Assignment = 4

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Section 2E — Thomas Calculus

Q1:

@ $\int_0^{\pi/2} \sin^n x \, dx$

$= \int \sin \quad dx$

$= \int \sin^{n-1} x \cdot \sin x \, dx.$

$= \sin^{n-1} x \int \sin x \, dx - \int \left(\int \sin x \, dx \right) \frac{d}{dx} \left(\sin^{n-1} x \right) dx$

$= \sin^{n-1} x (-\cos x) - \int (-\cos x)(n-1) \sin^{n-2} (\cos x) \, dx$

$= \sin^{n-1} x (-\cos x) + (n-1) \int \sin^{n-2} x \cdot \cos^2 x \, dx$

$= \sin^{n-1} x (-\cos x) + (n-1) \int \sin^{n-2} x (1 - \sin^2 x) \, dx$

$= \sin^{n-1} x (-\cos x) + (n-1) \int \sin^{n-2} x \, dx - (n-1) \int \sin^{n-2} x \cdot \sin^2 x \, dx$

$= \sin^{n-1} x (-\cos x) + (n-1) \int \sin^{n-2} x \, dx - (n-1) \int \sin^n x \, dx$

Now,

$1 \int \sin^n x \, dx + (n-1) \int \sin^n x \, dx = -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x \, dx$

$n \int \sin^n x \, dx = -\cos x \sin^{n-1} x + n-1 \int \sin^{n-2} x \, dx$

Now Applying Limit

$$\int_0^{\pi/2} \sin^n x\,dx = \left|\frac{-\cos x \sin^{n-1} x}{n}\right|_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$

(b):

$$\int_0^{\pi/2} \sin^6 x\,dx$$

$$\left|\frac{-\cos x \sin^{6-1} x}{6}\right|_0^{\pi/2} - \left|\frac{6-1}{6}\int_0^{\pi/2} \sin^4 dx\right.$$

$$= \left|\frac{-\cos x \sin^5 x}{6}\right|_0^{\pi/2} - \frac{5}{6}\left(\left|\frac{-\cos x \sin^{4-1}}{4}\right|_0^{\pi/2} - \frac{4-1}{4}\int_0^{\pi/2} \sin^2 x\,dx\right)$$

$$= \left|\frac{-\cos x \sin^5 x}{6}\right|_0^{\pi/2} - \frac{5}{6}\left(\left|\frac{-\cos x \sin^3 x}{4}\right|_0^{\pi/2} - \frac{3}{4}\left(\left|-\cos x \sin x\right|_0^{\pi/2} - \frac{2-1}{2}\int_0^{\pi/2} \sin^0 x\,dx\right)\right)$$

$$= \left|\frac{-\cos x \sin x}{6}\right|_0^{\pi/2} - \frac{5}{6}\left(\left|\frac{-\cos x \sin^3 x}{4}\right|_0^{\pi/2} - \frac{3}{4}\left(\left|\frac{-\cos x \sin x}{2}\right|_0^{\pi/2} - \frac{1}{2}\left(x\Big|_0^{\pi/2}\right)\right)\right)$$

$$= 0 - \frac{5}{6}\left(0 - \frac{3}{4}\left((0) - \frac{1}{2}\left(\pi/2 - 0\right)\right)\right)$$

$$= 0 - \frac{5}{6}\left(-\frac{3}{4}\right)\left(-\frac{\pi}{4}\right) \Rightarrow -\frac{5}{\cancel{6}_2}\left(\frac{\cancel{3}\pi}{16}\right)$$

$$= \frac{-5\pi}{32}.$$

Q2:- $\int_0^{\pi/2} \cos^n x\,dx$

(Solve without Limits)

$$\int \cos^n x\,dx = \int \cos^{n-1} x \cos x\,dx$$

$$\Rightarrow \cos^{n-1} x \int \cos x\,dx - \int \left(\left(\int \cos x\,dx\right) \frac{d}{dx}\left(\cos^{n-1} x\right) \right) dx$$

$$\cos^{n-1} x (\sin x) - \int \sin x (n-1) \cos^{n-2} x (-\sin x)\,dx$$

$$\cos^{n-1} x (\sin x) + (n-1) \int \cos^{n-2} x \sin^2 x\,dx$$

$$\cos^{n-1} x (\sin x) + (n-1) \int \cos^{n-2} x (1 - \cos^2 x)\,dx$$

$$\cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x\,dx \, (n-1) \int \cos^{n-2} x \cos^2 x\,dx$$

$$\cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x\,dx - (n-1) \int \cos^n x\,dx.$$

Now

$$1 \int \cos^n x\,dx + (n-1) \int \cos^n x\,dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x\,dx$$

$$n \int \cos^n x\,dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x\,dx$$

$$\int \cos^n x\,dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{(n-1)}{n} \int \cos^{n-2} x\,dx$$

(with Limits)

$$\left\{ \int_0^{\pi/2} \cos^n x\,dx = \left| \frac{\cos^{n-1} x \sin x}{n} \right|_0^{\pi/2} + \frac{(n-1)}{n} \int_0^{\pi/2} \cos^{n-2} x\,dx \right\}$$

(b)

$$\int_0^{\pi/2} \cos^5 x \, dx$$

(Using Reduction Formula)

$$= \left| \frac{\cos^4 x \sin x}{5} \right|_0^{\pi/2} + \frac{4}{5} \int_0^{\frac{\pi}{2}} \cos^3 x.$$

$$= \left| \frac{\cos^4 x \sin x}{5} \right|_0^{\pi/2} + \frac{4}{5}\left(\left| \frac{\cos^2 x \sin x}{3} \right|_0^{\pi/2} + \frac{2}{3} \int_0^{\pi/2} \cos x \, dx \right)$$

$$= \left| \frac{\cos^4 x \sin x}{5} \right| + \frac{4}{5}\left(\left| \frac{\cos^2 x \sin x}{3} \right|_0^{\pi/2} + \frac{2}{3} \left| \sin x \right|_0^{\pi/2}\right.$$

$$= 0 + \frac{4}{5}\left(0 + \frac{2}{3}\left(\sin \pi/2 - \sin(0)\right)\right)$$

$$= \frac{4}{5}\left(\frac{2}{3}\right) = \frac{8}{15}$$

Q3

ⓐ $\int \tan^n x \, dx$

$$\int \tan^{n-2} x \tan^2 x \, dx = \int (\tan^{n-2} x (\sec^2 x - 1)) dx$$

$$\int \tan^{n-2} x \sec^2 x \, dx - \int \tan^{n-2} x \, dx$$

$$\frac{\tan^{n-2+1} x}{n-2+1} - \int \tan^{n-2} x \, dx.$$

$$\Rightarrow \tan^{n-1} x / n-1 - \int \tan^{n-2} x \, dx + c.$$

(b) $\int \tan^7 x\,dx$

(Usiong Reduction Formula)

$\frac{\tan^6 x}{6} - \int \tan^5 x\,dx + C$

$\frac{\tan^6 x}{6} - \left(\frac{\tan^4 x}{4} - \int \tan^3 x\,dx\right) + C$

$\frac{\tan^6 x}{6} - \left(\frac{\tan^4 x}{4} - \left(\frac{\tan^2 x}{2} - \int \tan x\right)\right) + C$

$\frac{\tan^6 x}{6} - \left(\frac{\tan^4 x}{4} - \left(\frac{\tan^2 x}{2} - \int (\cos x)^{-1} \sin x\,dx\right)\right) + C$

$\frac{\tan^6 x}{6} - \left(\frac{\tan^4 x}{4} - \left(\frac{\tan^2 x}{2} - \ln|\cos x|\right)\right) + C$

$\frac{\tan^6 x}{6} - \frac{\tan^4 x}{4} + \frac{\tan^2 x}{2} - \ln|\cos x| + C.$

Q# 4:

$\int_0^{\pi/2} \sin^2 x \cos^6 x\,dx$

$\int_0^{\pi/2} (1-\cos^2 x)\cos^6 x\,dx$

$\int_0^{\pi/2} \cos^6 x - \int_0^{\pi/2} \cos^8 x\,dx,$

(Using Reduction Formula)

$$\left(\left|\frac{\cos^5 x \sin x}{6}\right|_0^{\pi/2} + \frac{5}{6}\int_0^{\frac{\pi}{2}} \cos^4 x\,dx\right) - \left(\left|\frac{\cos^7 x \sin x}{8}\right|_0^{\pi/2} + \frac{7}{8}\int_0^{\pi/2} \cos^6 x\,dx\right)$$

$$\left(0+\frac{5}{6}\left(\left(\frac{\cos^3 x \sin x}{4}\right)^{\pi/2} + \frac{3}{4}\int_0^{\pi/2}\cos^2 x\,dx\right)\right) - \left(0+\frac{7}{8}\left(\left|\frac{\cos^5 x \sin x}{6}\right|_0 + \frac{5}{6}\int_0^{\pi/2}\cos^4 x\,dx\right.\right.$$

$$\frac{5}{6}\left(0+\frac{3}{4}\left(\left|\frac{\cos x \sin x}{2}\right|_0^{\pi/2} + \frac{1}{2}\int_0^{\pi/2} dx\right)\right) - \left(\frac{7}{8}\left(0+\frac{5}{6}\left(\left|\frac{\cos^3 x \sin x}{4}\right|_0^{\pi/2} + \frac{3}{4}\int_0^{\pi/2}\cos^2 x\,dx\right)\right)\right)$$

$$= \frac{5}{6}\left(\frac{3}{4}\left(\frac{\pi}{4}\right)\right) - \frac{7}{8}\left(\frac{5}{6}\left(\frac{3}{4}\left(0+\frac{1}{2}\left(\frac{\pi}{2}\right)\right)\right)\right)$$

$$\Rightarrow \frac{5}{6}\left(\frac{3\pi}{16}\right) - \frac{7}{8}\left(\frac{5}{6}\left(\frac{3\pi}{16}\right)\right)$$

$$= \frac{15\pi}{96} - \frac{7}{8}\left(\frac{15\pi}{96}\right) \Rightarrow \frac{15\pi}{96}\left(1-\frac{7}{8}\right)$$

$$= \frac{15}{96}\pi\left(\frac{8-7}{8}\right)$$

$$= \frac{15\pi}{768}$$

Q5.

(a) $\int \frac{dx}{a^2-x^2}$

$$\frac{1}{a^2-x^2} = \frac{A}{(a+x)} + \frac{B}{(a-x)} \rightarrow (1)$$

Multiply By L.C.M, we get

$1 = A(a-x) + B(a+x) \longrightarrow (2)$

Put; $a-x=0$; $x=a$ in (2)

$1 = A(a-a) + B(a+a)$

$1 = 2aB$

$$\boxed{B = \frac{1}{2a}}$$

Put $a+x=0$; $x=-a$ in (2)

$1 = A(a-(-a)) + B(a-a)$

$1 = A(2a)$

$$\boxed{A = \frac{1}{2a}}$$

$$\frac{1}{a^2-x^2} = \frac{1}{2a(a+x)} + \frac{1}{2a(a-x)}$$

$$\int \frac{1}{a^2-x^2}dx = \int \frac{1}{2a(a+x)}dx + \int \frac{1}{(a-x)2a}dx.$$

$$= \frac{1}{2a}\left(\int \frac{1}{a+x}\,dx + \int \frac{1}{a-x}\,dx\right)$$

$$= \frac{1}{2a}\left(\int \frac{1}{a+x}\,dx - \int \frac{-1}{a-x}\,dx\right)$$

$$= \frac{1}{2a}\left(\ln(a+x) - \ln(a-x)\right) + C \longrightarrow (1)$$

(b))

$$= \int_0^2 \frac{5}{9-x^2}\,dx$$

$$= 5 \cdot \int_0^2 \frac{1}{9-x^2}\,dx$$

$$5\left(\frac{1}{2(3)}\left(\ln(3+x) - \ln(3-x)\right|_0^2\right)$$

$$\frac{5}{6}\left[\ln(3+2) - \ln(3-2) - \ln(3+0) + \ln(3-0)\right]$$

$$= \frac{5}{6}\left[\ln|5| - \ln|1| - \cancel{\ln|3|} + \cancel{\ln|3|}\right]$$

$$= \frac{5}{6}\left[\ln|5| + 0\right)$$

$$= \frac{5}{6}\ln(5)$$

Q6:

$$\int \frac{dx}{\sqrt{a^2-x^2}}$$

Put $x = a\sin\theta$

$= dx = a\cos\theta dx$ —— ①

$$= \int \frac{dx}{\sqrt{a^2 - a^2\sin^2\theta}}.$$

$$= \int \frac{dx}{\sqrt{a^2(1-\sin^2\theta)}}$$

$$\int \frac{dx}{\sqrt{a^2(\cos^2\theta)}} \Rightarrow \int \frac{dx}{a\cos\theta}$$

From eq ①:

$$\int \frac{\cancel{a\cos\theta}\, d\theta}{\cancel{a\cos\theta}} = \int d\theta \Rightarrow \theta + c \longrightarrow ②$$

$\therefore \sin\theta = \frac{x}{a}$

$\theta = \sin^{-1}\left(\frac{x}{a}\right)$

Put in ②

$\sin^{-1}\left(\frac{x}{a}\right) + c$

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Q7

$$\int_0^4 \sqrt{16-x^2}\,dx$$

Put $x = 4\sin\theta$ — ①

Put $x = 4$;

$4 = 4\sin\theta$.

$\sin\theta = 1$

$\theta = \sin^{-1}(1)$

$$\boxed{\theta = \frac{\pi}{2}}$$

when $x = 0$;

$4\sin\theta = 0$

$\sin\theta = 0$

$\theta = \sin^{-1}(0)$

$\theta = 0$

From 1

$x = 4\sin\theta$

$dx = 4\cos\theta\,d\theta$

(Putting values)

$$\int_0^{\pi/2} \sqrt{16-(4\sin\theta)^2}\;4\cos\theta\,d\theta.$$

$$4\int_0^{\pi/2} \sqrt{16-16\sin^2\theta}\;\cos\theta\,d\theta$$

$$4\int_0^{\pi/2}\sqrt{16(1-\sin^2\theta)}\cos\theta\, d\theta$$

$$4\int_0^{\pi/2} 4\cos^2\theta\, d\theta$$

$$16\int_0^{\pi/2}\cos^2\theta\, d\theta$$

$$= 16\int_0^{\pi/2}\frac{1}{2}(1+\cos 2\theta)\, d\theta.$$

$$= 8\int_0^{\frac{\pi}{2}}(1+\cos 2\theta)\, d\theta.$$

$$= 8\int_0^{\pi/2} 1\, d\theta + \cancel{8}\int_0^{\pi/2}\cos 2\theta\, d\theta$$

$$= \overset{4}{\cancel{8}}\left(\frac{\pi}{\cancel{2}}\right) + \cancel{8}\left|\frac{\sin 2\theta}{\cancel{2}}\right|_0^{\pi/2}$$

$$= 4\pi + 4\left|\sin 2\theta\right|_0^{\pi/2}$$

$$= 4\pi + 4\left[\sin 2(\pi/2) - \sin 2(0)\right]$$

$$= 4\pi + 4(\sin\pi)$$

$$= 4\pi + 0 = 4\pi$$

Q8

(a) $\int \frac{dx}{a^2+x^2}$

$\therefore$ put $x = a\tan\theta$; $dx = a\sec^2\theta \, d\theta$

$= \int \frac{dx}{a^2+(a\tan\theta)^2}$

$= \int \frac{a \cdot \sec^2\theta \, d\theta}{a^2 + a^2\tan^2\theta}$

$= \int \frac{a\sec^2\theta \, d\theta}{a^2(1+\tan^2\theta)} \Rightarrow \int \frac{a\sec^2\theta \, d\theta}{a^2\sec^2\theta}$.

$= \int \frac{\cancel{a}\cancel{\sec^2\theta} \, d\theta}{a^{\cancel{2}}\cancel{\sec^2\theta}} \Rightarrow \frac{1}{a}\int d\theta$

$\frac{1}{a}\theta + C \Rightarrow \tan\theta = \frac{x}{a}$

$\theta = \tan^{-1}(x/a)$

$\Rightarrow \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

(b) $\int_0^2 \frac{dx}{4+x^2}$

$\left(\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2 \Rightarrow \frac{1}{2}\left|\tan^{-1}\left(\frac{x}{2}\right)\right|_0^2$

$\frac{1}{2}\left[\tan^{-1}\left(\frac{2}{2}\right) - \tan^{-1}(0)\right]$

$$= \frac{1}{2}\left(\frac{\pi}{4} - 0\right) = \boxed{\frac{\pi}{8}}$$