CS 4072 - Topics in CS
Process Mining

Lecture # 10

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FAST - NUCES, CFD Campus

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Today’s Topics

- Alpha algorithm (continued)

α-algorithm

- The **α-algorithm** *scans the event log for particular patterns*.
- Four *log-based ordering relations* are defined to capture relevant patterns in an event log.
  - Direct succession (>)
  - Causality (→)
  - Parallelism (||)
  - Choice (#)

Table 6.1 Footprint of L_1: $a \#_{L_1} a$, $a \rightarrow_{L_1} b$, $a \rightarrow_{L_1} c$, etc.

	a	b	c	d	e
a	$\#_{L_1}$	$\rightarrow_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$	$\rightarrow_{L_1}$
b	$\leftarrow_{L_1}$	$\#_{L_1}$	$\Vert_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$
c	$\leftarrow_{L_1}$	$\Vert_{L_1}$	$\#_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$
d	$\#_{L_1}$	$\leftarrow_{L_1}$	$\leftarrow_{L_1}$	$\#_{L_1}$	$\leftarrow_{L_1}$
e	$\leftarrow_{L_1}$	$\#_{L_1}$	$\#_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$

Alpha Algorithm

- (α-algorithm): Let L be an event log over $T \subseteq \mathcal{A}$. $\alpha(L)$ is defined as follows:

1. $T_L = \{t \in T \mid \exists_{\sigma \in L}\ t \in \sigma\}$,
2. $T_I = \{t \in T \mid \exists_{\sigma \in L}\ t = \mathit{first}(\sigma)\}$,
3. $T_O = \{t \in T \mid \exists_{\sigma \in L}\ t = \mathit{last}(\sigma)\}$,
4. $X_L = \{(A, B) \mid A \subseteq T_L \ \wedge\ A \neq \emptyset \ \wedge\ B \subseteq T_L \ \wedge\ B \neq \emptyset \ \wedge$ $\forall_{a \in A} \forall_{b \in B}\ a \rightarrow_L b \ \wedge\ \forall_{a_1, a_2 \in A}\ a_1 \#_L a_2 \ \wedge\ \forall_{b_1, b_2 \in B}\ b_1 \#_L b_2\}$,
5. $Y_L = \{(A, B) \in X_L \mid \forall_{(A', B') \in X_L} A \subseteq A' \ \wedge B \subseteq B' \Longrightarrow (A, B) = (A', B')\}$,
6. $P_L = \{p_{(A,B)} \mid (A, B) \in Y_L\} \cup \{i_L, o_L\}$,
7. $F_L = \{(a, p_{(A,B)}) \mid (A, B) \in Y_L \ \wedge\ a \in A\} \cup \{(p_{(A,B)}, b) \mid (A, B) \in Y_L \ \wedge\ b \in B\} \cup \{(i_L, t) \mid t \in T_I\} \cup \{(t, o_L) \mid t \in T_O\}$, and
8. $\alpha(L) = (P_L, T_L, F_L)$.

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5. $Y_L = \{(A, B) \in X_L \mid \forall_{(A', B') \in X_L} A \subseteq A'\ \wedge B \subseteq B' \Longrightarrow (A, B) = (A', B')\}$,
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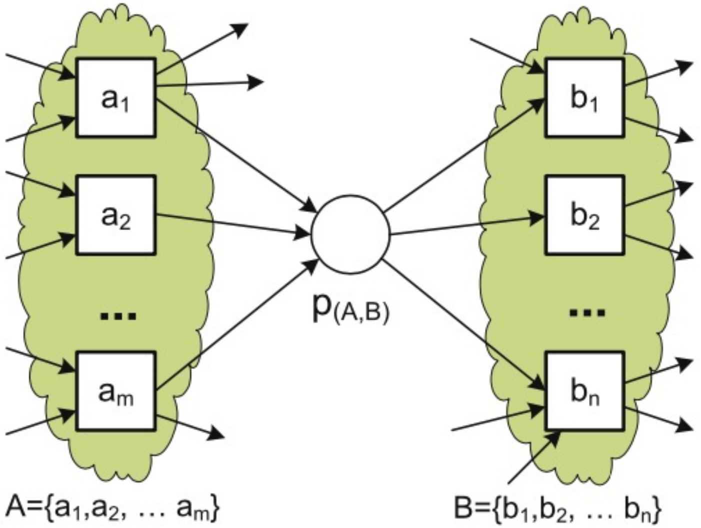
7. $F_L = \{(a, p_{(A,B)}) \mid (A, B) \in Y_L\ \wedge\ a \in A\}\ \cup\ \{(p_{(A,B)}, b) \mid (A, B) \in Y_L\ \wedge\ b \in B\}\ \cup \{(i_L, t) \mid t \in T_I\}\ \cup \{(t, o_L) \mid t \in T_O\}$, and
8. $\alpha(L) = (P_L, T_L, F_L)$.

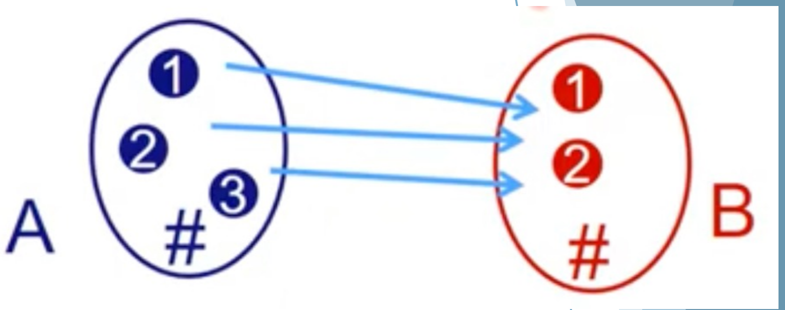
These are core steps of the α-algorithm.
The challenge is to to determine the places of the WF-net and their connections.

We aim at constructing places named p(A,B) such that A is the set of input transitions (•p(A,B) = A) and B is the set of output transitions (p(A,B)• = B) of p(A,B).

Alpha Algorithm

Fig. 6.7 Place $p_{(A,B)}$ connects the transitions in set A to the transitions in set B





Check for the four quadrants

Table 6.4 How to identify $(A, B) \in X_L$? Rearrange the rows and columns corresponding to $A = \{a_1, a_2, \dots, a_m\}$ and $B = \{b_1, b_2, \dots, b_n\}$ and remove the other rows and columns from the footprint

	a_1	a_2	$\dots$	a_m	b_1	b_2	$\dots$	b_n
a_1	#	#	$\dots$	#	$\rightarrow$	$\rightarrow$	$\dots$	$\rightarrow$
a_2	#	#	$\dots$	#	$\rightarrow$	$\rightarrow$	$\dots$	$\rightarrow$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
a_m	#	#	$\dots$	#	$\rightarrow$	$\rightarrow$	$\dots$	$\rightarrow$
b_1	$\leftarrow$	$\leftarrow$	$\dots$	$\leftarrow$	#	#	$\dots$	#
b_2	$\leftarrow$	$\leftarrow$	$\dots$	$\leftarrow$	#	#	$\dots$	#
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
b_n	$\leftarrow$	$\leftarrow$	$\dots$	$\leftarrow$	#	#	$\dots$	#

Example

$L_1 = [\langle a, b, c, d\rangle^3, \langle a, c, b, d\rangle^2, \langle a, e, d\rangle]$

Table 6.1 Footprint of L_1: $a \#_{L_1} a$, $a \rightarrow_{L_1} b$, $a \rightarrow_{L_1} c$, etc.

	a	b	c	d	e
a	$\#_{L_1}$	$\rightarrow_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$	$\rightarrow_{L_1}$
b	$\leftarrow_{L_1}$	$\#_{L_1}$	$\|_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$
c	$\leftarrow_{L_1}$	$\|_{L_1}$	$\#_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$
d	$\#_{L_1}$	$\leftarrow_{L_1}$	$\leftarrow_{L_1}$	$\#_{L_1}$	$\leftarrow_{L_1}$
e	$\leftarrow_{L_1}$	$\#_{L_1}$	$\#_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$

Following is the set of all pairs that meet the requirements of Step 4

$$X_{L_1} = \{(\{a\},\{b\}), (\{a\},\{c\}), (\{a\},\{e\}), (\{a\},\{b,e\}), (\{a\},\{c,e\}), (\{b\},\{d\}), (\{c\},\{d\}), (\{e\},\{d\}), (\{b,e\},\{d\}), (\{c,e\},\{d\})\}$$

Inserting a place for any element in X_{L1} would introduce too many places.

Therefore, only the "maximal pairs" (A,B) should be included.

Alpha Algorithm

- (α-algorithm): Let L be an event log over $T \subseteq \mathcal{A}$. α(L) is defined as follows:

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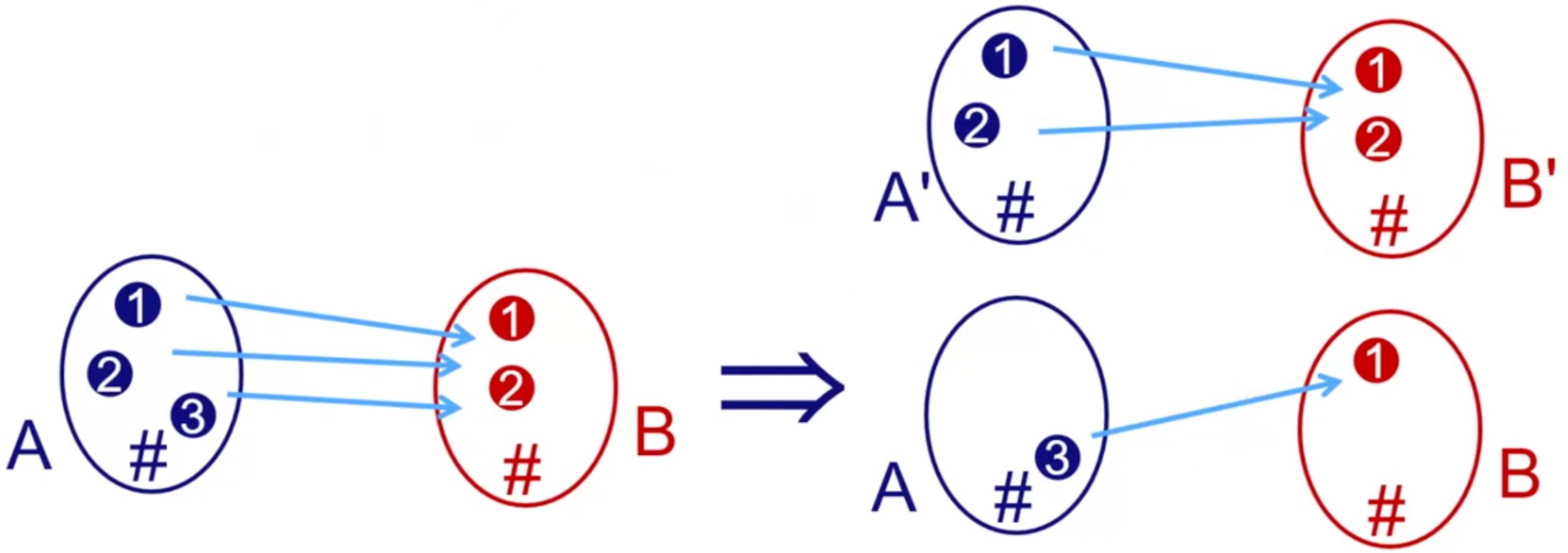
2. $T_I = \{t \in T \mid \exists_{\sigma \in L}\ t = \mathit{first}(\sigma)\}$,
3. $T_O = \{t \in T \mid \exists_{\sigma \in L}\ t = \mathit{last}(\sigma)\}$,
4. $X_L = \{(A, B) \mid A \subseteq T_L \ \wedge\ A \neq \emptyset \ \wedge\ B \subseteq T_L \ \wedge\ B \neq \emptyset \ \wedge$ $\forall_{a \in A} \forall_{b \in B}\ a \rightarrow_L b \ \wedge\ \forall_{a_1, a_2 \in A}\ a_1 \#_L a_2 \ \wedge\ \forall_{b_1, b_2 \in B}\ b_1 \#_L b_2\}$,
5. $Y_L = \{(A, B) \in X_L \mid \forall_{(A', B') \in X_L} A \subseteq A' \ \wedge B \subseteq B' \Longrightarrow (A, B) = (A', B')\}$,
6. $P_L = \{p_{(A,B)} \mid (A, B) \in Y_L\} \cup \{i_L, o_L\}$,
7. $F_L = \{(a, p_{(A,B)}) \mid (A, B) \in Y_L \ \wedge\ a \in A\} \cup \{(p_{(A,B)}, b) \mid (A, B) \in Y_L \ \wedge\ b \in B\} \cup \{(i_L, t) \mid t \in T_I\} \cup \{(t, o_L) \mid t \in T_O\}$, and
8. $\alpha(L) = (P_L, T_L, F_L)$.

Step 5 removes all non-maximal pairs.

Note that for any pair $(A, B) \in X_L$, non-empty set $A' \subseteq A$, and non-empty set $B' \subseteq B$, it is implied that $(A', B') \in X_L$.

Alpha Algorithm: removing non-maximal pairs



Note that for any pair (A, B) ∈ X_L, non-empty set A′ ⊆ A, and non-empty set B′ ⊆ B, it is implied that (A′, B′) ∈ X_L .

Example

$L_1 = [\langle a, b, c, d\rangle^3, \langle a, c, b, d\rangle^2, \langle a, e, d\rangle]$

Table 6.1 Footprint of L_1: $a \#_{L_1} a$, $a \rightarrow_{L_1} b$, $a \rightarrow_{L_1} c$, etc.

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a	$\#_{L_1}$	$\rightarrow_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$	$\rightarrow_{L_1}$
b	$\leftarrow_{L_1}$	$\#_{L_1}$	$\Vert_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$
c	$\leftarrow_{L_1}$	$\Vert_{L_1}$	$\#_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$
d	$\#_{L_1}$	$\leftarrow_{L_1}$	$\leftarrow_{L_1}$	$\#_{L_1}$	$\leftarrow_{L_1}$
e	$\leftarrow_{L_1}$	$\#_{L_1}$	$\#_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$

$$X_{L_1} = \{(\{a\},\{b\}), (\{a\},\{c\}), (\{a\},\{e\}), (\{a\},\{b,e\}), (\{a\},\{c,e\}),$$
$$(\{b\},\{d\}), (\{c\},\{d\}), (\{e\},\{d\}), (\{b,e\},\{d\}), (\{c,e\},\{d\})\}$$

X_{L1} after the removal of non-maximal pairs:

$$Y_{L_1} = \{(\{a\},\{b,e\}), (\{a\},\{c,e\}), (\{b,e\},\{d\}), (\{c,e\},\{d\})\}$$

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6. $\boxed{P_L = \{p_{(A,B)} \mid (A, B) \in Y_L\} \cup \{i_L, o_L\}}$,
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8. $\alpha(L) = (P_L, T_L, F_L)$.

Step 6: Every element of $(A, B) \in Y_L$ corresponds to a place $p_{(A,B)}$ connecting transitions A to transitions B (in union with a special input and output place).

Example

L_1 = [<a, b, c, d>^{3}, <a, c, b, d>^{2}, <a, e, d>]

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c	$\leftarrow_{L_1}$	$\Vert_{L_1}$	$\#_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$
d	$\#_{L_1}$	$\leftarrow_{L_1}$	$\leftarrow_{L_1}$	$\#_{L_1}$	$\leftarrow_{L_1}$
e	$\leftarrow_{L_1}$	$\#_{L_1}$	$\#_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$

$$X_{L_1} = \{(\{a\},\{b\}), (\{a\},\{c\}), (\{a\},\{e\}), (\{a\},\{b,e\}), (\{a\},\{c,e\}), (\{b\},\{d\}), (\{c\},\{d\}), (\{e\},\{d\}), (\{b,e\},\{d\}), (\{c,e\},\{d\})\}$$

$$Y_{L_1} = \{(\{a\},\{b,e\}), (\{a\},\{c,e\}), (\{b,e\},\{d\}), (\{c,e\},\{d\})\}$$

Every element of (A, B) ∈ Y_L corresponds to a place $p_{(A,B)}$ connecting transitions A to transitions B (in union with a special input and output place).

$P_L = \{p_{(\{a\}, \{b,e\})}, p_{(\{a\}, \{c,e\})}, p_{(\{b,e\}, \{d\})}, p_{(\{c,e\}, \{d\})}), i_L, o_L\}$

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8. $\alpha(L) = (P_L, T_L, F_L)$.

Step 7: The arcs of the WF-net are generated.
All places p(A,B) have A as input nodes and B as output nodes.
All start transitions in T_I have i_L as an input place and all end transitions T_O have o_L as output place.

Example

$L_1 = [\langle a, b, c, d\rangle^3, \langle a, c, b, d\rangle^2, \langle a, e, d\rangle]$

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b	$\leftarrow_{L_1}$	$\#_{L_1}$	$\parallel_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$
c	$\leftarrow_{L_1}$	$\parallel_{L_1}$	$\#_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$
d	$\#_{L_1}$	$\leftarrow_{L_1}$	$\leftarrow_{L_1}$	$\#_{L_1}$	$\leftarrow_{L_1}$
e	$\leftarrow_{L_1}$	$\#_{L_1}$	$\#_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$

$$Y_{L_1} = \{(\{a\},\{b,e\}), (\{a\},\{c,e\}), (\{b,e\},\{d\}), (\{c,e\},\{d\})\}$$

$T_L = \{a, b, c, d, e\}$
$T_I = \{a\}$
$T_O = \{d\}$

$P_L = \{p_{(\{a\}, \{b,e\})}, p_{(\{a\}, \{c,e\})}, p_{(\{b,e\}, \{d\})}, p_{(\{c,e\}, \{d\})}, i_L, o_L\}$

$F_L = \{(a,p_{(\{a\}, \{b,e\})}), (p_{(\{a\}, \{b,e\})},b), (p_{(\{a\}, \{b,e\})},e),$
$(a,p_{(\{a\}, \{c,e\})}), (p_{(\{a\}, \{c,e\})},c), (p_{(\{a\}, \{c,e\})},e),$
$(b,p_{(\{b,e\}, \{d\})}), (e,p_{(\{b,e\}, \{d\})}), (p_{(\{b,e\}, \{d\})},d),$
$(c,p_{(\{c,e\}, \{d\})}), (e,p_{(\{c,e\}, \{d\})}), (p_{(\{c,e\}, \{d\})},d),$
$(i_L,a),$
$(d,o_L)\}$

Step 7: The arcs of the WF-net are generated.
All places p(A,B) have A as input nodes and B as output nodes.
All start transitions in T_I have i_L as an input place and all end transitions T_O have o_L as output place.

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8. $\boxed{\alpha(L) = (P_L, T_L, F_L).}$

Step 8: resultant Petri-net that describes the behavior of input event log *L*.

Example

L_1 = [<a, b, c, d>^{3}, <a, c, b, d>^{2}, <a, e, d>]

Table 6.1 Footprint of L_1: $a \#_{L_1} a$, $a \rightarrow_{L_1} b$, $a \rightarrow_{L_1} c$, etc.

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a	$\#_{L_1}$	$\rightarrow_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$	$\rightarrow_{L_1}$
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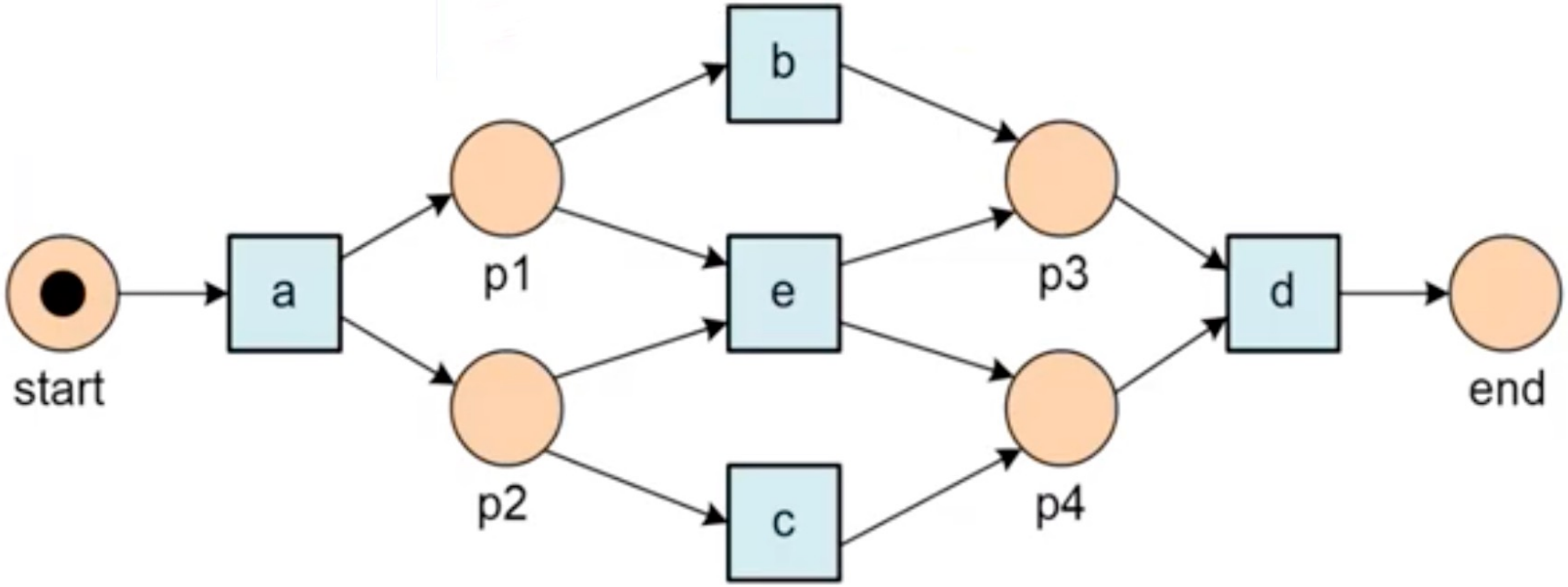
d	$\#_{L_1}$	$\leftarrow_{L_1}$	$\leftarrow_{L_1}$	$\#_{L_1}$	$\leftarrow_{L_1}$
e	$\leftarrow_{L_1}$	$\#_{L_1}$	$\#_{L_1}$	$\rightarrow_{L_1}$	$\#_{L_1}$

T_L = {a, b, c, d, e}
T_I = {a}
T_O = {d}

P_L = {$p_{(\{a\}, \{b,e\})}$, $p_{(\{a\}, \{c,e\})}$, $p_{(\{b,e\}, \{d\})}$, $p_{(\{c,e\}, \{d\})}$, i_L, o_L}

F_L = {(a,$p_{(\{a\}, \{b,e\})}$), ($p_{(\{a\}, \{b,e\})}$,b), ($p_{(\{a\}, \{b,e\})}$,e),
(a,$p_{(\{a\}, \{c,e\})}$), ($p_{(\{a\}, \{c,e\})}$,c), ($p_{(\{a\}, \{c,e\})}$,e),
(b,$p_{(\{b,e\}, \{d\})}$), (e,$p_{(\{b,e\}, \{d\})}$), ($p_{(\{b,e\}, \{d\})}$,d),
(c,$p_{(\{c,e\}, \{d\})}$), (e,$p_{(\{c,e\}, \{d\})}$), ($p_{(\{c,e\}, \{d\})}$,d),
(i_L,a),
(d,o_L)}



Example

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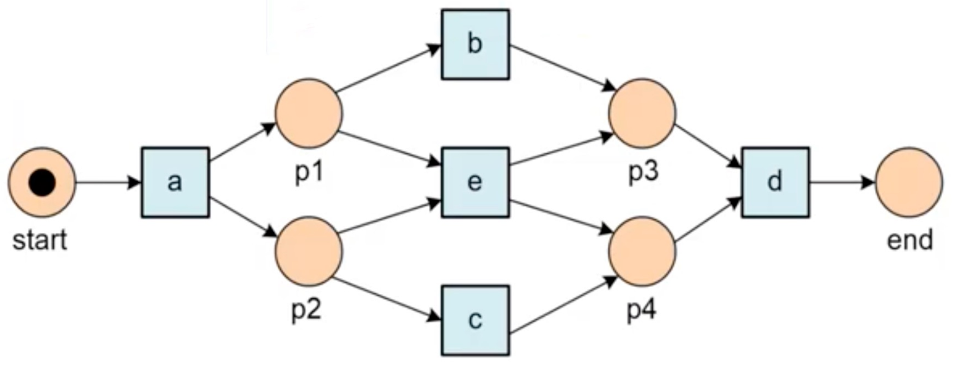

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1. $T_L = \{t \in T \mid \exists_{\sigma \in L}\; t \in \sigma\}$,
2. $T_I = \{t \in T \mid \exists_{\sigma \in L}\; t = \mathit{first}(\sigma)\}$,
3. $T_O = \{t \in T \mid \exists_{\sigma \in L}\; t = \mathit{last}(\sigma)\}$,
4. $X_L = \{(A, B) \mid A \subseteq T_L \;\wedge\; A \neq \emptyset \;\wedge\; B \subseteq T_L \;\wedge\; B \neq \emptyset \;\wedge\; \forall_{a \in A}\forall_{b \in B}\; a \rightarrow_L b \;\wedge\; \forall_{a_1, a_2 \in A}\; a_1 \#_L a_2 \;\wedge\; \forall_{b_1, b_2 \in B}\; b_1 \#_L b_2\}$,
5. $Y_L = \{(A, B) \in X_L \mid \forall_{(A', B') \in X_L} A \subseteq A' \wedge B \subseteq B' \Longrightarrow (A, B) = (A', B')\}$,
6. $P_L = \{p_{(A,B)} \mid (A, B) \in Y_L\} \cup \{i_L, o_L\}$,
7. $F_L = \{(a, p_{(A,B)}) \mid (A, B) \in Y_L \;\wedge\; a \in A\} \cup \{(p_{(A,B)}, b) \mid (A, B) \in Y_L \;\wedge\; b \in B\} \cup \{(i_L, t) \mid t \in T_I\} \cup \{(t, o_L) \mid t \in T_O\}$, and
8. $\alpha(L) = (P_L, T_L, F_L)$.

T_L = {a, b, c, d, e}
T_I = {a}
T_O = {d}

$$X_{L_1} = \{(\{a\},\{b\}), (\{a\},\{c\}), (\{a\},\{e\}), (\{a\},\{b,e\}), (\{a\},\{c,e\}), (\{b\},\{d\}), (\{c\},\{d\}), (\{e\},\{d\}), (\{b,e\},\{d\}), (\{c,e\},\{d\})\}$$

$$Y_{L_1} = \{(\{a\},\{b,e\}), (\{a\},\{c,e\}), (\{b,e\},\{d\}), (\{c,e\},\{d\})\}$$

$P_L = \{p_{(\{a\}, \{b,e\})}, p_{(\{a\}, \{c,e\})}, p_{(\{b,e\}, \{d\})}, p_{(\{c,e\}, \{d\})}), i_L, o_L\}$

$F_L = \{(a, p_{(\{a\}, \{b,e\})}), (p_{(\{a\}, \{b,e\})}, b), (p_{(\{a\}, \{b,e\})}, e),$
$(a, p_{(\{a\}, \{c,e\})}), (p_{(\{a\}, \{c,e\})}, c), (p_{(\{a\}, \{c,e\})}, e),$
$(b, p_{(\{b,e\}, \{d\})}), (e, p_{(\{b,e\}, \{d\})}), (p_{(\{b,e\}, \{d\})}, d),$
$(c, p_{(\{c,e\}, \{d\})}), (e, p_{(\{c,e\}, \{d\})}), (p_{(\{c,e\}, \{d\})}, d),$
$(i_L, a),$
$(d, o_L)\}$

Reading Material

- Chapter 6: Aalst