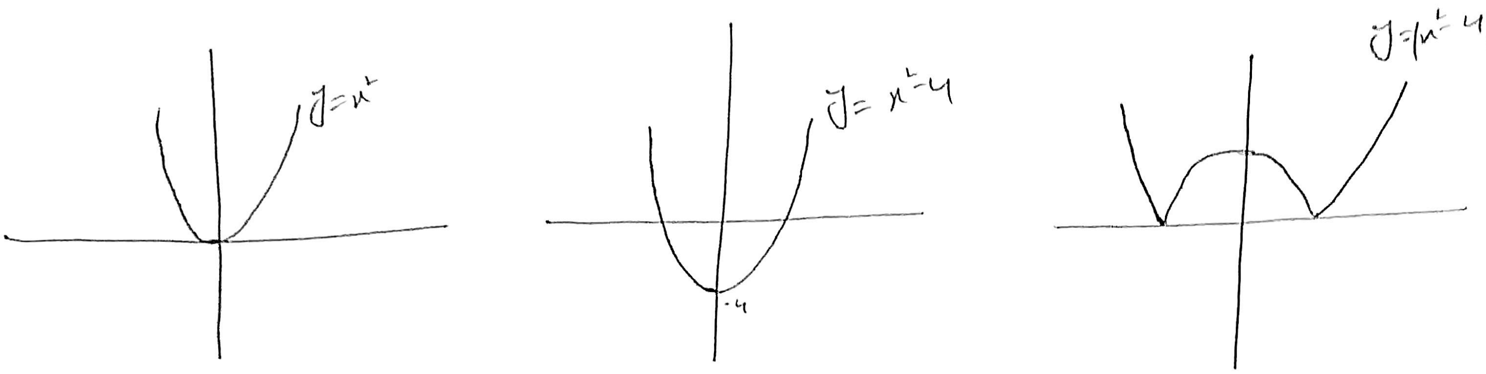
Exercise

Consider the function $f(x) = |x^2 - 4|$



As the function has no hole in its graph so it is continuous. Also the derivative at corner points does not exist. Let's check the differentiability at $x = 2$ and $x = -2$

By definition.

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{|x^2 - 4| - 0}{x - 2}$$

$$= \lim_{x \to 2} \frac{|x^2 - 4|}{x - 2}$$

$\Rightarrow$ Left limit

$$\lim_{x \to 2^-} \frac{-(x^2 - 4)}{x - 2}$$

$$= \lim_{x \to 2^-} \frac{-(x+2)\cancel{(x-2)}}{\cancel{(x-2)}}$$

Right limit

$$\lim_{x \to 2^+} \frac{x^2 - 4}{x - 2}$$

$$= \lim_{x \to 2^+} \frac{(x+2)\cancel{(x-2)}}{\cancel{(x-2)}}$$

$$= \lim_{x \to 2^-} -(x+2)$$

$$= -4$$

$$= \lim_{x \to 2^+} (x+2)$$

$$= 4$$

Since

$$\lim_{x \to 2^-} \frac{|x^2-4|}{x-2} \neq \lim_{x \to 2^+} \frac{|x^2-4|}{x-2}$$

So, $f(x)$ is not differentiable at $x=2$

Now at $x=-2$

$$f'(-2) = \lim_{x \to -2} \frac{f(x) - f(-2)}{x-(-2)}$$

$$= \lim_{x \to -2} \frac{|x^2-4| - 0}{x+2} = \lim_{x \to -2} \frac{|x^2-4|}{x+2}$$

$\Rightarrow$ Left Limit

$$\lim_{x \to -2^-} \frac{x^2-4}{x+2}$$

$$= \lim_{x \to -2^-} \frac{\cancel{(x+2)}(x-2)}{\cancel{(x+2)}}$$

$$= -2-2$$

$$= -4$$

Right limit.

$$\lim_{x \to -2^+} \frac{-(x^2-4)}{x+2}$$

$$= \lim_{x \to -2^+} \frac{-\cancel{(x+2)}(x-2)}{\cancel{(x+2)}}$$

$$= -(-2-2)$$

$$= 4$$

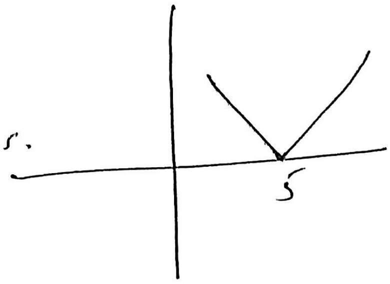
Since

$$\lim_{x \to -2^-} \frac{|x^2-4|}{x+2} \neq \lim_{x \to -2^+} \frac{|x^2-4|}{x+2}$$

So $f(x)$ is not differentiable at $x=-2$

Ex $\quad f(x) = |x-5|$

Check the differentiability at $x=5$, Function is continuous.



By definition

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x-a}$$

$$f'(5) = \lim_{x \to 5} \frac{f(x) - f(5)}{x-5}$$

$$= \lim_{x \to 5} \frac{|x-5| - 0}{x-5}$$

$$= \lim_{x \to 5} \frac{|x-5|}{x-5}$$

Left Limit

$$\lim_{x \to 5^-} = \frac{-(x-5)}{(x-5)}$$

$$= -1$$

Right Limit.

$$\lim_{x \to 5^+} \frac{(x-5)}{(x-5)}$$

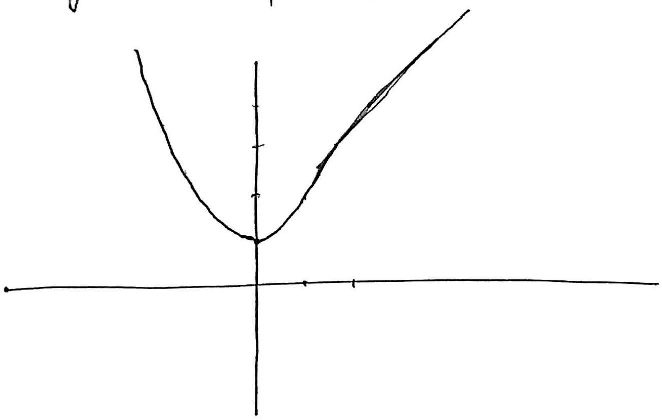
$$= 1$$

Since $\lim_{x \to 5^-} \frac{|x-5|}{x-5} \neq \lim_{x \to 5^+} \frac{|x-5|}{x-5}$

So, $f(x)$ is not diff. at $x = 5$.

Ex $f(x) = \begin{cases} x^2+1 & , x \leq 1 \\ 2x & , x > 1 \end{cases}$

is continuous and differentiable at $x=1$. sketch the graph of $\cancel{B}$. f.



For continuity at $x=1$

$f(1) = (1)^2+1 = 2$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2+1 = (1)^2+1 = 2$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2x = 2(1) = 2$

$\cancel{So}$, Since $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$

So, $f(x)$ is continuous at $x=1$

Now for diff.

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$$

Left Limit.

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1}$$

$$= \lim_{x \to 1^-} \frac{x^2 + 1 - 2}{x - 1}$$

$$= \lim_{x \to 1^-} \frac{x^2 - 1}{x - 1}$$

$$= \lim_{x \to 1^-} \frac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}}$$

$$= \lim_{x \to 1^-} (x+1)$$

$$= 1 + 1$$

$$= 2$$

Right Limit.

$$\lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1}$$

$$= \lim_{x \to 1^+} \frac{2x - 2}{x - 1}$$

$$= \lim_{x \to 1^+} \frac{2\cancel{(x-1)}}{\cancel{(x-1)}}$$

$$= 2$$

since Left Limit = Righ Limit = 2.

So, $f(x)$ is differentiable at $x = 1$.

Exercise:

$$f(x) = \begin{cases} x^2+2 & , \; x \le 1 \\ x+2 & , \; x > 1 \end{cases}$$

Check the continuity and differentiablity at $x=1$
Also sketch the graph.

Ex Find the point on curve where tangent line is horizontal.

$$y = x^4 - 6x^2 + 4$$

Tangents are horizontal at those points where

$$\frac{dy}{dx} = 0$$

So,

$$4x^3 - 12x = 0$$

$$4x(x^2-3) = 0$$

$4x = 0$ or $x^2 - 3 = 0$

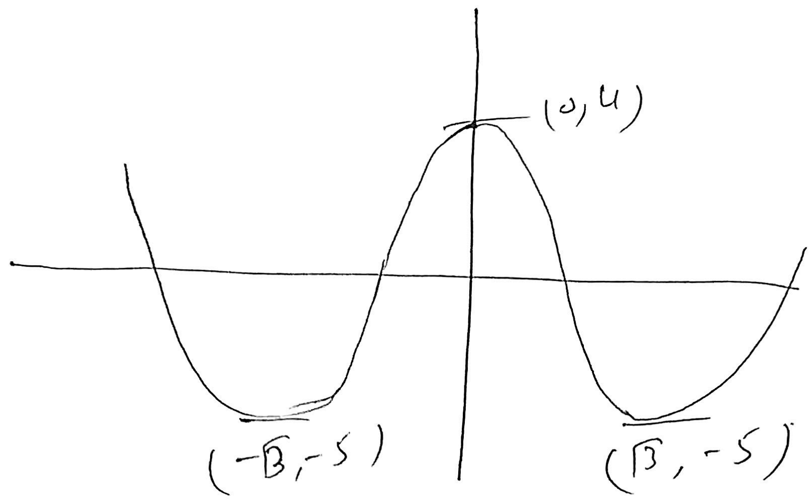
$x = 0$ | $x = \sqrt{3}$, $x = -\sqrt{3}$

When $x = 0$, $y = 4$

When $x = \sqrt{3}$, $y = -5$

When $x = -\sqrt{3}$, $y = -5$

So, $(0,4)$, $(\sqrt{3}, -5)$ and $(-\sqrt{3}, -5)$ are three points where tangent line is horizontal.



Ex:- At what point on hyperbola $xy = 12$ is the tangent line parallel to the line $3x + y = 0$

Slope of the given straigh line is -3.

As $xy = 12$

$$y = \frac{12}{x} = 12x^{-1}$$

$$\frac{dy}{dx} = -12x^{-2} = \frac{-12}{x^2}$$

By given condition.

$$\frac{-12}{x^2} = -3$$

$$x^2 = 4$$

$$\Rightarrow \quad x = \pm 2$$

when $x = 2$, $y = 6$

$x = -2$, $y = -6$

So, $(2, 6)$, $(-2, -6)$ are two points where tangent line is parallel to the line $3x + y = 0$

Differentiation Rules:

* Power Rule:-

$$\frac{d}{dx}(x^n) = n x^{n-1}$$

$$\frac{d}{dx}(f(x))^n = n(f(x))^{n-1} f'(x)$$

* Sum Rule:-

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$$

* Difference Rule:-

$$\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx} f(x) - \frac{d}{dx} g(x).$$

* Product Rule:-

$$\frac{d}{dx}(f(x) \cdot g(x)) = f(x) g'(x) + g(x) f'(x)$$

* Quotient Rule:-

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x) f'(x) - f(x) g'(x)}{(g(x))^2}$$