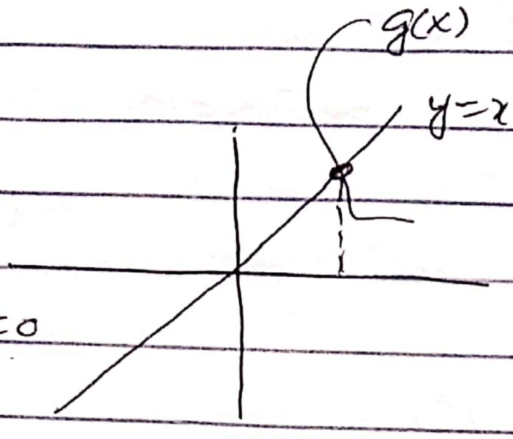
Fixed point iteration method :-

$x = g(x)$



$x - g(x) = 0$

$$f(x) = x - g(x) = 0$$

Fixed point : "A number at which the value of function does not change when function is applied."

The number p is a fixed point for a given function g if $g(p) = p$.

→ "By putting the value from Domain the range will also belongs to Domain."

Theorem :-

All the continous function on interval

i) If $g \in c[a,b]$ & $g(x) \in [a,b]$ for all $x \in [a,b]$ then g has atleast one fixed point in $[a,b]$.

ii) If in addition, $g'(x)$ exists on (a,b) such that $|g'(x)| \leq K < 1$, for all $x \in (a,b)$ then there is exactly one fixed point in $[a,b]$.

(unique fixed point)

Continuity defines on close interval.
Derivative defines on open interval.

Example:-

$$x^3+4x^2-10=0 \qquad [1,2]$$

$$x = g_1(x) = x - x^3 - 4x^2 + 10$$

$$x = g_2(x) = \left(\frac{10}{x} - 4x\right)^{1/2}$$

$$x = g_3(x) = \frac{1}{2}\left(10 - x^3\right)^{1/2}$$

$$x = g_4(x) = \left(\frac{10}{4+x}\right)^{1/2}$$

$$x = g_5(x) = x - \frac{x^3+4x^2-10}{3x^2+8x}$$

(newton's interation method OR newton-raphson method)

$$x_{n+1} = x_n - \frac{x_n^3+4x_n^2-10}{3x_n^2+8x_n}$$

$$x_{n+1} = \varphi(x_n)$$

" To approximate the fixed point of a function g, we choose an initial approximation P_0 and generate the sequence $\{P_n\}_{n=0}^{\infty}$ by letting $P_n = g(P_{n-1})$, for each $n \geq 1$. If the sequence converges to P & g is continous, then

$$P = \lim_{n\to\infty} g(P_n) = g\left(\lim_{n\to\infty} P_{n-1}\right) = g(P)$$

e.g $\quad x = x - \dfrac{x^3 + 4x^2 - 10}{3x^2 + 8x}$

to construct the corresponding iterative formula of the form

$$x_{n+1} = g(x_n)$$

the above expression becomes

$$x_{n+1} = x_n - \frac{x_n^3 + 4x_n^2 - 10}{3x_n^2 + 8x_n}$$

For $n = 0$

$$x_1 = x_0 - \frac{x_0^3 + 4x_0^2 - 10}{3x_0^2 + 8x_0}$$

$x_1 = 1.3733$

$$x_2 = x_1 - \frac{x_1^3 + 4x_1^2 - 10}{3x_1^2 + 8x_1}$$

$x_2 = 1.365262015$

$x_3 = 1.365230014$

$x_4 = 1.365230013$

$$x = \left(\frac{10}{4+x}\right)^{1/2}$$

$$x_{n+1} = \left(\frac{10}{4+x_n}\right)^{1/2} \quad \because x_0 = 1.5$$

$$x_1 = \left(\frac{10}{4+1.5}\right)^{1/2}$$

$$x_1 = 1.348399725$$

$$x_2 = \left(\frac{10}{4+1.348399725}\right)^{1/2}$$

$$x_2 = 1.367376372$$

$$x_3 = \left(\frac{10}{4+1.367376372}\right)^{1/2}$$

$$x_3 = 1.364957015$$

$$x_4 = \left(\frac{10}{4+1.367376372}\right)^{1/2}$$

$$x_4 = 1.365264748$$

$$x_5 = \left(\frac{10}{4+1.365264748}\right)^{1/2}$$

$x_5 = 1.365225594$

$$x_6 = \left(\frac{10}{4+1.365225594}\right)^{1/2}$$

$x_6 = 1.365230576$

$x_7 = 1.365229942$

$x_8 = 1.365230023$

$x_9 = 1.365230012$

$x_{10} = 1.365230014$

$x_{11} = 1.365230013$

Example: Find the real root of the equation $x^3+x^2-1=0$ by fixed point iteration method. $[0,1]$

$x = x + x^3 + x^2 - 1$

$x = \sqrt{-x^3+1}$

$$x = x - \frac{x^3+x^2-1}{3x^2+2x}$$

$x = (1-x^2)^{1/3}$

$x^2(x+1) - 1 = 0$

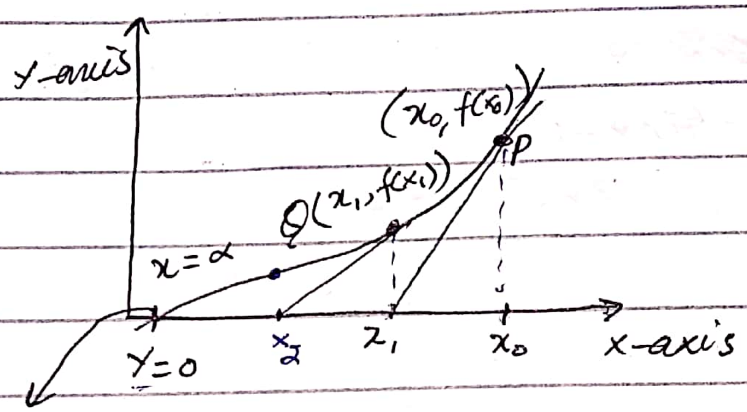
$$x = \frac{1}{\sqrt{x+1}} \quad \text{—— (i)}$$

1'st condition satisfied on eq (i)

Newton- Raphson Method:

Method of tangents.

$m = \frac{dy}{dx} = \tan\alpha$ (slope)



(Root)

$y - y_1 = m(x - x_1)$ (tangent line)

$m = \frac{y_2 - y_1}{x_2 - x_1}$ (slope)

$m = \frac{dy}{dx} = f'(x) = f'(x_0)$ when we have only one point given, So slope of point will be calculated through derivative.

$y - f(x_0) = f'(x_0)(x - x_0)$ ... (i)

Now, the TL (i) crosses the x-axis at point $x = x_1$, $y = 0$

$$0 - f(x_0) = f'(x_0)(x_1 - x_0)$$

$$x_1 = x_0 - \frac{f(x_0)}{f(x_0)'}$$

$$x_2 = x_1 - \frac{f(x_1)}{f(x_1)'}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f(x_n)'}$$ (Newton's iteration method)

$$f(x) = \cos x - x = 0 \qquad [0, \tfrac{\pi}{2})$$

$$f(0) = \cos 0 - 0 = 1 \quad +ve$$

$$f(\tfrac{\pi}{2}) = \cos\frac{\pi}{2} - \frac{\pi}{2} = 0 - \frac{\pi}{2} = -\frac{\pi}{2} \quad -ve$$

$$g(x) = x - \frac{\cos x - x}{-\sin x - 1}$$

$$x_n = x_{n-1} - \frac{\cos x_{n-1} - x_{n-1}}{-\sin x_{n-1} - 1}$$

$$x_0 = \frac{\pi}{4}$$

$$x_1 = \frac{\pi}{4} - \frac{\cos \pi/4 - \pi/4}{-\sin\frac{\pi}{4} - 1} =$$

$$r = 0.7390851332.$$

$$g(x) = x - \frac{f(x)}{f(x)'}$$

$$|g'(x)| = \left| 1 - \frac{f(x)'f(x)' - f(x)f''(x)}{[f(x)']^2} \right| < 1$$

$$= \left| \frac{\cancel{f(x)'^2} - \cancel{f(x)'^2} + f(x)f''(x)}{[f(x)']^2} \right| < 1$$

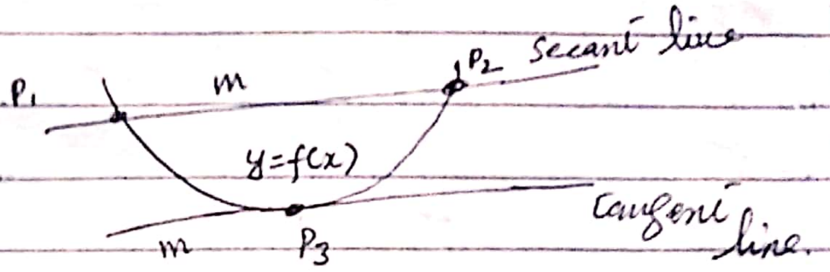
$$\frac{|f(x)f''(x)|}{[f(x)']^2} < 1$$

for newton-raphon method.

$|f(x)f''(x)| < (f'(x))^2$ Criterian

The Secant Method:

$m_1 = \frac{y_2 - y_1}{x_2 - x_1}$



To circumvent the problem of derivative evaluation in Newton's Method, we introduce a slight variation. By Definition:

$$f'(x_{n-1}) = \lim_{x \to x_{n-1}} \frac{f(x) - f(x_{n-1})}{x - x_{n-1}}$$

If x_{n-2} is very close to x_{n-1}, then

$$f'(x) \approx \frac{f(x_{n-2}) - f(x_{n-1})}{x_{n-2} - x_{n-1}} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f(x_{n-2}) - f(x_{n-1})} (x_{n-2} - x_{n-1})$$

$$x_n = x_{n-1} - f(x_{n-1}) \frac{(x_{n-1} - x_{n-2})}{f(x_{n-1}) - f(x_{n-2})}$$

$$x_n = x_{n-1} - \frac{(\cos x_{n-1} - x_{n-1})(x_1 - x_0)}{(\cos x_1 - x_1) - (\cos x_0 - x_0)} \quad n \geq 2$$

The Secant Method:

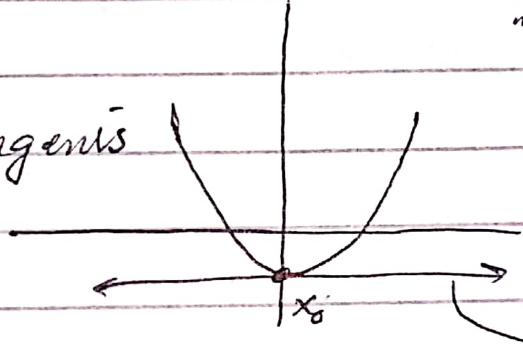
$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

~~New~~ Newton Method's Dangers:

Danger #1 x_0 is on the Stationary point.

"x_0 is on Stationary point"

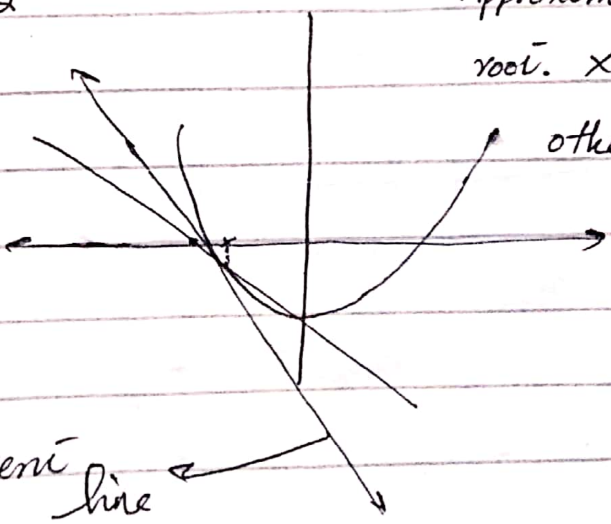
Horizontal tangents will not intersects the x-axis



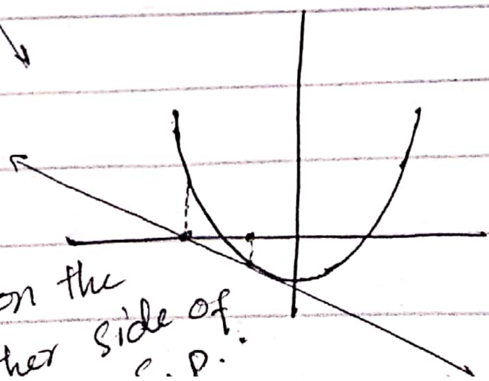
tangent line.

Danger #2

Approximate the wrong root. x_0 is on the other side of Stationary point.

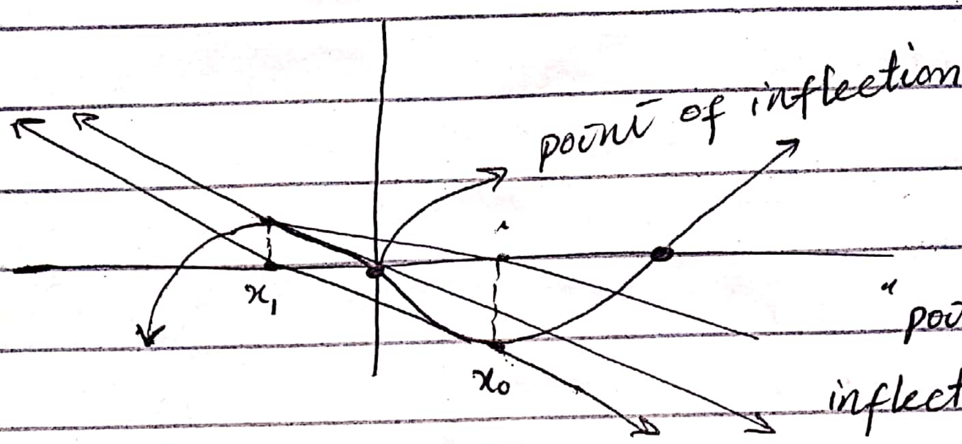


tangent line



"lowing on the other side of S.P.

Danger #3 Oscillating Sequence.



"point of inflection at the root."

imp: Note: "The ODD function."

e.g

$$f(x) = x^3 - 5x$$

$$f'(x) = 3x^2 - 5$$

$$x_0 = 1$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = -1$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1$$

Simultaneous Linear Algebraic Equations:

$$a_1x + a_2y + a_3z = d_1$$
$$b_1x + b_2y + b_3z = d_2$$
$$c_1x + c_2y + c_3z = d_3$$

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$AX = B$$

methods to solve system of linear equations

Direct Method:

→ Gauss elimination

→ Gauss – jordan

→ Factorization Method.

→ Crout's Method

Iterative Methods:

→ Guass-Jacobi

→ Guass-Seidel

"Factorization / Triangularisation Method"

eg

$$2x-3y+10z=3$$
$$-x+4y+2z=20$$
$$5x+2y+z=-12$$

$$A=\begin{bmatrix}2 & -3 & 10\\ -1 & 4 & 2\\ 5 & 2 & 1\end{bmatrix},\quad X=\begin{bmatrix}x\\ y\\ z\end{bmatrix},\quad B=\begin{bmatrix}3\\ 20\\ -12\end{bmatrix}$$

In this method, we use the fact that the square matrix A can be factorized into the form LU, where

L = unit lower triangular matrix

U = upper triangular matrix

eg
$$L=\begin{bmatrix}1 & 0 & 0\\ l_{21} & 1 & 0\\ l_{31} & l_{32} & 1\end{bmatrix}$$

$$U=\begin{bmatrix}u_{11} & u_{12} & u_{13}\\ 0 & u_{22} & u_{23}\\ 0 & 0 & u_{33}\end{bmatrix}$$

$$A = L\cdot U$$

(L, U Decomposition method)

$$\begin{bmatrix} 2 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$

By compairing the entries of both matrices,

$\Rightarrow \quad u_{11} = 2$

$u_{12} = -3$

$u_{13} = 10$

$l_{21}u_{11} \Rightarrow l_{21}(2) = -1 \Rightarrow \boxed{l_{21} = -1/2}$

$l_{21}u_{12} + u_{22} = 4 \Rightarrow (-\frac{1}{2})(-3) + u_{22} = 4$

$\boxed{u_{22} = 5/2}$

$l_{21}u_{13} + u_{23} = 2 \Rightarrow (-1/2)(10) + u_{23} = 2$

$\boxed{u_{23} = 7}$

$l_{31}u_{11} = 5 \Rightarrow l_{31}(2) = 5 \Rightarrow \boxed{l_{31} = 5/2}$

$l_{31}u_{12} + l_{32}u_{22} = 2 \Rightarrow (\frac{5}{2})(-3) + l_{32}(5/2) = 2$

$-\frac{15}{2} + \frac{5}{2} \cdot l_{32} = 2$

$\boxed{l_{32} = 19/5}$

$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 1$

$(5/2)(10) + (19/5)(7) + u_{33} = 1$

$25 + 133/5 + u_{33} = 1$

$$\boxed{u_{33} = -253/5}$$

As,

$AX = B$

$LUX = B$

$UX = Y$

$LY = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 5/2 & 19/5 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 20 \\ -12 \end{bmatrix}$$

$y_1 = 3$

$-1/2\, y_1 + y_2 = 20 \Rightarrow -\frac{1}{2}(3) + y_2 = 20$

$$\boxed{y_2 = 43/2}$$

$5/2\, y_1 + 19/5\, y_2 + y_3 = 12$

$5/2(3) + 19/5(43/2) + y_3 = 12$

$$\boxed{y_3 = -506/5}$$

$UX = Y$

$$\begin{bmatrix} 2 & -3 & 10 \\ 0 & 5/2 & 7 \\ 0 & 0 & -253/5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 43/2 \\ -\frac{506}{5} \end{bmatrix}$$

$$-\frac{253}{5} z = -\frac{506}{5}$$

$$z = -\frac{506}{\cancel{5}} \times \frac{\cancel{5}}{-253} \Rightarrow \boxed{z = 2}$$

$$\left(\frac{5}{2}\right) y + 7z = \frac{43}{2} \Rightarrow z = 2$$

$$\boxed{y = 3}$$

$$2x - 3y + z(10) = 3$$

$$2x - 3(3) + 10(2) = 3$$

$$\boxed{x = -4}$$

So, the Solution of the System of linear Algebraic equations is

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ 2 \end{bmatrix}$$

Crout's Method

$$AX = B$$

$$LUX = B$$

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow \quad A = LU$

"Iterative Methods"

→ Diagonally Dominant Systems.

The Solution of a system of linear equations will exist by iterative procedures only if the absolute value of the largest coefficient is greater than the sum of the absolute values of all the remaining co-efficients in each equation.
(Condition of Convergence)
Such systems are diagonally dominant.

jacobi iterative method

Consider the following system of equations.

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

let

$|a_1| > |b_1| + |c_1|$

$$|b_2| > |a_2| + |c_2|$$

$$|c_3| > |a_3| + |b_3|$$

that, is the system is diagonally dominant.

Solving for x, y, z:

$$x = \frac{1}{a_1}\left(d_1 - b_1 y - c_1 z\right)$$

$$y = \frac{1}{b_2}\left(d_2 - a_2 x - c_2 z\right)$$

$$z = \frac{1}{c_3}\left(d_3 - a_3 x - b_3 y\right)$$

The iterative formula will be:

$$x_{n+1} = \frac{1}{a_1}\left(d_1 - b_1 y_n - c_1 z_n\right)$$

$$y_{n+1} = \frac{1}{b_2}\left(d_2 - a_2 x_n - c_2 z_n\right)$$

$$z_{n+1} = \frac{1}{c_3}\left(d_3 - a_3 x_n - b_3 y_n\right)$$

for x, y, z : $n = 0$

$$x_1 = \frac{1}{a_1}\left(d_1 - b_1 y_0 - c_1 z_0\right)$$

$$y_1 = \frac{1}{b_2}\left(d_2 - a_2 x_0 - c_2 z_0\right)$$

$$z_1 = \frac{1}{c_3}\left(d_3 - a_3 x_0 - b_3 y_0\right)$$

$$x_0 = y_0 = z_0 = 0 \qquad \underline{\text{initial points}}$$

e.g.

$$8x - 3y + 2z = 20$$
$$6x + 3y + 12z = 35$$
$$4x + 11y - z = 33$$

Diagonally dominant system.

$$|8| > |-3| + |2|$$
$$|11| > |4| + |-1|$$
$$|12| > |6| + |3|$$

So, the iterative formulas would be:

$$x_{n+1} = \frac{1}{8}\left(20 + 3y_n - 2z_n\right)$$

$$y_{n+1} = \frac{1}{11}\left(33 - 4x_n - 1z_n\right)$$

$$z_{n+1} = \frac{1}{12}\left(35 - 6x_n - 3y_n\right)$$

put $n=0$ in above eq.

$$x_1 = \frac{1}{8}(20 + 3y_0 - 2z_0)$$

$$y_1 = \frac{1}{11}(33 - 4x_0 + z_0) \qquad x_0 = y_0 = z_0 = 0$$

$$z_1 = \frac{1}{12}(35 - 6x_0 - 3y_0)$$

1'st approximation.

$x_1 = 2.5$

$y_1 = 3$

$z_1 = 2.916666667$

put $n = 1$

$x_2 =$

put $n = 2$

$x_3 = 2.8125$, $y_3 = 2.181818182$, $z_3 = 0.729166666$

put $n = 3$

$x_4 = 2.8515625$, $y_4 = 2.404958678$, $z_3 = 2.369791667$

put $n = 4$

$x_5 = 2.856445313$, $y_5 = 2.344102179$, $z_3 = 1.139322917$

put $n = 5$

$x_6 = 2.857055664$, $y_6 = 2.360699406$, $z_3 = 2.062174479$

put $n = 6$

$x_7 = 2.857131958$, $y_7 = 2.356172889$, $z_7 = 1.370035807$

put $n = 7$

$x_8 = 2.857141495$, $y_8 = 2.357407394$, $z_8 = 1.889139811$

$, y_9 = 2.357070711$:

$z_9 = 1.666661367$

Gauss-Seidel method:

$$x_{n+1} = \frac{1}{a_1}\left(d_1 - b_1 y_n - c_1 z_n\right)$$

$$y_{n+1} = \frac{1}{b_2}\left(d_2 - a_2 x_{n+1} - c_2 z_n\right)$$

$$z_{n+1} = \frac{1}{c_3}\left(d_3 - a_3 x_{n+1} - b_3 y_{n+1}\right)$$

Finite Differences

→ limits
→ Countinuity $\qquad$ countinous set (a, b)
→ Derivatives
→ Integeration $\qquad$ → Discrete Sets $\{a, b\}$

Forward differences
Backward differences
Centeral differences

Argument (x) : $x_0 \quad x_1 = x_0 + h \quad x_2 = x_0 + 2h \; --- \; x_n = x_0 + nh$

Entity $y = f(x)$: $y_0 = f(x_0) \quad y_1 = f(x_0 + h) \; ----- \; y_n = f(x_0 + nh)$

let $y = f(x)$ be a discrete function
if $x_0, x_0 + h, x_0 + 2h, -----, x_0 + nh$ are successive values of x (arguments), where consective values of x differ by a quantity h, then the corresponding values of y are $y_0, y_1, y_2, --- y_n$.

The independent variables(x) are arguments.
The corresponding functional values are entities.

To determine the values of $f(x)$ or the derivatives of $f(x)$ or the derivative of $f(x)$ etc. for

Some intermediate arguments, the following 3 types of differences are usefull :

→ Forward differences

→ Backward differences

→ Centeral differences

Forward Differences

If we subtract from each value of y (except y_0) the preceding value of y we get $y_1 - y_0, y_2 - y_1, y_3 - y_2, \ldots, y_n - y_{n-1}$ respectively known as the 1st differences of y.

The results will be denoted by:

(forward y_0) $\Delta y_0 = \Delta y_1, \Delta y_2, \ldots, \Delta y_n$

$$\Delta y_0 = y_1 - y_0, \quad \Delta y_1 = y_2 - y_1, \quad \Delta y_2 = y_3 - y_1, \ldots$$

$$\ldots, \Delta y_{n-1} = y_n - y_{n-1}$$

where Δ represents an operation of forward differences.

The first forward differences are given by :

$$\Delta y_i = y_{i+1} - y_i \quad , i = 0, 1, 2, ---, n$$

Now, the second forward differences are defined as the differences of the first differences that is,

$$\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta(y_1 - y_0)$$

$$= \Delta y_1 - \Delta y_0 = y_2 - y_1 - (y_1 - y_0)$$

$$= y_2 - 2y_1 + y_0$$

$$\Delta^2 y_1 = \Delta(\Delta y_1) = \Delta(y_2 - y_1) = \Delta y_2 - \Delta y_1$$

$$= y_3 - y_2 - (y_2 - y_1)$$

$$= y_3 - 2y_2 + y_1$$

$$\Delta^2 y_2 = y_4 - 2y_3 + y_2$$

$$\Delta^2 y_n = y_{n+2} - 2y_{n+1} + y_n$$

$$\Delta^3 y_0 = ?$$

$$\Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0$$

In function notation, The forward differences are :

$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

$$\Delta f(x) = f(x+h) - f(x)$$

$$\Delta^2 f(x) = f(x+2h) - 2f(x+h) - f(x)$$

$$\Delta^3 f(x) = f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)$$

Forward difference Table

1st difference

$\Delta y_0 = y_1 - y_0$

x	$y = f(x)$	Δ	Δ^2	Δ^3	Δ^4	Δ^5
x_0	$y_0 = f(x_0)$					
		Δy_0				
$x_1 = x_0 + h$	$y_1 = f(x_1)$		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
$x_2 = x_0 + 2h$	$y_2 = f(x_2)$		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
$x_3 = x_0 + 3h$	$y_3 = f(x_3)$		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
$x_4 = x_0 + 4h$	$y_4 = f(x_4)$		$\Delta^2 y_3$			
		Δy_4				
$x_5 = x_0 + 5h$	$y_5 = f(x_5)$					

Observation: 1

We can express any higher order differences of y_0 in terms of $y_0, y_1, y_2, \ldots, y_n$.

from the definition:

$$\Delta^1 y_0 = y_1 - y_0$$
$$\Delta^2 y_0 = y_2 - 2y_1 + y_0$$
$$\Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0$$
$$\vdots$$

$^nC_r = \frac{n!}{(n-r)!\,r!}$

We can see that the co-efficients of the entries on the R.H.S are binomial co-efficients.

$$\Delta^n y_0 = y_n - {}^nC_1\, y_{n-1} + {}^nC_2\, y_{n-2} + \cdots + y_0$$

Observation 2

We can express any value of y in terms of leading entry y_0

we know

$$y_1 - y_0 = \Delta y_0 \Rightarrow y_1 = y_0 + \Delta y_0 = (1+\Delta)y_0$$

$$y_2 = y_1 + \Delta y_1 = (1+\Delta)y_1 = (1+\Delta)(1+\Delta)y_0$$
$$= (1+\Delta)^2 y_0$$

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\cdots$$

$$y_n = (1+\Delta)^n y_0$$

$$y_n = \left(1 - \binom{n}{c_1}\Delta + \binom{n}{c_2}\Delta^2 \cancel{+} \binom{n}{c_3}\Delta^3 + \cdots\right) y_0$$

$$y_n = y_0 + \binom{n}{c_1}\Delta y_0 + \binom{n}{c_2}\Delta^2 y_0 + \binom{n}{c_3}\Delta^3 y_0 + \cdots + \Delta^n y_0$$

Backward Differences:

The differences $y_1 - y_0, y_2 - y_1, y_3 - y_2, \ldots, y_n - y_{n-1}$ when denoted by $\nabla y_1, \nabla y_2, \nabla y_3, \ldots, \nabla y_n$, where ∇ is the backward difference operator called nabla operator.

$\nabla y_0 = y_0 - y_{-1}$ $\downarrow$ unk

$\nabla y_1 = y_1 - y_0, \; \nabla y_2 = y_2 - y_1, \ldots \nabla y_n = y_n - y_{n-1}$

is 1st Backward difference.

$$\rightarrow \nabla^2 y_2 = \nabla(\nabla y_2) = \nabla(y_2 - y_1) = \nabla y_2 - \nabla y_1$$

$$= y_2 - y_1 - (y_1 - y_0) \Rightarrow y_2 - 2y_1 + y_0$$

Note: $\quad$ " $\Delta^2 y_0 = y_2 - 2y_1 + y_0$ "

$$\rightarrow \nabla^2 y_3 = y_3 - 2y_2 + y_1$$

$$\rightarrow \nabla^3 y_3 = y_3 - 3y_2 + 3y_1 - y_0$$

In function notation:

$$\nabla f(x) = f(x) - f(x-h)$$

$$\rightarrow \nabla f(x+h) = f(x+h) - f(x)$$

$$\rightarrow \nabla^2 f(x+2h) = f(x+2h) - 2f(x+h) + f(x)$$

$$\rightarrow \nabla^3 f(x+3h) = f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)$$

Where "h" is the interval of differencing on values of "x".

Argument x	Entity $y=f(x)$	1st difference ∇	2nd difference ∇^2	∇^3	∇^4	∇^5
x_0	y_0					
		$y_1-y_0=\nabla y_1$				
$x_0+h=x_1$	y_1		$\nabla^2 y_2$			
				$\nabla^3 y_3$		
$x_0+2h=x_2$	y_2	$y_2-y_1=\nabla y_2$			$\nabla^4 y_4$	
			$\nabla^2 y_3$			$\nabla^5 y_5$
$x_0+3h=x_3$	y_3	$y_3-y_2=\nabla y_3$		$\nabla^3 y_4$		
$x_0+4h=x_4$	y_4	$y_4-y_3=\nabla y_4$	$\nabla^2 y_4$		$\nabla^4 y_5$	
				∇y_5		
$x_0+5h=x_5$	y_5	$y_5-y_4=\nabla y_5$	$\nabla^2 y_5$			

Observations:

Observation: 1

We can express any value of y in terms of y_n and the backward differences ∇y_n, $\nabla^2 y_n$ etc.

By defining $y_n - y_{n-1} = \nabla y_n \Rightarrow y_{n-1} = y_n - \nabla y_n$

$$y_{n-1} = (1-\nabla)y_n \quad \text{—— (i)}$$

Now $y_{n-2} = y_{n-1} - \nabla y_{n-1} = (1-\nabla)y_{n-1}$

$= (1-\nabla)(1-\nabla)y_n \Rightarrow (1-\nabla)^2 y_n$.

$\Rightarrow \boxed{y_{n-2} = (1-\nabla)^2 y_n}$

$\Rightarrow$ $y_{n-k} = (1-\nabla)^k y_n$ (General form)

Scanned with CamScanner

x	y_n	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1	$\Delta y_0 = 0.5$	$\Delta^2 y_0 = 0.2$	$\Delta^3 y_0 = 0$	$\underline{0.4}$
1	1.5	$\Delta y_1 = 0.7$	$\Delta^2 y_1 = 0.2$	$\Delta^3 y_1 = 0.4$	
2	2.2	$\Delta y_2 = 0.9$	$\Delta^2 y_2 = 0.6$		
3	3.1	$\Delta y_3 = 1.5$			
4	4.6				

(i) $\Delta^3 y_1 = ? \Rightarrow \underline{0.4}$ $\qquad \Delta^3 y_1 = y_4 - 3y_3 + 3y_2 - y_1$

(ii) $y_x = ?$

to

$$y_x = y_0 + {}^{x}C_1 \Delta y_0 + {}^{x}C_2 \Delta^2 y_0 + {}^{x}C_3 \Delta^3 y_0 + {}^{x}C_4 \Delta^4 y_0$$

Shift Operator: (E) The Translation Operator

If "h" is the interval of differencing ~~then~~ in the arguments the the operator E is defined as:

$$E f(x) = f(x+h)$$

$$E^2 f(x) = f(x+2h)$$

It is also called the translation Operator due to the reason that it results into the next value of the function.

$$E^2 f(x) = E(E(f(x))) = E(f(x+h))$$
$$= f(x+2h)$$
$$\vdots$$
$$E^n(f(x)) = f(x+nh)$$

$$E^n y_x = y_{x+nh}$$

Inverse Operator:

The inverse operator E^{-1} is defined as:

$$E^{-1} f(x) = f(x-h), \quad E^{-2} f(x) = f(x-2h)$$
$$\vdots$$
$$E^{-n} f(x) = f(x-nh)$$

Shift Operator.

Relation between Δ and E:

forward difference operator

$$\Delta f(x) = f(x+h) - f(x)$$

$$= E f(x) - f(x)$$

$$\Delta f(x) = (E-1) f(x)$$

$$\boxed{\begin{aligned}\Delta &= E-1 \\ \text{OR} \quad E &= \Delta+1\end{aligned}}$$

Relationship between ∇ and E :

$$\nabla f(x) = f(x) - f(x-h)$$
$$= f(x) - E^{-1} f(x)$$
$$\nabla f(x) = (1-E^{-1}) f(x)$$

$$\boxed{\nabla = 1 - E^{-1}}$$

OR $\quad \nabla = \dfrac{E-1}{E}$

Interpolation with Equal intervals:

(i) Gregory-Newton interpolation formula.

→ Forward.

Let $y=f(x)$ be a function which takes the values $y_0, y_1, y_2, \dots, y_n$ for $n+1$ values of x, that is x_0, x_1, x_2 of the ~~indepden~~ independent variable x (argument). let these values of x be equidistant i.e.

$$x_i = x_0 + ih, \text{ where } i = 0, 1, 2, \text{---}, n$$

let $y(x)$ be the polynomial in x of nth degree such that

$$y_i = f(x_i) = f(x_0 + ih)$$

Gregory-Newton Forward Interpolation Formula:

03-05-2021

Repetition <3

let "$y = f(x)$" be a function which assumes the values "$y_0, y_1, y_2, \text{---}, y_n$" for "$n+1$" values "$x_0, x_1, x_2, \text{---}$" of the independent variable "x" (argument)

let these values of "x" be equidistant (equally spaced) i.e

$$x_i = x_0 + ih \quad , i = 0, 1, 2, \text{---}, n$$

let "$y(x)$" be a polynomial in "x" of degree "n" such that:

$$y_i = f(x_i)$$

$$y(x) = A_0 + A_1(x_1 - x_0) + A_2(x - x_0)(x - x_1) +$$
$$A_3(x - x_0)(x - x_1)(x - x_2) + \text{---} +$$
$$A_n(x - x_0)(x - x_1)(x - x_2) \text{---} (x - x_n) \quad \text{--- } i$$

where A_i, $i = 0, 1, 2, \ldots, n$ are the const. Now we need to find these A_i's

By putting $x = x_0$

$$y(x_0) = A_0 + 0 + 0 + 0$$

$$\boxed{y_0 = A_0}$$

put $x = x_1$

$$y(x_1) = A_0 + A_1(x_1 - x_0) + 0 + 0 + 0 -$$

$$y_1 = A_0 + A_1 h \Rightarrow A_1 = \frac{y_1 - y_0}{h} = \frac{\Delta y_0}{h}$$

put $x = x_2$

$$y(x_2) = A_0 + A_1(x_2 - x_0) + A_2(x_2 - x_0)(x_2 - x_1)$$

$$= A_0 + A_1(2h) + A_2(2h)(h)$$

$$y_2 = y_0 + 2y_1 - 2y_0 + A_2(2h^2)$$

$$y_2 = y_0 + \frac{y_1 - y_0}{h}(2h) + A_2(2h^2)$$

$$A_2 = \frac{y_2 - y_0 - 2y_1 + 2y_0}{2h^2}$$

$$A_2 = \frac{\Delta^2 y_0}{2!\, h^2}$$

$\vdots$

$$A_3 = \frac{\Delta^3 y_0}{3!\, h^3}$$

$$A_n = \Delta^n y_0 / n!\, h^3$$

putting in eq (i)

$$y(x) = y_0 + \frac{\Delta y_0}{h}(x-x_0) + \frac{\Delta^2 y_0}{2!h^2}(x-x_0)(x-x_1)$$

$$+ \frac{\Delta^3 y_0}{3!h^3}(x-x_0)(x-x_1)(x-x_2) + \cdots$$

$$y(x) = y_0 + \Delta y_0 \frac{(x-x_1)}{h} + \frac{\Delta^2 y_0}{2!}\left(\frac{x-x_0}{h}\right)\left(\frac{x-x_1}{h}\right)$$

$$+ \frac{\Delta^3 y_0}{3!} \frac{(x-x_0)}{h} \cdot \frac{(x-x_1)}{h} \frac{(x-x_0)}{h} + \cdots$$

let $\frac{x-x_0}{h} = p \Rightarrow x = x_0 + ph$

$$\frac{x-x_1}{h} = \frac{x_0 + ph - x_1}{h} \Rightarrow \frac{-h+ph}{h} = p-1$$

$\vdots$

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 +$$

$$\frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots + \frac{p(p-1)\cdots(p-n-1)}{n!}\Delta^n y_0$$

(Newton Gregory forward interpolation formula)

OR

$$y_p = \left(1 + \binom{p}{c_1}\Delta + \binom{p}{c_2}\Delta^2 + \cdots + \binom{p}{c_n}\Delta^n\right) y_0$$

Interpolation with Unequal intervals:

2nd divided difference

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \longrightarrow \text{"1st divided difference"}$$

$$f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$

$$f(x_0, x_1, x_2, x_3) = \frac{f(x_1, x_2, x_3) - f(x_0, x_1, x_2)}{x_3 - x_0}$$

↓

3rd divided difference.

Argument	entry	1st DD	2nd DD	3rd DD	4th DD (order)
x	$y = f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	Δ^3 DD	Δ^4 DD
x_0	$f(x_0)$				
		$f(x_0, x_1)$			
x_1	$f(x_1)$		$f(x_0, x_2, x_3)$		
		$f(x_1, x_2)$		$f(x_0, x_1, x_2, x_3)$	
x_2	$f(x_2)$		$f(x_1, x_2, x_3)$		
		$f(x_2, x_3)$		$f(x_1, x_2, x_3, x_4)$	
x_3	$f(x_3)$		$f(x_2, x_3, x_4)$		
		$f(x_3, x_4)$			
x_4	$f(x_4)$				

max order $\Delta^4 DD = f(x_0, x_1, x_2, x_3, x_4)$

e.g

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
4	48					
5	100	$\frac{100-48}{5-4}=52$				
7	294	$\frac{294-100}{7-5}=97$	$\frac{97-52}{7-4}=15$			
10	900	$\frac{900-294}{10-7}=202$	$\frac{202-97}{10-5}=21$	1	0	
11	1210	$\frac{1210-900}{11-10}=310$	$\frac{310-202}{11-7}=27$	1		0
13	2028	$\frac{2028-1210}{13-11}=409$	$\frac{409-310}{13-10}=33$	1	0	

If a, b, c, d are the unequal arguments of $f(x) = \frac{1}{x}$, Show that

$$f(a,b,c,d) = \frac{1}{abcd}$$

Given: $f(x) = \frac{1}{x}$

$$\rightarrow f(a) = \frac{1}{a}, \quad f(b) = \frac{1}{b}, \quad f(c) = \frac{1}{c}, \quad f(d) = \frac{1}{d}$$

$$f(a,b) = \frac{f(b) - f(a)}{b-a} = \frac{1/b - 1/a}{b-a} = \frac{a-b}{ab} \times \frac{1}{b-a}$$

$$f(a,b) = \frac{-1}{ab}$$

$$f(a,b,c) = \frac{f(b,c) - f(a,b)}{c-a} = \frac{-1/bc + 1/ab}{c-a}$$

$$= \frac{1}{abc}$$

$$f(a,b,c,d) = \frac{f(b,c,d) - f(a,b,c)}{d-a}$$

$$= \frac{1}{bcd} - \frac{1}{abc} \Big/ d-a$$

$$\boxed{f(a,b,c,d) = \frac{-1}{abcd}}$$

Newton's divided difference formula:

Let $f(x_0), f(x_1), f(x_2), ---, f(x_n)$ be the functional values of $f(x)$ corresponding to non-equal spaced arguments $x_0, x_1, x_2, x_3, ---, x_n$ Now, from the definition of divided difference we have,

$$f(x, x_0) = f(x, x_0) = \frac{f(x) - f(x_0)}{x - x_0}$$

$$f(x) = f(x_0) + (x - x_0) f(x, x_0) \quad \text{---(i)}$$

$$f(x, x_0, x_1) = \frac{f(x, x_0) - f(x_0, x_1)}{x - x_1}$$

$$f(x, x_0) = f(x_0, x_1) + (x - x_1) f(x, x_0, x_1) \quad \text{(ii)}$$

Subsituting eq (ii) in eq (i)

$$f(x) = f(x_0) + (x-x_0)\left(f(x_0,x) + (x-x_1) f(x, x_0, x_1)\right)$$

$$f(x) = \underline{f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)}(x-x_1) f(\underline{x, x_0,} x_1)$$

Also, $f(x, x_0, x_1, x_2) = \dfrac{f(x, x_0, x_1) - f(x_0, x_1, x_2)}{x_2 - x}$

$$f(\underline{x, x_0}, x_1) = f(x_0, x_1, x_2) + (x-x_2) f(x, x_0, x_1, x_2)$$

Substituting

$$f(x) = f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1)$$
$$f(x_0, x_1, x_2) + (x-x_0)(x-x_1)(x-x_2) f(x_0, x_1, x_2, x_3)$$
$$+ \cdots + (x-x_0)(x-x_1) \cdots (x-x_n) f(x_0, x_1, \ldots x_n)$$

e.g

x	654	658	659	661
$f(x) = \log x$	2.8156	2.8182	2.8189	2.8202

using Newton's DD formula, find value of log 656?

The divided difference table is :

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
654	2.8156			
		0.00065		
658	2.8182		0.00001	-0.000003809
		0.0007		
659	2.8189		-0.000016666	
661	2.8202	0.00065		

The interpolation with unequal intervals in Numerical Analysis:

let $f(x_0), f(x_1), f(x_2), ---, f(x_n)$ be the entry values of $y=f(x)$ at arguments $x_0, x_1, x_2, ---, x_n$, which are not equally spaced.

for this new situation, we introduce the concept of Divided difference (DD). This DD is defined as:

"The difference between the two successive values of the entries divided by the difference b/w the corresponding values of arguments."

→ The 1st DD of $f(x)$ for x_0 & x_1 is:

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = [x_0, x_1] = \underset{x_1}{\Delta} f(x_0)$$

$$\underset{x_1}{\Delta} f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_0) - f(x_1)}{x_0 - x_1}$$

$$f(x_1, x_2) = \frac{f(x_1) - f(x_2)}{x_1 - x_2} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

2nd order DD:

$$f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$

25-5-2021

Find the polynomial satisfied by the following table.

x :	-4	-1	0	2	5
f(x) :	1245	33	5	9	1335

using Newton's divided difference formula.

Lagrange's Interpolation formula :

Let $y=f(x)$ be a function which takes the $n+1$ values $y_0, y_1, y_2, ---, y_n$ corresponding to nth degree in x. Let this polynomial be of the form:

$$y=f(x)=a_0(x-x_1)(x-x_2)----(x-x_n)+$$
$$a_1(x-x_0)(x-x_2)----(x-x_n)+$$
$$a_2(x-x_0)(x-x_1)(x-x_3)---(x-x_n)+$$
$$----+a_n(x-x_0)(x-x_1)---(x-x_{n-1}) \quad (i)$$

$a_0 = ?$ put $x=x_0$

$$y_0 = f(x_0) = a_0(x_0-x_1)(x_0-x_2)---$$

$$a_0 = \frac{y_0}{(x_0-x_1)(x_0-x_2)---(x_0-x_n)}$$

$a_1 = ?$ put, $x = x_1$

$$a_1 = \frac{y_1}{(x_1 - x_0)(x_1 - x_2) \cdots (x_1 - x_2)}$$

$\vdots$

$a_n = ?$ put

$$a_n = \frac{y_n}{(x_n - x_0)(x_n - x_1) \cdots (x_n - x_{n-1})}$$

put a's in eq (i)

$$y = f(x) = \frac{(x - x_1)(x - x_2) \cdots (x - x_n)}{(x_0 - x_1)(x_0 - x_2) \cdots (x_0 - x_n)} y_0 + \frac{(x - x_0)(x - x_2) \cdots (x - x_n)}{(x_1 - x_0)(x_1 - x_2) \cdots (x_1 - x_n)} y_1 + \cdots$$

$$\frac{(x - x_0)(x - x_1) \cdots (x - x_{n-1})}{(x_n - x_0)(x_n - x_1) \cdots (x_n - x_{n-1})} y_n \quad \text{——(ii)}$$

$\downarrow$

lagrange interpolation formula.

e.g Use the lagrange formula to find value of y when x = 10 if the value of x & y are given.

x	5	6	9	11
y	12	13	14	16

The corresponding formula is:

$$y = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} y_0 +$$

The population of a certain town is:

Year (x) : 1951 1961 1971 1981 1991

population : 19.96 36.65 58.81 77.21 94.61

$Y = f(x)$

find the rate of groth at 1981? at 1955?

Here, we have to find the derivative of

Year	population	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1951	19.96				
		16.69			
1961	36.65		5.47		
		22.16		-9.23	
1971	58.81		-3.76		11.99
		18.40		2.76	
1981	77.21		-1		
		17.40			
1991	94.61				

$$P = \frac{x - x_n}{h} = \frac{1981 - 1991}{10} = -1$$

$$Y = Y_n + P\nabla Y_n + \frac{P(P+1)}{2!}\nabla^2 Y_n + \frac{P(P+1)(P+2)}{3!}\nabla^3 Y_n + \frac{P(P+1)(P+2)(P+3)}{4!}\nabla^4 Y_n$$

$$\frac{dy}{dx} = \frac{dy}{dp} \times \frac{dp}{dx} = \frac{1}{h}\frac{dy}{dp}$$

$$\frac{dy}{dx} = \frac{1}{h}\left(\nabla y_n + \frac{2p+1}{2}\nabla^2 y_n + \frac{3p^2+6p+2}{6}\nabla^3 y_n + \frac{2p^3+9p^2+11p+3}{12}\nabla^4 y_n\right) \text{ ——— (i)}$$

putting values in eq (i)

$$\frac{dy}{dx} = \frac{1}{10}\left(17.40 + \frac{2(-1)+1}{2}(-1) + \frac{3(-1)^2+6(-1)+2}{6}(2.76) + \frac{2(-1)^3+9(-1)^2+11(-1)+3}{12}(11.99)\right)$$

$$\frac{dy}{dx} = \frac{1}{10}\left(17.40+0.5+(-0.46)+(-0.9991)\right)$$

$$\boxed{\frac{dy}{dx} = 1.64408333}$$

→ To find the rate of groth at 1951 we need to use forward interpolation formula.

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0.$$

$$p = \frac{x-x_0}{h} = \frac{1955-1951}{10} = 0.4$$

$$\frac{dy}{dx} = \frac{1}{h}\left(\Delta y_0 + \frac{2p+1}{2} \Delta^2 y_0 + \frac{3p^2+6p+2}{6} \Delta^3 y_0 \right.$$

$$\left. + \frac{2p^3+9p^2+11p+3}{12} \Delta^4 y_0 \right)$$

$$= \frac{1}{10}\left(16.96 + \frac{2(0.4)+1}{2}(5.47) + \right.$$

$$\frac{3(0.4)^2+6(0.4)+2}{6}(-9.23) +$$

$$\left. \frac{2(0.4)^3+9(0.4)^2+11(0.4)+3}{12}(11.99) \right)$$

$$\frac{dy}{dx} = \frac{1}{10}(16.96 + 4.923 - 7.587066667$$

$$8.96052667$$

$$\boxed{\frac{dy}{dx} = 2.333646}$$

OR $\quad f'(19.55) = 2.333646.$

Numerical Integrations

The process of finding / Computing $\int_a^b y\,dx$, where $y = f(x)$ is given by a set of tabulated values (x_i, y_i), $i = 0, 1, 2, ---, n$ such that $a = x_0$ & $b = x_n$, is called "numerical integeration."

→ Definite integerals (Anti derivative) "+c"
→ Indefinite integerals → "limits"

$$\frac{b-a}{n} = h \Rightarrow n \propto \frac{1}{h}$$

No. of segments

Since, $y = f(x)$ is a single variable function, the process in general is known as Quadrature.
Like that of numerical differentiation here we also replace $f(x)$ by an interpolation formula, and integrate it in between the given limits.
In this way, we can derive the quadrature formula for

approximate integration of a function defined by a set of numerical values.

General Quadrature formula

let $I = \int_a^b y\,dx$, where $y = f(x)$ takes the values $y_0, y_1, y_2, ----, y_n$ for $x_0, x_1, x_2, ----, x_n$. let us divide the interval $[a, b]$ into "n" equal parts of width "h" so that -

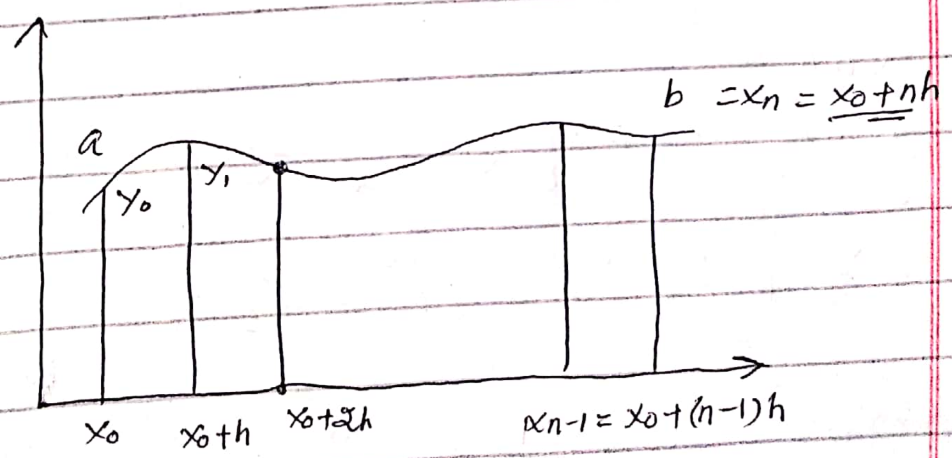
$a := x_0$

$x_1 = x_0 + h$

$x_2 = x_0 + 2h$

$\vdots$

$x_n = x_0 + nh = b$



Thus $\int_{x_0}^{x_0+nh} f(x)\,dx$

not const. variable involved,

putting $X = x_0 + nh$, $\frac{x - x_0}{h} = p$

$$= \int_0^n f(x_0+nh)\, h\,dp \qquad x = x_0 + hp$$

$$\frac{dx}{dp} = h$$

$$\boxed{dx = h\,dp}$$

$$= h\int_0^n y_p \, dp$$

Chang the limits interms of "p"

when,

$x = x_0 \Rightarrow p = 0$

$x = x_0 + hn \Rightarrow p = n$

$$= h\int_0^n \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots \right] dp.$$

Now, integrating w.r.t "p" & applying limits will give us the following Quadrature formula after some arrangements :)

$$\int_{x_0}^{x_0+nh} f(x)\,dx = h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{n^4}{4} - n^3 + n^2\right)\Delta^3 y_0 + \frac{1}{24}\left(\frac{n^5}{5} - \frac{3n^4}{2} + \frac{11n^3}{3} - \frac{3n^2}{2}\right)\Delta^4 y_0 + \right.$$

$$\frac{1}{120}\left(\frac{n^6}{6} - 2n^5 + \frac{35}{4}n^4 - \frac{50}{3}n^3 + 12n^2\right)\Delta^5 y_0 + \text{------}\Big].$$

"General Quadrature formula."

put $n = 1$

1st Trapozidal Rule:

$$= \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \text{---} + y_{n-1})\right]$$

"Trapozoidal Rule:"

put $n = 1$ in Quadrature formula

$$\int_{x_0}^{x_0+nh} f(x)\,dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right]$$

$$\int_{x_0}^{x_1} f(x)\,dx = h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right] = \frac{h}{2}\left[y_0 + y_1\right]$$

$$+$$

$$\int_{x_1}^{x_2} f(x)\,dx = \frac{h}{2}\left[y_1 + y_2\right]$$

$$+$$

$$\int_{x_2}^{x_3} f(x)\,dx = \frac{h}{2}\left[y_2 + y_3\right]$$

$$\vdots$$

$$\int_{a=x_0}^{b=x_0+nh} f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \text{-----} + y_{n-1})\right].$$

(General Trapozoidal Rule)

"Simpson's 1/3 Rule:"

put $n=2$ in General Quadrature formula:

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{3}\Big[(y_0+y_n)+4(y_1+y_3+\cdots+y_{n-1})$$

$$+2(y_2+y_4+\cdots+y_{n-2})\Big]$$

(General Simpson's 1/3 formula)

→ "n" should be even.

"Simpson's 3/8 Rule:"

put $n=3$ in Quadrature formula:

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3}{8}h\Big[(y_0+y_n)+3(y_1+y_2+y_4+$$

$$y_5+y_7+\cdots+y_{n-1})+2(y_3+y_6+y_9+\cdots+y_{n-3})\Big].$$

"Weddle's Rule:" $n=6$

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{3}{10}h\Big[(y_0+y_n)+(5y_1+y_2+6y_3+$$

$$y_4+5y_5)+(2y_6+5y_7+y_8+6y_9+y_{10}+5y_{11})$$

$$+\cdots+(2y_{n-6}+5y_{n-5}+y_{n-4}+6y_{n-3}+y_{n-2}+5y_{n-1})\Big]$$

$$= \frac{3h}{10}\left[(y_0 + y_n) + (5y_1 + y_2 + 6y_3 + y_4 + 5y_5) \right.$$

$$\left. + (2y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11}) + \cdots \right.$$

$$5y_1 + \underset{4}{5y_5} + \underset{2}{5y_7} + \underset{4}{5y_{11}} + \underset{2}{5y_{13}} \cdots$$

$$y_2 + \frac{y_4}{2} + \frac{y_8}{4} + \frac{y_{10}}{2} + \frac{y_{14}}{4} \cdots$$

$$6y_3 + \frac{6y_9}{6} + \frac{6y_{15}}{6}$$

e.g for $\underline{y_{18}}$:

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{10}\left[(y_0 + y_{18}) + 5(y_1 + y_5 + y_7 + y_{11} + y_{13} + y_{17}) + (y_2 + y_4 + y_8 + y_{10} + y_{14} + y_{16}) + 6(y_3 + y_9 + y_{15}) + 2(y_6 + y_{12}) \right].$$

Numerical Solution of differential Equation:

Euler Method:

Consider the diffrential equation

$$\frac{dy}{dx} = f(x, y) \quad \text{where } y(x_0) = y_0$$

Slope

Suppose that, we wish to find successively $y_1, y_2, y_3, ---, y_m$ where y_m is the value of y corresponding to $x = x_m$. Such that:

$$x_m = x_0 + mh \quad , m = 1, 2, 3 ---$$

$\Rightarrow$ where h is very small.

Here, we use the property that in a small interval, a curve is nearly a straight line.

Thus, in the interval x_0 to x_1 we approximate the curve by the tangent at the point (x_0, y_0). There fore, the equation of the tangent at (x_0, y_0) is:

$$y - y_0 = m(x - x_0)$$

$$y - y_0 = \frac{dy}{dx}(x - x_0)$$

$$y - y_0 = f(x_0, y_0)(x - x_0).$$

$$y = y_0 + (x - x_0) f(x_0, y_0)$$

Therefore at $x = x_1$, $y = y_1$

$$y_1 = y_0 + (x_1 - x_0) f(x_0, y_0)$$

$$y_1 = y_0 + h f(x_0, y_0)$$

$$y_2 = y_1 + h f(x_1, y_1)$$

$$\boxed{y_{m+1} = y_m + h f(x_m, y_m)}$$

Modified Euler Method:

$$y_{m+1} = y_m + h f\left(x_m + \frac{h}{2}, y_m + \frac{h}{2} f(x_m, y_m)\right)$$

e.g

Solve $\frac{dy}{dx} = 1 - y$, $y(0) = 0$ in the range $0 \leq x \leq 0.3$ using

(i) Euler's Method

(ii) Modified Euler Method.

Solution:-

Given: $f(x_m, y_m) = 1 - y_m$, $y_0 = 0$

$h = 0.1$

$$y_{m+1} = y_m + h f(x_m, y_m)$$

$$y_1 = y_0 + h f(x_0, y_0)$$

$$y_1 = y_0 + h(1 - y_0) \quad = 0.1$$

$$y_2 = y_1 + h f(x_1, y_1)$$

$$y_2 = 0.1 + (0.1)(1 - y_1)$$

$$y_2 = 0.1 + (0.1)(1 - 0.1)$$

$$\boxed{y_2 = 0.19}$$

$\vdots$

$$y_3 = 0.271$$

"Supline theory"

Using Modified euler Method:

$$y_{m+1} = y_m + h\left(f\left(x_m + \frac{h}{2}, y_m + \frac{h}{2} f(x_m, y_m)\right)\right)$$

put $m = 0$

$$y_1 = y_0 + h\left(f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right)\right)$$

$$y_1 = 0 + (0.1)\left(f\left(0 + \frac{0.1}{2}, 0 + \frac{0.1}{2} f(0, 0)\right)\right)$$

$$y_1 = 0 + 0.1 f(0.05, 0.05)$$

$$\boxed{y_1 = 0.095}$$