HMM Signal Source

EQ2341 Pattern Recognition and Machine Learning, Assignment 1

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A.1.2 Verify the MarkovChain and HMM Sources

Question 1

According to sum rule and product rule:

$$P(S_t = j) = \sum_{i=1}^{2} P(S_t = j|S_{t-1} = i) \cdot P(S_{t-1} = i)$$

When $t = 2$:

$$P(S_2 = 1) = 0.75 \times 0.99 + 0.25 \times 0.03 = 0.75 = P(S_1 = 1)$$

$$P(S_2 = 2) = 0.75 \times 0.01 + 0.25 \times 0.97 = 0.25 = P(S_1 = 2)$$

So by induction, it can be proved that

$$P(S_t = j) = P(S_{t-1} = j) = constant$$

Question 2

We have found $P(S_t = 1) = 0.78$, $P(S_t = 2) = 0.22$, which are approximately equal to the theoretical probabilities.

Question 3

According to the conditional expectation formulas:

$$\mu_x = 0.75 \times 0 + 0.25 \times 3 = 0.75$$

$$var[x] = 0.75 \times 1 + 0.25 \times 4 + (0.75 \times (0 - 0.75)^2 + 0.25 \times (3 - 0.75)^2) = 3.44$$

We have found that the mean is 0.78 and the variance is 3.46, which are approximately equal to the theoretical values.

Question 4

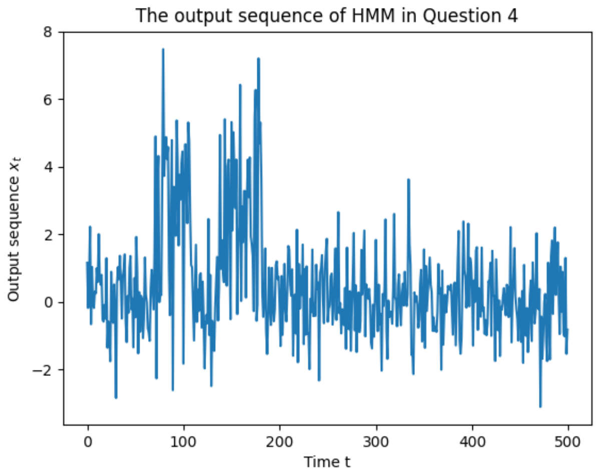

Figure 1: The output sequence of HMM.

We can obviously see that there are two kinds of output distributions: one with the mean of 0 and the variance of 1, and the other with the mean of 3 and the variance of 2, corresponding to State 1 and State 2, respectively. We can also see that the output of zero mean corresponds to longer time, which is consistent with $P_{1|2} < P_{2|1}$, i.e., State 2 is more likely to transition to State 1.

Question 5

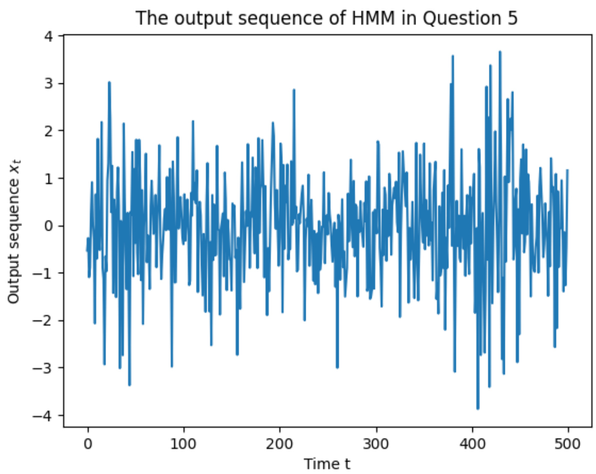

Figure 2: The output sequence of HMM.

It is still true that the output of different statistical characteristics corresponds to different states. In this question, we adjust the output distribution corresponding to State 2 to have a mean of 0 and a variance of 2. We can see the output follows two kinds of distributions correspondingly, where the value of x from State 1 with smaller variance fluctuates in a relatively narrower band than that from State 2. Therefore, we can estimate the state sequence based on the statistical characteristics of the output at different time.

Question 6

We define the new finite-duration HMM $\lambda = q, A, B$ as

$$q = \begin{pmatrix} 0.75 \\ 0.25 \end{pmatrix}; \quad A = \begin{pmatrix} 0.7 & 0.3 & 0 \\ 0.65 & 0.25 & 0.1 \end{pmatrix}; \quad B = \begin{pmatrix} b_1(x) \\ b_2(x) \end{pmatrix}.$$

where $b_1(x)$ is a scalar Gaussian density function with mean $\mu_1 = 0$ and standard deviation $\sigma_1 = 1$, and $b_2(x)$ is a similar distribution with mean $\mu_2 = 3$ and standard deviation $\sigma_2 = 2$. In order to maintain consistency with the original infinite-duration HMM, we set T to 500 as before. The result is shown in Figure 3 and it can be seen that the length of generated sequence is shorter than 500 since the exit state is reached before $T = 500$.

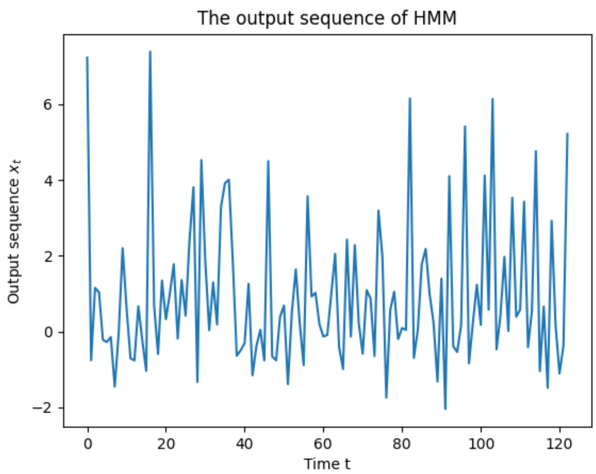

Figure 3: The output sequence of HMM.

Question 7

In this question, we still utilize the HMM with the same q and A while $b_1(x)$ and $b_2(x)$ follow the Gaussian distributions with

$$\mu_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \Sigma_1 = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix};$$

$$\mu_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \Sigma_2 = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}.$$

We set T to 6 and get the result:
[-2.23718966 0.18241097 2.68111375 -1.44143201 -0.01907204 0.87073679]
[-1.71466221 -0.05053822 -0.3662666 -2.6599358 1.03219622 0.18801309]
The statistical characteristics of the output are consistent with the theoretical values.