Recursion

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Objectives

- become familiar with the idea of recursion
- learn to use recursion as a programming tool
- become familiar with the binary search algorithm as an example of recursion
- become familiar with the merge sort algorithm as an example of recursion

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How do you look up a name in the phone book?

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One Possible Way

```
Search:
    middle page = (first page + last page)/2
    Go to middle page;
    If (name is on middle page)
        done;  //this is the base case
    else if (name is alphabetically before middle page)
        last page = middle page  //redefine search area to front half
        Search  //same process on reduced number of pages
    else  //name must be after middle page
        first page = middle page  //redefine search area to back half
        Search  //same process on reduced number of pages
```

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Overview

Recursion: a definition in terms of itself.

Recursion in algorithms:

- Natural approach to **some** (not all) problems
- A *recursive algorithm* uses itself to solve one or more smaller identical problems

Recursion in Java:

- Recursive methods implement recursive algorithms
- A *recursive method* includes a call to itself

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Recursive Methods Must Eventually Terminate

A recursive method must have at least one base, or stopping, case.

- A base case does not execute a recursive call
  - stops the recursion
- Each successive call to itself must be a "smaller version of itself"
  - an argument that describes a smaller problem
  - a base case is eventually reached

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Key Components of a Recursive Algorithm Design

1. What is a smaller ***identical*** problem(s)?
   - Decomposition
2. How are the answers to smaller problems combined to form the answer to the larger problem?
   - Composition
3. Which is the smallest problem that can be solved easily (without further decomposition)?
   - Base/stopping case

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Examples in Recursion

- Usually quite confusing the first time
- Start with some simple examples
  - recursive algorithms might not be best
- Later with inherently recursive algorithms
  - harder to implement otherwise

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Factorial (*N*!)

- *N*! = (*N*-1)! * N [for *N* > 1]
- 1! = 1
- 3!
  = 2! * 3
  = (1! * 2) * 3
  = 1 * 2 * 3
- Recursive design:
  - Decomposition: (*N*-1)!
  - Composition: * *N*
  - Base case: 1!

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`factorial` Method

```
public static int factorial(int n)
{
  int fact;
  if (n > 1) // recursive case (decomposition)
    fact = factorial(n - 1) * n; // composition
  else // base case
    fact = 1;

  return fact;
}
```

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```
public static int factorial(int 3)
{
  int fact;
  if (n > 1)
    fact = factorial(2) * 3;
  else
    fact = 1;
  return fact;
}
```

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```
public static int factorial(int 3)
{
  int fact;
  if (n > 1)
    fact = factorial(2) * 3;
  else
    fact = 1;
  return fact;
}
```

```
public static int factorial(int 2)
{
  int fact;
  if (n > 1)
    fact = factorial(1) * 2;
  else
    fact = 1;
  return fact;
}
```

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```
public static int factorial(int 3)
{
  int fact;
  if (n > 1)
    fact = factorial(2) * 3;
  else
    fact = 1;
  return fact;
}
```

```
public static int factorial(int 2)
{
  int fact;
  if (n > 1)
    fact = factorial(1) * 2;
  else
    fact = 1;
  return fact;
}
```

```
public static int factorial(int 1)
{
  int fact;
  if (n > 1)
    fact = factorial(n - 1) * n;
  else
    fact = 1;
  return fact;
}
```

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```
public static int factorial(int 3)
{
  int fact;
  if (n > 1)
    fact = factorial(2) * 3;
  else
    fact = 1;
  return fact;
}
```

```
public static int factorial(int 2)
{
  int fact;
  if (n > 1)
    fact = factorial(1) * 2;
  else
    fact = 1;
  return fact;
}
```

```
public static int factorial(int 1)
{
  int fact;
  if (n > 1)
    fact = factorial(n - 1) * n;
  else
    fact = 1;
  return 1;
}
```

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```
public static int factorial(int 3)
{
  int fact;
  if (n > 1)
    fact = factorial(2) * 3;
  else
    fact = 1;
  return fact;
}
```

```
public static int factorial(int 2)
{
  int fact;
  if (n > 1)
    fact = 1 * 2;
  else
    fact = 1;
  return fact;
}
```

```
public static int factorial(int 1)
{
  int fact;
  if (n > 1)
    fact = factorial(n - 1) * n;
  else
    fact = 1;
  return 1;
}
```

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```
public static int factorial(int 3)
{
  int fact;
  if (n > 1)
    fact = factorial(2) * 3;
  else
    fact = 1;
  return fact;
}
```

```
public static int factorial(int 2)
{
  int fact;
  if (n > 1)
    fact = 1 * 2;
  else
    fact = 1;
  return 2;
}
```

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```
public static int factorial(int 3)
{
  int fact;
  if (n > 1)
    fact = 2 * 3;
  else
    fact = 1;
  return fact;
}
```

```
public static int factorial(int 2)
{
  int fact;
  if (n > 1)
    fact = 1 * 2;
  else
    fact = 1;
  return 2;
}
```

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```
public static int factorial(int 3)
{
  int fact;
  if (n > 1)
    fact = 2 * 3;
  else
    fact = 1;
  return 6;
}
```

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Execution Trace (decomposition)

```
public static int factorial(int n)
{
  int fact;
  if (n > 1) // recursive case (decomposition)
    fact = factorial(n - 1) * n; (composition)
  else // base case
    fact = 1;
  return fact;
}
```

factorial(4)

factorial(3)   4

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Execution Trace (decomposition)

```
public static int factorial(int n)
{
  int fact;
  if (n > 1) // recursive case (decomposition)
    fact = factorial(n - 1) * n; (composition)
  else // base case
    fact = 1;
  return fact;
}
```

factorial(4)

factorial(3)   4

factorial(2)   3

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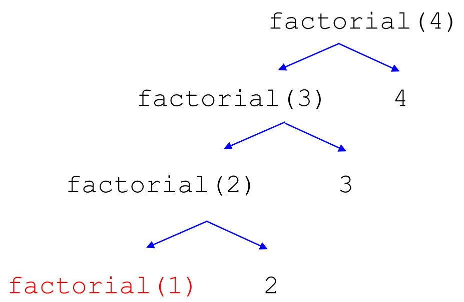
Execution Trace (decomposition)

```
public static int factorial(int n)
{
  int fact;
  if (n > 1) // recursive case (decomposition)
    fact = factorial(n – 1) * n; (composition)
  else // base case
    fact = 1;
  return fact;
}
```



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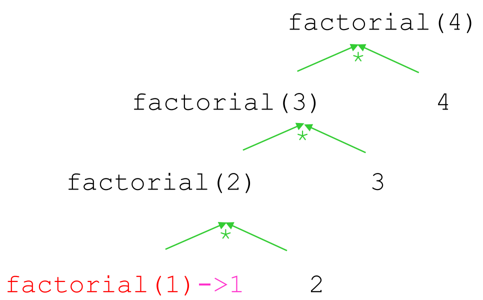
Execution Trace (composition)

```
public static int factorial(int n)
{
  int fact;
  if (n > 1) // recursive case (decomposition)
    fact = factorial(n – 1) * n; (composition)
  else // base case
    fact = 1;
  return fact;
}
```



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Execution Trace (composition)

```
public static int factorial(int n)
{
  int fact;
  if (n > 1) // recursive case (decomposition)
    fact = factorial(n - 1) * n; (composition)
  else // base case
    fact = 1;
  return fact;
}
```

```
            factorial(4)
                 *
     factorial(3)        4
            *
factorial(2)->2      3
```

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Execution Trace (composition)

```
public static int factorial(int n)
{
  int fact;
  if (n > 1) // recursive case (decomposition)
    fact = factorial(n - 1) * n; (composition)
  else // base case
    fact = 1;
  return fact;
}
```

```
            factorial(4)
                 *
     factorial(3)->6     4
```

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Execution Trace (composition)

```
public static int factorial(int n)
{
  int fact;
  if (n > 1) // recursive case (decomposition)
    fact = factorial(n - 1) * n; (composition)
  else // base case
    fact = 1;
  return fact;
}
```

`factorial(4)->24`

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Improved `factorial` Method

```
public static int factorial(int n)
{
  int fact=1;  // base case value

  if (n > 1)   // recursive case (decomposition)
    fact = factorial(n - 1) * n; // composition
  // else do nothing; base case

  return fact;
}
```

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Fibonacci Numbers

- The *N*th Fibonacci number is the sum of the previous two Fibonacci numbers
- 0, 1, 1, 2, 3, 5, 8, 13, …
- Recursive Design:
  - Decomposition & Composition
    - fibonacci(n) = fibonacci(n-1) + fibonacci(n-2)
  - Base case:
    - fibonacci(1) = 0
    - fibonacci(2) = 1

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`fibonacci` Method

```
public static int fibonacci(int n)
{
   int fib;
   if (n > 2)
     fib = fibonacci(n-1) + fibonacci(n-2);
   else if (n == 2)
     fib = 1;
   else
     fib = 0;
   return fib;
}
```

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Execution Trace (decomposition)

```
            fibonacci(4)

fibonacci(3)            fibonacci(2)
```

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Execution Trace (decomposition)

```
                        fibonacci(4)

            fibonacci(3)            fibonacci(2)

fibonacci(2)        fibonacci(1)
```

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Execution Trace (composition)

```
              fibonacci(4)
                   +
      fibonacci(3)         fibonacci(2)
            +
fibonacci(2)->1   fibonacci(1)->0
```

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Execution Trace (composition)

```
              fibonacci(4)
                   +
      fibonacci(3)->1      fibonacci(2)->1
```

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Execution Trace (composition)

```
fibonacci(4)->2
```

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Remember:
Key to Successful Recursion

- if-else statement (or some other branching statement)
- Some branches: recursive call
  - "smaller" arguments or solve "smaller" versions of the same task (*decomposition*)
  - Combine the results (*composition*) [if necessary]
- Other branches: no recursive calls
  - stopping cases or base cases

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Template

```
… method(…)
{
   if ( … )// base case
   {
   }
   else // decomposition & composition
   {
   }
   return … ; // if not void method
}
```

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Template (only one base case)

```
… method(…)
{
   … result = …   ;//base case

   if ( … ) // not base case
   { //decomposition & composition
     result = …
   }

   return result;
}
```

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What Happens Here?

```
public static int factorial(int n)
{
  int fact=1;

  if (n > 1)
    fact = factorial(n) * n;

  return fact;
}
```

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What Happens Here?

```
public static int factorial(int n)
{
  return factorial(n - 1) * n;
}
```

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Warning: Infinite Recursion May Cause a Stack Overflow Error

- Infinite Recursion
  - Problem not getting smaller (no/bad decomposition)
  - Base case exists, but not reachable (bad base case and/or decomposition)
  - No base case
- *Stack:* keeps track of recursive calls by JVM (OS)
  - Method begins: add data onto the stack
  - Method ends: remove data from the stack
- Recursion never stops; stack eventually runs out of space
  - **Stack overflow error**

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Mistakes in recursion

- No composition -> ?
- Bad composition -> ?

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Number of Zeros in a Number

- Example: 2030 has 2 zeros
- If `n` has two or more digits
  - the number of zeros is the number of zeros in `n` with the last digit removed
  - plus an additional 1 if the last digit is zero
- Examples:
  - number of zeros in `20030` is number of zeros in `2003` plus 1
  - number of zeros in `20031` is number of zeros in `2003` plus 0

recursive

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numberOfZeros Recursive Design

- numberOfZeros in the number *N*
- *K* = number of digits in *N*
- Decomposition:
  - numberOfZeros in the first *K* - 1 digits
  - Last digit
- Composition:
  - Add:
    - numberOfZeros in the first *K* - 1digits
    - 1 if the last digit is zero
- Base case:
  - *N* has one digit (*K* = 1)

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`numberOfZeros` method

```
public static int numberOfZeros(int n)
{
  int zeroCount;
  if (n==0)
     zeroCount = 1;
  else if (n < 10)   // and not 0
     zeroCount = 0;   // 0 for no zeros
  else if (n%10 == 0)
     zeroCount = numberOfZeros(n/10) + 1;
  else   // n%10 != 0
     zeroCount = numberOfZeros(n/10);
  return zeroCount;
}
```

Which is (are) the base case(s)? Why?

Decompostion, Why?

Composition, why?

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Execution Trace (decomposition)

Each method invocation will execute one of the `if-else` cases shown at right.

```
public static int numberOfZeros(int n)
{
  int zeroCount;
  if (n==0)
     zeroCount = 1;
  else if (n < 10)   // and not 0
     zeroCount = 0;   // 0 for no zeros
  else if (n%10 == 0)
     zeroCount = numberOfZeros(n/10) + 1;
  else   // n%10 != 0
     zeroCount = numberOfZeros(n/10);
  return zeroCount;
}
```

```
                    numberOfZeros(2005)
           numberOfZeros(200)          5
      numberOfZeros(20)            0
numberOfZeros(2)            0
```

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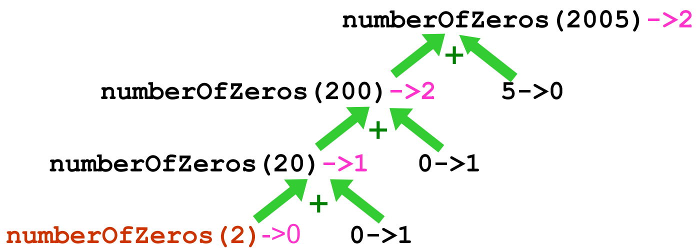
Execution Trace (composition)

Recursive calls return

```
public static int numberOfZeros(int n)
{
  int zeroCount;
  if (n==0)
      zeroCount = 1;
  else if (n < 10)  // and not 0
      zeroCount = 0;  // 0 for no zeros
  else if (n%10 == 0)
      zeroCount = numberOfZeros(n/10) + 1;
  else  // n%10 != 0
      zeroCount = numberOfZeros(n/10);
  return zeroCount;
}
```



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Number in English Words

- Process an integer and print out its digits in words
  - Input: 123
  - Output: "one two three"
- RecursionDemo class

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inWords Resursive Design

- inWords prints a number *N* in English words
- *K* = number of digits in *N*
- Decomposition:
  - inWords for the first *K* – 1 digits
  - Print the last digit
- Composition:
  - Execution order of composed steps [more later]
- Base case:
  - *N* has one digit (*K* = 1)

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`inWords` method

Base case executes when only 1 digit is left

Size of problem is reduced for each recursive call

```
public static void inWords(int numeral)
{
    if (numeral < 10)
        System.out.print(digitWord(numeral) + " ");
    else //numeral has two or more digits
    {
        inWords(numeral/10);
        System.out.print(digitWord(numeral%10) + " ");
    }
}
```

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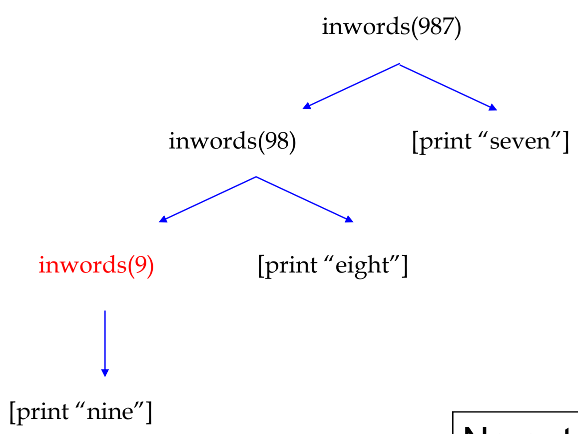
Execution Trace (decomposition)



No output yet, why?

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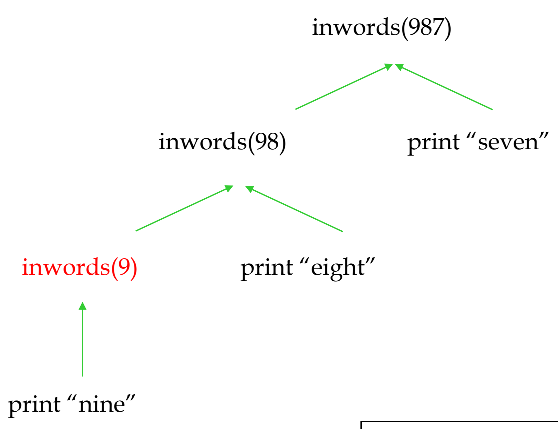
Execution Trace (composition)



Output: nine eight seven

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```
inWords(987)
if (987 < 10)
    // print digit here
else //two or more digits left
{
   inWords(987/10);
   // print digit here
}
```

1

What Happens with a Recursive Call

- `inWords` (slightly simplified) with argument `987`

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```
inWords(987)
if (987 < 10)
    // print digit here
else //two or more digits left
{
   inWords(987/10);
   // print digit here
}
```

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```
inWords(98)
if (98 < 10)
    // print digit here
else //two or more digits left
{
   inWords(98/10);
   // print digit here
}
```

Execution Trace

The argument is getting shorter and will eventually get to the base case.

Computation waits here until recursive call returns

- The `if` condition is false
- recursive call to `inWords`, with `987/10` or `98` as the argument

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Execution Trace

```
inWords(987)
if (987 < 10)
    // print digit here
else //two or more digits left
{
   inWords(987/10);
   // print digit here
}
```

```
inWords(98)
if (98 < 10)
    // print digit here
else //two or more digits left
{
   inWords(98/10);
   // print digit here
}
```

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```
inWords(9)
if (9 < 10)
    // print digit here
else //two or more digits left
{
   inWords(numeral/10);
   // print digit here
}
```

- the if condition is false
- another recursive call is made.

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Execution Trace

Output: nine

```
inWords(987)
if (987 < 10)
    // print digit here
else //two or more digits left
{
   inWords(987/10);
   // print digit here
}
```

```
inWords(98)
if (98 < 10)
    // print digit here
else //two or more digits left
{
   inWords(98/10);
   // print 98 % 10
}
```

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```
if (9 < 10)
    // print nine
else //two or more digits left
{
   inWords(numeral/10);
   // print digit here
}
```

- if condition is true (base case)
- prints nine and returns
- no recursive call

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```
inWords(987)
if (987 < 10)
    // print out digit here
else //two or more digits left
{
   inWords(987/10);
   // print digit here
}
```

Execution Trace

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```
if (98 < 10)
    // print out digit here
else //two or more digits left
{
   inWords(98/10);
   // print out 98 % 10 here
}
```

Output: `nine eight`

- executes the next statement after the recursive call
- prints eight and then returns

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```
inWords(987)
if (987 < 10)
    // print out digit here
else //two or more digits left
{
   inWords(987/10);
   // print 987 % 10
}
```

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Execution Trace

Output: `nine eight seven`

- executes the next statement after the recursive method call.
- prints seven and returns

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Composition Matters

Print before making the recursive call

```
public static void inWords(int numeral)
{
   if (numeral < 10)
      System.out.print(digitWord(numeral) + " ");
   else //numeral has two or more digits
   {
      System.out.print(digitWord(numeral%10) + " ");
      inWords(numeral/10);
   }
}
```

Recursive Design:

1. Print the last digit
2. inWords for the first *K* – 1 digits

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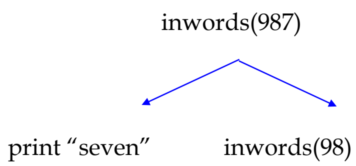
Execution Trace (decomposition)



Output: seven

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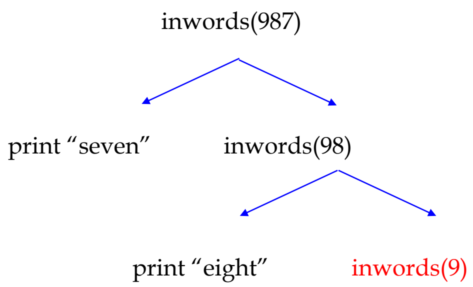
Execution Trace (decomposition)



Output: seven eight

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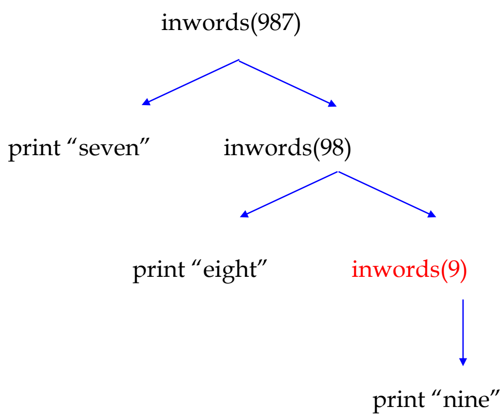
Execution Trace (decomposition)



Output: seven eight nine

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Execution Trace (composition)

inwords(987)

print “seven” inwords(98)

print “eight” inwords(9)

print “nine”

No additional output

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"Name in the Phone Book" Revisited

```
Search:
  middle page = (first page + last page)/2
  Go to middle page;
  If (name is on middle page)
        done;//this is the base case
  else if (name is alphabetically before middle page)
        last page = middle page//redefine to front half
        Search//recursive call
  else //name must be after middle page
        first page = middle page//redefine to back half
        Search//recursive call
```

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Binary Search Algorithm

- Searching a list for a particular value
  - *sequential* and *binary* are two common algorithms
- *Sequential search* (aka *linear search*):
  - Not very efficient
  - Easy to understand and program
- *Binary search*:
  - more efficient than sequential
  - but *the list must be sorted first!*

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Why Is It Called "Binary" Search?

Compare sequential and binary search algorithms:
How many elements are eliminated from the list each time a value is read from the list and it is not the "target" value?

Sequential search: *only one item*

Binary search: *half the list*!

That is why it is called *binary* -
each unsuccessful test for the target value
reduces the remaining search list by 1/2.

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Binary Search Method

- `public find(target)` calls `private search(target, first, last)`
- returns the index of the entry if the target value is found or -1 if it is not found
- Compare it to the pseudocode for the "name in the phone book" problem

```
private int search(int target, int first, int last)
 {
    int location = -1; // not found

    if (first <= last) // range is not empty
    {
       int mid = (first + last)/2;

       if (target == a[mid])
          location = mid;
       else if (target < a[mid]) // first half
          location = search(target, first, mid - 1);
       else //(target > a[mid]) second half
          location = search(target, mid + 1, last);
    }

    return location;
 }
```

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Where is the composition?

- If no items
  - not found (-1)
- Else if target is in the middle
  - middle location
- Else
  - location found by search(first half) or search(second half)

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Binary Search Example

`target` is `33`

The array **a** looks like this:

Indices	0	1	2	3	4	5	6	7	8	9
Contents	5	7	9	13	32	33	42	54	56	88

`mid = (0 + 9) / 2` (which is `4`)

`33 > a[mid]` (that is, `33 > a[4]`)

So, if `33` is in the array, then `33` is one of:

					5	6	7	8	9
					33	42	54	56	88

Eliminated half of the remaining elements from consideration because array elements are sorted.

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Binary Search Example

`target` is `33`

The array **a** looks like this:

Indexes	0	1	2	3	4	5	6	7	8	9
Contents	5	7	9	13	32	33	42	54	56	88

`mid = (5 + 9) / 2` (which is `7`)

`33 < a[mid]` (that is, `33 < a[7]`)

So, if `33` is in the array, then `33` is one of:

					5	6			
					33	42			

Eliminate half of the remaining elements

`mid = (5 + 6) / 2` (which is `5`)

`33 == a[mid]`

So we found `33` at index `5`:

					5				
					33				

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Tips

- Don't throw away answers (return values)--need to compose the answers
  - Common programming mistake: not capturing and composing answers (return values)
- Only one return statement at the end
  - Easier to keep track of and debug return values
  - "One entry, one exit"
- www.cs.fit.edu/~pkc/classes/cse1001/BinarySearch/BinarySearch.java

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Worst-case Analysis

- Item not in the array (size N)
- $T(N)$ = number of comparisons with array elements
- $T(1) = 1$
- $T(N) = 1 + T(N / 2)$

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Worst-case Analysis

- Item not in the array (size N)
- $T(N)$ = number of comparisons with array elements
- $T(1) = 1$
- $T(N) = 1 + T(N / 2)$

  $= 1 + [1 + T(N / 4)]$

71

Worst-case Analysis

- Item not in the array (size N)
- $T(N)$ = number of comparisons with array elements
- $T(1) = 1$
- $T(N) = 1 + T(N / 2)$

  $= 1 + [1 + T(N / 4)]$

  $= 2 + T(N / 4)$

  $= 2 + [1 + T(N / 8)]$

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Worst-case Analysis

- Item not in the array (size N)
- $T(N)$ = number of comparisons with array elements
- $T(1) = 1$
- $T(N) = 1 + T(N / 2)$ ←

  $= 1 + [1 + T(N / 4)]$

  $= 2 + T(N / 4)$ ←

  $= 2 + [1 + T(N / 8)]$

  $= 3 + T(N / 8)$ ←

  $= \ldots$

73

Worst-case Analysis

- Item not in the array (size N)
- $T(N)$ = number of comparisons with array elements
- $T(1) = 1$
- $T(N) = 1 + T(N / 2)$ ←

  $= 1 + [1 + T(N / 4)]$

  $= 2 + T(N / 4)$ ←

  $= 2 + [1 + T(N / 8)]$

  $= 3 + T(N / 8)$ ←

  $= \ldots$

  $= k + T(N / 2^k)$ [1]

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Worst-case Analysis

- $T(N) = k + T(N / 2^k)$ [1]
- $T(N / 2^k)$ gets smaller until the base case: T(1)
  - $2^k = N$
  - $k = \log_2 N$
- Replace terms with k in [1]:

$$T(N) = \log_2 N + T(N / N)$$
$$= \log_2 N + T(1)$$
$$= \log_2 N + 1$$

- "$\log_2 N$" algorithm
- We used *recurrence equations*

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Main steps for analysis

- Set up the recurrence equations for the recursive algorithm
- Expand the equations a few times
- Look for a pattern
- Introduce a variable to describe the pattern
- Find the value for the variable via the base case
- Get rid of the variable via substitution

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Binary vs. Sequential Search

- Binary Search
  - $\log_2 N + 1$ comparisons (worst case)
- Sequential/Linear Search
  - N comparisons (worst case)
- Binary Search is faster ***but***
  - array is assumed to be sorted beforehand
- Faster searching algorithms for "non-sorted arrays"
  - More sophisticated *data structures* than arrays
  - Later courses

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Recursive Versus Iterative Methods

All recursive algorithms/methods can be rewritten without recursion.

- *Iterative* methods use loops instead of recursion

- Iterative methods generally run faster and use less memory--less overhead in keeping track of method calls

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So When Should You Use Recursion?

- Solutions/algorithms for some problems are inherently recursive
  - iterative implementation could be more complicated
- When efficiency is less important
  - it might make the code easier to understand
- *Bottom line is about:*
  - *Algorithm design*
  - *Tradeoff between readability and efficiency*

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Pages 807 NOT a good tip [*Programming Tip*: Ask Until the User Gets It Right]

- Recursion continues until user enters valid input.

```
public void getCount()
{
   System.out.println("Enter a positive number:");
   count = SavitchIn.readLineInt();
   if (count <= 0)
   {
      System.out.println("Input must be positive.
      System.out.println("Try again.");
      getCount();  //start over
   }
}
```

read a number

Use a recursive call to get another number.

- *No notion of a smaller problem for recursive design*
- *Easily implemented using iteration without loss of readability*

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Merge Sort—
A Recursive Sorting Algorithm

- Example of *divide and conquer* algorithm
- Recursive design:
  - Divides array in half and *merge sorts* the halves (decomposition)
  - Combines two sorted halves (composition)
  - Array has only one element (base case)
- Harder to implement iteratively

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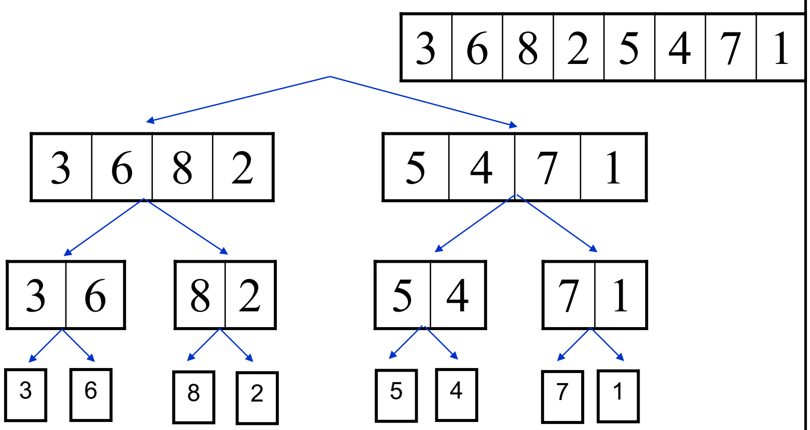
Execution Trace (decomposition)



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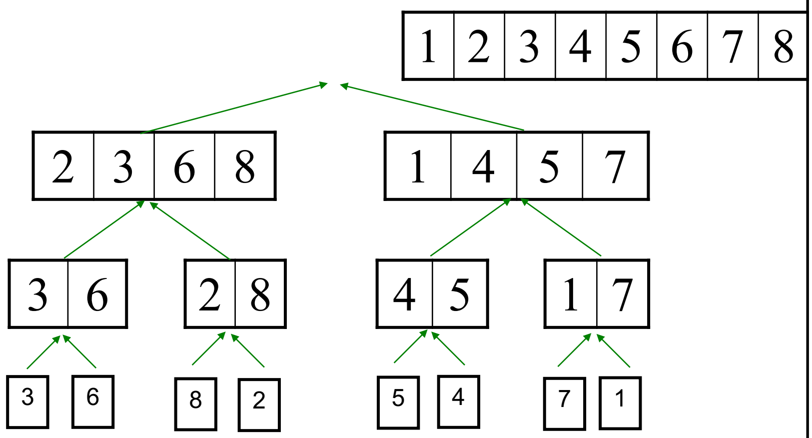
Execution Trace (composition)



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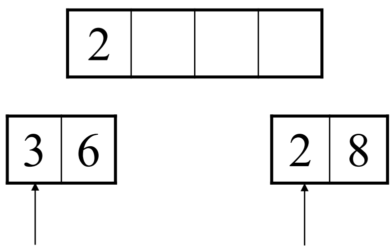
Merging Two Sorted Arrays



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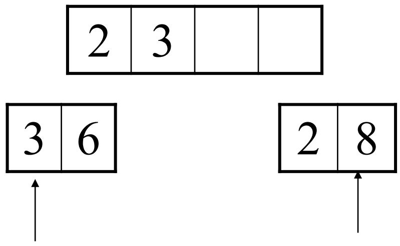
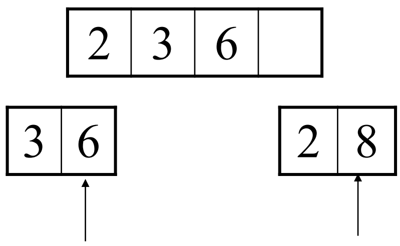
Merging Two Sorted Arrays



85

Merging Two Sorted Arrays



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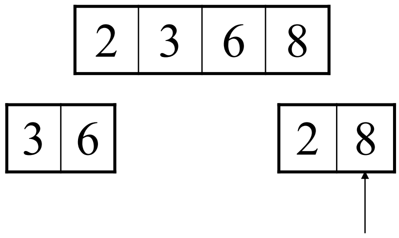
Merging Two Sorted Arrays



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Merge Sort Algorithm

1. If array a has more than one element:
   a. Copy the first half of the elements in *a* to array *front*
   b. Copy the rest of the elements in *a* to array *tail*
   c. Merge Sort *front*
   d. Merge Sort *tail*
   e. Merge the elements in *front* and *tail* into *a*
2. Otherwise, do nothing

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Merge Sort

```
public static void sort(int[] a)
{
   if (a.length >= 2)
   {
      int halfLength = a.length / 2;
      int[] front = new int[halfLength];
      int[] tail = new int[a.length – halfLength];
      divide(a, front, tail);
      sort(front);
      sort(tail);
      merge(a, front, tail);
   }
   // else do nothing.
}
```

do recursive case if true, base case if false

recursive calls

make "smaller" problems by dividing array

Combine the two sorted arrays

base case: a.length == 1 so a is sorted and no recursive call is necessary.

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Worst-case Theoretical Analysis

- Comparisons of array elements
- None during decomposition
- Only during merging two sorted arrays (composition)
  - To get an array of size *N* from two sorted arrays of size *N*/2
  - *N* - 1 comparisons (worst case: the largest two elements are in different halves)

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Analysis: Array of size N

- Let $T(N)$ be the number of comparisons
- $T(1) = 0$
- $T(N) = 2\ T(N / 2) + (N - 1)$

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Analysis: Array of size N

- Let $T(N)$ be the number of comparisons
- $T(1) = 0$
- $T(N) = 2\ T(N / 2) + (N - 1)$

  $= 2\ [2\ T(N / 4) + (N / 2 - 1)] + (N - 1)$

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Analysis: Array of size N

- Let $T(N)$ be the number of comparisons
- $T(1) = 0$
- $T(N) = 2\ T(N / 2) + (N - 1)$

$$= 2\ [2\ T(N / 4) + (N / 2 - 1)] + (N - 1)$$
$$= 4\ T(N / 4) + (N - 2) + (N - 1)$$
$$= 4\ [\ 2\ T(N / 8) + (N / 4 - 1)\ ] + (N - 2) + (N - 1)$$

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Analysis: Array of size N

- Let $T(N)$ be the number of comparisons
- $T(1) = 0$
- $T(N) = 2\ T(N / 2) + (N - 1) \quad \leftarrow$

$$= 2\ [2\ T(N / 4) + (N / 2 - 1)] + (N - 1)$$
$$= 4\ T(N / 4) + (N - 2) + (N - 1) \quad \leftarrow$$
$$= 4\ [\ 2\ T(N / 8) + (N / 4 - 1)\ ] + (N - 2) + (N - 1)$$
$$= 8\ T(N / 8) + (N - 4) + (N - 2) + (N - 1) \quad \leftarrow$$

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Analysis: Array of size N

- Let $T(N)$ be the number of comparisons
- $T(1) = 0$
- $T(N) = 2\ T(N / 2) + (N - 1)$ ←

$= 2\ [2\ T(N / 4) + (N / 2 - 1)] + (N - 1)$

$= 4\ T(N / 4) + (N - 2) + (N - 1)$ ←

$= 4\ [\ 2\ T(N / 8) + (N / 4 - 1)\ ] + (N - 2) + (N - 1)$

$= 8\ T(N / 8) + (N - 4) + (N - 2) + (N - 1)$ ←

$= 8\ T(N / 8) + 3N - (1 + 2 + 4)$

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Analysis: Array of size N

- Let $T(N)$ be the number of comparisons
- $T(1) = 0$
- $T(N) = 2\ T(N / 2) + (N - 1)$ ←

$= 2\ [2\ T(N / 4) + (N / 2 - 1)] + (N - 1)$

$= 4\ T(N / 4) + (N - 2) + (N - 1)$ ←

$= 4\ [\ 2\ T(N / 8) + (N / 4 - 1)\ ] + (N - 2) + (N - 1)$

$= 8\ T(N / 8) + (N - 4) + (N - 2) + (N - 1)$ ←

$= 8\ T(N / 8) + 3N - (1 + 2 + 4)$

$= \ldots$

$= 2^k\ T(N / 2^k) + kN - (1 + 2 + \ldots 2^{k-1})$ [1]

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Analysis Continued

- $T(N) = 2^k T(N / 2^k) + kN – (1 + 2 + \dots 2^{k-1})$ [1]
  $= 2^k T(N / 2^k) + kN – (2^k - 1)$ [2]
- $T(N / 2^k)$ gets smaller until the base case $T(1)$:
  - $2^k = N$
  - $k = \log_2 N$
- Replace terms with k in [2]:
  $T(N) = N\,T(N / N) + \log_2 N * N – (N – 1)$
  $= N\,T(1) + N\log_2 N – (N – 1)$
  $= N\log_2 N – N + 1$
- "$N\log_2 N$" algorithm

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Geometric Series and Sum

- $1 + 2 + 4 + 8 + \dots + 2^k$
  - $1 + 2 = 3$
  - $1 + 2 + 4 = 7$
  - $1 + 2 + 4 + 8 = 15$

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Geometric Series and Sum

- $1 + 2 + 4 + 8 + \ldots + 2^k$
  - $1 + 2 = 3$ $(4 – 1)$
  - $1 + 2 + 4 = 7$ $(8 – 1)$
  - $1 + 2 + 4 + 8 = 15$ $(16 – 1)$

99

Geometric Series and Sum

- $1 + 2 + 4 + 8 + \ldots + 2^k$
  - $1 + 2 = 3$ $(4 – 1)$
  - $1 + 2 + 4 = 7$ $(8 – 1)$
  - $1 + 2 + 4 + 8 = 15$ $(16 – 1)$
- $1 + 2 + 4 + 8 + \ldots + 2^k$
  $= 2^{k+1} - 1$
- $1 + r + r^2 + r^3 + \ldots + r^k$
  $= r^0 + r^1 + r^2 + r^3 + \ldots + r^k$
  $= (r^{k+1} – 1) / (r – 1)$ [for $r > 1$]

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Merge Sort Vs. Selection/Insertion/Bubble Sort

- Merge Sort
  - "*N*log*N*" algorithm (in comparisons)
- Selection/Insertion/Bubble Sort
  - "N^2" algorithm (in comparisons)
- "*N*log*N*" is "optimal" for sorting
  - Proven that the sorting problem cannot be solved with fewer comparisons
  - Other *N*log*N* algorithms exist, many are recursive

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Real Data Set: Web Server Log

- http://www.cs.fit.edu/~pkc/classes/writing/data/jan99.log
- 4.6 MB (44057 entries)
- Example entry in log:

  ip195.dca.primenet.com - - [04/Jan/1999:09:16:51 -0500] "GET / HTTP/1.0" 200 762
- Extracted features
  - remote-host names (strings)
  - file-size (integers)
- List size - 100 to 44000 entries

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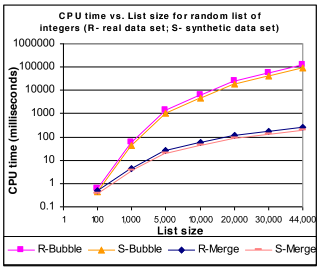
CPU Time: Randomly Ordered Integers



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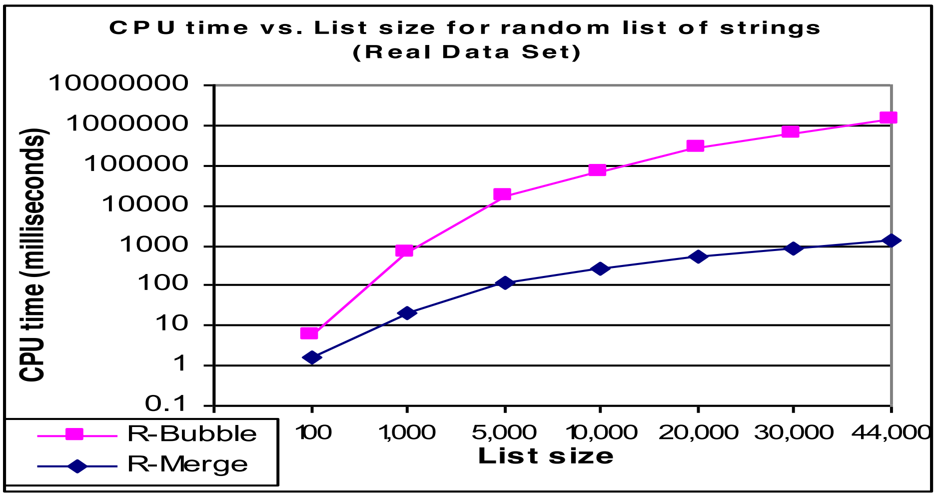
CPU Time: Randomly Ordered Strings



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Google's PageRank (1998)

- PageRank(*x*) depends on:
  1. How many pages (*y's*) linking to *x*
     - how many incoming links (citations) from *y's* to *x*
  2. How important those pages (*y*'s) are:
     - PageRank(*y*)'s
- How to determine PageRank(*y*)'s?
- What is the base case?

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Summary

- *Recursive call*: a method that calls itself
- Powerful for algorithm design at times
- Recursive algorithm design:
    - Decomposition (smaller identical problems)
    - Composition (combine results)
    - Base case(s) (smallest problem, no recursive calls)
- Implementation
  - Conditional (e.g. if) statements to separate different cases
  - Avoid infinite recursion
    - Problem is getting smaller (decomposition)
    - Base case exists and reachable
  - Composition could be tricky

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Summary

- Binary Search
  - Given an ordered list
  - "log*N*" algorithm (in comparisons)
  - "Optimal"
- Merge Sort
  - Recursive sorting algorithm
  - *"N*log*N"* algorithm (in comparisons)
  - "Optimal"

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