Chapter 3: Arithmetic for Computers

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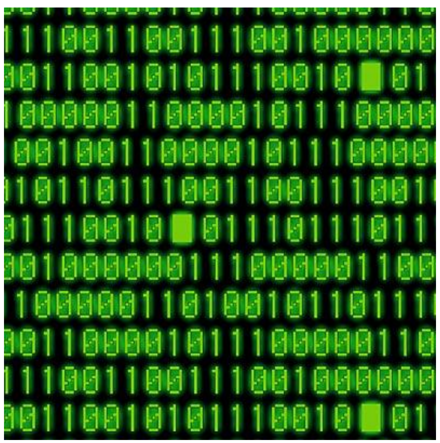
[with materials from *Computer Organization and Design, 4th Edition*,
Patterson & Hennessy, © 2008, MK
and M.J. Irwin's presentation, PSU 2008]

Content

- Integer representation and arithmetic
- Floating point number representation and arithmetic

Overview

- Computers store data as sequences of 1s and 0s
- How these sequences can be converted/displayed as audio, image, photo,…?



Type Name	Size
bool	8 bit unsigned
byte, unsigned char	8 bit unsigned
char	8 bit signed
unsigned int	16 bit unsigned
short, int	16 bit unsigned
unsigned long	32 bit unsigned
long	32 bit signed
float	32 bit floating point value
string	Array of byte

In this chapter: How to represent complicated data types in binary

Sign and unsigned integer

- Unsigned integer

$$x = x_{n-1}x_{n-2}\ldots x_1x_0$$
$$= x_{n-1}2^{n-1} + x_{n-2}2^{n-2} + \cdots + x_1 2^1 + x_0 2^0$$

- 32-bit unsigned numbers

```
0000 0000 0000 0000 0000 0000 0000 0000two = 0ten
0000 0000 0000 0000 0000 0000 0000 0001two = 1ten
...
0111 1111 1111 1111 1111 1111 1111 1110two = 2,147,483,646ten
0111 1111 1111 1111 1111 1111 1111 1111two = 2,147,483,647ten
1000 0000 0000 0000 0000 0000 0000 0000two = 2,147,483,648ten
1000 0000 0000 0000 0000 0000 0000 0001two = 2,147,483,649ten
...
1111 1111 1111 1111 1111 1111 1111 1110two = 4,294,967,294ten
MSB 1111 1111 1111 1111 1111 1111 1111 1111two = 4,294,967,295ten
LSB
```

Sign and unsigned integer

- Signed integer

$$x = x_{n-1}x_{n-2}\ldots x_1x_0$$
$$= -x_{n-1}2^{n-1} + x_{n-2}2^{n-2} + \cdots + x_1 2^1 + x_0 2^0$$

- 32-bit signed numbers (2's complement):

```
0000 0000 0000 0000 0000 0000 0000 0000two = 0ten
0000 0000 0000 0000 0000 0000 0000 0001two = + 1ten
...

0111 1111 1111 1111 1111 1111 1111 1110two = + 2,147,483,646ten
0111 1111 1111 1111 1111 1111 1111 1111two = + 2,147,483,647ten
1000 0000 0000 0000 0000 0000 0000 0000two = – 2,147,483,648ten
1000 0000 0000 0000 0000 0000 0000 0001two = – 2,147,483,647ten
...

1111 1111 1111 1111 1111 1111 1111 1110two = – 2ten
1111 1111 1111 1111 1111 1111 1111 1111two = – 1ten
```

maxint

minint

MSB

LSB

sign bit

Addition and subtraction

- Addition
  - Similar to what you do to add two numbers by hand
  - Digits are added bit by bit from right to left
  - Carries passed to the next digit to the left
- Subtraction
  - Negate the second operand then add to the first operand

$$
\begin{array}{rl}
 & 0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0111_{two} = 7_{ten} \\
+ & 0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0110_{two} = 6_{ten} \\
\hline
 & 0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 1101_{two} = 13_{ten}
\end{array}
$$

Examples

- All numbers are 8-bit signed integer

```
 12 + 8 =
122 + 8 =
122 + 80 =
```

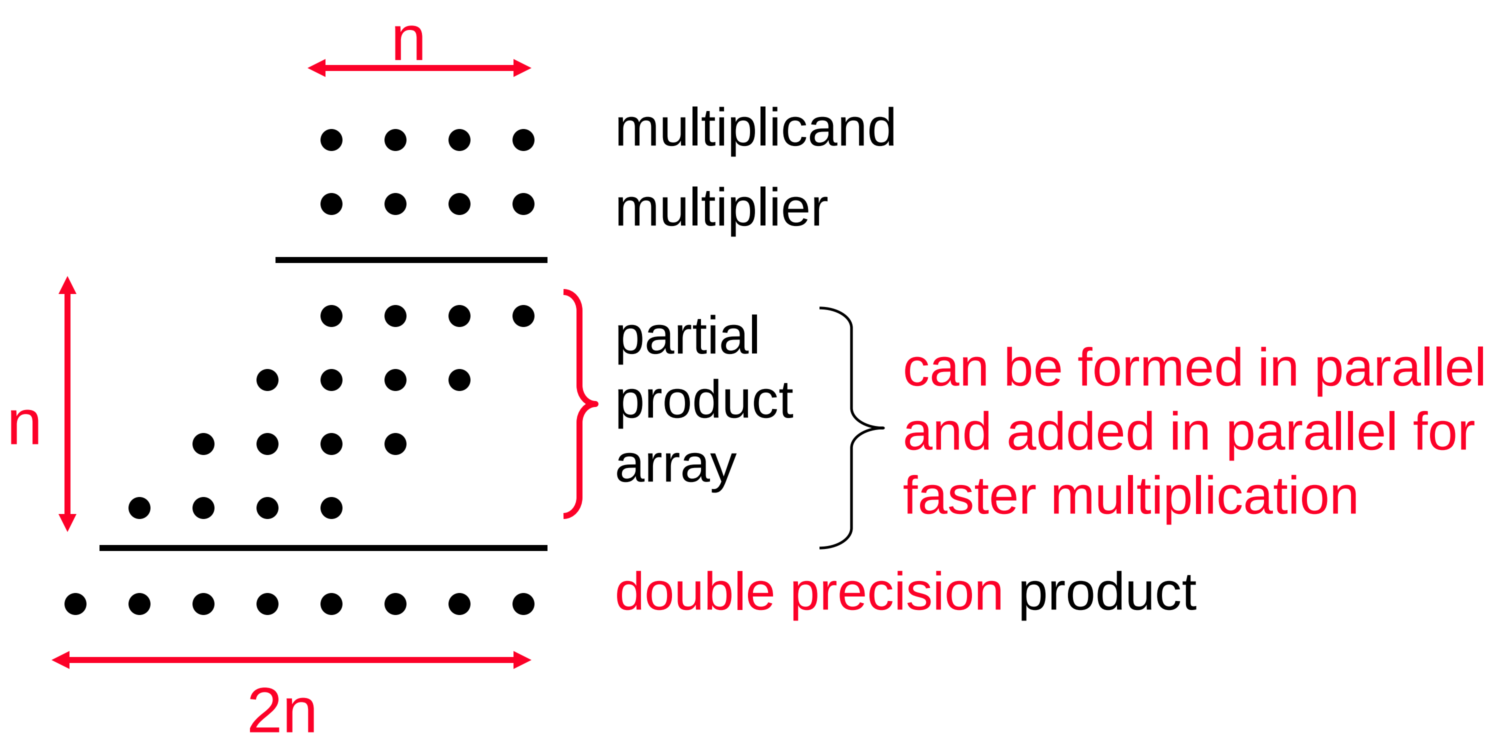
Dealing with Overflow

- Overflow occurs when the result of an operation cannot be represented in 32-bits, i.e., when the sign bit contains a value bit of the result and not the proper sign bit
  - When adding operands with different signs or when subtracting operands with the same sign, overflow can *never* occur

Operation	Operand A	Operand B	Result indicating overflow
A + B	≥ 0	≥ 0	< 0
A + B	< 0	< 0	≥ 0
A - B	≥ 0	< 0	< 0
A - B	< 0	≥ 0	≥ 0

Multiply

- Binary multiplication is just a *bunch* of right shifts and adds



n-bit multiplicand and multiplier → 2n-bit product

Example

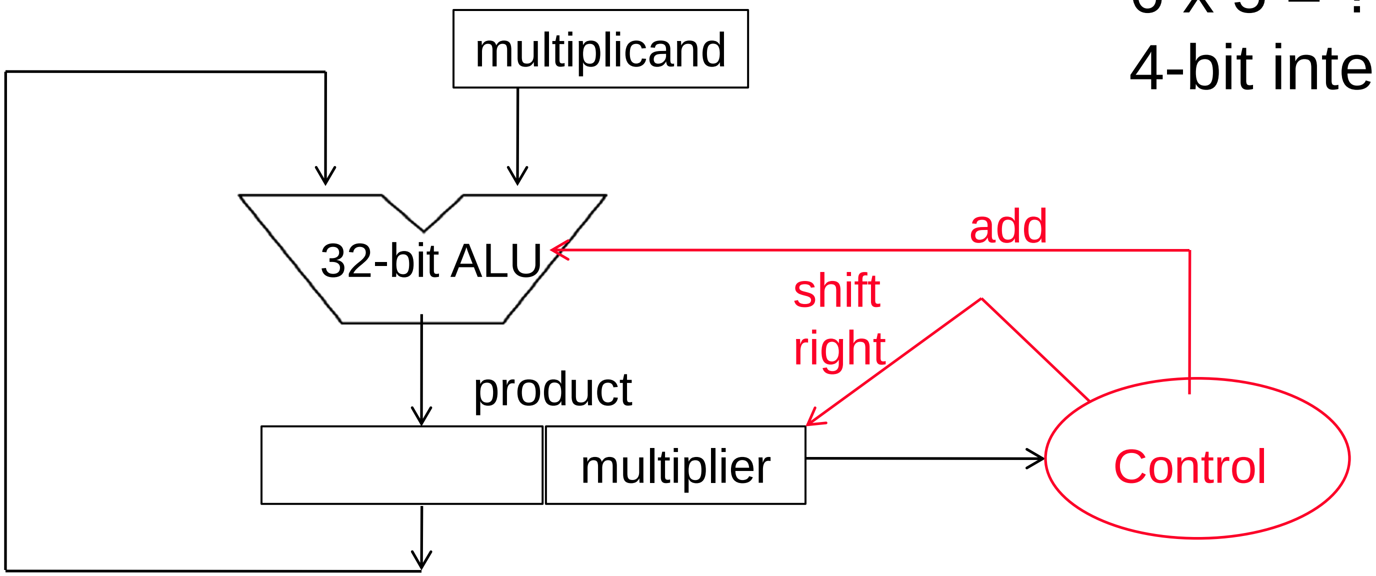
$$
\begin{array}{llr}
\text{Multiplicand} & & 1000_{\text{ten}} \\
\text{Multiplier} & \times & 1001_{\text{ten}} \\
\hline
 & & 1000 \\
 & & 0000 \\
 & & 0000 \\
 & & 1000 \\
\hline
\text{Product} & & 1001000_{\text{ten}}
\end{array}
$$

Add and Right Shift Multiplier Hardware

0 1 1 0 = 6

6 x 5 = ?

4-bit integer



	0 0 0 0	0 1 0 1	= 5
add	0 1 1 0	0 1 0 1	LSB=1 → add multiplicand
	0 0 1 1 →	0 0 1 0	shift right
add	0 0 1 1	0 0 1 0	LSB=0 → no change
	0 0 0 1 →	1 0 0 1	shift right
add	0 1 1 1	1 0 0 1	LSB=1 → add multiplicand
	0 0 1 1 →	1 1 0 0	shift right
add	0 0 1 1	1 1 0 0	LSB=0 → no change
	0 0 0 1 →	1 1 1 0	shift right = 30

MIPS Multiply Instruction

- Multiply (`mult` and `multu`) produces a double precision product (2 x 32 bit)

```
mult   $s0, $s1      # hi||lo = $s0 * $s1
```

0	16	17	0	0	0x18

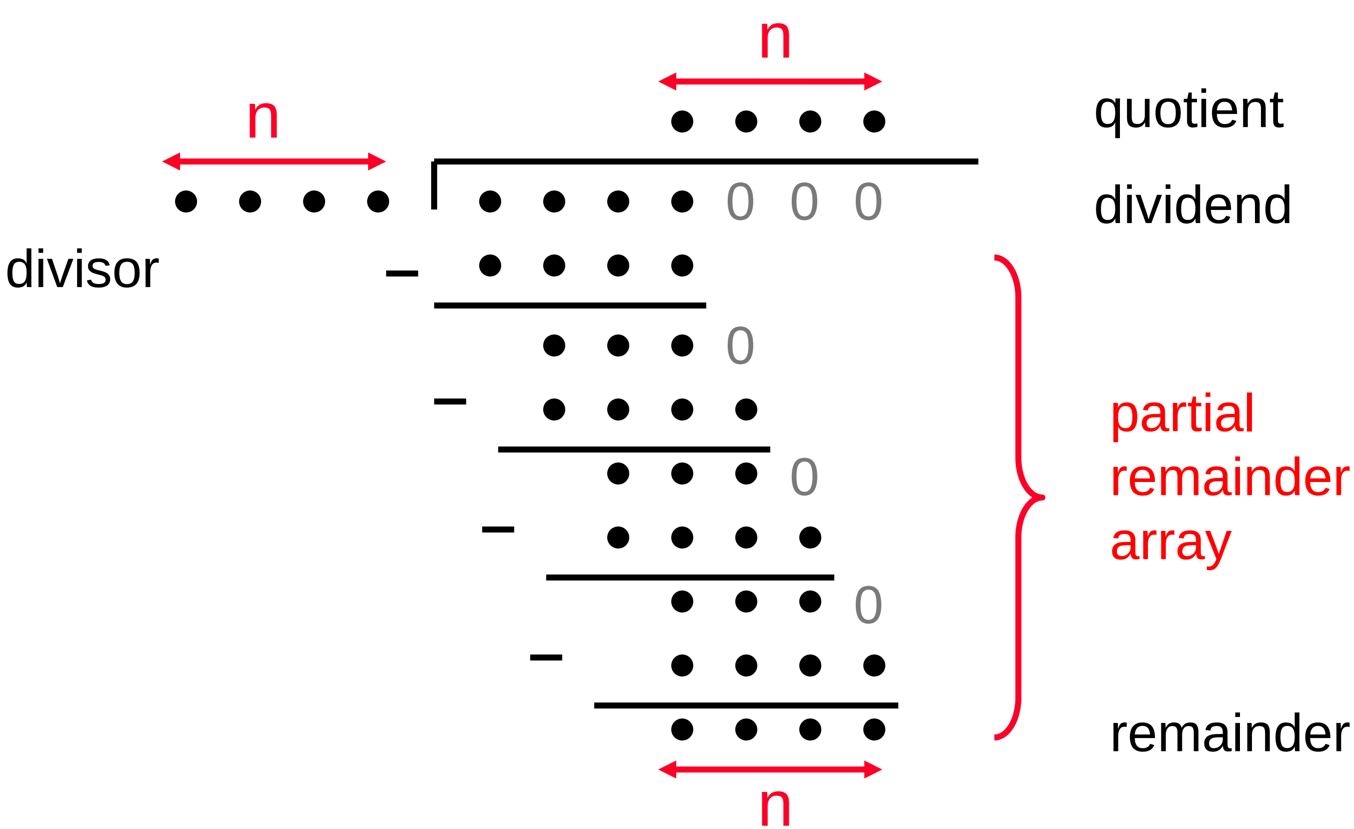
  - Two additional registers: **hi** and **lo**
  - Low-order word of the product is left in processor register `lo` and the high-order word is left in register `hi`
  - Instructions `mfhi rd` and `mflo rd` are provided to move the product to (user accessible) registers in the register file

- Multiplies are usually done by fast, dedicated hardware and are much more complex (and slower) than adders

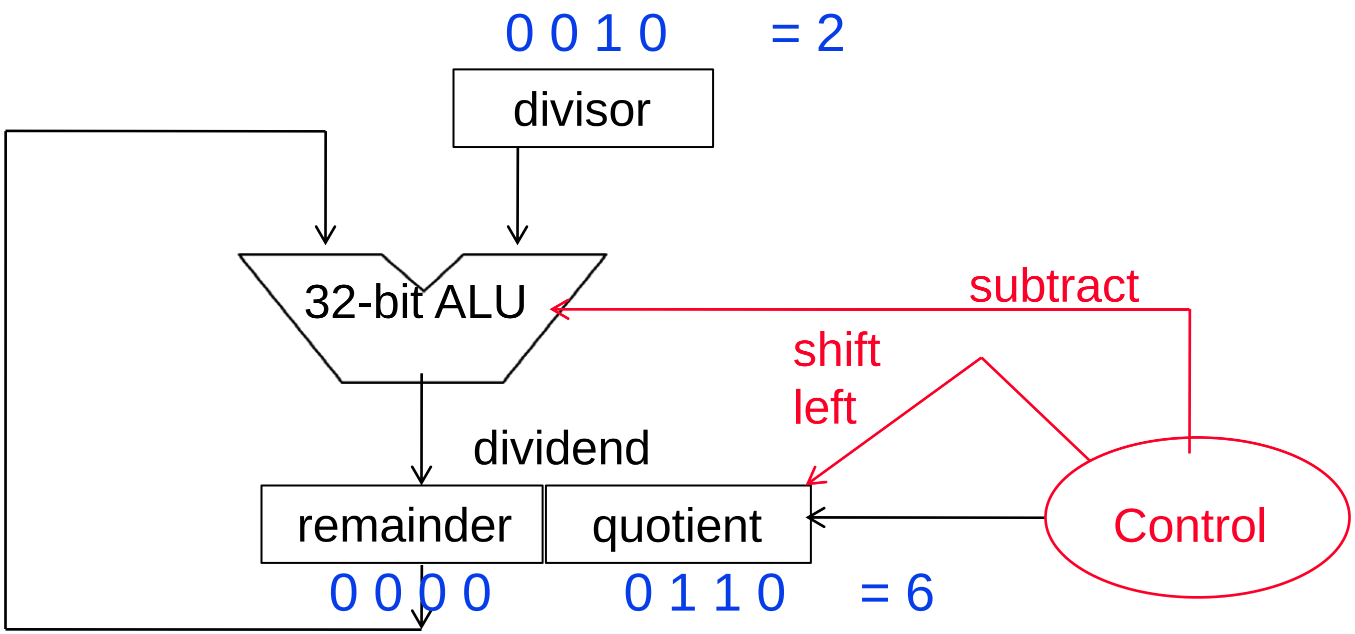
Division

- Division is just a *bunch* of quotient digit guesses and left shifts and subtracts

dividend = quotient x divisor + remainder



Left Shift and Subtract Division Hardware



	remainder	quotient	
	0 0 0 0 ←	1 1 0 0	
sub	1 1 1 0	1 1 0 0	rem neg, so 'ient bit = 0
	0 0 0 0	1 1 0 0	restore remainder
	0 0 0 1 ←	1 0 0 0	
sub	1 1 1 1	1 1 0 0	rem neg, so 'ient bit = 0
	0 0 0 1	1 0 0 0	restore remainder
	0 0 1 1 ←	0 0 0 0	
sub	0 0 0 1	0 0 0 1	rem pos, so 'ient bit = 1
	0 0 1 0 ←	0 0 1 0	
sub	0 0 0 0	0 0 1 1	rem pos, so 'ient bit = 1

= 3 with 0 remainder

MIPS Divide Instruction

- Divide (`div` and `divu`) generates the reminder in `hi` and the quotient in `lo`

```
div   $s0, $s1        # lo = $s0 / $s1
                      # hi = $s0 mod $s1
```

0	16	17	0	0	0x1A

  - Instructions `mfhi rd` and `mflo rd` are provided to move the quotient and reminder to (user accessible) registers in the register file

- As with multiply, divide ignores overflow so software must determine if the quotient is too large. Software must also check the divisor to avoid division by 0.

Representing Big (and Small) Numbers

- What if we want to encode the approx. age of the earth?

  4,600,000,000 or 4.6×10^9

  or the weight in kg of one a.m.u. (atomic mass unit)

  0.000000000000000000000000166 or 1.6×10^{-27}

  or a famous number

  PI = 3.14159….

  There is no way we can encode either of the above in a 32-bit integer.

- We need reals or floating point numbers!
- Floating point numbers in decimal:
  - 1000
  - 1×10^3
  - 0.1×10^4

Floating point number

- In decimal system

$$2013.1228 = 201.31228 * 10$$
$$= 20.131228 * 10^2$$
$$= 2.0131228 * 10^3$$
$$= 20131228 * 10^{-4}$$

- What is the "standard" form?

$$2.0131228 * 10^3 = \underbrace{2.0131228}_{\text{mantissa}}\text{E+}\underbrace{03}_{\text{exponent}}$$

- In binary $X = \pm 1.xxxxx * 2^{yyyy}$
- ***Sign, mantissa, and exponent need to be represented***

Floating point number

- Floating point representation in binary

$$(-1)^{sign} \times \; 1.F \times 2^{E\text{-}bias}$$

  - Still have to fit everything in 32 bits (single precision)
  - Bias = 127 with single precision floating point number

s	E (exponent)	F (fraction)
1 sign bit	8 bits	23 bits

- Defined by the IEEE 754-1985 standard
  - Single precision: 32 bit
  - Double precision: 64 bit
  - Correspond to float and double in C

Examples

- Ex1: convert X into decimal value

X = 1100 0001 0101 0110 0000 0000 0000 0000

sign = 1 $\rightarrow$ X is negative

E = 1000 0010 = 130

F = 10101100...00

$\rightarrow$ X = $(-1)^1 \times 1.101011000..00 \times 2^{130-127}$

$= -1.101011 \times 2^3 = -1101.011$

$= -13.375$

Example

- Ex2: find decimal value of X

X = 0011 1111 1000 0000 0000 0000 0000 0000

sign = 0

e = 0111 1111 = 127

m = 000…0000 (23 bit 0)

$X = (-1)^0 \times 1.00…000 \times 2^{127-127} = 1.0$

Example

- Ex3: find binary representation of X = 9.6875 in IEEE 754 single precision

Converting X to plain binary

$9_{10} = 1001_2$

0.6875 x 2 = 1.375 → get bit 1
0.375 x 2 = 0.75 → get bit 0
0.75 x 2 = 1.5 → get bit 1
0.5 x = 1.0 → get bit 1

➔ $9.6875_{10} = 1001.1011_2$

Example

- Ex3: find binary representation of X = 9.6875 in IEEE 754 single precision

$X = 9.6875_{(10)} = 1001.1011_{(2)} = 1.0011011 \times 2^3$

Then

S = 0

$e = 127 + 3 = 130_{(10)} = 1000\ 0010_{(2)}$

m = 001101100...00 (23 bit)

Finally

X = 0100 0001 0001 1011 0000 0000 0000 0000

Examples

- $1.0_2 \times 2^{-1} =$
- $100.75_{10} =$

Some special values

- Smallest+: 0 00000001 1.00000000000000000000000

  $= 1 \times 2^{1-127}$

- Zero: 0 00000000 00000000000000000000000

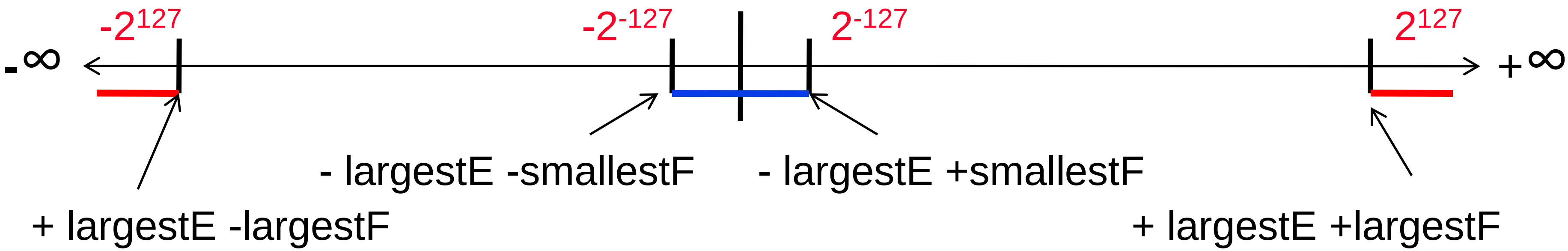
  = true 0

- Largest+: 0 11111110 1.11111111111111111111111

  $= (2-2^{-23}) \times 2^{254-127}$

Too large or too small values

- Overflow (floating point) happens when a positive exponent becomes too large to fit in the exponent field
- Underflow (floating point) happens when a negative exponent becomes too large to fit in the exponent field



- One way to reduce the chance of underflow or overflow is to offer another format that has a larger exponent field
  - Double precision – takes two MIPS words

s	E (exponent)	F (fraction)
1 bit	11 bits	20 bits

F (fraction continued)
32 bits

IEEE 754 FP Standard Encoding

- Special encodings are used to represent unusual events
  - ± infinity for division by zero
  - NAN (not a number) for the results of invalid operations such as 0/0
  - True zero is the bit string all zero

Single Precision		Double Precision		Object Represented
E (8)	F (23)	E (11)	F (52)	
0000 0000	0	0000 … 0000	0	true zero (0)
0000 0000	nonzero	0000 … 0000	nonzero	± denormalized number
0111 1111 to +127,-126	anything	0111 …1111 to +1023,-1022	anything	± floating point number
1111 1111	+ 0	1111 … 1111	- 0	± infinity
1111 1111	nonzero	1111 … 1111	nonzero	not a number (NaN)

Floating Point Addition

- Addition (and subtraction)

$$(\pm F1 \times 2^{E1}) + (\pm F2 \times 2^{E2}) = \pm F3 \times 2^{E3}$$

- Step 0: Restore the hidden bit in F1 and in F2
- Step 1: Align fractions by right shifting F2 by E1 - E2 positions (assuming E1 $\geq$ E2) keeping track of (three of) the bits shifted out in G R and S
- Step 2: Add the resulting F2 to F1 to form F3
- Step 3: Normalize F3 (so it is in the form 1.XXXXX …)
  - If F1 and F2 have the same sign $\rightarrow$ F3 $\in[1,4)$ $\rightarrow$ 1 bit right shift F3 and increment E3 (check for overflow)
  - If F1 and F2 have different signs $\rightarrow$ F3 may require *many* left shifts each time decrementing E3 (check for underflow)
- Step 4: Round F3 and possibly normalize F3 again
- Step 5: Rehide the most significant bit of F3 before storing the result

Floating Point Addition Example

- Add: 0.5 + (-0.4375) = ?

  $(0.5 = 1.0000 \times 2^{-1}) + (-0.4375 = -1.1100 \times 2^{-2})$

  - Step 0: Hidden bits restored in the representation above
  - Step 1: Shift significand with the smaller exponent (1.1100) right until its exponent matches the larger exponent (so once)
  - Step 2: Add significands

    1.0000 + (-0.111) = 1.0000 – 0.111 = 0.001
  - Step 3: Normalize the sum, checking for exponent over/underflow

    $0.001 \times 2^{-1} = 0.010 \times 2^{-2} = .. = 1.000 \times 2^{-4}$
  - Step 4: The sum is already rounded, so we're done
  - Step 5: Rehide the hidden bit before storing

Floating Point Multiplication

- Multiplication

$$(\pm F1 \times 2^{E1}) \text{ x } (\pm F2 \times 2^{E2}) = \pm F3 \times 2^{E3}$$

- Step 0: Restore the hidden bit in F1 and in F2
- Step 1: Add the two (biased) exponents and subtract the bias from the sum, so E1 + E2 – 127 = E3

  also determine the sign of the product (which depends on the sign of the operands (most significant bits))
- Step 2: Multiply F1 by F2 to form a double precision F3
- Step 3: Normalize F3 (so it is in the form 1.XXXXX …)
  - Since F1 and F2 come in normalized $\rightarrow$ F3 $\in [1,4)$ $\rightarrow$ 1 bit right shift F3 and increment E3
  - Check for overflow/underflow
- Step 4: Round F3 and possibly normalize F3 again
- Step 5: Rehide the most significant bit of F3 before storing the result

Floating Point Multiplication Example

- Multiply

$$(0.5 = 1.0000 \times 2^{-1}) \text{ x } (-0.4375 = -1.1100 \times 2^{-2})$$

- Step 0: Hidden bits restored in the representation above
- Step 1: Add the exponents (not in bias would be -1 + (-2) = -3 and in bias would be (-1+127) + (-2+127) – 127 = (-1 -2) + (127+127-127) = -3 + 127 = 124
- Step 2: Multiply the significands
  1.0000 x 1.110 = 1.110000
- Step 3: Normalized the product, checking for exp over/underflow
  1.110000 x 2^{-3} is already normalized
- Step 4: The product is already rounded, so we're done
- Step 5: Rehide the hidden bit before storing

MIPS Arithmetic Logic Unit (ALU)

- Must support the Arithmetic/Logic operations of the ISA

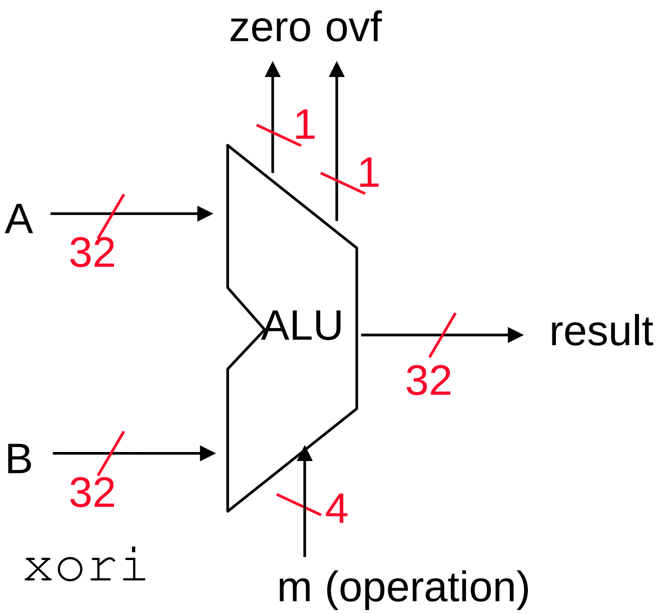
  `add, addi, addiu, addu`

  `sub, subu`

  `mult, multu, div, divu`

  `and, andi, nor, or, ori, xor, xori`

  `beq, bne, slt, slti, sltiu, sltu`



- With special handling for
  - sign extend – `addi, addiu, slti, sltiu`
  - zero extend – `andi, ori, xori`
  - overflow detection – `add, addi, sub`