Equation:

A 1st order D.E of the form $\frac{dy}{dx} = g(x)\,h(y)$ or

$$\frac{dy}{dx} - g(x)\,h(y) = 0$$

is said to be seperable or to have seperable variable.

Method of Solution:

$$\frac{dy}{dx} = g(x)\,h(y)$$

$$\frac{1}{h(y)}\,dy = g(x)\,dx \quad (S.E)$$

Apply integeration on b.s.

$$\int \frac{1}{h(y)}\,dy = \int g(x)\,dx$$

Say: $\int \frac{1}{h(y)}\,dy = H(y)$

& $\int g(x)\,dx = G(x)$

$$H(y) = G(x) + c$$

Note:

$H(y) = G(x) + c$ is also called family of solution or General solution.

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If $c = 1$

$$H(y) = G(x) + 1 \longrightarrow \text{perticular Solution.}$$

Example : 1

$$\frac{dy}{dx} = -\frac{x}{y} \; ; \; y(4) = -3$$

initial condition

$$y\,dy = -x\,dx$$

apply integeration.

$$\int y\,dy = -\int x\,dx$$

$$\boxed{y^2 = -x^2 + c} \longrightarrow \text{General Solution.}$$

To find the perticular Solution we have to apply initial Condition.

$$y(4) = -3$$

$$x^2 + y^2 = c$$

$$(4)^2 + (-3)^2 = c$$

$$\boxed{c = 25}$$

Now, $x^2 + y^2 = 25$

perticular solution

$$(e^{2y} - y) \cos x \frac{dy}{dx} - e^y \sin^2 x = 0 \; , \; (y(0) = 0) \text{ I.C}$$

$$(e^{2y} - y) \cos x \frac{dy}{dx} = e^y \sin^2 x$$

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$$\left(\frac{e^{2y}-y}{e^{y}}\right) dy = \frac{\cancel{e^{y}}}{\cancel{e^{y}}}\frac{\sin 2x}{\cos x}\, dx$$

$$dy\,(e^{y} - y e^{-y}) = \frac{2\sin x \cancel{\cos x}}{\cancel{\cos x}}\, dx$$

$$dy\,(e^{y} - y e^{-y}) = 2\sin x\, dx$$

Apply integeration on b.s

$$\int (e^{y} - y e^{-y})\, dy = 2\int \sin x\, dx$$

$$e^{y} - (-y e^{-y} - e^{-y}) = -2\cos x + c$$

$$e^{y} + y e^{-y} + e^{-y} = -2\cos x + c \quad \text{(i)}$$

General Solution

Apply initial condition on (i) $y(0)=0$

$$e^{0} + (0)e^{-0} + e^{0} = -2\cos(0) + c$$

$$1 + 0 + 1 = -2 + c$$

$$\boxed{c = 4}$$

The perticular Solution would be :

$$\boxed{e^{y} + y e^{-y} + e^{-y} = -2\cos x + 4}$$

Bernoulli's Equations

$$\frac{dy}{dx} + P(x)y = f(x)y^n \quad \text{where } n \neq 0, 1$$

$n = 0$

$$\frac{dy}{dx} + P(x)y = f(x) \quad \text{standard form}$$

of Linear D-equation.

$n = 1$

$$\frac{dy}{dx} + P(x)y = f(x)y$$

$$\frac{dy}{dx} + P(x)y - f(x)y = 0$$

$$\frac{dy}{dx} + (P(x) - f(x))y = 0$$

$\because f(x) = 0,\ a(x) = \text{function}.$

$$\frac{dy}{dx} + a(x)y = 0 \longrightarrow \text{Standard form}$$

of Linear D.equation.

Method of Solution:

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

$\div$ing by y^n

$$\frac{1}{y^n}\frac{dy}{dx} + P(x)\frac{y}{y^n} = f(x)$$

$$\frac{1}{y^n}\frac{dy}{dx} + P(x)y^{1-n} = f(x) \quad \text{---(i)}$$

~~Substitution~~ Let $u = y^{1-n}$

$$\frac{du}{dx} = (1-n)y^{-n}\frac{dy}{dx}.$$

$$\boxed{\frac{dy}{dx} = \frac{1}{1-n}y^n\frac{du}{dx}} \quad \text{---(ii)}$$

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$$\frac{1}{y^{n}} \cdot \frac{1}{1-n} \cdot y^{n} \frac{du}{dx} + p(x)u = f(x)$$

$$\frac{1}{1-n}\frac{du}{dx} + p(x)u = f(x).$$

General form of D. Linear eq with dependant u

Example: $$x\frac{dy}{dx} + y = \frac{1}{y^2}$$

$$x\frac{dy}{dx} + y = y^{-2}$$

÷ing by x

$$\frac{dy}{dx} + \frac{1}{x}y = \frac{1}{x}y^{-2} \quad \text{—(i)} \qquad \frac{dy}{dx} + p(x)y = f(x)y^{n}$$

÷ing by y^{-2}

$$\frac{1}{y^{-2}}\left(\frac{dy}{dx}\right) + \frac{1}{x}y^{3} = \frac{1}{x} \quad \text{—(ii)}$$

Let $u = y^3$

$$\frac{du}{dx} = 3y^2\frac{dy}{dx}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{3y^2}\frac{du}{dx}}$$

put in eq (ii)

$$\frac{1}{y^{-2}} \cdot \frac{1}{3y^2}\frac{du}{dx} + \frac{1}{x}u = \frac{1}{x}$$

$$\frac{1}{3}\frac{du}{dx} + \frac{1}{x}u = \frac{1}{x} \quad \text{—(iii)}$$

General form of Linear D-eq.

$$\frac{du}{dx} + \frac{3}{x}u = \frac{3}{x} \quad \text{——(iii)}$$

$$\Rightarrow \quad P(x) = \frac{3}{x}$$

$$\int P(x) = \int \frac{3}{x} = 3\ln x$$

$$I.F = e^{\int P(x)} = e^{3\ln x} \Rightarrow e^{\ln x^3} \Rightarrow x^3$$

xing (iii) by I.F

$$x^3\frac{du}{dx} + 3x^2 u = 3x^2$$

$$\frac{d}{dx}\left[\underline{\underline{x^3 \cdot u}}\right] = 3x^2$$

Apply integral

$$\int \frac{d}{dx}\left(x^3 \cdot u\right) = 3\int x^2$$

$$x^3 u = \frac{\cancel{3}x^3}{\cancel{3}} + c$$

$$x^3 u = x^3 + c$$

<u>OR</u> $u = 1 + \frac{c}{x^3}$ General Solution of Linear eq.

As, $u = y^3$

$$y^3 = 1 + \frac{c}{x^3}$$

<u>OR</u>

$$\boxed{y = \left(1 + \frac{c}{x^3}\right)^{1/3}}$$ The general Solution of Bernoulli's equation.

25 Date
2-15-22

Q:20 $3(1+t^2)\frac{dy}{dt} = 2ty(y^3-1)$

$$3(1+t^2)\frac{dy}{dt} = 2ty^4 - 2ty$$

$$3(1+t^2)\frac{dy}{dt} + 2ty = 2ty^4$$

Bernoulli's Equation

$\div$ing by y^4

$$3(1+t^2)\frac{1}{y^4}\frac{dy}{dt} + 2ty^{-3} = 2t \quad \text{--- (i)}$$

Let $y^{-3} = u$

$$\frac{du}{dt} = -3y^{-4}\frac{dy}{dt}$$

$$\boxed{\frac{dy}{dt} = \frac{1}{-3}y^4\frac{du}{dt}}$$ put in eq (i)

$$\cancel{3}(1+t^2)\frac{1}{\cancel{y^4}}\cdot\frac{1}{-\cancel{3}}\cancel{y^4}\frac{du}{dt} + 2tu = 2t$$

$$-(1+t^2)\frac{du}{dt} + 2tu = 2t \quad \text{--- (ii)}$$

General form of Bernoulli's eq with dependant variable u.

Convert (ii) into standard form.

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$$\frac{du}{dt} - \frac{2t}{1+t^2} u = \frac{-2t}{1+t^2} \quad \text{---(iii)}$$

here $\Rightarrow p(x) = \frac{-2t}{1+t^2}$ $\quad \underline{2.5}$

15-22

$$\int p(x) = -\ln(1+t^2) \quad \underline{or} \quad \ln(1+t^2)^{-1}$$

$$I.F = e^{\int p(x)} = e^{\ln(1+t^2)^{-1}} = (1+t^2)^{-1}$$

xing I.F by (iii)

$$\frac{1}{1+t^2}\frac{du}{dt} - \frac{2t}{(1+t^2)^2} u = \frac{-2t}{(1+t^2)^2}$$

$$\frac{d}{dt}\left(\frac{1}{1+t^2} \cdot u\right) = \frac{-2t}{(1+t^2)^2}$$

Dependant variable.

$$\frac{1}{1+t^2} \cdot u = -\int \frac{2t}{(1+t^2)^2} dt$$

$$\frac{1}{1+t^2} u = -\int 2t(1+t^2)^{-2} dt$$

$$\frac{1}{1+t^2} u = \frac{+(1+t^2)^{-1}}{+1} + c$$

$$u = 1 + c(1+t^2)$$

As, $u = y^{-3}$

$$y^{-3} = 1 + c(1+t^2)$$

General Solution of Given B.eq.

$\underline{OR}$ $\quad y^3 = \frac{1}{1+c(1+t^2)}$ $\quad \underline{or} \quad y = \frac{1}{(1+c(1+t^2))^{1/3}}$

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Homogeneous Equation:

A 1st order D.Equation of the form $M(x,y)dx + N(x,y)dy = 0$ is called Homogeneous Equation if:
Both functions $M(x,y)$ & $N(x,y)$ are homogeneous function of the same degree.

Explaination:

$$f(x,y) = 2x^3 + y^3 + 3x^2y$$

$f(x,y)$ is a homogeneous function of degree 3.

A function $f(x,y)$ is called Homogeneous function of degree n if it can be written in the form

$$f(tx, ty) = t^n f(x,y).$$

e.g

$$f(tx, ty) = 2(tx)^3 + (ty)^3 + 3(tx)^2(ty)$$

$$f(tx, ty) = t^3(2x^3 + y^3 + 3x^2y)$$

$$\boxed{f(tx, ty) = t^3 f(x,y)}$$

Degree of H.F.

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Q: $(2x^2y)dx + (xy^2)dy = 0$

$M(x,y)dx + N(x,y)dy = 0$

$\Rightarrow \quad M(x,y) = 2x^2y$

$N(x,y) = xy^2$

Both $M(x,y)$ & $N(x,y)$ are homogenous equation of degree 3.

Method of Solution:

Given a homogeneous eq

2.5 : 14

$M(x,y)dx + N(x,y)dy = 0$ —— (i)

By using the substitution:

$y = ux$ or $x = vy$

We can convert eq (i) into seperable eq.

Q:6 $(y^2+yx)dx + x^2dy = 0$ —— (i)

So, the given eq. is a Homogenous D.eq.

$y = u(x)$ $\quad \frac{dy}{dx} = \frac{d}{dx}(ux)$

Let $y = ux \Rightarrow \frac{dy}{dx} = u + x\frac{du}{dx}$ $\quad u(1) + x\frac{du}{dx}$

$dy = udx + xdu$

put in eq (i)

$(u^2x^2 + ux^2)dx + x^2[udx + xdu] = 0$

$$x^2(u^2+u)\,dx + x^2[u\,dx + x\,du] = 0$$

$$x^2[(u^2+u)\,dx + u\,dx + x\,du] = 0$$

$$x^2 \neq 0$$

$$(u^2+u)\,dx + u\,dx + x\,du = 0$$

$$dx[u^2+u+u] + du(x) = 0$$

$$(u^2+2u)\,dx + x\,du = 0$$

$$(u^2+2u)\,dx = -x\,du.$$

$$-\frac{1}{x}\,dx = \frac{1}{u^2+2u}\,du \quad (\text{Seperable eq})$$

Apply integeral on both sides.

$$-\int \frac{1}{x}\,dx = \int \frac{1}{u^2+2u}\,du.$$

$$\frac{1}{u(u+2)} = \frac{A}{u} + \frac{B}{u+2} \Rightarrow \frac{1}{2u} - \frac{1}{2u+4}$$

$$1 = A(u+2) + Bu$$

$$\boxed{B = -1/2}$$

$$\boxed{A = 1/2}$$

$$-\int \frac{1}{x}\,dx = \frac{1}{2}\int \left(\frac{1}{u} - \frac{1}{u+2}\right) du.$$

$$-\ln x = \frac{1}{2}[\ln u - \ln(u+2)] + c$$

By Substituting Back again.

$$-\ln x = \frac{1}{2}\left[\ln\left(\frac{y}{x}\right) - \ln\left(\frac{y}{x} + 2\right)\right] + c$$

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Partial Derivatives:

$$f(x,y) = 2x^2y^3$$

$\frac{\partial f}{\delta x}$ = partial derivative of $f(x,y)$ w.r.t "x"

$\frac{\delta f}{\delta y}$ = partial derivative of $f(x,y)$ w.r.t "y"

$$\frac{\delta f}{\delta x} = \frac{\delta}{\delta x}(2x^2y^3) = 2y^3(2x) = 4xy^3.$$

$$\frac{\delta f}{\delta x} = \frac{\delta}{\delta y}(2x^2y^3) = 2x^2 3y^2 \Rightarrow 6x^2y^2$$

$$z = f(x,y)$$

$$d(z) = d[f(x,y)]$$

$$d(z) = \frac{\delta f}{\delta x}dx + \frac{\delta f}{\delta y}dy$$

$$d(f(x,y)) = dx\,4xy^3 + 6x^2y^2dy.$$

Exact Differential:

An expression of the form

$M(x,y)\,dx + N(x,y)\,dy$

is called exact differential if it is the ~~form~~ (exact) derivative of same function $f(x,y)$.

e.g → $d(x,y) = ydx + xdy$ ✓

$d(f(x,y)) = Mdx + Ndy$

$ydx + xdy$ is called an exact differential because it is the derivative of

$$f(x,y) = xy.$$

<u>Exact Equation:</u>

A 1st order D-equation of the form $M(x,y)dx + N(x,y)dy = 0$ is called an exact equation if the expression on L.H.S is an exact differential of same function $f(x,y)$.

eg $ydx + xdy = 0$ is an exact D.equation because $ydx + xdy$ is an exact differential.

$$d(xy) = ydx + xdy$$

$x^2y^3dx + y^3x^3dy = 0$ is an exact D.equation bcz L.H.S is exact differential.

Criterion for an <u>Exact Differential:</u>

An expression is exact differential iff: $\frac{\delta M}{\delta y} = \frac{\delta N}{\delta x}$

eg $(2x - 5y)dx + (-5x + 3y)dy = 0$

$M(x,y) = 2x - 5y$, $N(x,y) = -5x + 3y$

$\frac{\delta M}{\delta y} = \frac{\delta}{\delta y}(2x - 5y)$

$= -5$

$\frac{\delta N}{\delta x} = -5 + 0$

$= -5$

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$$\Rightarrow \frac{\delta M}{\delta y} = \frac{\delta N}{\delta x}$$

So, given differential eq. is an exact differential eq.

Example of Solving exact D-eq.

$$2xy\,dx + (x^2-1)\,dy = 0 \quad \text{——— (i)}$$

$\Rightarrow M(x,y) = 2xy$, $N(x,y) = x^2-1$

$\frac{\delta M}{\delta x} = 2x$ $\qquad \frac{\delta N}{\delta x} = 2x$

$$\Rightarrow \frac{\delta M}{\delta x} = \frac{\delta N}{\delta x}$$

Hence, eq(i) is an exact D-equation.

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$$\frac{\delta f}{\delta x}\,dx + \frac{\delta f}{\delta y}\,dy = 0$$

$M(x,y) = \frac{\delta f}{\delta x}$, $N(x,y) = \frac{\delta f}{\delta y}$

$\frac{\delta f}{\delta x} = 2xy$ ——— (i) $\qquad \frac{\delta f}{\delta y} = x^2-1$ ——— (ii)

(i) $\Rightarrow$ Apply partial integration w.r.t "x"

$$f(x,y) = \int 2xy\,\delta x + h(y)$$

$$= 2y\int x\,\delta x + h(y)$$

$$f(x,y) = x^2y + h(y) \quad \text{——— (iii)}$$

(ii) $\Rightarrow \frac{\delta f}{\delta y} = x^2-1$

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$$\frac{\delta}{\delta y}\left(x^2y + h(y)\right) = x^2 - 1$$

$\because \frac{d}{dy} h(y) = h'(y)$

partial & ordinary are same always in this case.

$$x^2 + h'(y) = x^2 - 1$$

$$h'(y) = -1$$

$$\frac{d}{dy}(h(y)) = -1$$

Apply integeration w.r.t "y"

$$h(y) = -y + c$$

So, $f(x,y) = x^2y + h(y)$

$$\boxed{f(x,y) = x^2y - y + c}$$

$x^2y - y + c$ is the solution of exact differential equation.

OR $\int M\,dx + \int N\,dy = c$

(y- Const) (Not containing x term)

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Q: $(\sin y - y\sin x)dx + (\cos x + x\cos y - y)dy = 0$

$M(x,y)dx + N(x,y)dy = 0$ (General form)

$\Rightarrow M(x,y) = \sin y - y\sin x$

$$\frac{\delta M}{\delta y} = \frac{\delta}{\delta y}(\sin y - y\sin x) \text{ --- (ii)}$$

$$\boxed{\frac{\delta M}{\delta y} = \cos y - \sin x} \text{ --- (iii)}$$

$$\frac{\delta N}{\delta x} = \frac{\delta}{\delta x}(\cos x + x\cos y - y) \text{ --- (i)}$$

$$\boxed{\frac{\delta N}{\delta x} = -\sin x + \cos y} \text{ --- (iv)}$$

$$\Rightarrow \frac{\delta M}{\delta y} = \frac{\delta N}{\delta x}$$

Hence, given equation is an exact eq.
Apply partial integration w.r.t x on eq (ii)

$$f(x,y) = \int (\sin y - y\sin x)\,\delta x + h(y)$$

$$= \int \sin y\,\delta x - \int y.\sin x\,\delta x + h(y)$$

$$= \sin y(x) - y(-\cos x) + h(y)$$

$$f(x,y) = x\sin y + y\cos x + h(y) \text{ --- (v)}$$

$$(i) \Rightarrow \frac{\delta f}{\delta y} = \cos x + x\cos y - y$$

$$\frac{\delta}{\delta y}(x\sin y + y\cos x + h(y)) = \cos x + x\cos y - y$$

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$$x\frac{\delta}{\delta y}(\sin y)+\cos x\frac{\delta}{\delta y}(y)+\frac{\delta}{\delta y}h(y)=$$

$$\cos x+x\cos y-y$$

$$\cancel{x\cos y}+\cancel{\cos x}+h'(y)=\cancel{\cos x}+\cancel{x\cos y}-y$$

$$\boxed{h'(y)=-y} \quad \text{or} \quad \frac{dh}{dy}=-y$$

Apply integration w.r.t "y"

$$\boxed{h(y)=-\frac{y^2}{2}+c}$$

hence,

$$f(x,y)=x\sin y+y\cos x-\frac{y^2}{2}+c$$

finally the solution of given D.eq is:

$$f(x,y)=c$$

$$\boxed{x\sin y+y\cos x-\frac{y^2}{2}=c}$$

Verification:

$$d\left(x\sin y+y\cos x-\frac{y^2}{2}\right)=d(c)$$

$$\frac{\delta f}{\delta x}dx+\frac{\delta f}{\delta y}dy=0 \quad \because$$

$$f(x,y)=c$$

$$d(f(x,y))=d(c)$$

$$\frac{\delta f}{\delta x}dx+\frac{\delta f}{\delta y}dy=0$$

$$f(x,y)=x\sin y+y\cos x-\frac{y^2}{2}$$

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$$\frac{\delta f}{\delta x} = \sin y \frac{\delta}{\delta x}(x) + y\frac{\delta}{\delta x}(\cos x)$$

$$\boxed{\frac{\delta f}{\delta x} = \sin y + y(-\sin x)}$$

$$\cancel{\frac{\delta f}{\delta x} = x\cos y \quad \sin y \frac{\delta}{\delta}}$$

$$\frac{\delta f}{\delta y} = x\frac{\delta}{\delta y}(\sin y) + \frac{\delta}{\delta y}(y)\cos x - \frac{1}{2}\frac{\delta y^2}{\delta y}$$

$$\boxed{\frac{\delta f}{\delta y} = x\cos y + \cos x - y}$$

$$(\sin y - y\sin x)dx + (\cos x + x\cos y - y)dy = 0$$