Computation of Miura surfaces

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Abstract

Miura surfaces are the solutions of a system of nonlinear elliptic equations. This system is derived by homogenization from the Miura fold, which is a type of origami fold with multiple applications in engineering. A previous inquiry, gave suboptimal conditions for existence of solutions and proposed an H^2-conformal finite element method to approximate them. In this paper, further insight into Miura surfaces is presented along with a proof of existence and uniqueness when using appropriate boundary conditions. A numerical method based on a least-squares formulation, $\mathbb{P}^1$–Lagrange finite elements and a Newton method is introduced to approximate them. The numerical method presents an improved convergence rate with respect to previous work and it is more efficient. Finally, numerical tests are performed to demonstrate the robustness of the method.

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1 Introduction

The ancient art of origami has attracted a lot of attention in recent years. The monograph [11] presents a very complete description of many different types of origami patterns as well as some questions surrounding them. Origami have found many applications in engineering. Origami folds are, for instance, used to ensure that airbags inflate correctly from their folded state [1]. Applications in aerospace engineering are as varied as radiators, panels for radio telescopes [15] or soft robots [19]. More recently, origami have been studied with a view to produce metamaterials [18, 2].

The Miura fold was introduced in [14]. It has later found applications in solar panels for satellites and in biology [10]. Recent applications also include the design of metamaterials [24]. The Miura ori is flat when completely unfolded and can be fully folded in a very compact form, hence its application in aerospace engineering. However, the Miura fold can also assume different shapes when partially unfolded. [21] and [23] provided a description of these partially unfolded states through a computation of the Poisson ratio of the Miura fold, which happens to be negative. A homogenization procedure for origami folds was proposed in [17] and [16] and then applied to the Miura fold in [12]. The authors obtained a constrained nonlinear elliptic equation describing the parametrized limit surface, which is called a Miura surface. The constraints are both equality and inequality constraints. In [13], existence and uniqueness of solutions was proved but only for the unconstrained equation and under restrictive assumptions. [13] also proposed an H^2-conformal Finite Element Method (FEM) to compute solutions to the problem, which is computationally involved.

In this paper, existence and uniqueness of solutions of the constrained equation is proved under optimal boundary conditions. Thereafter, a least-squares formulation is proposed coupled to a Newton method and $\mathbb{P}^1$–Lagrange finite elements to approximate the solutions. The method is much more economical than the previously proposed one and its robustness is demonstrated on several test cases.

2 Continuous equations

Let $\Omega \subset \mathbb{R}^2$ be a bounded convex polygon that can be perfectly fitted by triangular meshes. Note that, due to the convexity hypothesis, the boundary $\partial\Omega$ is Lipschitz [9] and thus verifies an exterior sphere condition.

2.1 Strong form equations

Let $\varphi : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a parametrization of the homogenized surface constructed from a Miura tessellation. The coordinates of φ are written as φ^i, for $i \in \{1, 2, 3\}$. As proved in [12], φ is a solution of the following strong form equation:

$$p(\varphi_x)\varphi_{xx} + q(\varphi_y)\varphi_{yy} = 0 \in \mathbb{R}^3, \tag{1a}$$

$$p(\varphi_x)q(\varphi_y) = 4, \tag{1b}$$

$$\varphi_x \cdot \varphi_y = 0, \tag{1c}$$

$$0 < |\varphi_x|^2 \leq 3, \quad 1 < |\varphi_y|^2 \leq 4, \tag{1d}$$

where

$$p(\varphi_x) = \frac{4}{4 - |\varphi_x|^2}, \quad q(\varphi_y) = \frac{4}{|\varphi_y|^2},$$

and the subscripts x and y stand respectively for ∂_x and ∂_y. The existence of solutions to (1a) has been proved in [13] which is not enough by itself to compute Miura surfaces. In the following, we investigate under what supplementary assumptions the solutions of (1a) verify the equality constraints (1b) and (1c) and the inequality constraints (1d). Note that the constraint $1 < |\varphi_y|^2$ in (1d) is particularly challenging as it is non-convex.

2.2 Main result

We introduce the Hilbert space $V := \left(H^2(\Omega)\right)^3 \cap \left(W^{1,\infty}(\Omega)\right)^3$. V is equipped with the usual $\left(H^2(\Omega)\right)^3$ Sobolev norm. Note that due to Rellich–Kondrachov theorem [4, Theorem 9.16], $V \subset \left(\mathcal{C}^0(\bar{\Omega})\right)^3$. For $\varphi \in V$, let $\mathcal{A}(\varphi) : V \mapsto \mathbb{R}^3$, be the operator defined for $\psi \in V$ as

$$\mathcal{A}(\varphi)\psi := p(\varphi_x)\psi_{xx} + q(\varphi_y)\psi_{yy} \in \mathbb{R}^3. \tag{2}$$

Solving Equation (1a) consists in finding $\varphi \in V$ such that

$$\mathcal{A}(\varphi)\varphi = 0 \text{ in } \Omega. \tag{3}$$

Note that (3) admits a translation invariance. Therefore, we consider the convex subset $\mathcal{V} := \{\varphi \in V \mid \int_\Omega \varphi = 0\}$, as our solution space to remove the translation invariance. The operator $A(\varphi)$ is not well adapted to obtain a constrained solution of (1a). Equation (3) is thus reformulated into an equation on $\mathcal{G} \equiv \nabla\varphi$. Let $W := H^1(\Omega)^{3\times 2}$ equipped with the usual $H^1(\Omega)$ norm. Let $G_D \in H^{\frac{3}{2}}(\partial\Omega)^{3\times 2}$ be the imposed Dirichlet boundary condition on $\partial\Omega$. We consider the convex subset $W_D := \{G \in W \mid G = G_D \text{ on } \partial\Omega\}$ as our solution space and $W_0 := \{G \in W \mid G = 0 \text{ on } \partial\Omega\}$

as the associated homogeneous space. Let $A(\mathcal{G}) : W \mapsto \mathbb{R}^6$ be the operator such that, for $G = (G^x, G^y) \in W$,

$$A(\mathcal{G})G := \begin{pmatrix} p(\mathcal{G}^x)G^x_x + q(\mathcal{G}^y)G^y_y \\ G^x_y - G^y_x \end{pmatrix}.$$

Equation (3) is restated as find $\mathcal{G} \in W_D$,

$$A(\mathcal{G})\mathcal{G} = 0 \text{ in } \Omega. \tag{4}$$

The second line in (4) translates the fact that $\mathcal{G}$ should be a gradient and thus verify a generalization of Clairault's theorem. $\mathcal{G}^x$ and $\mathcal{G}^y$ are interpreted as the noncolinear tangent vectors to the computed Miura surface.

The main result of this paper is Theorem 8 which proves the existence of solutions to (1) under appropriate Dirichlet boundary conditions. Let us describe the boundary conditions that will be used. We write the unit normal to a Miura surface as

$$\mathcal{N} := \frac{\mathcal{G}^x \wedge \mathcal{G}^y}{|\mathcal{G}^x \wedge \mathcal{G}^y|} \neq 0.$$

As shown in Figure 1, we impose on $\partial\Omega$, $\mathcal{N}$, $|\mathcal{G}^x|$, $|\mathcal{G}^y|$ and $\mathcal{G}^x \cdot \mathcal{G}^y = 0$. A final degree of freedom needs to be fixed in the form of an angle of rotation around $\mathcal{N}$, written α.

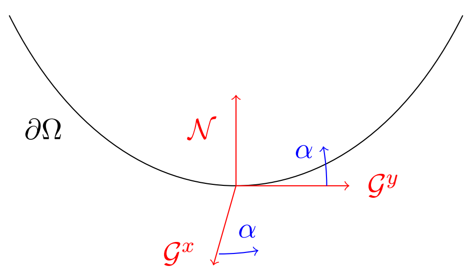

Figure 1: Imposed boundary conditions

This translates to G_D being chosen so that $|G^x_D|$ and $|G^y_D|$ verify (1b) and (1d) on $\partial\Omega$. $G^x_D \cdot G^y_D = 0$ is similarly set to verify (1c) on $\partial\Omega$. Note that this consists in imposing the first fundamental form [5] associated to φ on $\partial\Omega$.

2.3 Linearized problem

In this entire subsection, $\mathcal{G} \in W$ is assumed to be $\mathcal{C}^0(\bar{\Omega})$. Equation (1d) indicates that we are not interested in the values of $\nabla\varphi$ when $|\varphi_x| > 3$, $|\varphi_y| > 4$ and $|\varphi_y| < 1$. We thus define the Lipschitz cut-offs,

$$\bar{p}(\mathcal{G}^x) := \begin{cases} 4 & \text{if } |\mathcal{G}^x|^2 \geq 3 \\ p(\mathcal{G}^x) & \text{otherwise} \end{cases}, \quad \bar{q}(\mathcal{G}^y) := \begin{cases} 4 & \text{if } |\mathcal{G}^y|^2 \leq 1 \\ 1 & \text{if } |\mathcal{G}^y|^2 \geq 4 \\ q(\mathcal{G}^y) & \text{otherwise} \end{cases}.$$

Therefore, $1 \leq \bar{p}(\mathcal{G}^x) \leq 4$ and $1 \leq \bar{q}(\mathcal{G}^y) \leq 4$. The operator $\bar{A}(\mathcal{G})$ is defined for $G \in W$ as,

$$\bar{A}(\mathcal{G})G := \begin{pmatrix} \bar{p}(\mathcal{G}^x)G^x_x + \bar{q}(\mathcal{G}^y)G^y_y \\ G^x_y - G^y_x \end{pmatrix} \in \mathbb{R}^6. \tag{5}$$

The linearized equation we want to solve is thus, find $G \in W_D$,

$$\bar{A}(\mathcal{G})G = 0 \text{ in } \Omega. \tag{6}$$

Lemma 1. *Equation (6) admits a unique solution $G \in W_D$. There exists $C > 0$, independent from $\mathcal{G}$,*

$$\|G\|_{H^1(\Omega)} \leq C\|G_D\|_{H^{\frac{3}{2}}(\partial\Omega)}.$$

Proof. As $G_D \in H^{\frac{3}{2}}(\partial\Omega)^{3\times 2}$, there exists $\hat{G}_D \in H^2(\Omega)^{3\times 2}$, $\hat{G}_D = G_D$ on $\partial\Omega$ and

$$\|\hat{G}_D\|_{H^2(\Omega)} \leq C\|G_D\|_{H^{\frac{3}{2}}(\partial\Omega)},$$

where $C > 0$ is a constant independent from $\mathcal{G}$. We define

$$f := \bar{A}(\mathcal{G})\hat{G}_D \in H^1(\Omega)^6.$$

We now search for $\hat{G} := G - \hat{G}_D$ solution of (6) with homogeneous Dirichlet boundary conditions. The solution space is thus W_0 and the test space is defined as $Y := L^2(\Omega)^6$. Let $\hat{G} \in W_0$ and $\tilde{G} \in Y$, Equation (6) is written in weak form as

$$a(\hat{G}, \tilde{G}) := \int_\Omega \bar{A}(\mathcal{G})\hat{G} \cdot \tilde{G} = \int_\Omega f \cdot \tilde{G} =: b(\tilde{G}),$$

where $a \in \mathcal{L}(W_0 \times Y)$ and $b \in \mathcal{L}(Y)$ are continuous bilinear and linear forms. The non-zero singular values of $\bar{A}(\mathcal{G})$ are $\sqrt{\bar{p}^2 + \bar{q}^2}$ and $\sqrt{2}$. Thus, one has $|\bar{A}(\mathcal{G})| \geq \sqrt{2}$, in operator norm. Also, note that $\bar{A}(\mathcal{G})$ has maximal rank. We use the BNB lemma, see [7, Theorem 2.6, p. 85]. Using the equality case in the Cauchy–Schwarz inequality, one has

$$\sup_{\tilde{G} \in Y \setminus \{0\}} \frac{|a(\hat{G}, \tilde{G})|}{\|\tilde{G}\|_Y} = \|\bar{A}(\mathcal{G})\hat{G}\|_{L^2}.$$

Therefore

$$\inf_{\hat{G} \in W_0 \setminus \{0\}} \sup_{\tilde{G} \in Y \setminus \{0\}} \frac{|a(\hat{G}, \tilde{G})|}{\|\tilde{G}\|_Y \|\hat{G}\|_{W_0}} \geq \sqrt{2},$$

Note that the infimum is non-zero because $\bar{A}(\mathcal{G})$ has maximal rank. Therefore, the first condition of the BNB theorem is verified. The second is proved by assuming

$$\forall \hat{G} \in W_0, \quad a(\hat{G}, \tilde{G}) = 0.$$

We first consider $\hat{G} \in W_0$ such that,

$$\begin{cases} \hat{G}^x_x = \dfrac{\tilde{G}^x}{\bar{p}}, \\ \hat{G}^y = 0. \end{cases}$$

Therefore, $\int_\Omega |\tilde{G}^x|^2 = 0$. Then, we consider $\hat{G} \in W_0$ such that,

$$\begin{cases} \hat{G}^x_y = \tilde{G}^y, \\ \hat{G}^y = 0. \end{cases}$$

Therefore, $\int_\Omega |\tilde{G}^y|^2 = 0$ and thus $\int_\Omega |\tilde{G}|^2 = 0$, which proves the second condition of the BNB theorem. The BNB theorem and a trace inequality give

$$\|G\|_{H^1(\Omega)} \leq \|\hat{G}_D\|_{H^1(\Omega)} + \frac{1}{\sqrt{2}}\|\bar{A}(\mathcal{G})\hat{G}_D\|_{L^2(\Omega)} \leq C\|G_D\|_{H^{\frac{3}{2}}(\partial\Omega)},$$

where $C > 0$ is independent of $\mathcal{G}$. □

Lemma 2. *The solution $G \in W_D$ of Equation (6) is actually more regular as $G \in H^2(\Omega)^{3\times 2} \subset \mathcal{C}^0(\bar{\Omega})^{3\times 2}$. Also, one has*

$$\|G\|_{H^2(\Omega)} \leq C\|G_D\|_{H^{\frac{3}{2}}(\partial\Omega)},$$

where $C > 0$ is independent of $\mathcal{G}$.

Proof. We derive the first line in the right-hand side of (6) with respect to both variables. As $f \in H^1(\Omega)^6$, one gets

$$\begin{cases} \bar{p}G^x_{xx} + \bar{q}G^x_{yy} = f_x - \bar{p}_x G^x_x - \bar{q}_x G^y_y \in L^2(\Omega)^3, \\ \bar{p}G^y_{xx} + \bar{q}G^y_{yy} = f_y - \bar{p}_y G^x_x - \bar{q}_y G^y_y \in L^2(\Omega)^3, \end{cases}$$

where $\bar{p} \equiv \bar{p}(\mathcal{G}^x)$ and $\bar{q} \equiv \bar{q}(\mathcal{G}^y)$. These are two independent linear uniformly elliptic equations in G^x and G^y. It is proved in [22, 13], that such equations have a unique solution in $H^2(\Omega)^3 \cap H^1_0(\Omega)^3$. Therefore $G \in H^2(\Omega)^{3\times 2} \subset \mathcal{C}^0(\bar{\Omega})^{3\times 2}$ and

$$\begin{cases} \|G^x\|_{H^2(\Omega)} \leq C\|f_x - \bar{p}_x G^x_x - \bar{q}_x G^y_y\|_{L^2(\Omega)} \leq C\|G_D\|_{H^{\frac{3}{2}}(\partial\Omega)}, \\ \|G^y\|_{H^2(\Omega)} \leq C\|f_y - \bar{p}_y G^x_x - \bar{q}_x G^y_y\|_{L^2(\Omega)} \leq C\|G_D\|_{H^{\frac{3}{2}}(\partial\Omega)}, \end{cases}$$

where $C > 0$ is independent from $\mathcal{G}$. □

2.4 Nonlinear problem

The existence of a solution to (4) is proved through a fixed point method. Note that the definition of $\bar{A}$ as a Lipschitz cut-off makes the operator uniformly elliptic. Adding a positive lower bound on $|\mathcal{G}^y|$ was necessary as methods to deal with non-uniformly elliptic equations require all coefficients to be defined on all of $\mathbb{R}^{3\times 2}$ which is not the case for $q(\mathcal{G}^y)$.

Let $T : \mathcal{C}^0(\bar{\Omega})^{3\times 2} \ni \mathcal{G} \mapsto G(\mathcal{G}) \in W_D \cap H^2(\Omega)^{3\times 2}$ be the map that, given a $\mathcal{G}$, maps to the unique solution of (6). We define the following subset

$$B = \left\{ \tilde{G} \in H^2(\Omega)^{3\times 2}; \|\tilde{G}\|_{H^2(\Omega)} \leq C\|G_D\|_{H^{\frac{3}{2}}(\partial\Omega)} \right\}, \tag{7}$$

where $C > 0$ is the constant from Proposition 2.

Proposition 3. *The map T admits a fixed point $\mathcal{G} \in B$ which verifies*

$$\begin{cases} \bar{A}(\mathcal{G})\mathcal{G} = 0 \ \text{in } \Omega, \\ \mathcal{G} = G_D \ \text{on } \partial\Omega. \end{cases}$$

Proof. Let us show that B is stable by T. Let $\mathcal{G} \in B$. Letting $G := T\mathcal{G}$, one thus has $G \in B$, as a consequence of Lemma 2. Also, $B \subset \mathcal{C}^0(\bar{\Omega})^{3\times 2}$ with a compact Sobolev embedding and $\mathcal{C}^0(\bar{\Omega})^{3\times 2}$ equipped with its usual norm is a Banach space. Thus, B is a closed convex subset of the Banach space $\mathcal{C}^0(\bar{\Omega})^{3\times 2}$ and TB is precompact in $\mathcal{C}^0(\bar{\Omega})^{3\times 2}$.

We prove that T is continuous over $\mathcal{C}^0(\bar{\Omega})^{3\times 2}$. Let $(\mathcal{G}_n)_n$ be a sequence of B such that $\mathcal{G}_n \underset{n\to+\infty}{\longrightarrow} \mathcal{G} \in B$, for $\|\cdot\|_{\mathcal{C}^0(\bar{\Omega})}$ Let $G := T\mathcal{G}$ and $G_n := T\mathcal{G}_n$, for $n \in \mathbb{N}$. We want to prove that $G_n \underset{n\to+\infty}{\longrightarrow} G$ for $\|\cdot\|_{\mathcal{C}^0(\bar{\Omega})}$. As for all $n \in \mathbb{N}$, $G_n \in B$, $(\psi_n)_n$ is bounded in $H^2(\Omega)^{3\times 2}$. Therefore, there exists $\hat{G} \in H^2(\Omega)^{3\times 2}$, up to a subsequence, $G_n \underset{n\to+\infty}{\longrightarrow} \hat{G}$. Let us show that $\hat{G}$ is a solution of (6). By definition, one has

$$\begin{aligned} \|\bar{A}(\mathcal{G})G_n\|_{L^2(\Omega)} &= \|\bar{A}(\mathcal{G})G_n - \bar{A}(\mathcal{G}_n)G_n\|_{L^2(\Omega)} \\ &\leq |||\bar{A}(\mathcal{G}) - \bar{A}(\mathcal{G}_n)||| \ \|G_n\|_{H^2(\Omega)} \\ &\leq C|||\bar{A}(\mathcal{G}) - \bar{A}(\mathcal{G}_n)||| \underset{n\to\infty}{\longrightarrow} 0, \end{aligned}$$

where $|||\cdot|||$ is the operator norm and $(G_n)_n$ is bounded in $H^2(\Omega)^{3\times 2}$. The convergence in operator norm is possible because $\bar{p}$ and $\bar{q}$ are Lipschitz continuous with respect to their arguments. The left-hand side converges to $\|\bar{A}(\mathcal{G})\hat{G}\|_{L^2(\Omega)}$ when $n \to +\infty$. As for all $n \in \mathbb{N}$, $G_n \in W_D$, then $\hat{G} \in W_D$ and $\hat{G} = G_D$ on $\partial\Omega$. Therefore, $\hat{G}$ solves (6). Also, as $H^2(\Omega)^{3\times 2} \subset \mathcal{C}^0(\bar{\Omega})^{3\times 2}$, up to a subsequence,

$$\|G_n - \hat{G}\|_{\mathcal{C}^0(\bar{\Omega})} \leq \|G_n - \hat{G}\|_{H^2(\Omega)} \underset{n\to\infty}{\longrightarrow} 0.$$

Using Lemma 1, the solution of (6) is unique in W_D. Thus the full sequence $(G_n)_n$ converges towards $\hat{G} = G = T\mathcal{G}$ in $\mathcal{C}^0(\bar{\Omega})^{3\times 2}$. We conclude with the Schauder fixed point Theorem, see Corollary 11.2 of [8], that T admits a fixed point. □

Lemma 4 (Uniqueness)**.** *There exists a unique $\mathcal{G} \in W_D$ solution of* (4)*.*

Proof. Let $\mathcal{G}, \tilde{\mathcal{G}} \in W_D$ solutions of (4) and $\hat{G}_D \in H^2(\Omega)^{3\times 2}$, $\hat{G}_D = G_D$ on $\partial\Omega$. Let $G := \mathcal{G} - \hat{G}_D \in H_0^1(\Omega)^{3\times 2}$ and $\tilde{G} := \tilde{\mathcal{G}} - \hat{G}_D \in H_0^1(\Omega)^{3\times 2}$. One thus has,

$$\begin{cases} \bar{A}(\mathcal{G})G = -\bar{A}(\mathcal{G})\hat{G}_D, \\ \bar{A}(\tilde{\mathcal{G}})\tilde{G} = -\bar{A}(\tilde{\mathcal{G}})\hat{G}_D, \end{cases}$$

Therefore,

$$\begin{aligned} &\bar{A}(\tilde{\mathcal{G}})\tilde{G} - \bar{A}(\mathcal{G})G = (\bar{A}(\mathcal{G}) - \bar{A}(\tilde{\mathcal{G}}))\hat{G}_D, \\ &\bar{A}(\tilde{\mathcal{G}})\tilde{G} - \bar{A}(\tilde{\mathcal{G}})G + \bar{A}(\tilde{\mathcal{G}})G - \bar{A}(\mathcal{G})G = (\bar{A}(\mathcal{G}) - \bar{A}(\tilde{\mathcal{G}}))\hat{G}_D, \\ &\bar{A}(\tilde{\mathcal{G}})(\tilde{G} - G) = (\bar{A}(\mathcal{G}) - \bar{A}(\tilde{\mathcal{G}}))(\hat{G}_D + G). \end{aligned}$$

Using the inequality from the BNB Lemma, one has

$$\begin{aligned} \|G - \tilde{G}\|_{H^1(\Omega)} &\leq \frac{1}{\sqrt{2}}\|(\bar{A}(\mathcal{G}) - \bar{A}(\tilde{\mathcal{G}}))(\hat{G}_D + G)\|_{L^2(\Omega)}, \\ &\leq C|||\bar{A}(\mathcal{G}) - \bar{A}(\tilde{\mathcal{G}})|||, \\ &\leq C\|\mathcal{G} - \tilde{\mathcal{G}}\|_{\mathcal{C}^0(\bar{\Omega})}, \end{aligned}$$

where $C > 0$ is independent from G and $\tilde{G}$, because $\hat{G}_D + G$ is bounded in $H^1(\Omega)^{3\times 2}$ and $\bar{A}$ is Lipschitz continuous. Therefore, T is Lipschiz and the solution of (4) is unique. □

The following proposition is destined to get solutions of (3) from solutions of (4).

Proposition 5. *Let $\mathcal{G} \in W_D \cap H^2(\Omega)^{3\times 2}$ solution of* (4)*. There exists a unique $\varphi \in \mathcal{V} \cap H^3(\Omega)^3$,*

$$\nabla\varphi = \mathcal{G} \text{ in } \Omega.$$

There exists $\alpha \in (0,1)$, $\varphi \in \mathcal{C}^1(\bar{\Omega})^3 \cap \mathcal{C}^{2,\alpha}(\Omega)$ and φ is a solution of

$$\bar{p}(\varphi_x)\varphi_{xx} + \bar{q}(\varphi_y)\varphi_{yy} = 0.$$

Proof. We use again the BNB lemma. Let $X = \{\psi \in H^1(\Omega)^3 \mid \int_\Omega \psi = 0\}$ and $Y = L^2(\Omega)^{3\times 2}$. Let $\varphi \in X$, $\tilde{\psi} \in Y$ and

$$a(\varphi, \tilde{\psi}) := \int_\Omega \nabla\varphi \cdot \tilde{\psi} = \int_\Omega \mathcal{G} \cdot \tilde{\psi} =: b(\tilde{\psi}),$$

where $a \in \mathcal{L}(X \times Y)$ and $b \in \mathcal{L}(Y)$. Using the Poincaré inequality, there exists $C > 0$,

$$\|\varphi\|^2_{H^1(\Omega)} \leq C\|\nabla\varphi\|^2_{L^2(\Omega)}.$$

Therefore,

$$\sup_{\tilde{\psi}\in Y\backslash\{0\}} \frac{|a(\varphi,\tilde{\psi})|}{\|\varphi\|_X\|\tilde{\psi}\|_Y} = \frac{\|\nabla\varphi\|_{L^2(\Omega)}}{\|\varphi\|_{H^1(\Omega)}} \geq \frac{1}{\sqrt{1+C}}.$$

One thus has,

$$\inf_{\varphi\in X\backslash\{0\}} \sup_{\tilde{\psi}\in Y\backslash\{0\}} \frac{|a(\varphi,\tilde{\psi})|}{\|\varphi\|_X\|\tilde{\psi}\|_Y} \geq \frac{1}{\sqrt{1+C}}.$$

Let assume that,

$$\forall\varphi\in X, \quad a(\varphi,\tilde{\psi}) = 0.$$

We consider $\nabla\varphi = \tilde{\psi}$ and thus $\int_\Omega |\tilde{\psi}| = 0$ and $\tilde{\psi} = 0$. Applying the BNB lemma, there exists a unique $\varphi \in X$, $\nabla\varphi = \mathcal{G} \in \mathcal{V}\cap H^2(\Omega)^{3\times 2}$ and thus $\varphi \in H^3(\Omega)$. Using a classical Sobolev embedding, one has $\varphi \in \mathcal{C}^1(\bar{\Omega})^3$. Also, one has

$$\bar{A}(\nabla\varphi)\nabla\varphi = 0 \text{ in } \Omega,$$

and therefore

$$\bar{p}(\varphi_x)\varphi_{xx} + \bar{q}(\varphi_y)\varphi_{yy} = 0 \text{ in } \Omega.$$

Note that, as $\varphi \in \mathcal{C}^0(\bar{\Omega})$, one can consider $\varphi_{|\partial\Omega}$, the restriction of φ to $\partial\Omega$, which is continuous on $\partial\Omega$, as a Dirichlet boundary condition for

$$\bar{p}(\varphi_x)\psi_{xx} + \bar{q}(\varphi_y)\psi_{yy} = 0 \text{ in } \Omega,$$

which is a linear equation in $\psi \in H^2(\Omega)^3$ and has for unique solution φ. Note that because there are no cross derivative terms in the expression, the maximum principle can be applied to each individual component. Applying Theorem 6.13 of [8], there exists $\alpha \in (0,1)$, $\varphi \in \mathcal{C}^{2,\alpha}(\Omega)^3$, as done in [13]. $\square$

2.5 General problem

The previous section has proved the existence of a unique solution $\mathcal{G}$ of (1a) under a general Dirichlet boundary condition $\mathcal{G} = G_D$ on $\partial\Omega$. This section gives some restrictions on G_D so as to prove that (1b) - (1d) are verified in a subset of Ω.

Lemma 6. *Let $\varphi \in \mathcal{V}$ be a solution of Equation (1a), and let*

$$\begin{cases} u(\varphi) := \varphi_x \cdot \varphi_y, \\ v(\varphi) := \log\left((1 - \frac{1}{4}|\varphi_x|^2)|\varphi_y|^2\right). \end{cases}$$

We consider $\bar{u}(\varphi)$ and $\bar{v}(\varphi)$ as the Lipschitz cut-offs with constant functions of $u(\varphi)$ and $v(\varphi)$ when the constraints $|\varphi_x|^2 \leq 3$ and $1 < |\varphi_y|^2 \leq 4$ are not verified in Ω. Then $(\bar{u},\bar{v}) \in H^1(\Omega)^2$ is a solution of

$$\begin{cases} \bar{p}\bar{u}_x + 2\bar{v}_y = 0 \\ \bar{q}\bar{u}_y - 2\bar{v}_x = 0, \end{cases} \tag{8}$$

where $\bar{p} \equiv \bar{p}(\varphi_x)$ and $\bar{q} \equiv \bar{q}(\varphi_y)$.

Proof. Equation (1a) is projected onto φ_x and φ_y. One thus has

$$\begin{cases} \bar{p}\varphi_{xx}\cdot\varphi_x + \bar{q}\varphi_{yy}\cdot\varphi_x = 0, \\ \bar{p}\varphi_{xx}\cdot\varphi_y + \bar{q}\varphi_{yy}\cdot\varphi_y = 0. \end{cases}$$

When $|\varphi_y|^2 \leq 4$ and $1 < |\varphi_y|^2 \leq 3$, one has

$$\begin{cases} \bar{u}_x = \varphi_{xx} \cdot \varphi_y + \varphi_x \cdot \varphi_{xy}, \\ \bar{u}_y = \varphi_{xy} \cdot \varphi_y + \varphi_x \cdot \varphi_{yy}, \end{cases}$$

and

$$\begin{cases} \bar{v}_x = -\dfrac{\bar{p}}{2}\varphi_{xx} \cdot \varphi_x + \dfrac{\bar{q}}{2}\varphi_y \cdot \varphi_{yx}, \\ \bar{v}_y = -\dfrac{\bar{p}}{2}\varphi_{xy} \cdot \varphi_x + \dfrac{\bar{q}}{2}\varphi_y \cdot \varphi_{yy}. \end{cases}$$

Otherwise, $\bar{u}(\varphi)$ and $\bar{v}(\varphi)$ are constant and thus $\nabla\bar{u}(\varphi) = 0$ and $\nabla\bar{v}(\varphi) = 0$. Finally,

$$\begin{cases} \bar{p}\bar{u}_x = -2\bar{v}_y, \\ \bar{q}\bar{u}_y = 2\bar{v}_x. \end{cases}$$

□

Proposition 7. *The unique solution of* (8) *equipped with homogeneous Dirichlet boundary conditions is the pair* $(0, 0)$.

Proof. We define the solution space $X = H_0^1(\Omega)^2$ and the test space $Y = L^2(\Omega)^2$. Let $\mathcal{C} := \begin{pmatrix} \bar{p} & 0 & 0 & 2 \\ 0 & \bar{q} & -2 & 0 \end{pmatrix}$, $z = (\bar{u}, \bar{v}) \in X$ and $\tilde{z} \in Y$. $\mathcal{C}$ has rank 2, which is maximal. The non-zero singular values of $\mathcal{C}$ are $\sqrt{\bar{p}^2 + 4}$ and $\sqrt{\bar{q}^2 + 4}$. As $1 \leq \bar{p}$ and $1 \leq \bar{q}$, one has $|\mathcal{C}| \geq \sqrt{5}$. We define over $X \times Y$ the bilinear form

$$a(z, \tilde{z}) := \int_\Omega \mathcal{C}\nabla z \cdot \tilde{z},$$

where $\nabla z := \begin{pmatrix} \bar{u}_x \\ \bar{u}_y \\ \bar{v}_x \\ \bar{v}_y \end{pmatrix}$ and $a \in \mathcal{L}(X \times Y)$. We use the BNB lemma, see [7, Theorem 2.6, p. 85]. Using the equality case in the Cauchy–Schwarz inequality, one has

$$\sup_{\tilde{z} \in Y \setminus \{0\}} \frac{|a(z, \tilde{z})|}{\|\tilde{z}\|_Y} = \|\mathcal{C}\nabla z\|_{L^2}.$$

Therefore

$$\inf_{z \in X \setminus \{0\}} \sup_{\tilde{z} \in Y \setminus \{0\}} \frac{|a(z, \tilde{z})|}{\|\tilde{z}\|_Y \|z\|_X} \geq \sqrt{5}.$$

Therefore, the first condition of the BNB theorem is verified. Note that the infimum is non-zero because $\mathcal{C}$ has maximal rank. The second is proved by assuming

$$\forall z \in X, \quad a(z, \tilde{z}) = 0.$$

We consider $z \in H_0^1(\Omega)^2$ such that,

$$\nabla z = \begin{pmatrix} \tilde{z}^x / \bar{p} \\ \tilde{z}^y / \bar{q} \\ 0 \\ 0 \end{pmatrix}.$$

Therefore, $0 = \int_\Omega |\tilde{z}|^2$, and thus $\tilde{z} = 0$, which proves the second condition of the BNB theorem. Owing to the BNB theorem, there exists a unique solution to the equation and owing to the homogeneous Dirichlet boundary conditions, $z = 0$. □

The main result of this section is stated thereafter.

Theorem 8. *Let $G_D \in H^{\frac{3}{2}}(\partial\Omega)^{3\times 2}$, such that a.e. on $\partial\Omega$,*

$$\begin{cases} 0 < |G_D^x|^2 \leq 3, \\ |G_D^y|^2 = \dfrac{4}{4-|G_D^x|^2}, \\ G_D^x \cdot G_D^y = 0. \end{cases}$$

Let $\mathcal{G} \in W_D$ be the solution of (4) and $\varphi \in \mathcal{V} \cap H^3(\Omega)^3$ such that $\nabla\varphi = \mathcal{G}$ in Ω. There exists $\Omega' \subset \Omega$, open set in $\mathbb{R}^2$ of non-zero Lebesgues measure, such that $1 < |\varphi_y|^2$ a.e. in Ω'. Then, φ is the unique solution of Equation (1) in Ω'.

Remark 9. *φ is in general not a solution of (1) in all of Ω.*

Proof. On $\partial\Omega$, let

$$\begin{cases} |\varphi_x|^2 = e, \\ |\varphi_y|^2 = \dfrac{4}{4-e} > 1, \\ \varphi_x \cdot \varphi_y = 0, \end{cases}$$

with $0 < e \leq 3$. As $\nabla\varphi$ is continuous and $1 < |\varphi_y|$ on $\partial\Omega$, there exists an open domain $\Omega' \subset \Omega$ such that $1 < |\varphi_y|$ in Ω'. Also, $\partial\Omega' \supset \partial\Omega$. Due to the imposed boundary conditions, $(1-\frac{1}{4}|\varphi_x|^2)|\varphi_y|^2 = 1$ on $\partial\Omega$ and $\bar{u} = 0$ and $\bar{v} = 0$ on $\partial\Omega$ in the notations of (8). Using Proposition 7, one has $(\bar{u}, \bar{v}) = (0, 0)$ a.e. in Ω and thus in Ω'. Therefore,

$$\varphi_x \cdot \varphi_y = 0 \text{ and } p(\varphi_x)q(\varphi_y) = 4 \text{ a.e. in } \Omega'.$$

Thus, (1b) and (1c) are verified in Ω'. Let us now turn to (1d). Similarly as in the proof of Proposition 2, φ_x verifies

$$\bar{p}(\varphi_x)_{xx} + \bar{q}(\varphi_x)_{yy} = -\bar{p}_x\psi_{xx} - \bar{q}_x\psi_{yy} \in L^2(\Omega)^3.$$

Applying the maximum principle, see Theorem 9.1 of [8], to each component of φ_x, one has

$$\max_{\Omega} |\varphi_x| \leq \max_{\partial\Omega} |\varphi_x| = \max_{\partial\Omega} e \leq 3.$$

Similarly, φ_y verifies

$$\bar{p}(\varphi_y)_{xx} + \bar{q}(\varphi_y)_{yy} = -\bar{p}_y\psi_{xx} - \bar{q}_y\psi_{yy} \in L^2(\Omega)^3.$$

Applying the maximum principle, one has

$$\max_{\Omega} |\varphi_y| \leq \max_{\partial\Omega} |\varphi_y| \leq 4.$$

By definition of Ω', one has $|\varphi_y| > 1$ in Ω'. As $p(\varphi_x)q(\varphi_y) = 4$ a.e. in Ω', then $|\varphi_x| > 0$ a.e. in Ω' and (1d) is verified in Ω'. $\square$

3 Numerical method

In [13], an H^2-conformal finite element method was used to approximate the solutions of (1a). In this paper, a simpler approximation, based on a least-squares approximation of (4), and using $\mathbb{P}^1$–Lagrange finite elements is proposed for an improved efficiency. Indeed, the new method requires much less degrees of freedom (dofs) to operate.

3.1 Discrete Setting

Let $(\mathcal{T}_h)_h$ be a family of quasi-uniform and shape regular triangulations [7], perfectly fitting Ω. For a cell $c \in \mathcal{T}_h$, let $h_c := \mathrm{diam}(c)$ be the diameter of c. Then, we define $h := \max_{c\in\mathcal{T}_h} h_c$ as the mesh parameter for a given triangulation $\mathcal{T}_h$. Let $W_h := \mathbb{P}^1(\mathcal{T}_h)^{3\times 2}$. The corresponding interpolator is written as $\mathcal{I}_h$. We also define the following solution space,

$$W_{hD} := \{\psi_h \in W_h; \psi_h = \mathcal{I}_h G_D \text{ on } \partial\Omega\},$$

and its associated homogeneous space

$$W_{h0} := \{\psi_h \in W_h; \psi_h = 0 \text{ on } \partial\Omega\}.$$

3.2 Discrete problem

We will approximate a solution of (4) by constructing succesive approximations of (6). (6) is a first order system in G and therefore a direct discretization of the bilinear forms used in the proof of Lemma 1 would be unstable as the coercivity constant would vanish as $h \to 0$ as explained in Chapter 5 of [7]. The least-squares method is used to avoid this issue, see Chapter 5 of [7], for instance. We define the discrete form for $\mathcal{G}_h, \tilde{G}_h \in W_h$, such that,

$$a(\mathcal{G}_h, \tilde{G}_h) := \int_\Omega \bar{A}(\mathcal{G}_h)\mathcal{G}_h \cdot \bar{A}(\mathcal{G}_h)\tilde{G}_h. \tag{9}$$

We define for $\mathcal{G}_h, G_h \in W_h$, the following auxiliary continuous bilinear form, such that for all $\tilde{G}_h \in W_h$,

$$\mathfrak{a}(\mathcal{G}_h; G_h, \tilde{G}_h) := \int_\Omega \bar{A}(\mathcal{G}_h)G_h \cdot \bar{A}(\mathcal{G}_h)\tilde{G}_h. \tag{10}$$

Lemma 10. *Given $\mathcal{G}_h \in W_{hD}$, the equation search for $G_h \in W_{hD}$ such that*

$$\mathfrak{a}(\mathcal{G}_h; G_h, \tilde{G}_h) = 0, \quad \forall \tilde{G}_h \in W_{h0}, \tag{11a}$$

$$G_h = \mathcal{I}_h G_D \text{ on } \partial\Omega, \tag{11b}$$

admits a unique solution. This solution verifies

$$\|G_h\|_{H^1(\Omega)} \le C\|G_D\|_{H^{\frac{3}{2}}(\partial\Omega)}, \tag{12}$$

where $C > 0$ is a constant independent of $\mathcal{G}_h$.

Proof. There exists $\hat{G}_D \in H^2(\Omega)^{3\times 2} \subset \mathcal{C}^0(\Omega)^{3\times 2}$ such that $\hat{G}_D = G_D$ on $\partial\Omega$. We define the linear form for $\tilde{G}_h \in W_{h0}$,

$$L(\mathcal{G}_h; \tilde{G}_h) := \int_\Omega \bar{A}(\mathcal{G}_h)\mathcal{I}_h\hat{G}_D \cdot \bar{A}(\mathcal{G}_h)\tilde{G}_h,$$

which is bounded on W_{h0}, independently of $\mathcal{G}_h$. Therefore, we search for $\hat{G}_h \in W_{h0}$,

$$\mathfrak{a}(\mathcal{G}_h; \hat{G}_h, \tilde{G}_h) = L(\mathcal{G}_h; \tilde{G}_h),$$

where $G_h = \hat{G}_h + \mathcal{I}_h\hat{G}_D$. With an argument similar to the proof of Lemma 1, one can show that the bilinear form $\mathfrak{a}(\mathcal{G}_h)$ is coercive over W_{h0}^2,

$$\sqrt{2}\|\hat{G}_h\|^2_{H^1(\Omega)} \le \mathfrak{a}(\mathcal{G}_h; \hat{G}_h, \hat{G}_h),$$

with a coercivity constant independent from $\mathcal{G}_h$. Therefore, one has

$$\|G_h\|_{H^1(\Omega)} \le \|\mathcal{I}_h\hat{G}_D\|_{H^1(\Omega)} + \|\hat{G}_h\|_{H^1(\Omega)} \le C\|\mathcal{I}_h\hat{G}_D\|_{H^1(\Omega)} \le C\|G_D\|_{H^{\frac{3}{2}}(\partial\Omega)},$$

where $C > 0$ is independent from $\mathcal{G}_h$. $\square$

Proposition 11. *There exists a unique solution $\mathcal{G}_h \in W_{hD}$ of*

$$a(\mathcal{G}_h, \tilde{G}_h) = 0, \quad \forall \tilde{G}_h \in W_{h0}. \tag{13}$$

Proof. Let us define

$$B_h := \left\{ G_h \in W_{hD}; \|G_h\|_{H^1(\Omega)} \leq C \|G_D\|_{H^{\frac{3}{2}}(\partial\Omega)} \right\}.$$

Let $T_h : W_{hD} \to W_{hD}$ be such that $T_h : \mathcal{G}_h \mapsto G_h$, where G_h is the unique solution of (11). (12) shows that $T_h B_h \subset B_h$. We prove that T_h is Lipschitz. First, we mention that using Proposition 5.23 from [7], searching for $G_h \in W_{hD}$,

$$\mathfrak{a}(\mathcal{G}_h; G_h, \tilde{G}_h) = 0, \quad \forall \tilde{G}_h \in W_{h0},$$

is equivalent to searching for $G_h \in W_{hD}$,

$$\int_\Omega \bar{A}(\mathcal{G}_h) G_h \cdot x_h = 0, \quad \forall x_h \in X_h,$$

where $X_h := \mathbb{P}^1(\mathcal{T}_h)^6$. Let $x_h \in X_h$, $\mathcal{G}_h, \hat{\mathcal{G}}_h \in W_{hD}$, $G_h := T_h \mathcal{G}_h$ and $\hat{G}_h := T_h \hat{\mathcal{G}}_h$. One has

$$\int_\Omega (\bar{A}(\mathcal{G}_h) - \bar{A}(\hat{\mathcal{G}}_h)) G_h \cdot x_h = \int_\Omega \bar{A}(\hat{\mathcal{G}}_h)(G_h - \hat{G}_h) \cdot x_h.$$

Choosing $x_h := \bar{A}(\hat{G}_h)(G_h - \hat{G}_h)$, using the coercivity of $\mathfrak{a}(\hat{\mathcal{G}}_h)$, one has

$$\sqrt{2}\|G_h - \hat{G}_h\|_{H^1(\Omega)} \leq C |||\bar{A}(\mathcal{G}_h) - \bar{A}(\hat{\mathcal{G}}_h)||| \leq C \|\mathcal{G}_h - \hat{\mathcal{G}}_h\|_{H^1(\Omega)},$$

because $\bar{p}(\cdot)$ and $\bar{q}(\cdot)$ are globally Lipschitz. The existence of the fixed point is provided by the Brouwer fixed point theorem, see [4] and the uniqueness by the fact that T_h is Lipschitz. $\square$

3.3 Recomputing φ_h

The approximation space is written as

$$V_h := \left\{ \psi_h \in \mathbb{P}^1(\mathcal{T}_h)^3; \int_\Omega \psi_h = 0 \right\}.$$

The least-squares method is used again. The bilinear form $\mathtt{a} \in \mathcal{L}(V_h^2)$ is defined as

$$\mathtt{a}(\varphi_h, \tilde{\varphi}_h) := \int_\Omega \nabla \varphi_h \cdot \nabla \tilde{\varphi}_h.$$

It is classically coercive over V_h. The linear form $\mathtt{b} \in \mathcal{L}(V_h)$ is defined as

$$\mathtt{b}(\tilde{\varphi}_h) := \int_\Omega \mathcal{G}_h \cdot \nabla \tilde{\varphi}_h.$$

Lemma 12. *There exists a unique $\varphi_h \in V_h$ solution of*

$$\mathtt{a}(\varphi_h, \tilde{\varphi}_h) = \mathtt{b}(\tilde{\varphi}_h), \quad \forall \tilde{\varphi}_h \in V_h. \tag{14}$$

The proof is omitted for concision, as this is a classical FEM result.

3.4 Convergence of the FEM

Theorem 13. *The sequence $(\varphi_h)_h \in V_h^{\mathbb{N}}$ of solutions of (13) converges towards $\varphi \in \mathcal{V}$, solution of (1a), with the following convergence estimate,*

$$\|\varphi - \varphi_h\|_{H^1(\Omega)} \leq Ch|G_D|_{H^{\frac{3}{2}}(\partial\Omega)}, \tag{15}$$

where $C > 0$ is a constant independent of φ.

Proof. Using a poincaré inequality, there exists $C > 0$,

$$\begin{aligned}
\|\varphi - \varphi_h\|_{H^1(\Omega)} &\leq C\|\nabla\varphi - \nabla\varphi_h\|_{L^2(\Omega)}, \\
&\leq C\|\nabla\varphi - \mathcal{G}\|_{L^2(\Omega)} + C\|\mathcal{G} - \mathcal{G}_h\|_{L^2(\Omega)} + C\|\mathcal{G}_h - \nabla\varphi_h\|_{L^2(\Omega)}, \\
&= C\|\mathcal{G} - \mathcal{G}_h\|_{L^2(\Omega)}.
\end{aligned}$$

Therefore, (15) is proved by getting a convergence rate on $\mathcal{G}_h \to \mathcal{G}$. Let us prove that. First, for $G, \hat{G} \in W$, one has

$$\|\bar{A}(G)\hat{G}\|_{L^2(\Omega)} \leq \|\bar{p}(G)\hat{G}^x_x\|_{L^2(\Omega)} + \|\bar{q}(G)\hat{G}^y_y\|_{L^2(\Omega)} \leq 8\|\hat{G}\|_{H^1(\Omega)},$$

by definition of $\bar{p}$ and $\bar{q}$. Therefore, $|||\bar{A}(G)||| \leq 8$. Let $K > 0$ to be determined later. Let us define the following scalar product, for $G, \tilde{G} \in W_0$,

$$(G, \tilde{G})_h := \frac{1}{K}\mathfrak{a}(\mathcal{G}_h; G, \tilde{G}).$$

We write $\|G\|_h := \sqrt{(G, G)_h}$ the norm over W_0 associated to the scalar product. Thus,

$$\frac{\sqrt{K}}{8}\|G\|_h \leq \|G\|_{H^1(\Omega)}.$$

(13) is recast as,

$$(\mathcal{G}_h, \tilde{G}_h)_h = 0, \quad \forall \tilde{G}_h \in W_h,$$

As $\bar{A}(\mathcal{G})\mathcal{G} = 0$ and for all $x_h \in X_h$, $\int_\Omega \bar{A}(\mathcal{G}_h)\mathcal{G}_h \cdot x_h = 0$, one has

$$\int_\Omega \bar{A}(\mathcal{G}_h)\mathcal{G} \cdot x_h = \int_\Omega (\bar{A}(\mathcal{G}_h) - \bar{A}(\mathcal{G}))\mathcal{G} \cdot x_h.$$

Choosing $x_h := \bar{A}(\mathcal{G}_h)\tilde{G}_h$, (4) is recast as,

$$(\mathcal{G}, \tilde{G}_h)_h = \frac{1}{K}\int_\Omega (\bar{A}(\mathcal{G}_h) - \bar{A}(\mathcal{G}))\mathcal{G} \cdot \bar{A}(\mathcal{G}_h)\tilde{G}_h, \quad \forall \tilde{G}_h \in W_h.$$

Letting $\tilde{G}_h \in W_h$, one thus has,

$$(\mathcal{I}_h\mathcal{G}, \tilde{G}_h)_h = (\mathcal{I}_h\mathcal{G} - \mathcal{G}, \tilde{G}_h)_h + \frac{1}{K}\int_\Omega (\bar{A}(\mathcal{G}_h) - \bar{A}(\mathcal{G}))\mathcal{G} \cdot \bar{A}(\mathcal{G}_h)\tilde{G}_h.$$

Taking $\tilde{G}_h = \mathcal{I}_h\mathcal{G} - \mathcal{G}_h$, one has

$$\|\mathcal{I}_h\mathcal{G} - \mathcal{G}_h\|_h^2 = (\mathcal{I}_h\mathcal{G} - \mathcal{G}, \mathcal{I}_h\mathcal{G} - \mathcal{G}_h)_h + \frac{1}{K}\int_\Omega (\bar{A}(\mathcal{G}_h) - \bar{A}(\mathcal{G}))\mathcal{G} \cdot \bar{A}(\mathcal{G}_h)(\mathcal{I}_h\mathcal{G} - \mathcal{G}_h).$$

The first term in the right-hand side of the previous equation, written T_1, is bounded as such,

$$T_1 \leq \|\mathcal{I}_h\mathcal{G} - \mathcal{G}\|_h\|\mathcal{I}_h\mathcal{G} - \mathcal{G}_h\|_h.$$

Regarding the second, written T_2, one has

$$T_2 \leq \frac{C}{K}\|\mathcal{G}_h-\mathcal{G}\|_{H^1(\Omega)}\|\mathcal{I}_h\mathcal{G}-\mathcal{G}_h\|_{H^1(\Omega)} \leq \frac{C}{K}\left(\|\mathcal{I}_h\mathcal{G}-\mathcal{G}\|_{H^1(\Omega)}\|\mathcal{I}_h\mathcal{G}-\mathcal{G}_h\|_{H^1(\Omega)} + \|\mathcal{I}_h\mathcal{G}-\mathcal{G}_h\|^2_{H^1(\Omega)}\right).$$

Also, there exists $C' > 0$, one has for $\hat{G} \in W_0$,

$$\|\hat{G}\|^2_{H^1(\Omega)} \leq C'\|\hat{G}\|^2_h.$$

Therefore,

$$\|\mathcal{I}_h\mathcal{G}-\mathcal{G}_h\|^2_h \leq \left(1+\frac{CC'}{K}\right)\|\mathcal{I}_h\mathcal{G}-\mathcal{G}\|_h\|\mathcal{I}_h\mathcal{G}-\mathcal{G}_h\|_h + \frac{CC'}{K}\|\mathcal{I}_h\mathcal{G}-\mathcal{G}_h\|_h$$

The constant $K > 0$ is chosen so that $CC' < K$. One thus has,

$$\|\mathcal{I}_h\mathcal{G}-\mathcal{G}_h\|_h \leq \frac{1+K'}{1-K'}\|\mathcal{I}_h\mathcal{G}-\mathcal{G}\|_h \leq \frac{1+K'}{1-K'}\frac{8}{\sqrt{K}}\|\mathcal{I}_h\mathcal{G}-\mathcal{G}\|_{H^1(\Omega)},$$

where $K' := \frac{CC'}{K}$. Thus,

$$\|\mathcal{G}-\mathcal{G}_h\|_{H^1(\Omega)} \leq \left(1+\frac{1+K'}{1-K'}\frac{8C'}{\sqrt{K}}\right)\|\mathcal{I}_h\mathcal{G}-\mathcal{G}\|_{H^1(\Omega)}.$$

As $\mathcal{G} \in \left(H^2(\Omega)\right)^3$, one has only the classical interpolation error [3],

$$\|\mathcal{I}_h\mathcal{G}-\mathcal{G}\|_{H^1(\Omega)} \leq Ch|\mathcal{G}|_{H^2(\Omega)},$$

where $C > 0$ is independent of $\mathcal{G}$. Finally,

$$\|\mathcal{G}_h-\mathcal{G}\|_{H^1(\Omega)} \leq Ch|\mathcal{G}|_{H^2(\Omega)}.$$

□

Remark 14. *We cannot prove that φ_h, solution of (14) will verify a discrete equivalent of (1d) or give a convergence rate for a discrete equivalent of (u, v) solution of (8) but we explore these aspects numerically in the following.*

Remark 15. *Theorem 13 proves that the proposed method converges at order one in h, similarly to the method of [13]. However, numerically, we observe an improved convergence rate of two. This suggests that (15) could be improved to*

$$\|\varphi-\varphi_h\| \leq Ch^2|\mathcal{G}|_{H^1(\Omega)}.$$

Such a result might be proved using an Aubin–Nitsche trick, as $\mathfrak{a}(\mathcal{G}_h)$ is symmetric, but in a nonlinear setting. See [6], for instance.

4 Numerical examples

The method uses the automatic differentiation of Firedrake [20] to solve (13) with a Newton method. The boundary condition $G_h = \mathcal{I}_hG_D$ on $\partial\Omega$ is imposed strongly. The stopping criterion based on the relative residual in H^1-norm is set to $\varepsilon := 10^{-8}$. The initial guess $\mathcal{G}_{h,0} \in W_{hD}$ for the Newton method is computed as the solution of

$$\int_\Omega \nabla\mathcal{G}_{h,0}\cdot\nabla\tilde{G}_h = 0, \quad \tilde{G}_h \in W_{h0}.$$

To handle in practice the constraint $\int_\Omega \varphi_h = 0$ in computing a solution of (14), we follow [7, Remark 3.83] and fix the position of one node of the mesh to zero.

4.1 Hyperboloid

This test case comes from [12, 13]. The reference solution is

$$\phi(x,y) = \left(\rho(x)\cos(\alpha y), \rho(x)\sin(\alpha y), z(x)\right)^{\mathsf{T}}, \text{ where } \begin{cases} \rho(x) = \sqrt{4c_0^2x^2+1}, \\ z(x) = 2s_0x, \\ \quad \alpha = \left(1-s_0^2\right)^{-1/2}, \end{cases}$$

$c_0 = \cos(\frac{\theta}{2})$, $s_0 = \sin\left(\frac{\theta}{2}\right)$ and $\theta \in \left(0, \frac{2\pi}{3}\right)$. The domain is $\Omega = [-s_0^*, s_0^*] \times [0, \frac{2\pi}{\alpha}]$, where $s_0^* = \sin\left(0.5\cos^{-1}\left(0.5\cos\left(\frac{\theta}{2}\right)^{-1}\right)\right)$. Structured meshes, periodic in y, are used. This translates into the fact that the dofs on the lines of equation $y=0$ and $y = \frac{2\pi}{\alpha}$ of $\partial\Omega$ are one and the same, but remain unknown. $\nabla\phi$ is used as Dirichlet boundary condition on the lines of equations $x = -s_0^*$ and $x = s_0^*$. A convergence test is performed for $\theta = \frac{\pi}{2}$. Table 1 contains the errors and estimated convergence rate. The convergence rate is estimated using the formula

h	nb dofs	H_0^1-error	rate	nb iterations
0.178	$7,650$	3.917e-02	-	4
0.00889	$30,300$	9.492e-03	2.06	4
0.00444	$120,600$	2.356e-03	2.02	4
0.00222	$481,200$	5.880e-04	2.01	4

Table 1: Hyperboloid: estimated convergence rate and number of Newton iterations.

$$\log\left(\frac{e_1}{e_2}\right)\log\left(\frac{\mathrm{card}(\mathcal{T}_{h_1})}{\mathrm{card}(\mathcal{T}_{h_2})}\right)^{-1},$$

where e_1 and e_2 are the errors. The errors are measured with respect to φ_h and ϕ. The number of Newton iterations is also reported. The results in Table 1 are compatible with the second order rate announced in Remark 15. Let

$$\bar{u}_h := \varphi_{h,x} \cdot \varphi_{h,y}, \quad \bar{v}_h := \log\left(\left(1 - \frac{1}{4}|\varphi_{h,x}|^2\right)|\varphi_{h,y}|^2\right).$$

One notices that $\bar{u}_h \simeq 0$ in Ω, for all computations. Regarding $\bar{v}_h$, Table 2 gives the estimated convergence rate. It seems that $\bar{v}_h$ converges at order two towards zero in L^2-norm. Regarding

h	nb dofs	$\bar{v}_h$ H_0^1-error	rate
0.178	$7,650$	4.481e-02	-
0.0702	$30,300$	1.068e-02	2.08
0.0351	$139,482$	2.640e-03	2.02
0.0175	$550,242$	6.581e-04	2.01

Table 2: Hyperboloid: estimated convergence rates.

(1d), all computations verify a discrete equivalent of it. Figure 2 shows $|\varphi_{h,x}|$ and $|\varphi_{h,y}|$ on the coarsest mesh.

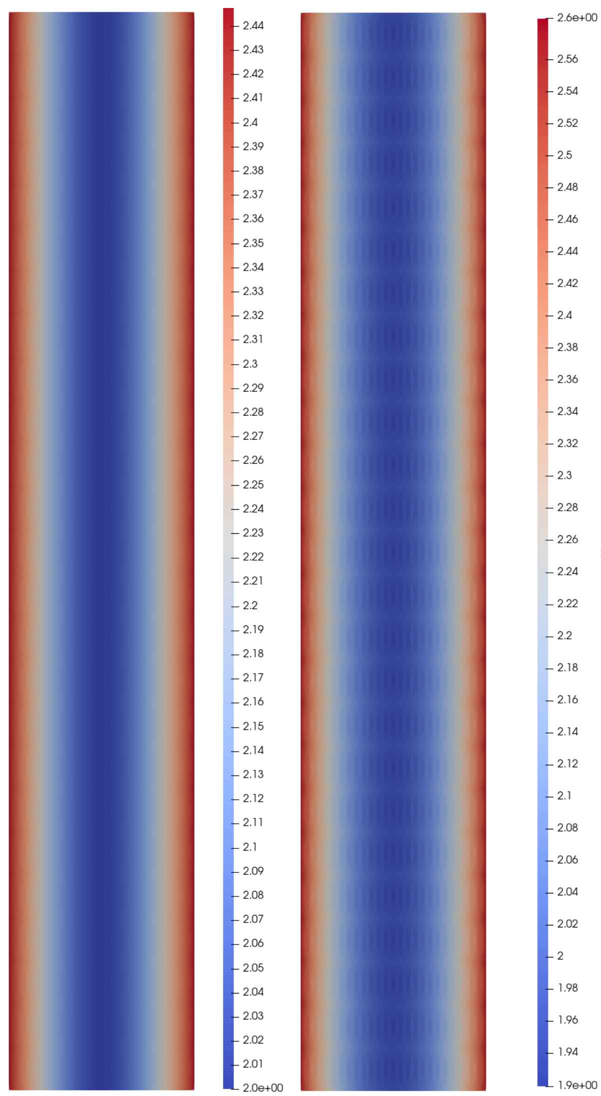

Figure 2: Hyperboloid: left: $|\varphi_{h,x}|$, right: $|\varphi_{h,y}|$.

4.2 Annulus

The domain is $\Omega = \left[0, \frac{3}{4}\right] \times [0, 2\pi]$. A structured mesh, periodic in the y direction, is used to mesh Ω. The mesh has a size $h = 0.015$ and contains $181,800$ dofs. The Dirichlet boundary conditions are

$$\begin{cases} G_D^x = (ax+1)e_r, \\ G_D^y = \dfrac{4}{4-|G_D^x|^2} e_\theta, \end{cases}$$

where (e_r, e_θ) is the polar basis of $\mathbb{R}^2$ and $a = 0.675$. The Newton method converges after 8 iterations. The resulting surface is presented in Figure 3. It is shown in [12], that the inner and

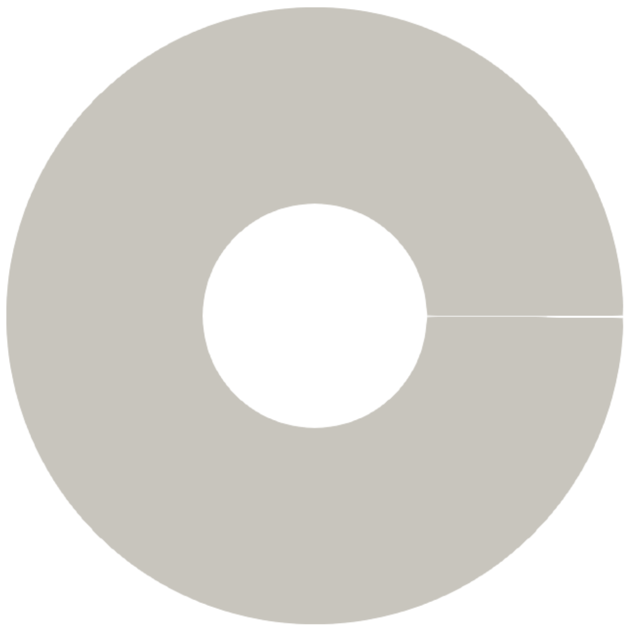

Figure 3: Annulus: Computed surface.

outer radius of the annulus cannot be arbitrarily chosen. This can be observed in Figure 4, where one can observe that $\Omega' \not\subseteq \Omega$. One can see in Figure 5 that the upper bounds in (1d) are verified in all Ω. $\bar{u}_h \simeq 0$ where as $v_h \simeq 0$ only in Ω' as shown in Figure 6.

4.3 Saddle shape

The domain is $\Omega = [-1, 1]^2$. It is meshed with an unstructured triangular mesh of size $h = 0.025$ which contains $51,360$ dofs. The Dirichlet boundary conditions for this test case derive from a hyperbolic paraboloid for which

$$f(x,y) = \left(x, y, \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{\mathsf{T}}.$$

The chosen Dirichlet boundary conditions are

$$\begin{cases} G_D^x = f_x, \\ G_D^y = \dfrac{4}{4-|f_x|^2} \dfrac{\mathfrak{f}}{|\mathfrak{f}|}, \end{cases}$$

Figure 4: Annulus: red: Ω', blue: $\Omega \setminus \Omega'$.

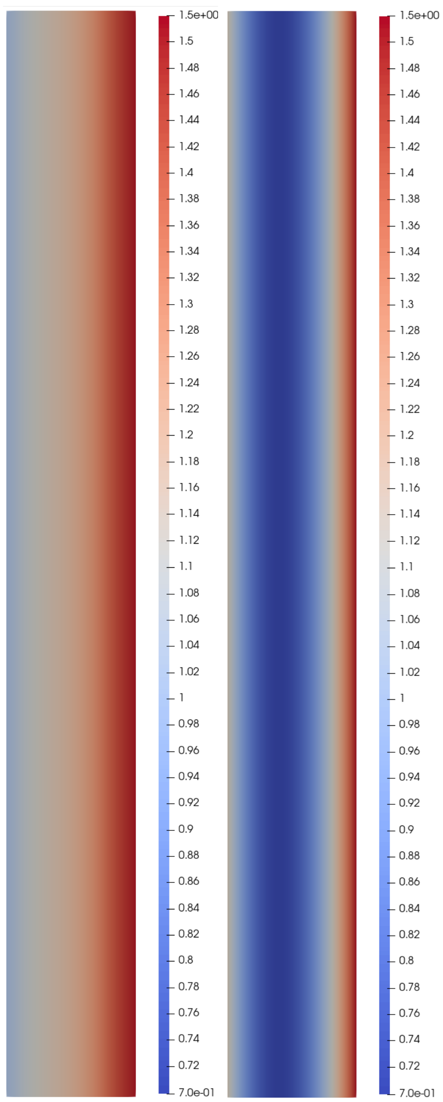

Figure 5: Annulus: left: $|\varphi_{h,x}|$ and right: $|\varphi_{h,y}|$.

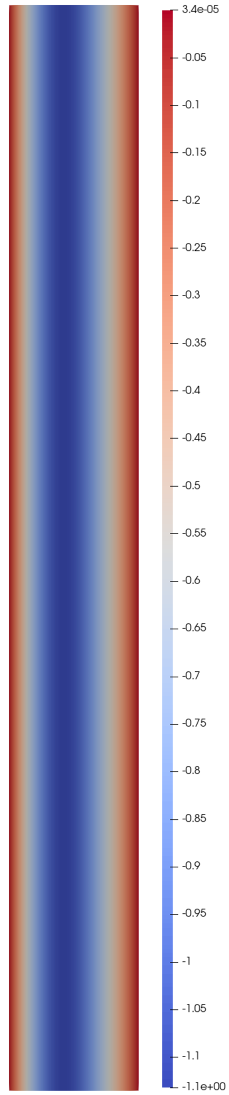

Figure 6: Annulus: v_h.

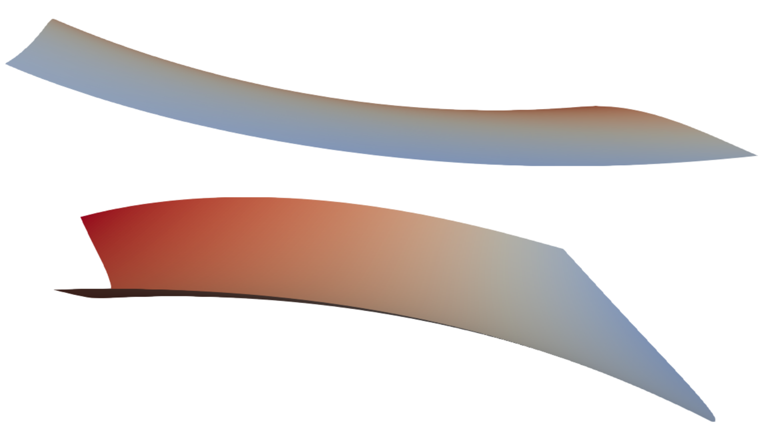

Figure 7: Saddle shape: Computed surface.

where $a = 2$, $b = \frac{4}{3}$ and $\mathfrak{f} := f_y - \frac{f_y \cdot f_x}{|f_x|^2} f_x$. The Newton method converges after 4 iterations. Figure 7 shows the resulting Miura surface. One has $|\varphi_{h,y}| > 1$ in Ω and thus $\Omega' = \Omega$. The inequality constraints are verified at the discrete level as shown in Figure 8. However, the equality constraints

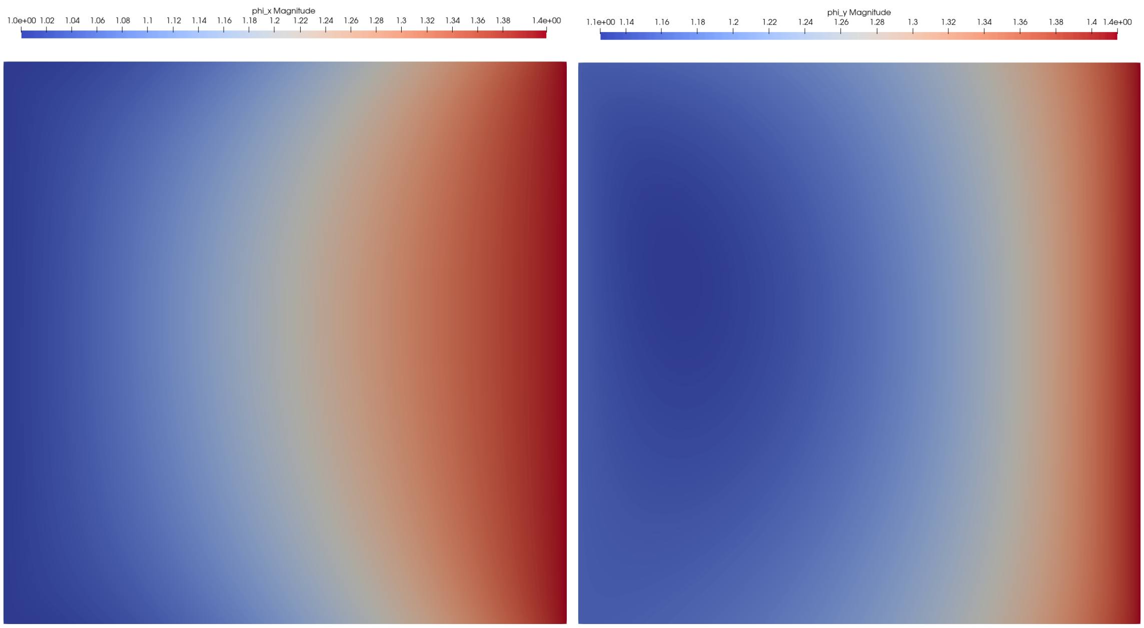

Figure 8: Saddle shape: left: $|\varphi_{h,x}|$, right: $|\varphi_{h,y}|$.

are not exactly verified at the local level as shown in Figure 9. That is not a problem since Theorem

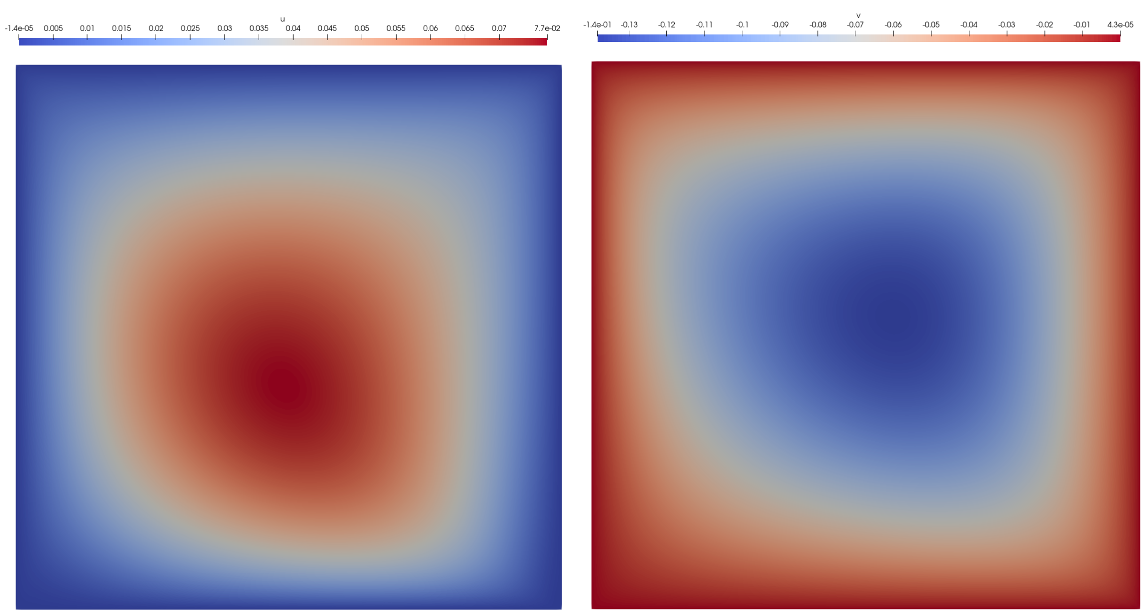

Figure 9: Saddle shape: left: $\bar{u}_h$, right: $\bar{v}_h$.

8 ensures that (1b) and (1c) are verified in Ω'. Also, $\bar{u}_h$ and $\bar{v}_h$ are small in Ω.

5 Conclusion

In this paper, the existence and uniqueness of solutions of the system of equations describing a Miura surface are proved under specific boundary conditions that still leave some freedom for design choices, specifically α and $\mathcal{N}$ in Figure 1. Then, a numerical method based on tensor $\mathbb{P}^1$–Lagrange elements, a Newton method and a least-squares formulation are introduced to approximate Miura surfaces. The method is proved to converge at order one in the space discretization parameter h and an order two is observed in practice. Some numerical tests are performed and allow to compute some non analytical Miura surfaces.

Future work include being able to compute the Miura surface "closest" to a given target surface and to be able to produce Miura tessellations with a given pattern size from Miura surfaces.

Code availability

The code is available at `https://github.com/marazzaf/Miura.git`

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