Machine Learning

4771

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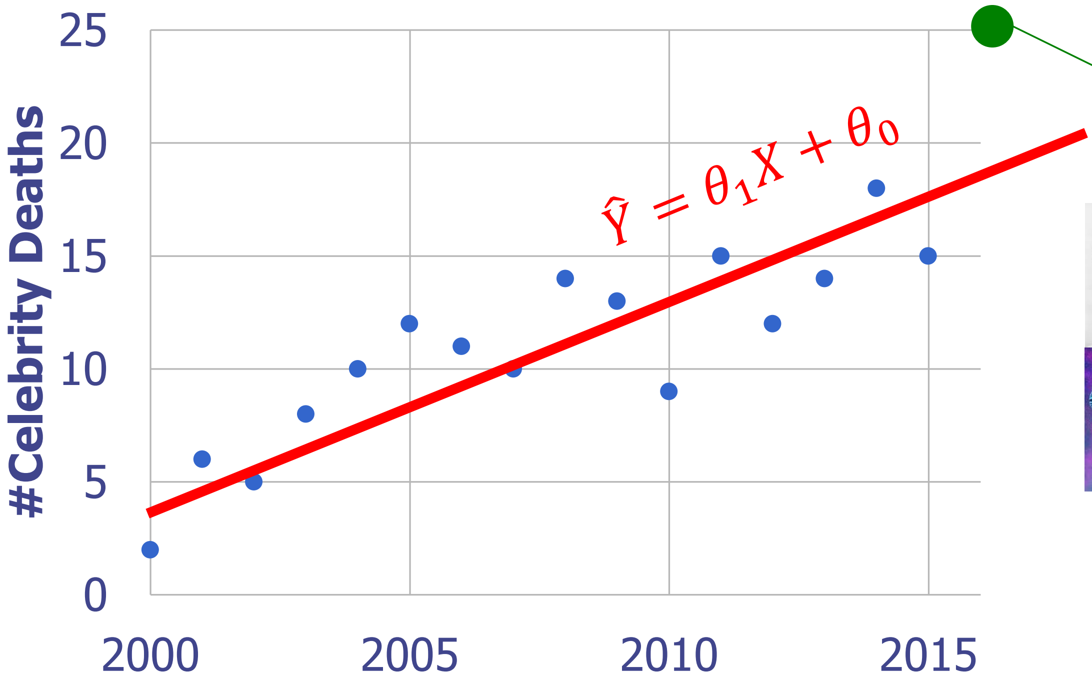
Reminder: Fitting = Maximizing Likelihood

- Regression to fit the Poisson rate



Probability Review

- Definitions
- Distributions
- Moments
- Theorems

Definition: Sample Space

- *Sample space* : Ω all possible outcomes

Definition: Events

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  Examples: deaths ’17, weather Thu, 2 dice
- *Event* : subset of outcomes

Definition: Probability

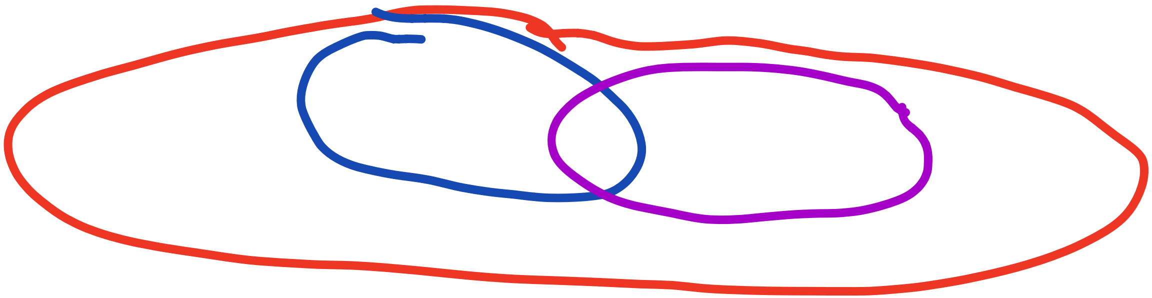
- *Sample space* : Ω all possible outcomes
  Examples: deaths '17, weather Thu, 2 dice
- *Event* : subset of outcomes
  Examples: I die Feb, snow Thu, sum dice<10
- *Probability function*: $\text{Prob}: \Omega \to [0,1]$, additive, $\text{Prob}(\Omega) = 1$

Definition: Random Variables

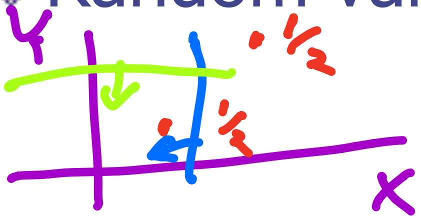
- *Sample space* : Ω all possible outcomes
  Examples: deaths '17, weather Thu, 2 dice
- *Event* : subset of outcomes
  Examples: I die Feb, snow Thu, sum dice<10
- *Probability function*: $\text{Prob}: \Omega \to [0,1]$, additive, $\text{Prob}(\Omega) = 1$
  Examples: forecast, $p([i,j]) = \frac{1}{36}$
- *Random variable*: $X: \Omega \to \boldsymbol{R}$ or $\boldsymbol{R}^D$
  Example: #deaths, percip.[mm], sum dice

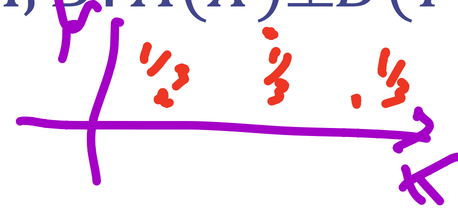
Definition: Independence

- Events $A \perp B$: $\mathrm{Prob}\,(A \cap B) = \mathrm{Prob}\,(A)\,\mathrm{Prob}\,(B)$



- Random variables: $X \perp Y$ if $\forall A, B$: $A(X) \perp B(Y)$





Definition: Independence

- Events $A \perp B$: $\mathrm{Prob}\,(A \cap B) = \mathrm{Prob}\,(A)\mathrm{Prob}\,(B)$
  Examples: coin flips, winning MI/PA
- Random variables $X \perp Y$ if $\forall A, B: A(X) \perp B(Y)$
  Examples: dice, results MI/PA, height/GPA

Definition: Conditional Probability

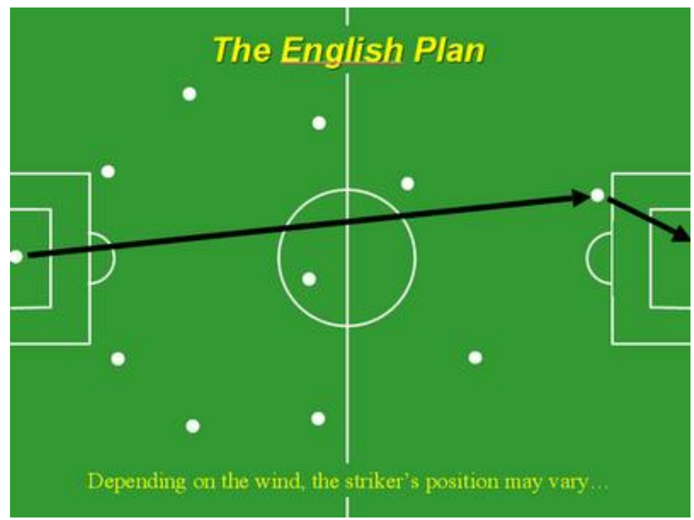
- $\text{Prob}\,(A \cap B) = \dfrac{\text{Prob}\,(A \cap B)}{\text{Prob}\,(B)}$

Examples: low pass possession & goal

80% of goals
≤ 3 passes

Prob(≤3 | G) = .8

Prob(≤3) = .9



Distributions

- Discrete
- Continuous

Distributions

- Discrete
  - Bernoulli, Binomial, Multinomial, Poisson Geometric
- Continuous

Bernoulli Distribution

- Bernoulli

x=0	x=1
0.95	0.05

Bernoulli Distribution

- Bernoulli(α): binary (coin flip) probability, just 1x2 table

$p(x) = \alpha^x(1-\alpha)^{1-x}$ $\alpha \in [0,1], x \in \{0,1\}$

x=0	x=1
0.95	α=0.05

- Multidimensional Bernoulli:

Bernoulli Distribution

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- Multidimensional Bernoulli: multiple binary events

$p(x_1, x_2)$

	x_2=0	x_2=1
x_1=0	0.4	0.1
x_1=1	0.3	0.2

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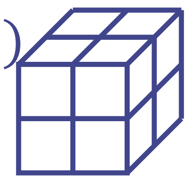
$p(x_1, x_2, x_3)$



Binomial Distribution

- Bernoulli(α): recall binary (coin flip) probability, 1x2 table

$$p(x) = \alpha^{x}(1-\alpha)^{1-x} \qquad \alpha \in [0,1], x \in \{0,1\}$$

iid

x=0	x=1
$1-\alpha$	α

- Binomial(n, α): sum of n identical, independent coin flips

$$p(x) = \binom{n}{x} \alpha^{x}(1-\alpha)^{n-x} \qquad \alpha \in [0,1], x \in \{0,1\} \sim n$$

Poisson Distribution

- Bernoulli(α): recall binary (coin flip) probability, 1x2 table

$$p(x) = \alpha^x (1-\alpha)^{1-x} \qquad \alpha \in [0,1], x \in \{0,1\}$$

x=0	x=1
$1-\alpha$	α

- Binomial(n, α): sum of n identical, independent coin flips

$$p(x) = \binom{n}{x} \alpha^x (1-\alpha)^{n-x} \qquad \alpha \in [0,1], x \in \{0, \ldots, n\}$$

- Poisson(λ): $\lim_{n \to \infty} Binomial\left(n, \frac{\lambda}{n}\right)$ sum of many rare iid coins

$$p(x) = \frac{\lambda^x e^{-\lambda}}{x!} \qquad \lambda \in \boldsymbol{R}^+, x \in \boldsymbol{Z}^+$$

Geometric Distribution

- Bernoulli(α): recall binary (coin flip) probability, 1x2 table

$$p(x) = \alpha^x (1-\alpha)^{1-x} \qquad \alpha \in [0,1], x \in \{0,1\}$$

x=0	x=1
$1-\alpha$	α

- Binomial(n, α): sum of n identical, independent coin flips

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$$p(x) = \frac{\lambda^x e^{-\lambda}}{x!} \qquad \lambda \in \boldsymbol{R}^+, x \in \boldsymbol{Z}^+$$

- Geometric(α): number of iid flips till first success

$$p(x) = (1-\alpha)^{x-1} \alpha \qquad \alpha \in [0,1], x \in \boldsymbol{Z}^+$$

Multinomial Distribution

- Multinomial$(\vec{\alpha})$: beyond binary multi-category event (dice)

1	2	3	4	5	6
$\vec{\alpha}(1)$	$\vec{\alpha}(2)$	$\vec{\alpha}(3)$	$\vec{\alpha}(4)$	$\vec{\alpha}(5)$	$\vec{\alpha}(6)$

$$p(x) = \prod_{m=1}^{M} \vec{\alpha}(m)^{\vec{x}(m)} \qquad \sum_{m} \vec{\alpha}(m) = 1$$

$\vec{x}(1)$	$\vec{x}(2)$	$\vec{x}(3)$	$\vec{x}(4)$	$\vec{x}(5)$	$\vec{x}(6)$

Expectation

- $E(X) = \sum_x xp(X = x)$

What is your best guess for X?

Example:

E(Bernoulli(α)) $= 0 \cdot (1-\alpha) + 1 \cdot \alpha = \alpha$

E(dice) $=$ Avg(1...6)

Avg(1...20)

Expectation

- $E(X) = \sum_x xp(X = x)$
- Important thms:
  - Linearity: $E(X+Y) = E(X) + E(Y)$

    $E(aX) = aE(X)$
  - Law of large numbers:

    $\{\mathrm{X}_1, \dots\}\ \mathrm{i.i.d.}, \mathrm{then}\ S_n = \frac{\sum_{i=1}^{n} x_i}{n} \xrightarrow[n\to\infty]{} E(X)$

Variance

- $Var(X) = E\left(\left(X - E(X)\right)^2\right)$

How wide is X 's distribution around $E(X)$?

Variance

- $\mathrm{Var}(X) = E\left(\left(X - E(X)\right)^2\right)$
- Quadratic scaling: $\mathrm{Var}(aX) = a^2\mathrm{Var}(X)$
- Standard deviation: $Std(X) = \sqrt{\mathrm{Var}(X)}$
- Covariance

$$Cov(X,Y) = E\left(\left(X - E(X)\right)\left(Y - E(Y)\right)\right)$$

Continuous Probability Models

- Probabilities can have both discrete & continuous variables
- We will discuss:
  1) discrete probability tables

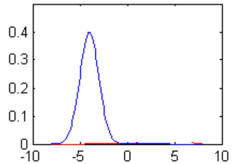
x=T	x=H
0.4	0.6

x=1	x=2	x=3	x=4	x=5	x=6
0.1	0.1	0.1	0.1	0.1	0.5

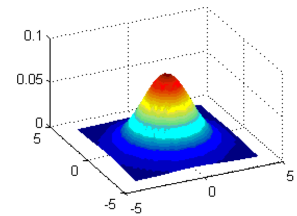
  2) continuous probability distributions

$p(x) =$ probability density function, not probability mass function

$cdf(x) = \int_{-\infty}^{x} p(t)dt$ gives actual probabilities



$$\int_{-\infty}^{\infty} p(x)dx = 1$$



Continuous Distributions: Uniform

- Uniform(a, b) :

$$p(x) = \frac{1}{b-a} \qquad a < b \in \boldsymbol{R}\,, x \in [a, b]$$

Exponential Distribution

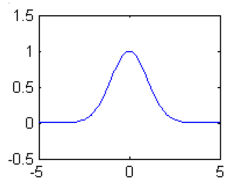
- Exponential(λ) : Time till next Poisson arrival, $\lim_{n \to \infty} \frac{Geometric\left(\frac{\lambda}{n}\right)}{n}$

  $p(x) = \lambda e^{-\lambda} \qquad \lambda \in \boldsymbol{R}^{+}, x \in \boldsymbol{R}^{+}$

Std. Gaussian (Normal) Distribution

- Bell shape curve

$$p(x) = \frac{1}{\sqrt{2\pi}} exp\left(-\frac{x^2}{2}\right)$$



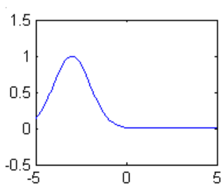
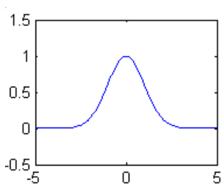
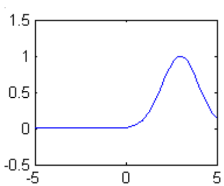
Central Limit Theorem

- $\{\mathrm{X}_1, \dots\}$ i.i.d., $S_n = \frac{\sum_{i=1}^{n} x_i}{n}$
  $E(X) = \mu, Var(X) = \sigma^2$
  then $\sqrt{n}\frac{S_n - \mu}{\sigma} \xrightarrow[n \to \infty]{}$ std. normal

Gaussian Distribution

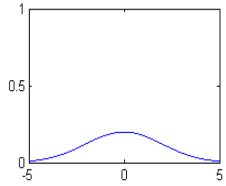
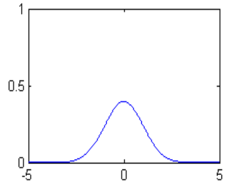
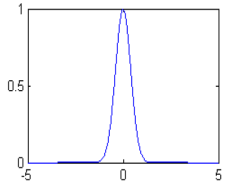
- 1-dimensional Gaussian with mean parameter μ translates Gaussian left & right

$$p(x|\mu) = \frac{1}{\sqrt{2\pi}} exp\left(-\frac{1}{2}(x-\mu)^2\right)$$

- Variance parameter σ^2 controls the width of the Gaussian

$$p(x|\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$

Note: $\int_{-\infty}^{\infty} p(x)dx = 1$

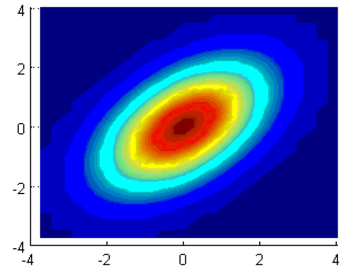
Multivariate Gaussian

- Gaussian can extend to D-dimensions

- Gaussian mean parameter μ vector, it translates the bump
- Covariance matrix Σ stretches and rotates bump

$$p(\vec{x}|\vec{\mu},\Sigma) = \frac{1}{(2\pi)^{\frac{D}{2}}\sqrt{|\Sigma|}} exp\left(-\frac{1}{2}(\vec{x}-\vec{\mu})\Sigma^{-1}(\vec{x}-\vec{\mu})\right)$$

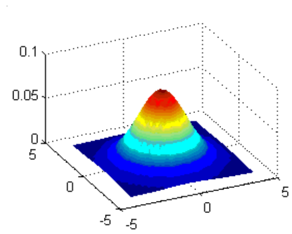
- Mean is any real vector
- Max and expectation = μ
- Variance parameter is now Σ matrix
- Covariance matrix is positive definite
- Covariance matrix is symmetric
- Need matrix inverse (inv)
- Need matrix determinant (det)
- Need matrix trace operator (trace)

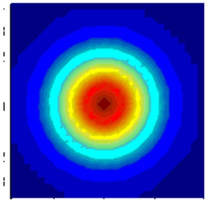


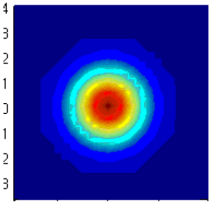
Multivariate Gaussian

- Spherical:

$$\Sigma = \sigma^2 I = \begin{bmatrix} \sigma^2 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & \sigma^2 \end{bmatrix}$$







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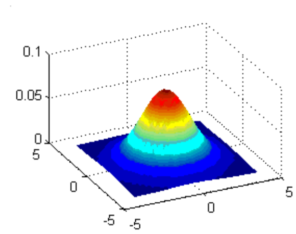
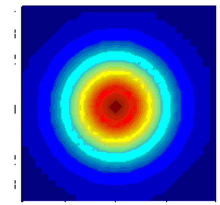
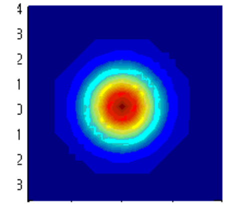
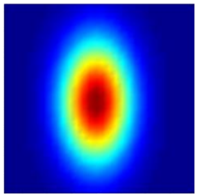


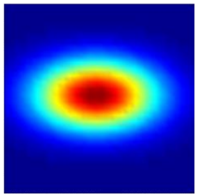




- Diagonal Covariance: dimensions of x are independent product of multiple 1d Gaussians

$\hat{Y}$





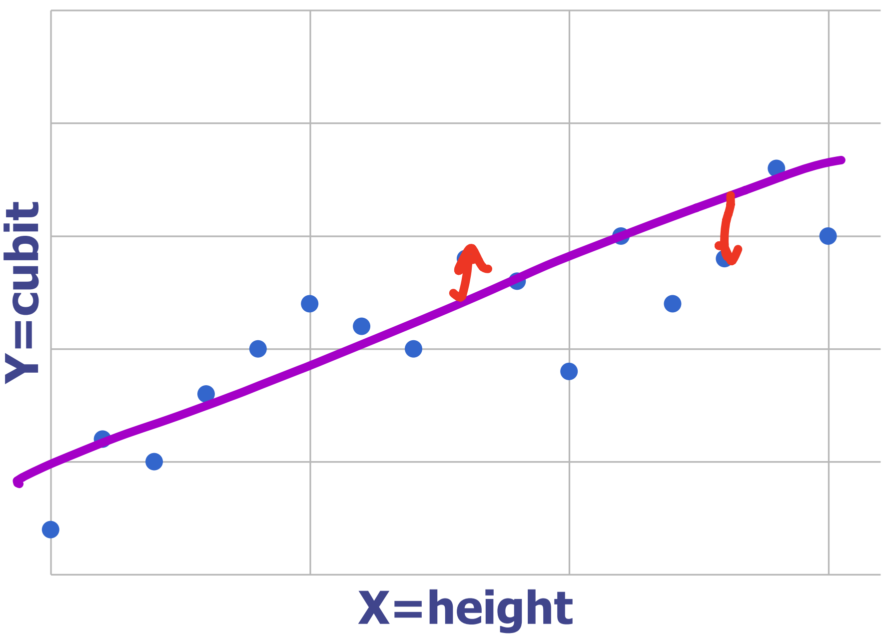
$$p(\vec{x}|\vec{\mu},\Sigma) = \prod_{d=1}^{D} \frac{1}{\sqrt{2\pi}\vec{\sigma}(d)} exp\left(-\frac{\left(\vec{x}(d)-\vec{\mu}(d)\right)^2}{2\vec{\sigma}(d)^2}\right)$$

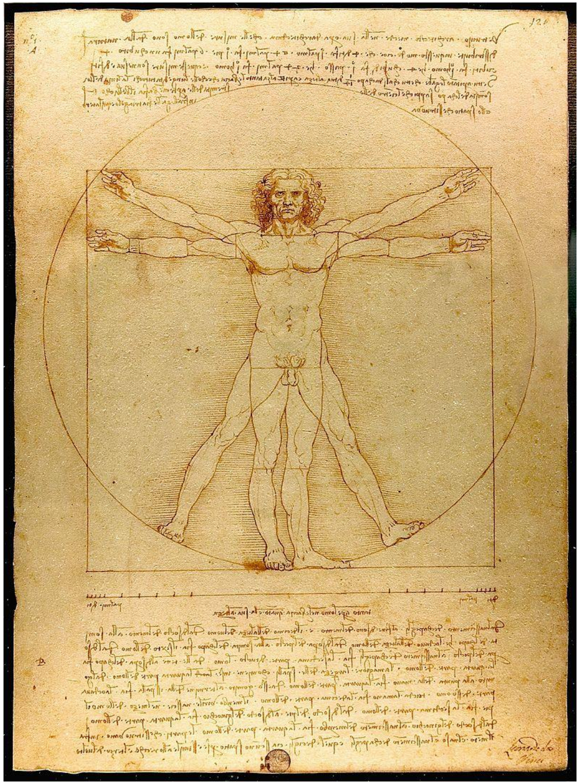
$$\Sigma = \begin{bmatrix} \vec{\sigma}(1)^2 & 0 & 0 & 0 \\ 0 & \vec{\sigma}(2)^2 & 0 & 0 \\ 0 & 0 & \vec{\sigma}(3)^2 & 0 \\ 0 & 0 & 0 & \vec{\sigma}(4)^2 \end{bmatrix}$$

Regression and Gaussians

Vitruvian Man: cubit = ¼ height

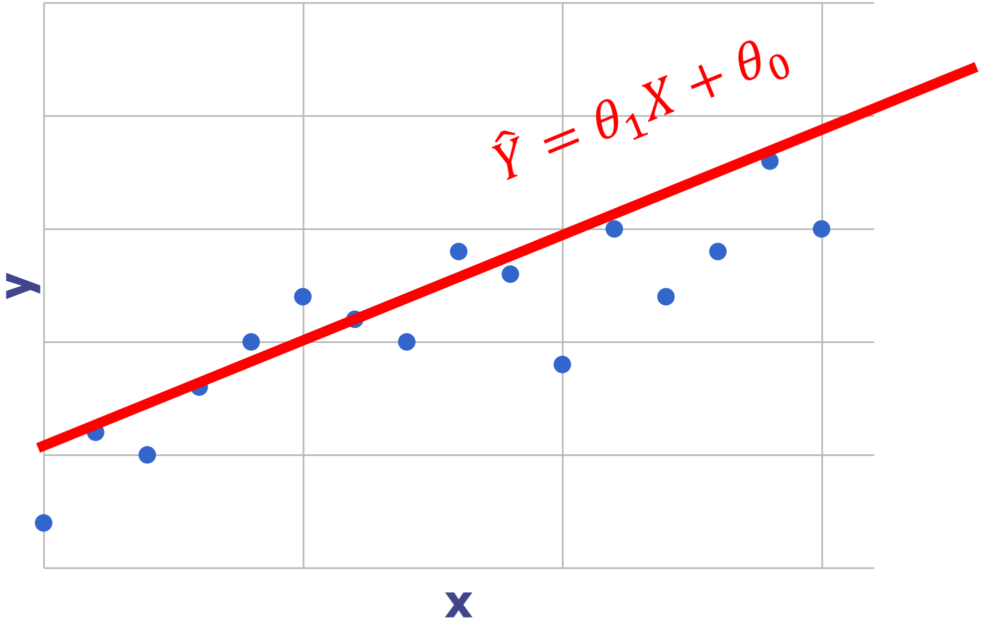
Reality : cubit = ¼ height + noise





Regression and Gaussians

Assume y_i is supposed to be $\hat{y}_i$ but many iid, small sources of error



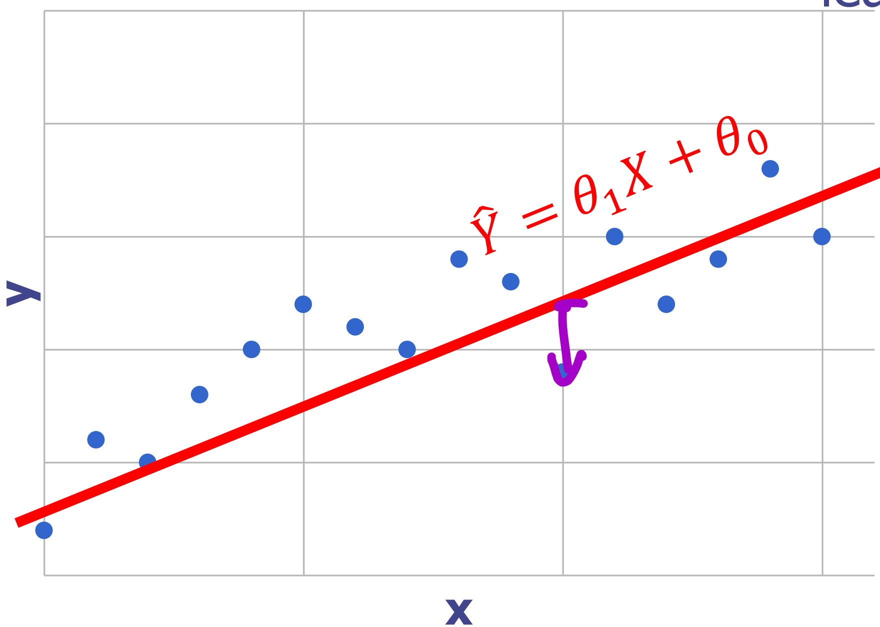
Regression and Gaussians

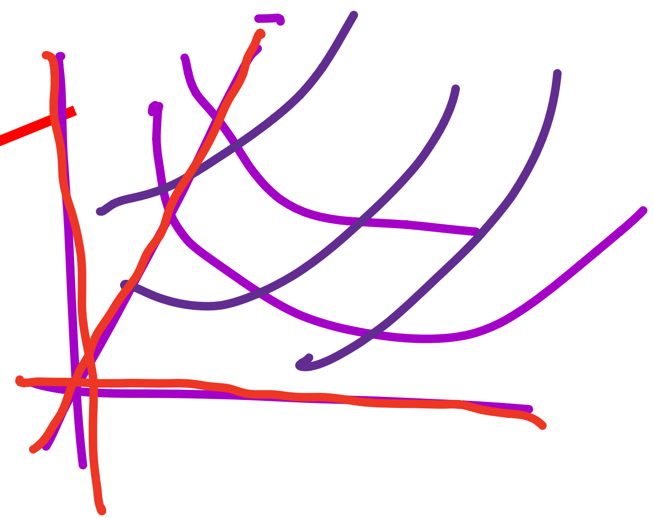
Assume y_i is supposed to be $\hat{y}_i$ but many iid, small sources of error $y_i \sim Normal(\hat{y}_i, \sigma^2)$

log-likelihood:

$$l(Y) = \log \prod_i \text{Prob}(y_i | \hat{y}_i, \sigma^2) = C - \frac{1}{2\sigma^2} \sum_i (y_i - \hat{y}_i)^2$$

least squares=max likelihood





Summary

- Probability definitions, distributions, moments, theorems
- Gaussians motivate least squares