Machine Learning

4771

Instructor: Itsik Pe’er

Reminder: Ensembles

Many weak classifiers → a powerful one




(Classification) models

Parametric

Estimate parameters of the distribution of the data

Non-parametric

Reason about data assuming unknown distribution

(Classification) models

Parametric

Estimate parameters of the distribution of the data

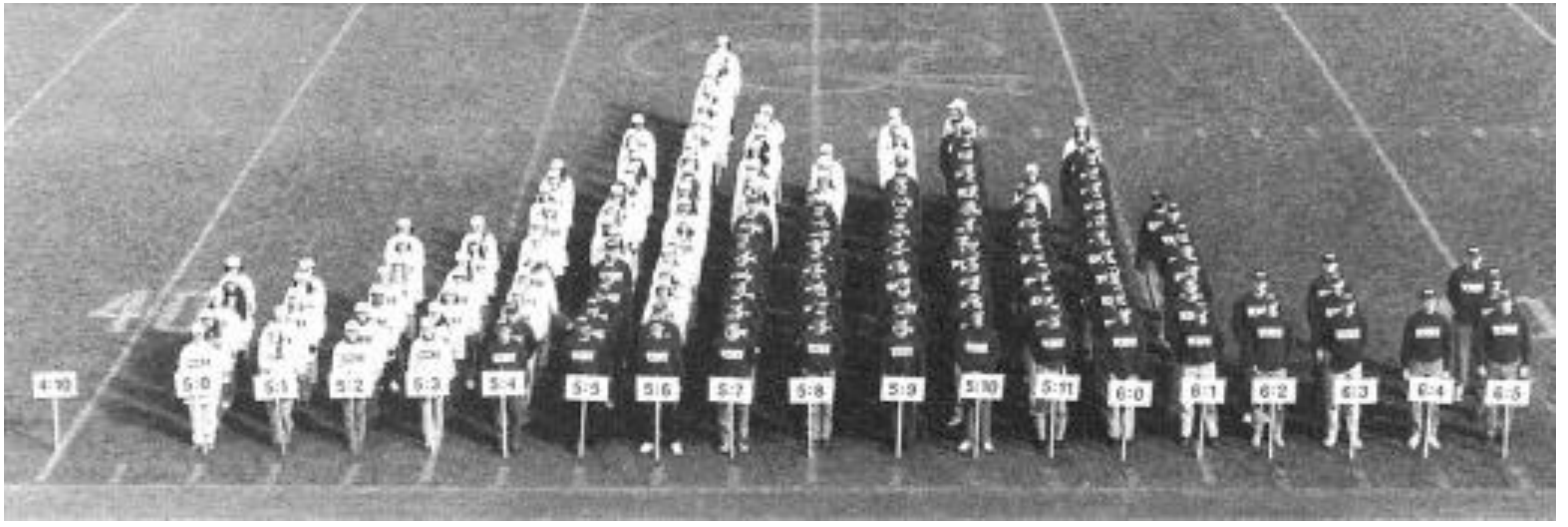
- Logistic regression
- Least squares regression

Non-parametric

Reason about data assuming unknown distribution

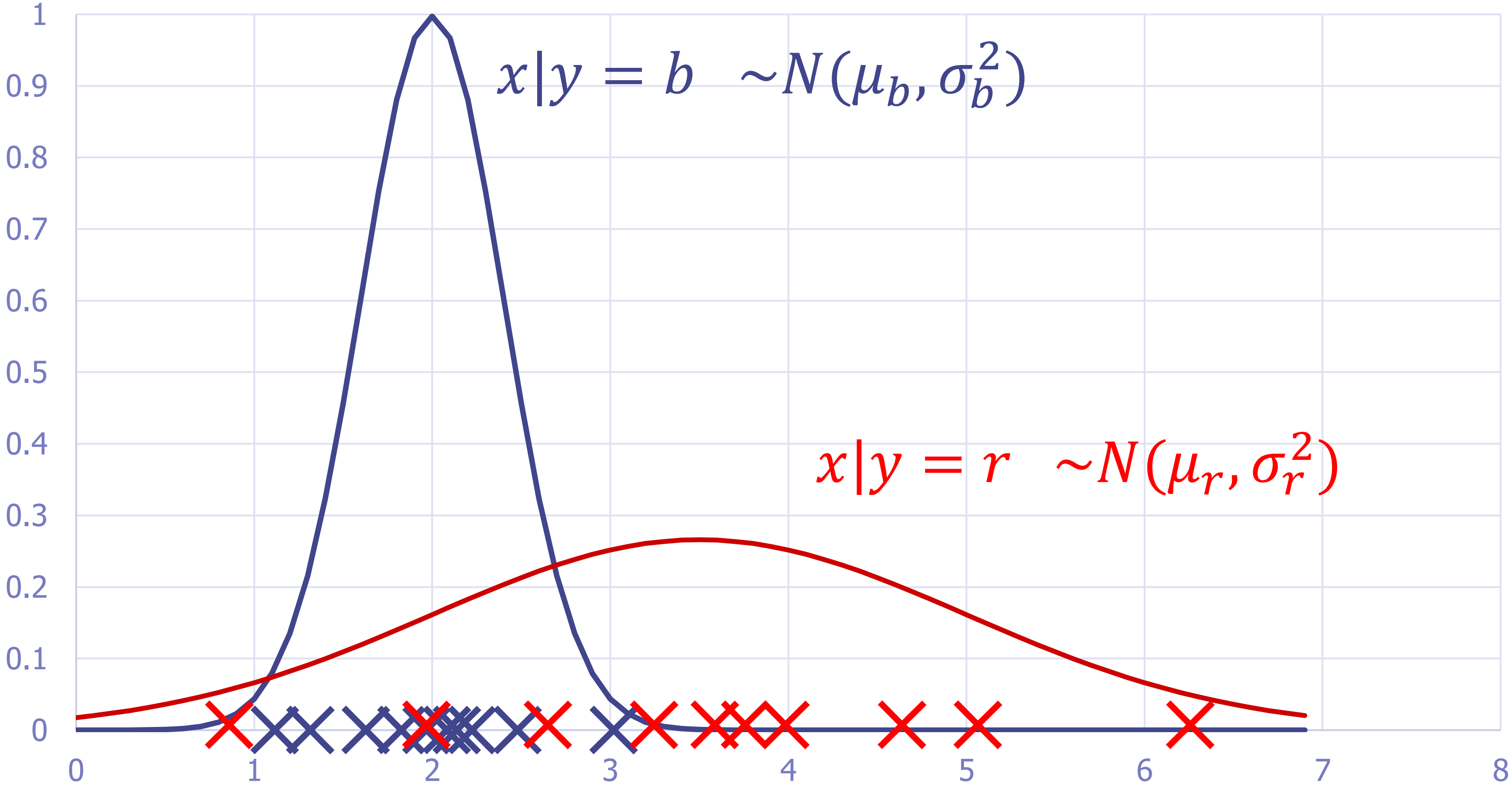
- Nearest neighbors
- Decision trees
- SVM
- RBF regression

Bayesian Classification

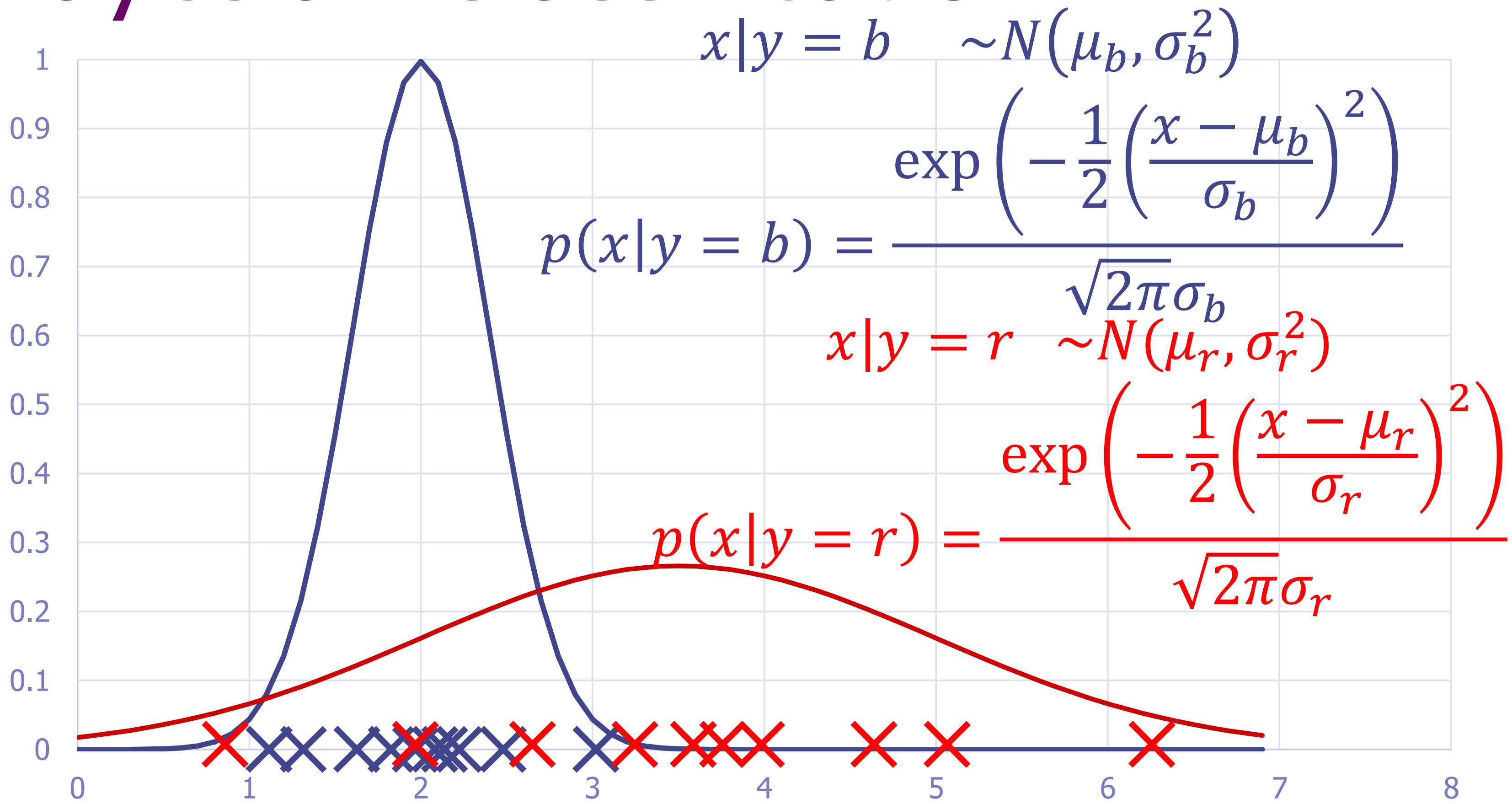


If distribution of each cluster known/inferable:
Can be used for classification

Bayesian Classification



Bayesian Classification



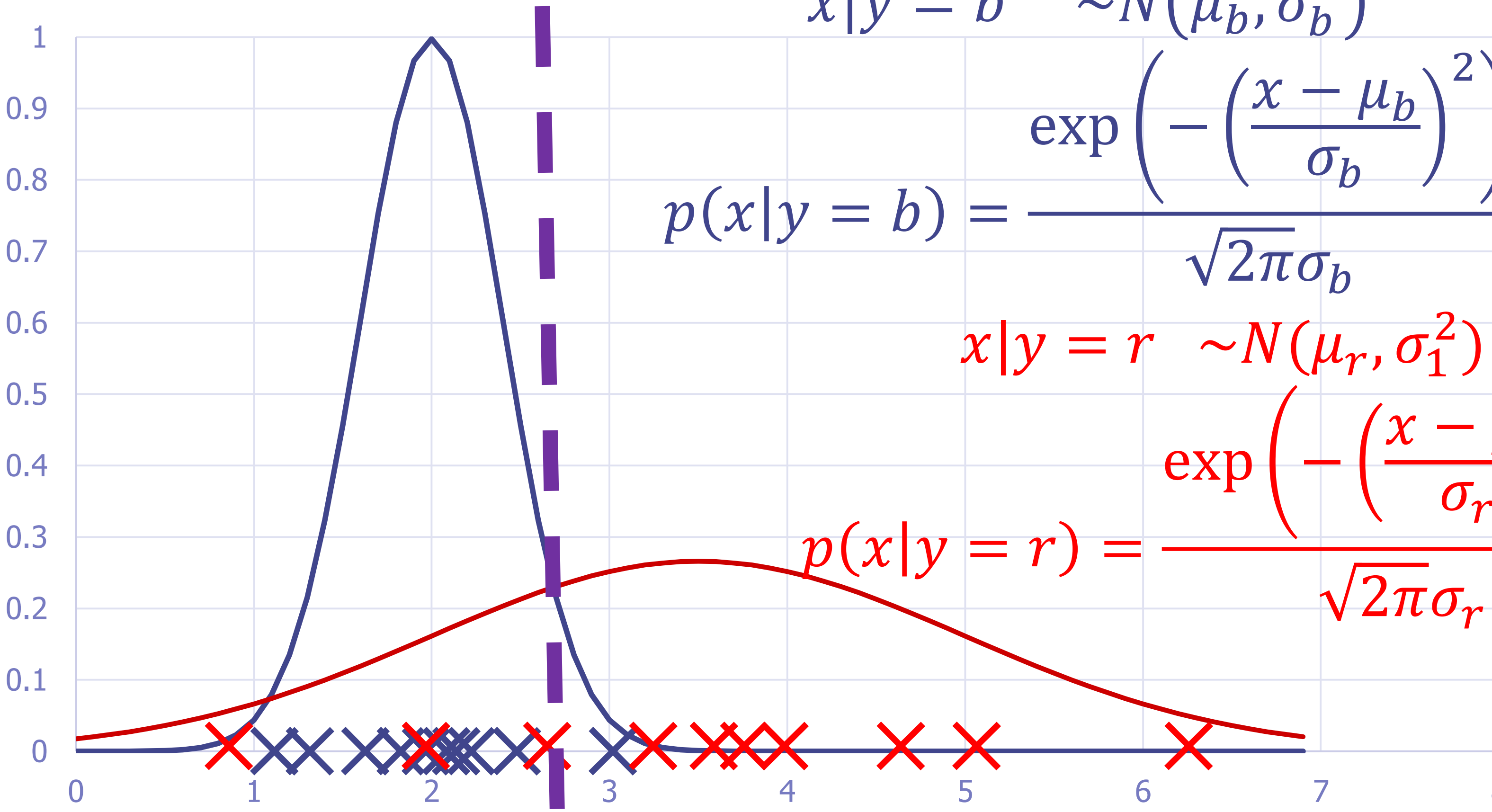
Bayesian Classification

$$x|y=b \quad \sim N\left(\mu_b,\sigma_b^2\right)$$

$$p(x|y=b)=\frac{\exp\left(-\left(\frac{x-\mu_b}{\sigma_b}\right)^2\right)}{\sqrt{2\pi}\sigma_b}$$

$$x|y=r \quad \sim N\left(\mu_r,\sigma_1^2\right)$$

$$p(x|y=r)=\frac{\exp\left(-\left(\frac{x-\mu_r}{\sigma_r}\right)^2\right)}{\sqrt{2\pi}\sigma_r}$$



$$p(y_i=b|x_i)=\frac{p(x_i|y_i=b)p(y_i=b)}{\sum_{y=\{b,r\}}p(x_i|y_i=y)p(y_i=y)}$$

If uniform prior: $p(y_i=b|x_i)=Cp(x_i|y_i=b)$ so max likelihood

High Dimensional Data

- Text classification: simplest model

$\in \begin{cases} religion \\ politics \end{cases}$

- 10^5-10^6 words in English
- Each document is D=10^5 dimensional binary vector $\vec{x}_i$
- Each dimension is a word, set to 1 if word in the document

Dim1: "we" = 1
Dim2: "hello" = 0
Dim3: "people" = 1
Dim4: "justice" = 1
...

$$p(\vec{x}) = p\left(\vec{x}(1), \vec{x}(2), \dots, \vec{x}(D)\right)$$

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...

$$p(\vec{x}) = p\left(\vec{x}(1), \vec{x}(2), \dots, \vec{x}(D)\right)$$

- Each 1 dimensional $\vec{x}(d)$ is a Bernoulli variable
- $\vec{x}$ is multivariate Bernoulli

High Dimensional Data

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...

- Naïve Bayes: assumes each word is independent

$$p(\vec{x}) = p\left(\vec{x}(1), \vec{x}(2), \dots, \vec{x}(D)\right) = \prod_{d=1}^{D} p\left(\vec{x}(d)\right)$$

$$= \prod_{d=1}^{D} \vec{\theta}(d)^{\vec{x}(d)} \left(1 - \vec{\theta}(d)\right)^{1-\vec{x}(d)}$$

Text: Naïve Bayes

- Maximum likelihood: assume we have several IID vectors
- Have N documents, each a 100,000 dimension binary vector
- Each dimension is a word, set to 1 if word in the document

			$\vec{x}_1$	$\vec{x}_2$	$\vec{x}_3$	$\vec{x}_4$
Dim1:	**“the”**	**=**	**1**	**0**	**1**	**1**
Dim2:	**“hello”**	**=**	**0**	**1**	**0**	**1**
Dim3:	**“and”**	**=**	**1**	**1**	**0**	**1**
Dim4:	**“happy”**	**=**	**1**	**0**	**0**	**1**

- Likelihood=

Text: Naïve Bayes

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- Likelihood= $\prod_{i=1}^{N} p\left(\vec{x}_i \middle| \vec{\theta}\right) = \prod_{i=1}^{N} \prod_{d=1}^{D} \vec{\theta}(d)^{\vec{x}_i(d)} \left(1 - \vec{\theta}(d)\right)^{\left(1-\vec{x}_i(d)\right)}$
- Max likelihood solution:

Text: Naïve Bayes

- Likelihood= $\prod_{i=1}^{N} p\left(\vec{x}_i \middle| \vec{\theta}\right) = \prod_{i=1}^{N} \prod_{d=1}^{D} \vec{\theta}(d)^{\vec{x}_i(d)} \left(1 - \vec{\theta}(d)\right)^{\left(1-\vec{x}_i(d)\right)}$
- Max likelihood solution: for each word d count $\vec{\theta}(d) = \frac{N_d}{N}$
- Assuming beta-prior: $p\left(\vec{\theta}(d)\right) \sim Beta(1,1)$

posterior: $p\left(\vec{\theta}(d) | data\right) \sim Beta(N_d + 1, (N - N_d) + 1)$

- $\mathrm{EAP}(\vec{\theta}(d)) = \frac{N_d+1}{N+2}$

Text: Naïve Bayes

- Likelihood$= \prod_{i=1}^{N} p\left(\vec{x}_i \middle| \vec{\theta}\right) = \prod_{i=1}^{N} \prod_{d=1}^{D} \vec{\theta}(d)^{\vec{x}_i(d)} \left(1 - \vec{\theta}(d)\right)^{\left(1-\vec{x}_i(d)\right)}$
- Max likelihood solution: for each word d count $\vec{\theta}(d) = \frac{N_d}{N}$
- Assuming (conjugate) beta-prior:$p\left(\vec{\theta}(d)\right) \sim Beta(\alpha, \beta)$

posterior: $p\left(\vec{\theta}(d) | data\right) \sim Beta(N_d + \alpha, (N - N_d) + \beta)$

- $\mathrm{EAP}(\vec{\theta}(d)) = \frac{N_d+\alpha}{N+\alpha+\beta}$
- To classify new document $\vec{x}_{new}$, build two models $\vec{\theta}_{religion}, \vec{\theta}_{politics}$

  Compare: $prediction = argmax_{y \in \{religion, politics\}} p\left(\vec{x}_{new} \middle| \vec{\theta}_y\right)$

Text: Naïve Bayes

- Likelihood= $\prod_{i=1}^{N} p\left(\vec{x}_i \middle| \vec{\theta}\right) = \prod_{i=1}^{N} \prod_{d=1}^{D} \vec{\theta}(d)^{\vec{x}_i(d)} \left(1 - \vec{\theta}(d)\right)^{\left(1-\vec{x}_i(d)\right)}$
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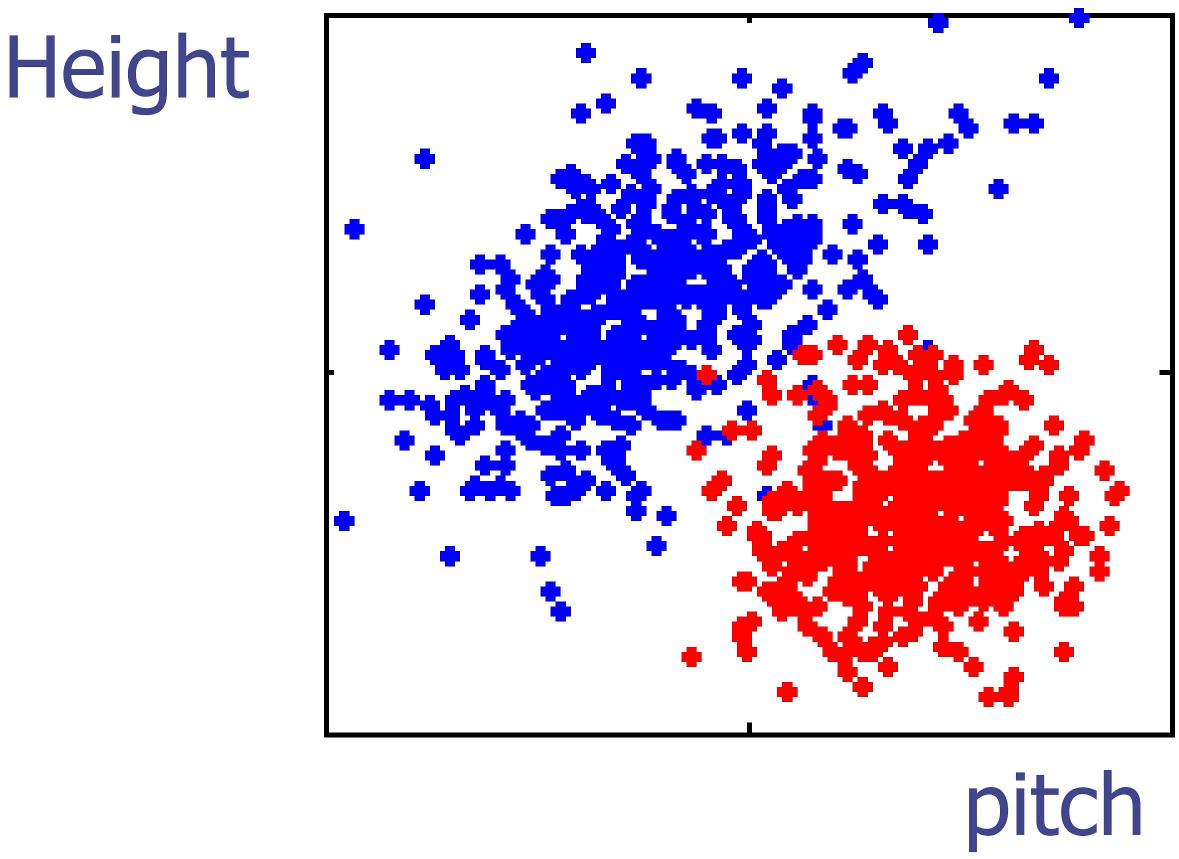
- To classify new document $\vec{x}_{new}$, build two models $\vec{\theta}_{religion}, \vec{\theta}_{politics}$

Compare: $prediction = argmax_{y \in \{religion, politics\}} \log p\left(\vec{x}_{new} \middle| \vec{\theta}_y\right) =$

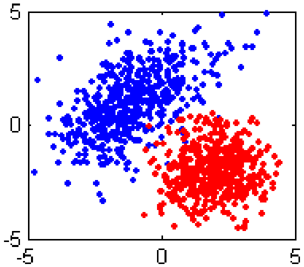
$$argmax_y \sum_{d=1}^{D} \left(\vec{x}_{new}(d) \log \vec{\theta}_y(d) + \left(1 - \vec{x}_{new}(d)\right) \log\left(1 - \vec{\theta}_y(d)\right)\right)$$

$$= argmax_y \sum_{d=1}^{D} \vec{x}_{new}(d) \log \frac{\vec{\theta}_y(d)}{1 - \vec{\theta}_y(d)}$$

Handling Dependencies: Two 2D Gaussians



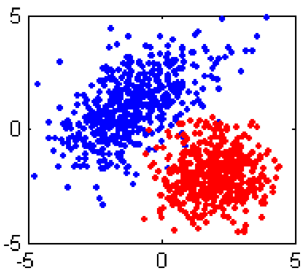
Classification with Gaussians



- Have two classes, each with their own Gaussian:

$$\{(x_1, y_1), \ldots, (x_N, y_N)\} \quad x \in \boldsymbol{R}^D, y \in \{0,1\}$$

Classification with Gaussians



- Have two classes, each with their own Gaussian:

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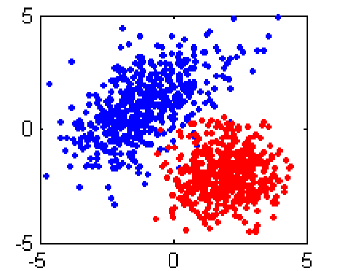
- Given parameters $\theta = \{\alpha, \mu_0, \Sigma_0, \mu_1, \Sigma_1\}$ we can generate iid data from $p(x, y|\theta) = p(y|\theta)p(x|y,\theta)$ by:

  1) flipping a coin to get y via Bernoulli $p(y|\theta) = \alpha^y(1-\alpha)^{1-y}$
  2) sampling an x from y'th Gaussian $p(x|y,\theta) = N(\mu_y, \Sigma_y)$

- Recover parameters from data using maximum likelihood $l(\theta)$

Classification with Gaussians



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- Recover parameters from data using maximum likelihood $l(\theta)$

$$\log p(data|\theta) = \sum_{i=1}^{N} \log p(x_i, y_i|\theta) = \sum_{i=1}^{N} \log p(y_i|\theta) + \sum_{i=1}^{N} p(x_i|y_i, \theta)$$

$$= \sum_{i=1}^{N} \log p(y_i|\alpha) + \sum_{y_i \in 0}^{N} p(x_i|\mu_0, \Sigma_0) + \sum_{y_i \in 1}^{N} p(x_i|\mu_1, \Sigma_1)$$

Classification with Gaussians

- Max Likelihood can be done separately for the 3 terms

$$l(\theta) = \sum_{i=1}^{N} \log p(y_i|\alpha) + \sum_{y_i \in 0}^{N} p(x_i|\mu_0, \Sigma_0) + \sum_{y_i \in 1}^{N} p(x_i|\mu_1, \Sigma_1)$$

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- Count # of pos & neg examples (class prior): $\alpha = \frac{N_1}{N_0 + N_1}$
- Get mean & cov of negatives and mean & cov of positives:

$$\mu_0 = \bar{x}\Big|_{y_i=0} = \frac{1}{N_0}\sum_{y_i=0} x_i \quad \Sigma_0 = \frac{1}{N_0}\sum_{y_i=0}(x_i - \mu_0)(x_i - \mu_0)^T$$

$$\mu_1 = \bar{x}\Big|_{y_i=1} = \frac{1}{N_1}\sum_{y_i=1} x_i \quad \Sigma_1 = \frac{1}{N_1}\sum_{y_i=1}(x_i - \mu_1)(x_i - \mu_1)^T$$

Classification with Gaussians

- Max Likelihood can be done separately for the 3 terms

$l(\theta) = \sum_{i=1}^{N} \log p(y_i|\alpha) + \sum_{y_i \in 0}^{N} p(x_i|\mu_0, \Sigma_0) + \sum_{y_i \in 1}^{N} p(x_i|\mu_1, \Sigma_1)$

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$$\mu_1 = \bar{x}_1 = \bar{x}\Big|_{y_i=1} = \frac{1}{N_1}\sum_{y_i=1} x_i \quad \Sigma_1 = \frac{1}{N_1}\sum_{y_i=1}(x_i - \mu_1)(x_i - \mu_1)^T$$

Posterior μ_y if (conjugate) prior is $N(\mu_p, \Sigma_p)$ and known Σ_y:

$\mu_y \sim N(\mu_{post}, \Sigma_{post})$

where : $\mu_{post} = \Sigma_{post}(\Sigma_p^{-1}\mu_p + N\Sigma_y \bar{x}_y), \Sigma_{post} = (\Sigma_p^{-1} + N\Sigma_y^{-1})^{-1}$

Classification with Gaussians

- Max Likelihood can be done separately for the 3 terms

$$l(\theta) = \sum_{i=1}^{N} \log p(y_i|\alpha) + \sum_{y_i \in 0}^{N} p(x_i|\mu_0, \Sigma_0) + \sum_{y_i \in 1}^{N} p(x_i|\mu_1, \Sigma_1)$$

- Given (x,y) pair, can now compute likelihood $p(x, y)$
- Bayesian classification:
  - Without x, can compute prior guess for y: $p(y)$
  - Give me x, want y, I need posterior $p(y|x)$
  - Bayes optimal decision: $\hat{y} = argmax_{y \in \{0,1\}} p(y|x)$
  - Optimal if we have true probability

Deciding between Gaussians

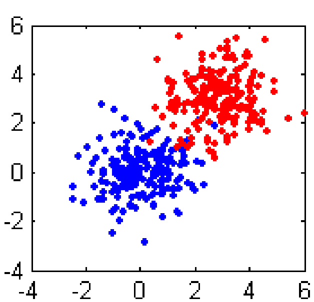
$$p(y=1|x) = \frac{p(x,y=1)}{p(x,y=0)+p(x,y=1)}$$

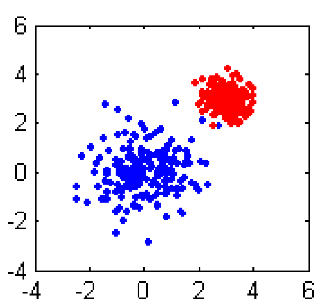
$$= \frac{\alpha N(x|\mu_1,\Sigma_1)}{(1-\alpha)N(x|\mu_0,\Sigma_0)+\alpha N(x|\mu_1,\Sigma_1)}$$

Mahalanobis Distance

$$p(y=1|x) = \frac{p(x,y=1)}{p(x,y=0)+p(x,y=1)}$$
$$= \frac{\alpha N(x|\mu_1,\Sigma_1)}{(1-\alpha)N(x|\mu_0,\Sigma_0)+\alpha N(x|\mu_1,\Sigma_1)}$$

$$\log p(y_{new}=y|x_{new}) = C - \underbrace{\left(x_{new}-\mu_y\right)^T \Sigma_y^{-1}\left(x_{new}-\mu_y\right)}_{\text{Mahalanobis Distance}\left(x_{new},\mu_y\right)}$$
$$C = C_{\alpha,x_{new}} + C_{\alpha,\Sigma_y}$$

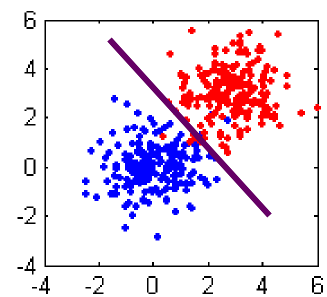




Linear or Quadratic Decisions

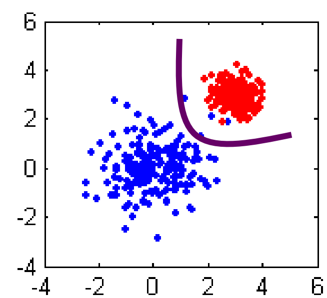
- Example cases, plotting decision boundary when = 0.5

$$p(y=1|x) = \frac{p(x,y=1)}{p(x,y=0)+p(x,y=1)}$$
$$= \frac{\alpha N(x|\mu_1,\Sigma_1)}{(1-\alpha)N(x|\mu_0,\Sigma_0)+\alpha N(x|\mu_1,\Sigma_1)}$$



- If covariances are equal:

  linear decision



- If covariances are different:

  quadratic decision

Summary

- Naïve Bayes:
  - Assuming independence of features
- Classifying Gaussians:
  - Bayesian
  - Mahalanobis Distance