Coding Physics

Gravity and the Three Body Problem

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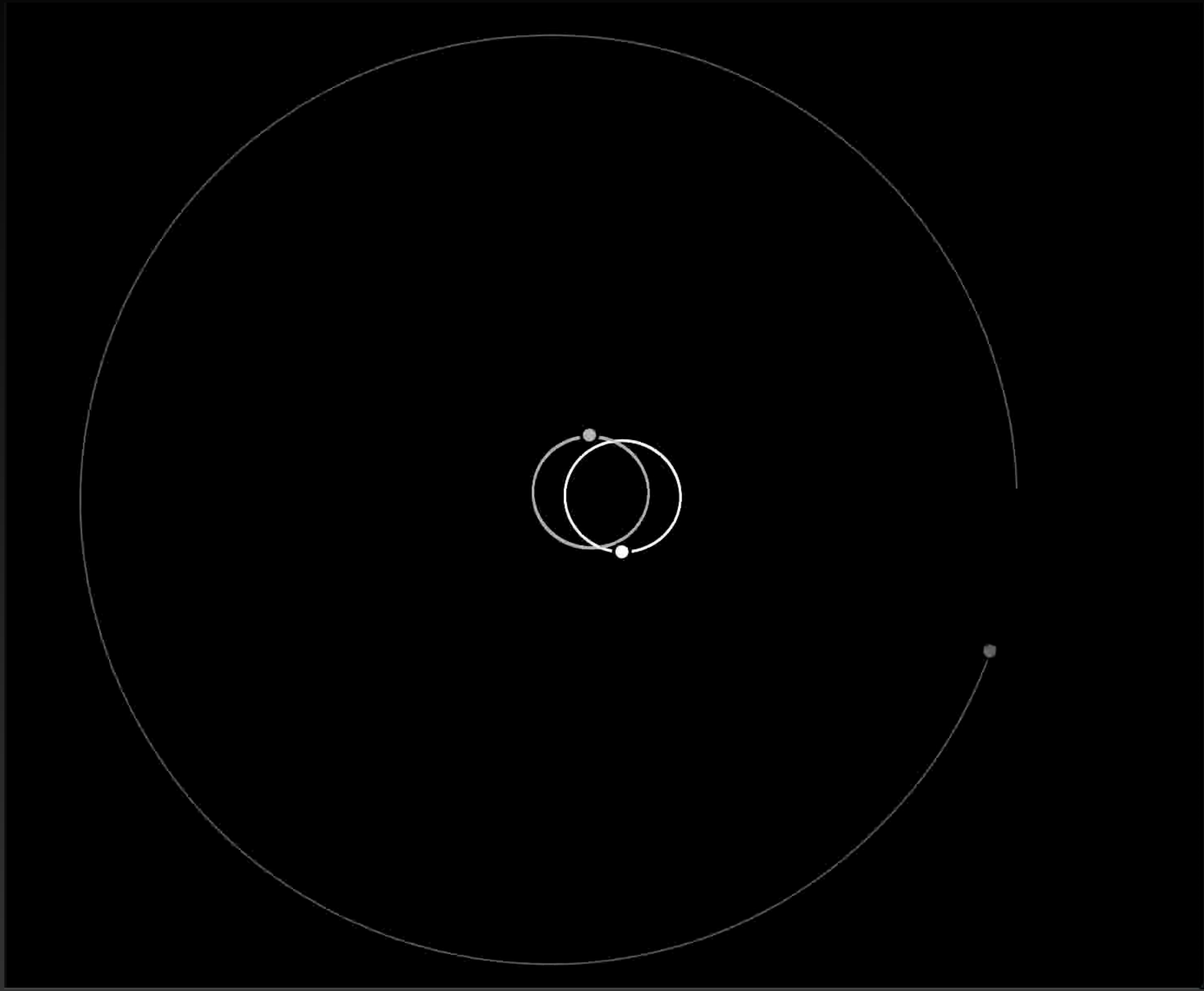
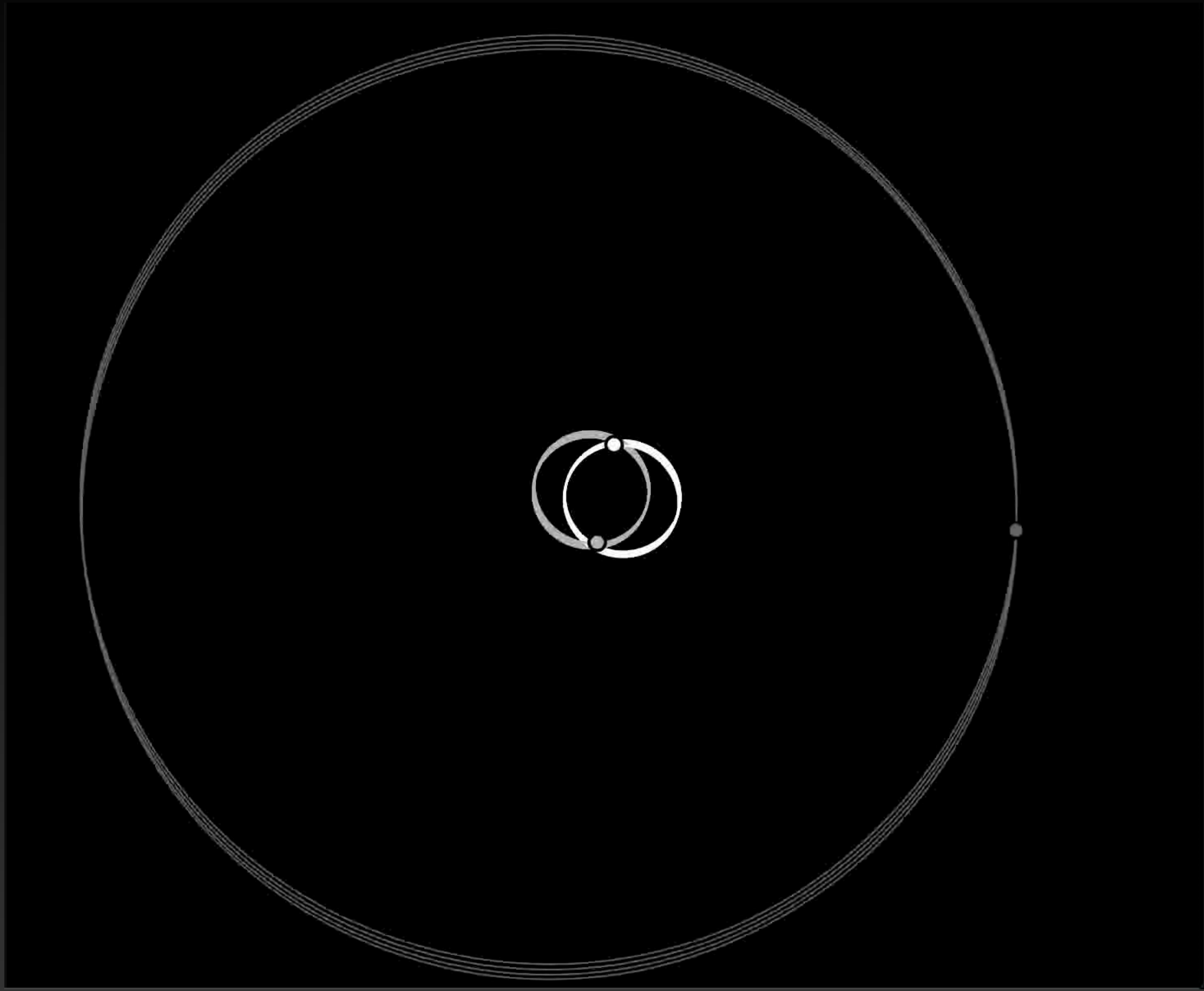
Why should we solve Physics Problems with Computer Code?

- Some experiments cannot be done in a physical lab.
  - Simulating the universe or the planets.
- Some experiments are too costly to practically run thousands of times.
- Machine Learning methods and AI can help us see new patterns



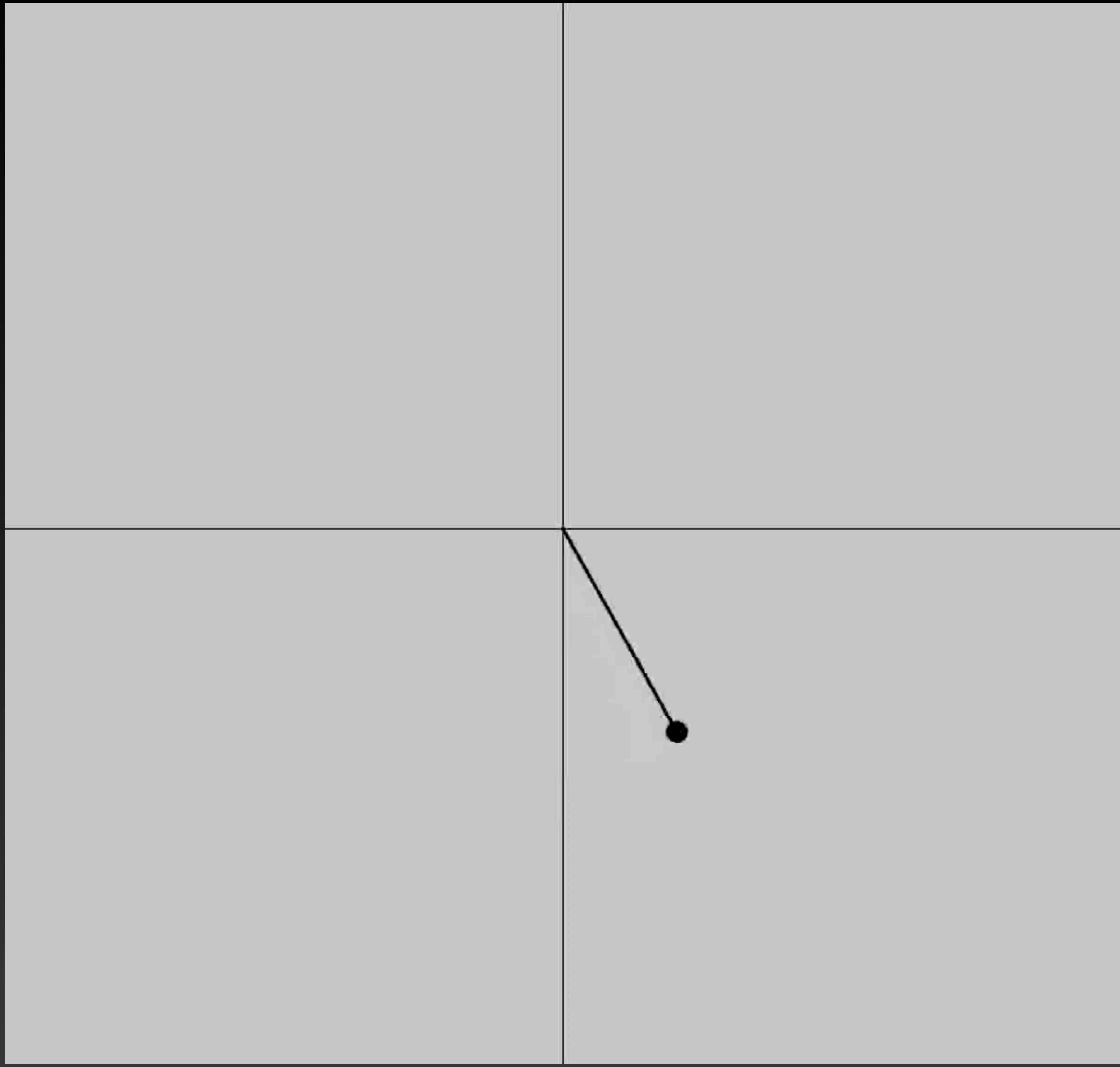
How do we solve Physics Problems with Computer Code?

- First, we need to be able to describe the motion.
- For many problems this can be done with Newton's 2nd law.
  - $F = ma$ and $\tau = I\alpha$
- We will do this for 3 examples:
  - The simple pendulum
  - Newton's Law of Gravity: The 2 and 3 body problems.



The Pendulum

- Many physics models are based on the motion of the simple pendulum.
- It is often used to model many other physical phenomena, and is used by analogy to describe many more.



The Simple Pendulum

Follow along by writing the code yourself.

- Go to https://trinket.io/library/trinkets/e455871d49
- Don’t worry about understanding how the animations are done right now, all of the code used to animate the pendulum is already there. Just find the part that says,

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#***********************************
    Code Solution Here:
#***********************************

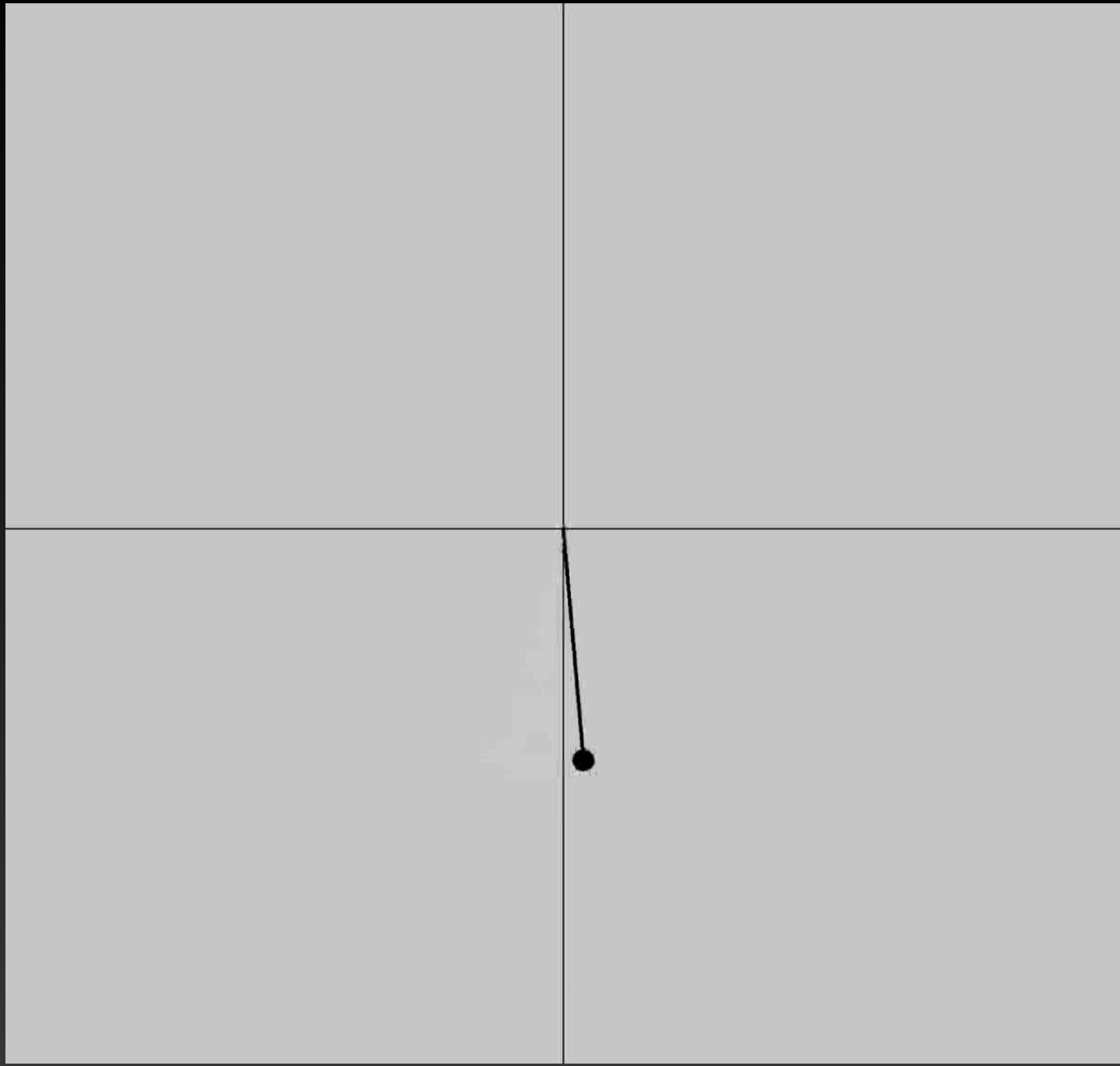

#***********************************
```

- and code up your solution between the lines. When you think you’ve got it, hit the play button at the top of the page and see if it worked! Also, play around with the initial conditions and physical constants.

The Pendulum

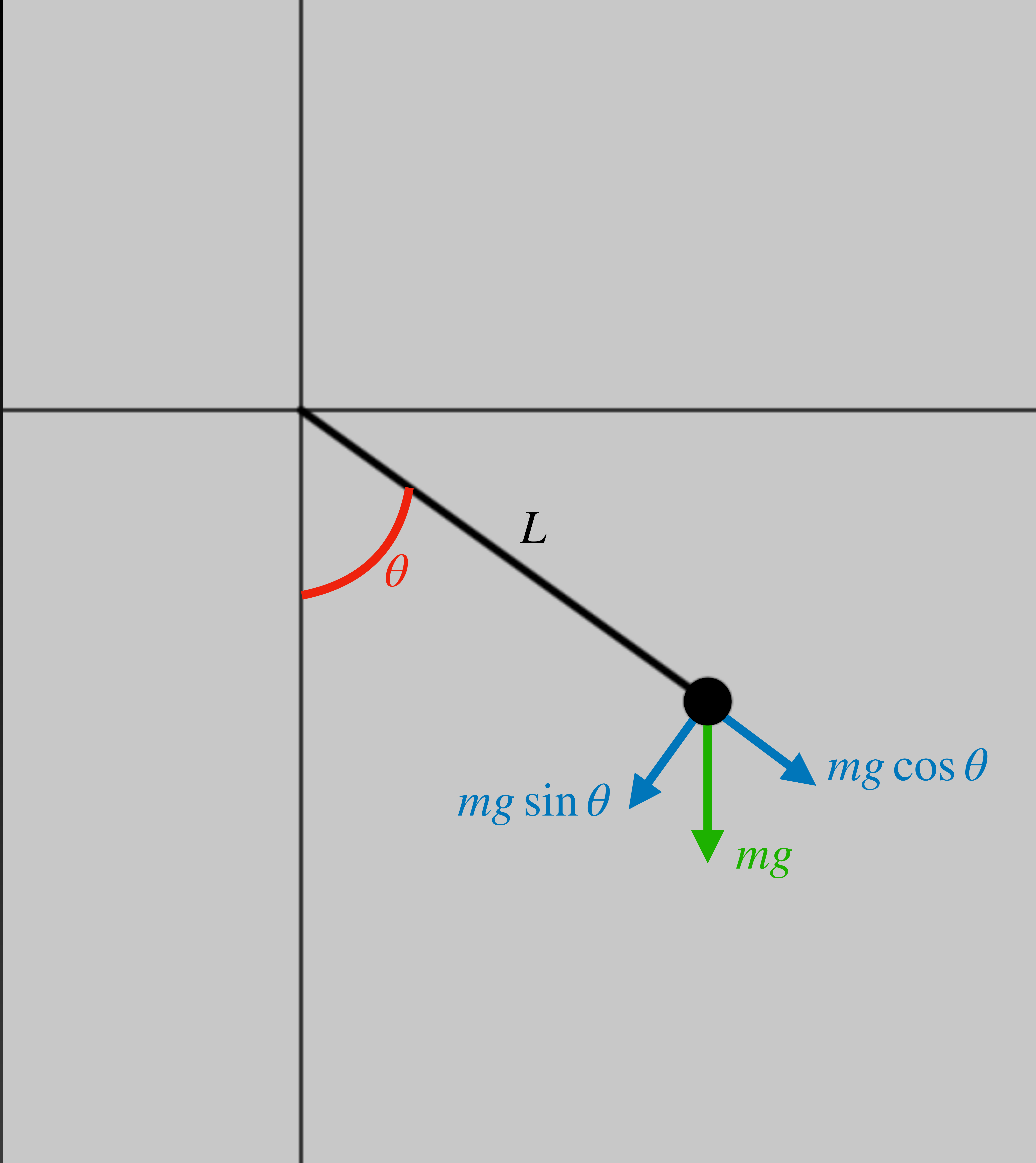
- We begin with Newton’s 2nd law for rotational systems.
- Often this is written as:
  - $\tau = I\alpha$

where τ is the torque, I is the moment of inertia, and α is the angular acceleration.



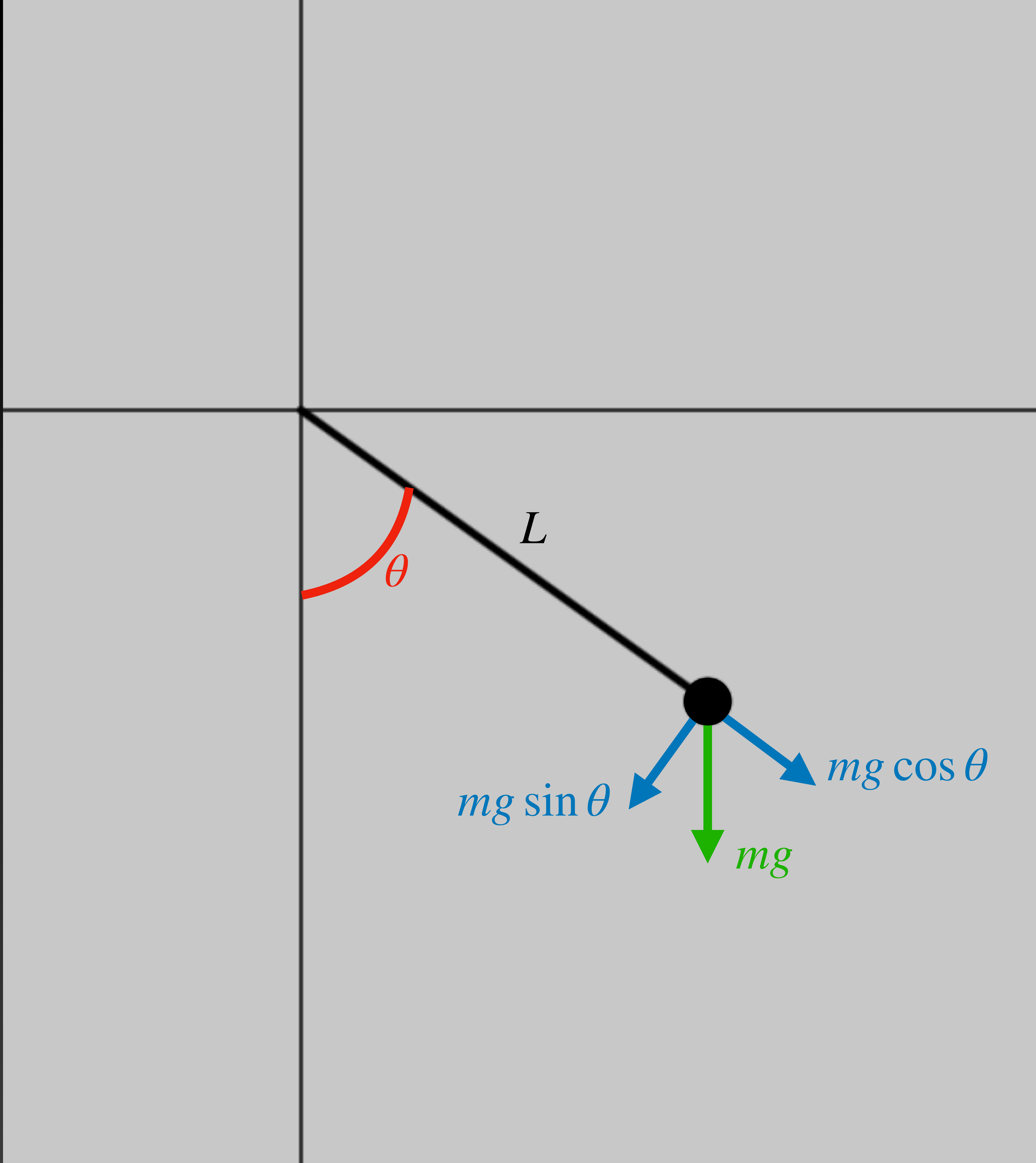
The Pendulum

- The position of the mass is determined only by θ, and gravity is the only force causing a torque on the mass.
- This torque is $\tau = -mgL\sin\theta$.
- The moment of inertia for the mass is $I = mL^2$
- The angular acceleration of the mass is simply α.



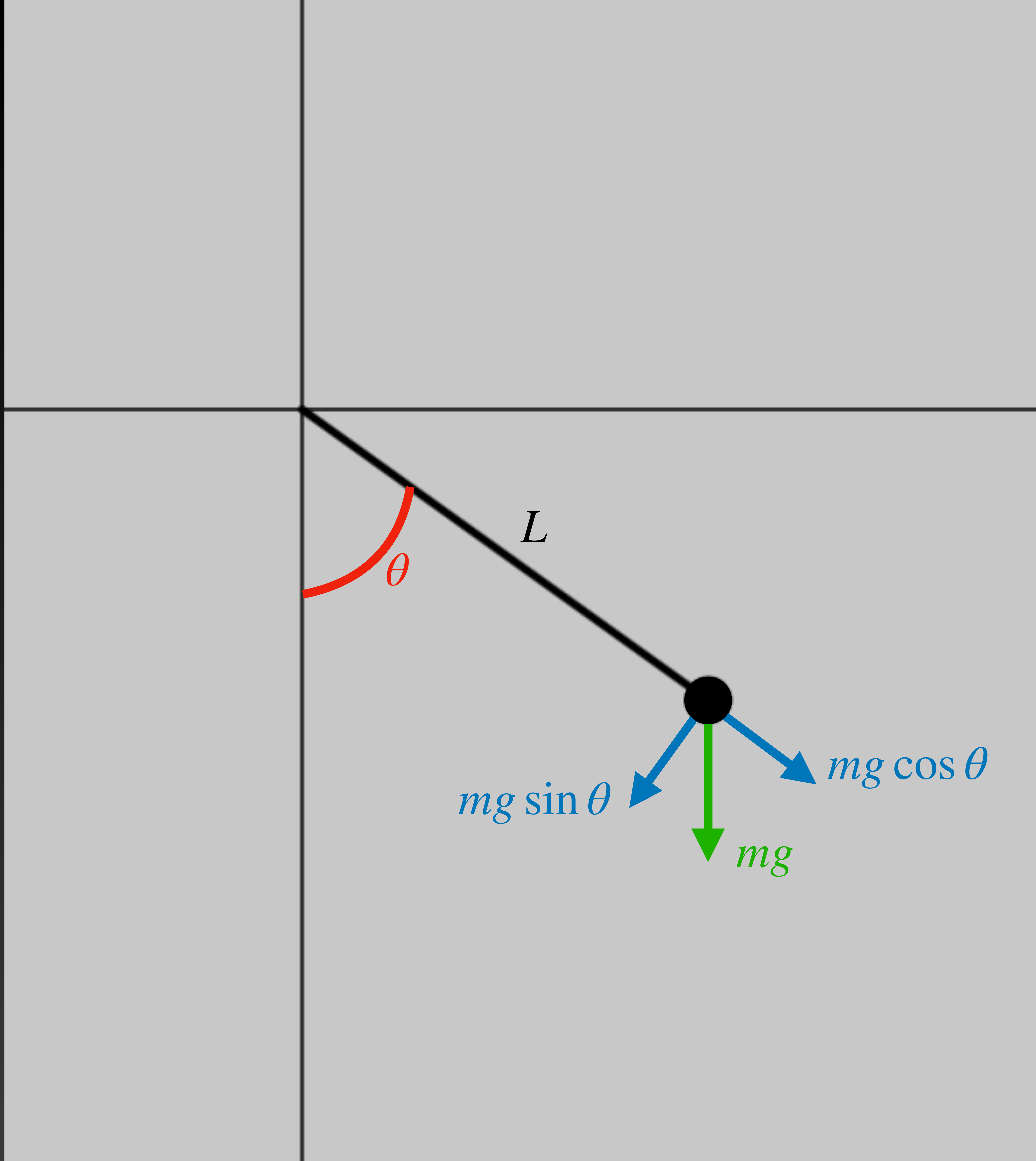
The Pendulum

- Bringing this all together gives,
  - $\tau = I\alpha$
  - $-mgL\sin\theta = mL^2\alpha$
- Now rearrange the equation to better see how the angular acceleration (α) is related to the angular position (θ).
  - $\alpha = -\dfrac{g}{L}\sin\theta$



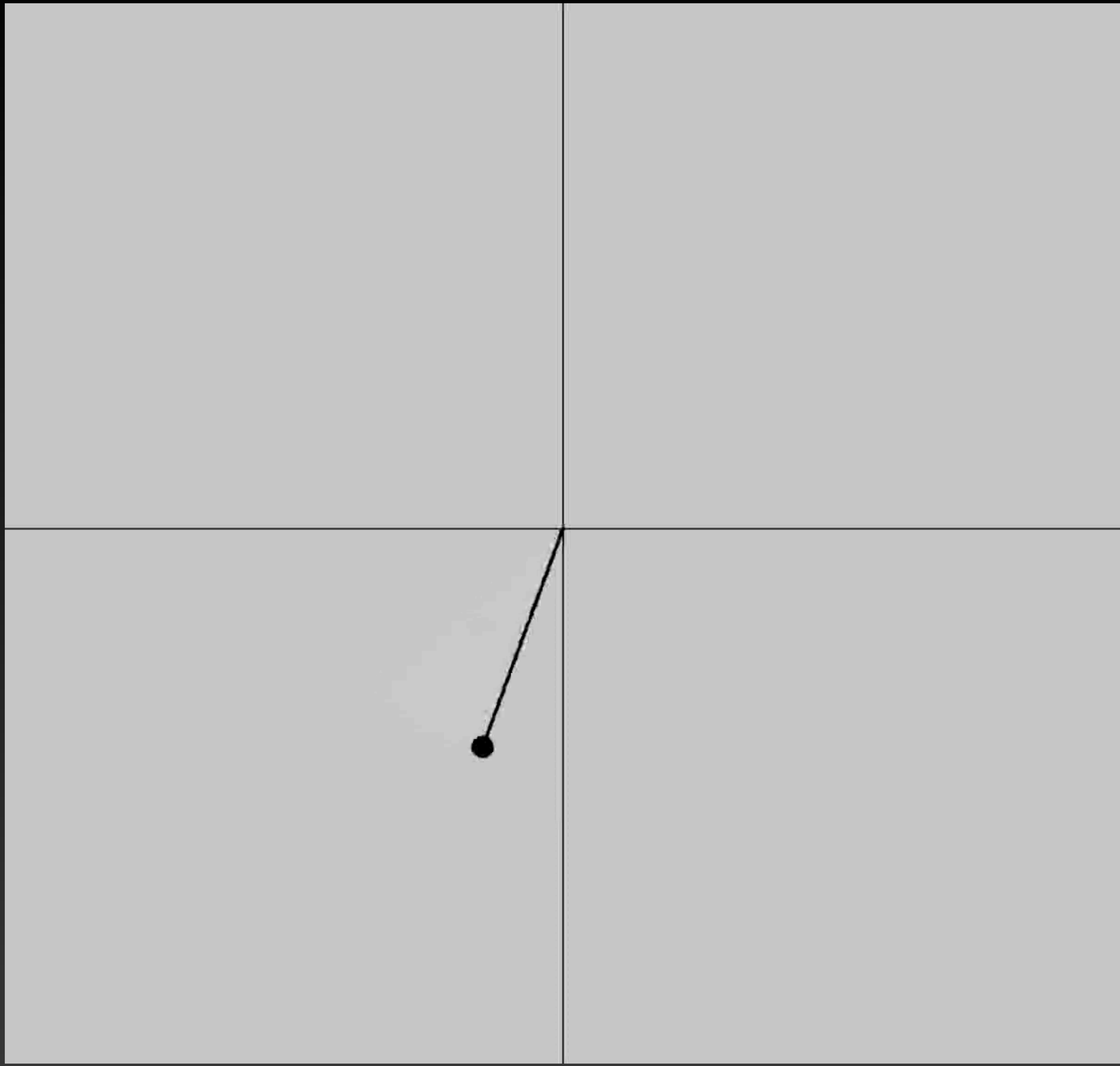
The Pendulum

- Recall how the angular position of a mass changes due to the angular acceleration.
- We know that the angular position and angular velocity are related by $\omega = \Delta\theta/\Delta t$, so that the angular velocity is the change in angular position divided by the change in time.



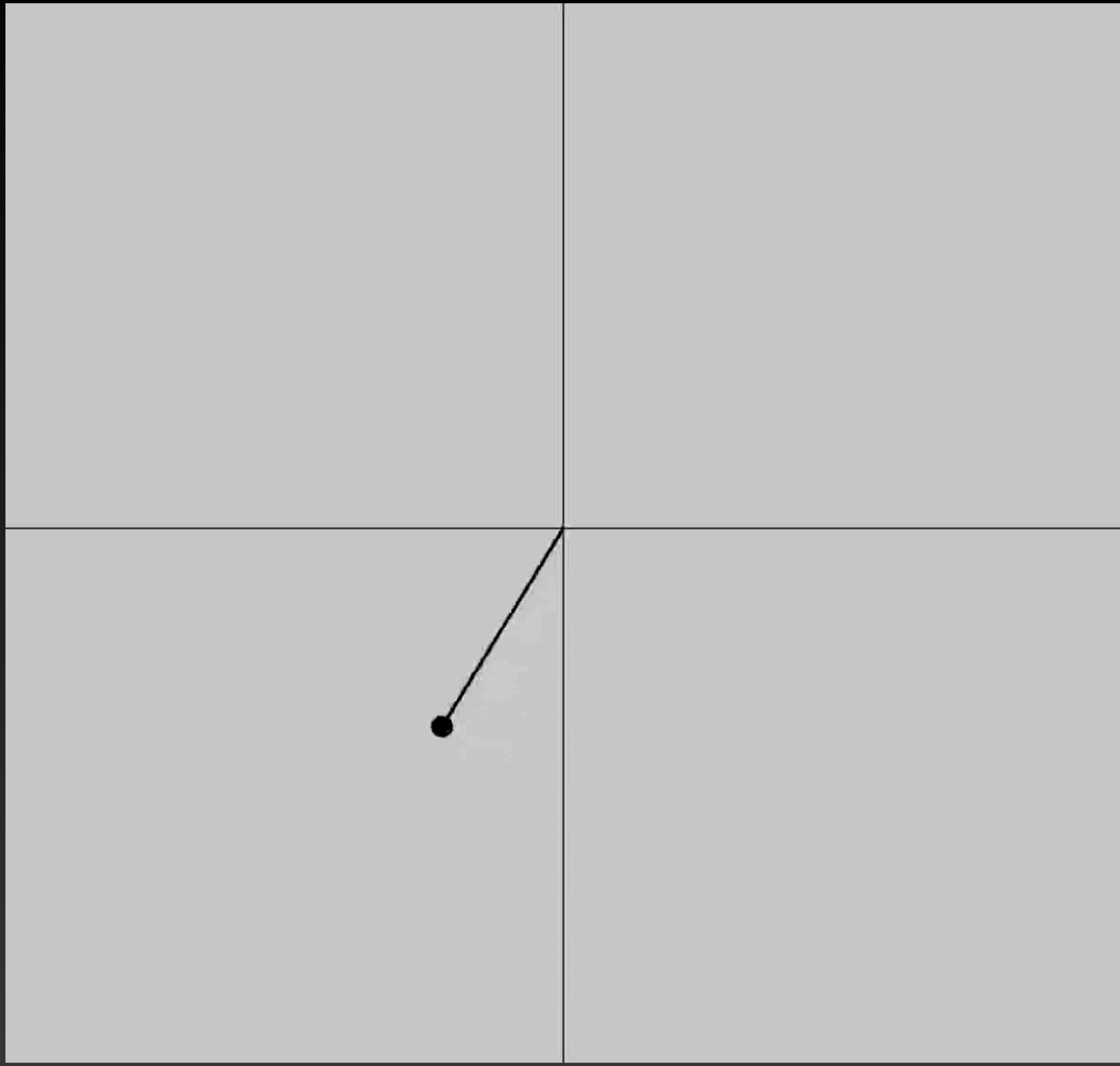
The Pendulum

- We can update the angular velocity based on the calculated angular acceleration.
  - Using $\alpha = \Delta\omega / \Delta t$, we can rearrange the equation for ω_f
    - $\Delta\omega = \alpha\Delta t$
    - $\omega_f - \omega_i = \alpha\Delta t$
    - $\omega_f = \omega_i + \alpha\Delta t$



The Pendulum

- If we similarly solve for the final angular position and put it all together we have,
  - $\alpha = -\dfrac{g}{L}\sin\theta$
  - $\omega_f = \omega_i + \alpha\Delta t$
  - $\theta_f = \theta_i + \omega_f\Delta t$
- Then if we calculate these 3 equations over and over again we can animate the position of the mass over time.



The Simple Pendulum
Play Around!

- If you haven't already, go to https://trinket.io/library/trinkets/e455871d49

#************************************
Code Solution Here:
#************************************

$$\alpha = -\frac{g}{L}\sin\theta$$

$$\omega_f = \omega_i + \alpha\Delta t$$

$$\theta_f = \theta_i + \omega_f\Delta t$$

#************************************

- Also, play around with the initial conditions and physical constants.
- If you get stuck, the solution can be found at https://trinket.io/library/trinkets/77ae8a6382
- As a bonus, see if you can add damping!

The Two Body Problem
Follow along!

- Go to https://trinket.io/library/trinkets/0b17a8381c
- Once again, find the part that says,

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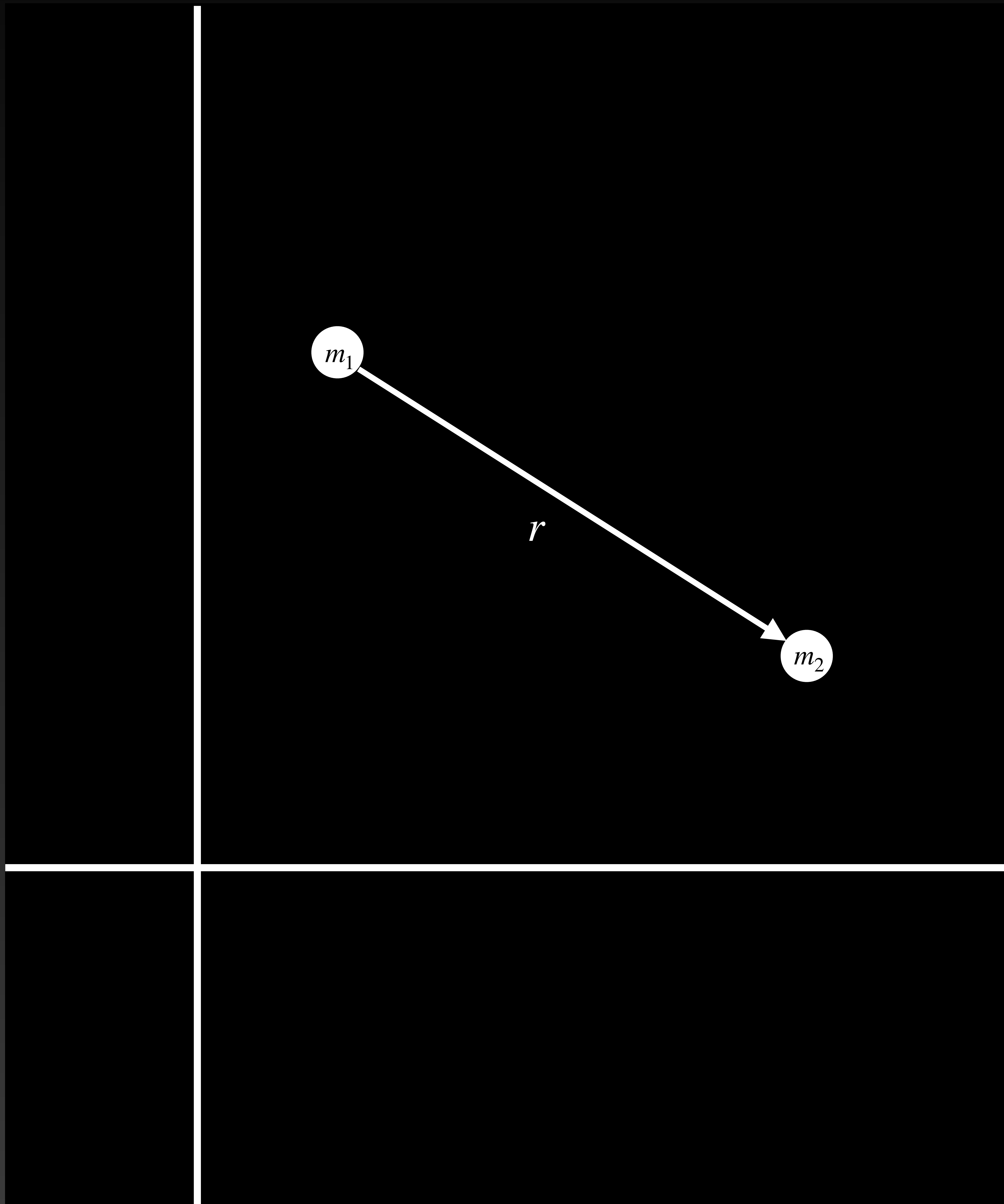
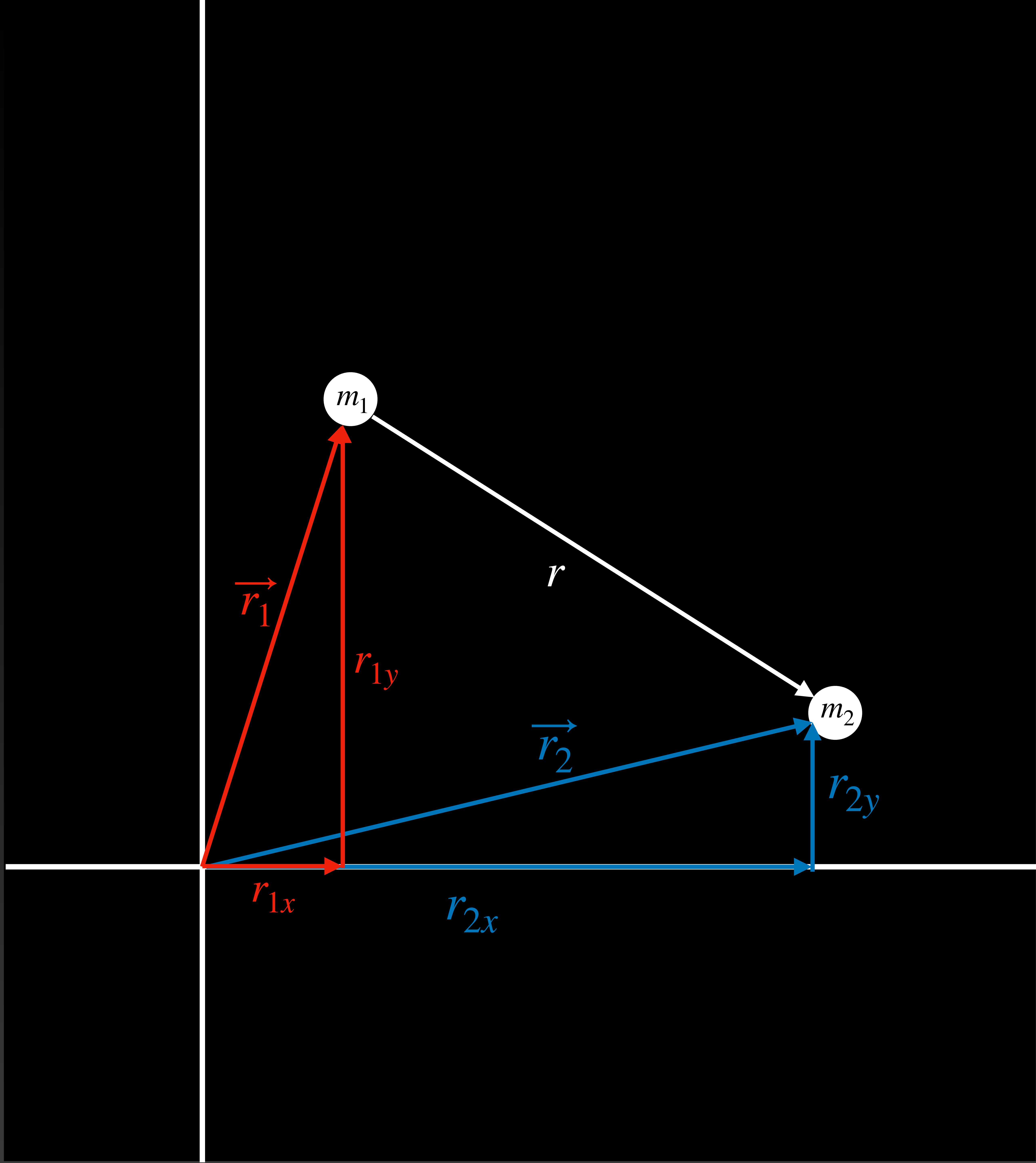


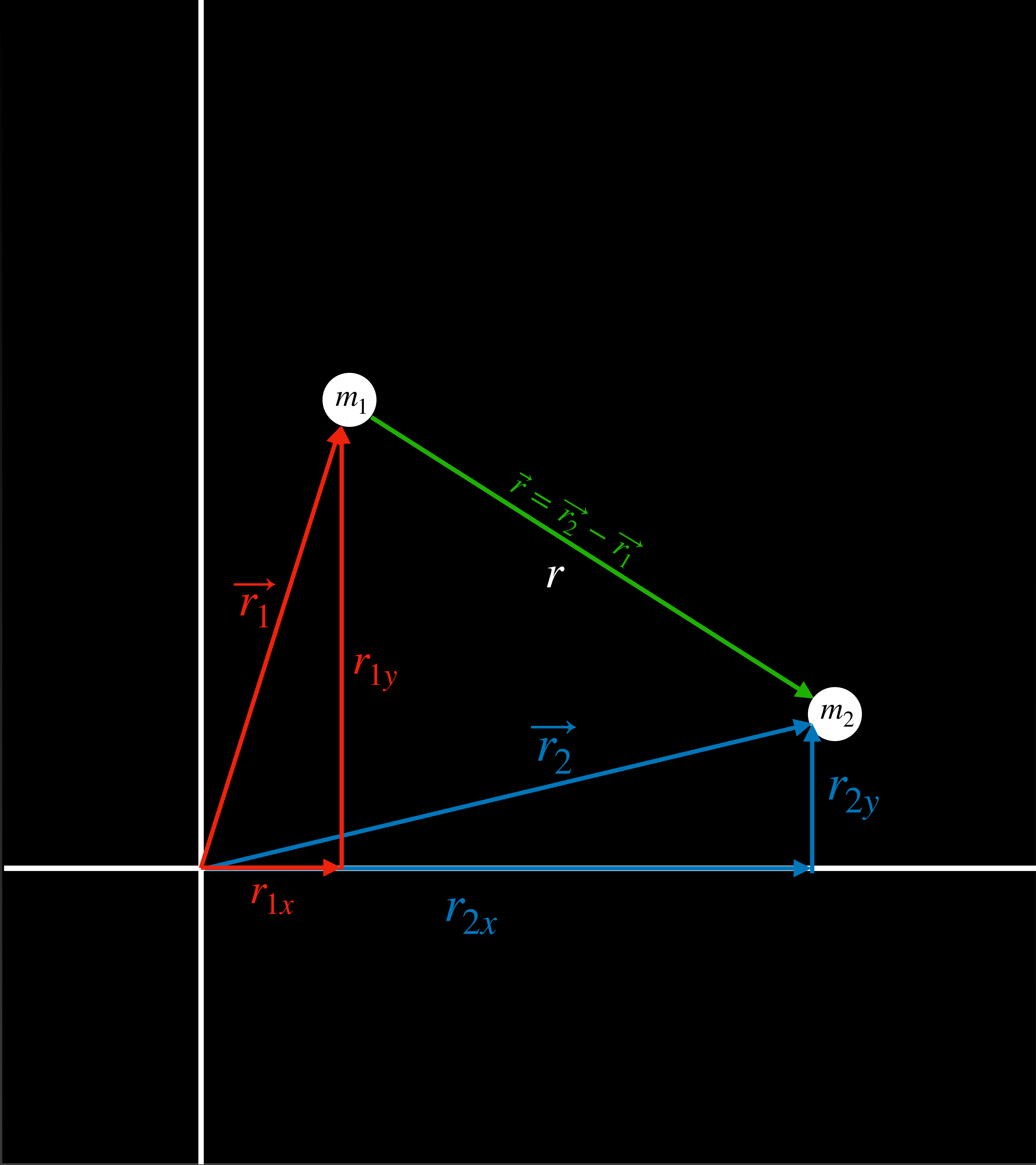
Gravity
Vector Review

- Solving the Two and Three Body Problems will require vectors.
- Let's do a quick review.
  - If we have a vectors $\overrightarrow{r_1}$ and $\overrightarrow{r_2}$ they can be split into their x,y components r_{1x}, r_{1y}, r_{2x}, and r_{2y}.



Gravity
Vector Review

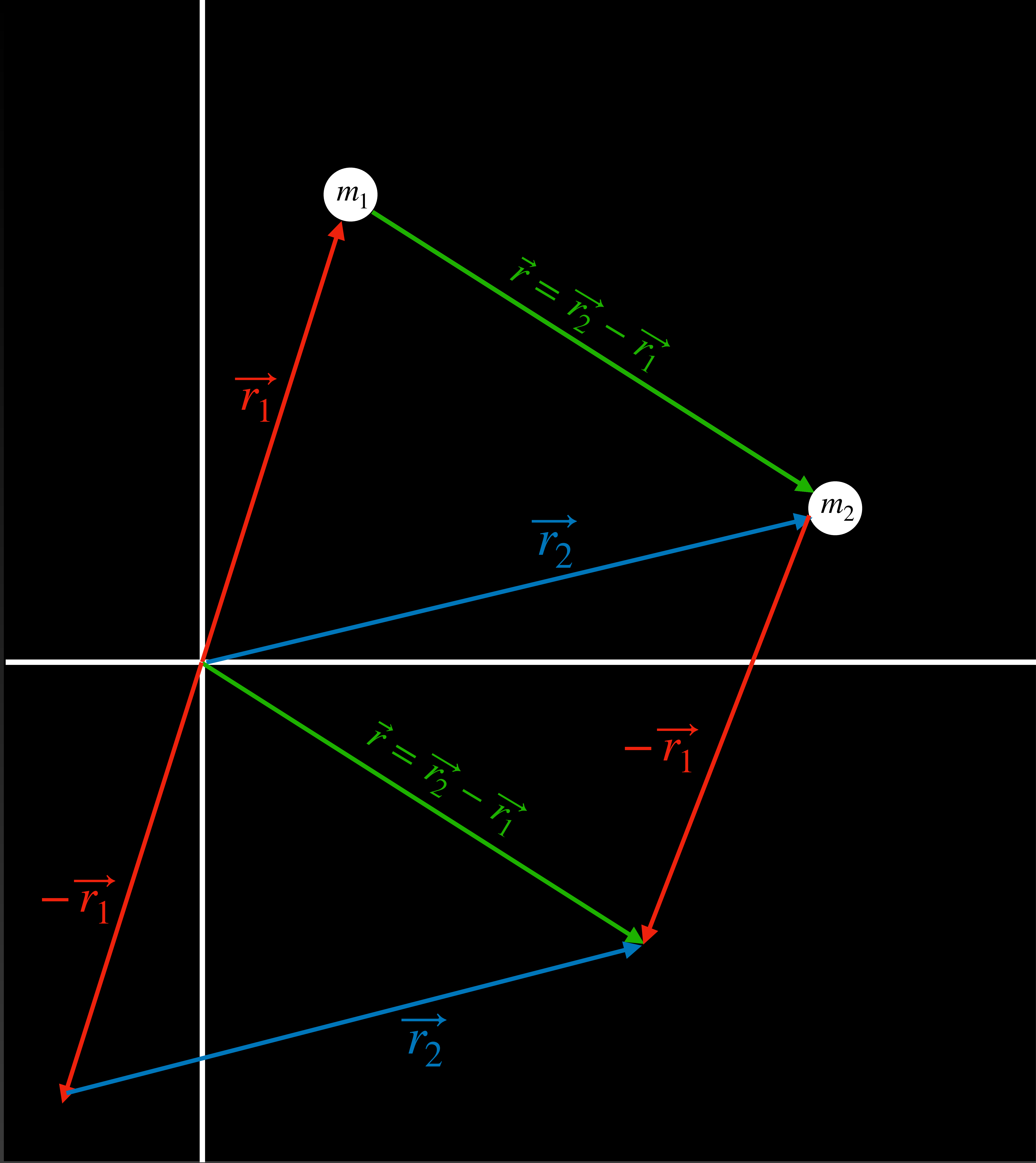
- When we add or subtract vectors like $\vec{r_1}$ and $\vec{r_2}$ the result is the same adding or subtracting their x,y respective components.
- For example, $\vec{r} = \vec{r_2} - \vec{r_1}$, or $r_x = r_{2x} - r_{1x}$ and $r_y = r_{2y} - r_{1y}$.



Gravity

Vector Review

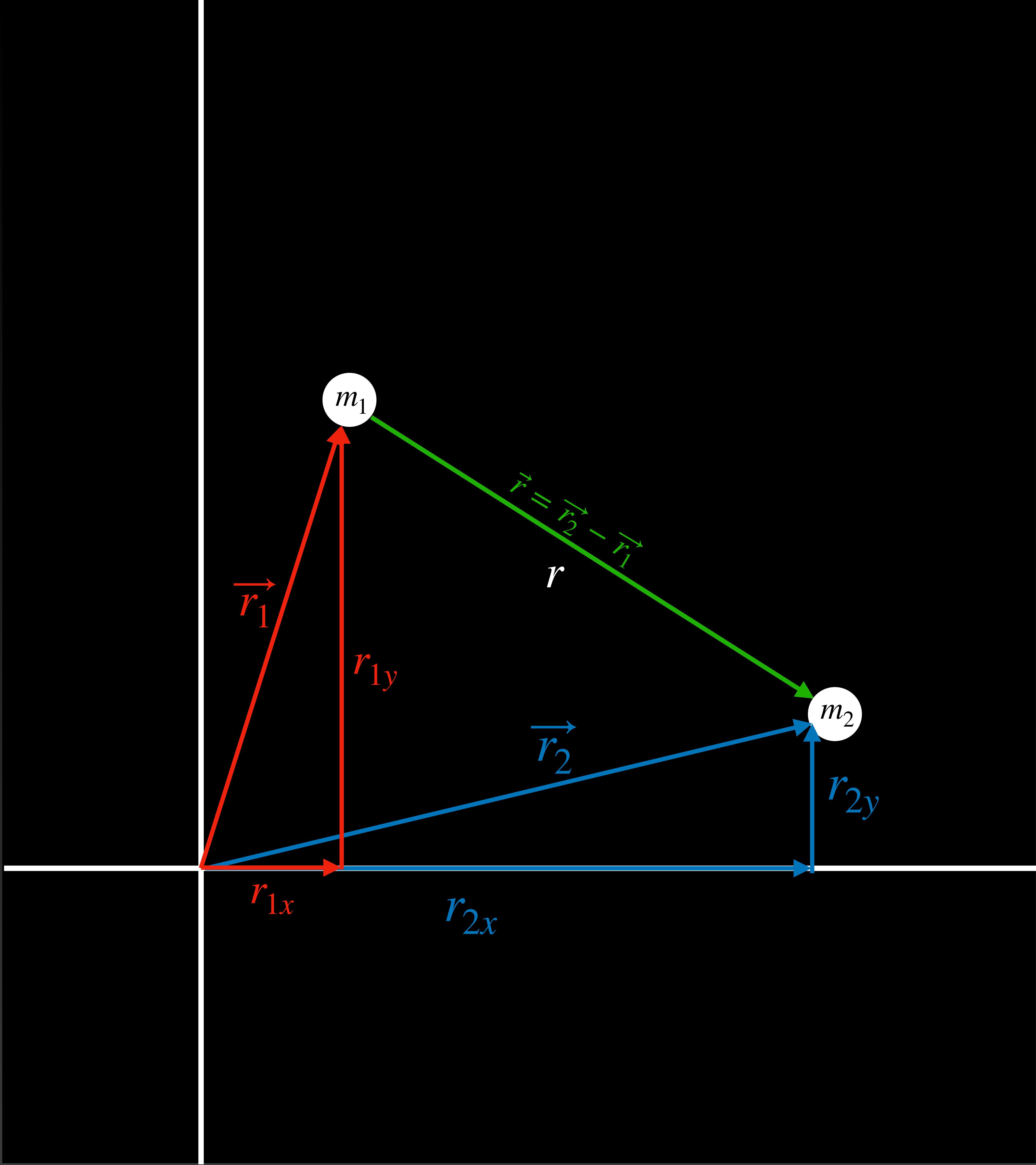
- Vector Addition/Subtraction:
  - place the end of one vector at the tip of the other, $\vec{r} = \vec{r_2} - \vec{r_1}$.



Gravity

Vector Review

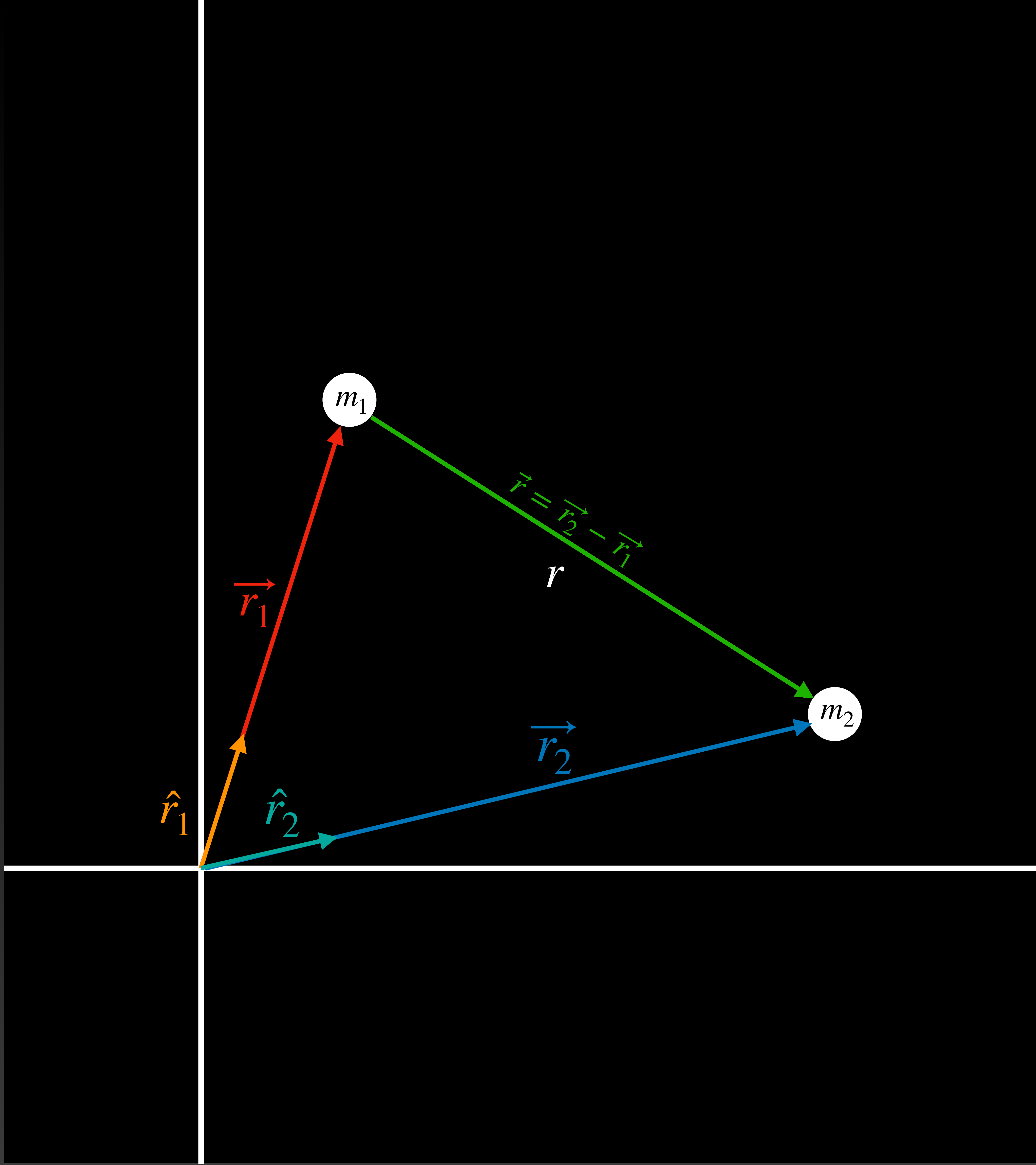
- Vectors have a magnitude and a direction.
- We can obtain the magnitude of the vector by using Pythagoras' Theorem.
- For example, the magnitude of $\vec{r}$, usually written as $|\vec{r}|$ or r, is
$$r = \sqrt{r_x^2 + r_y^2}$$
- For 3D vectors $r = \sqrt{r_x^2 + r_y^2 + r_z^2}$



Gravity

Vector Review

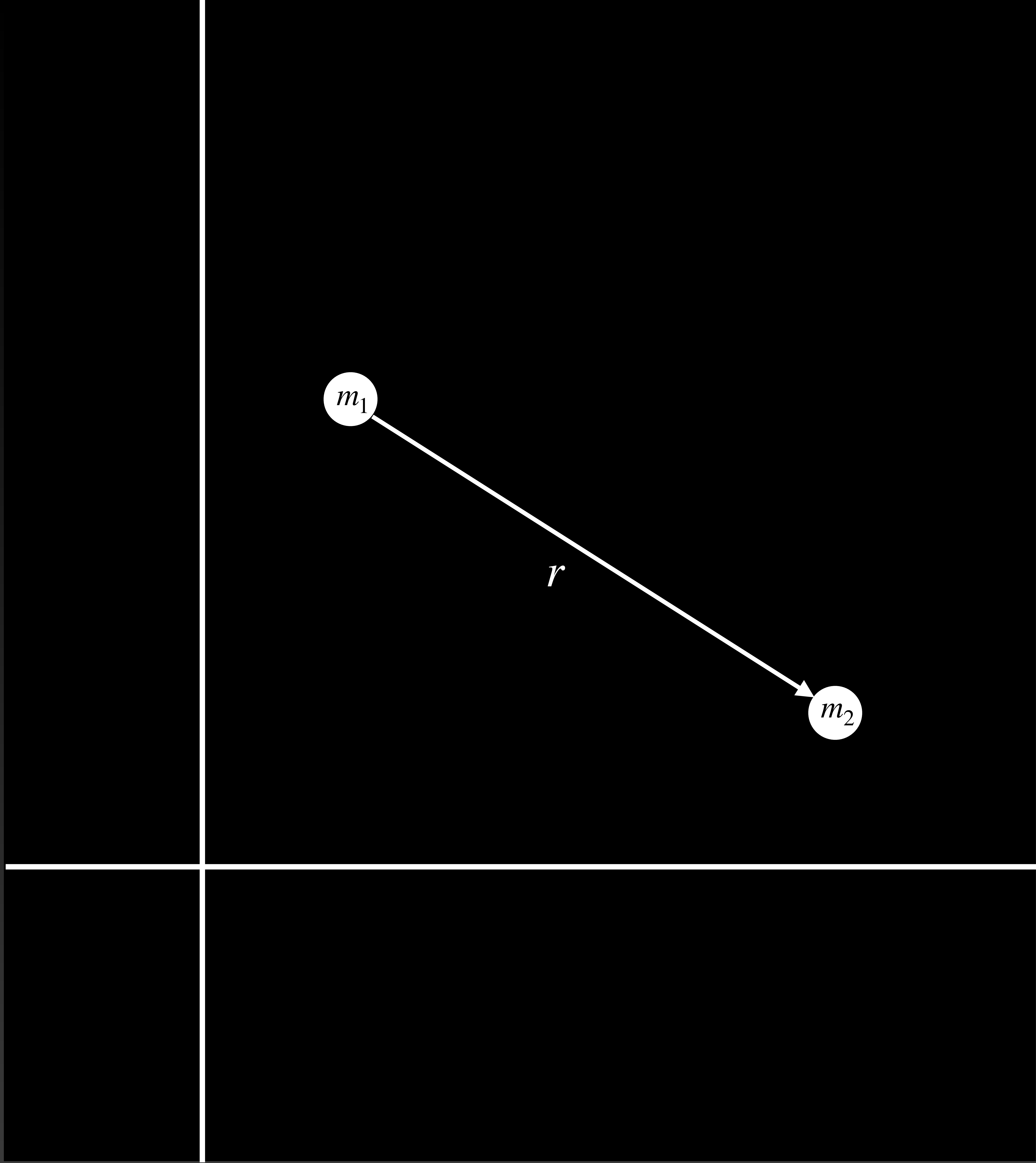
- We can obtain the direction of $\vec{r}$, by calculating its unit vector.
- A unit vector is a vector divided by its magnitude, $\hat{r} = \dfrac{\vec{r}}{r}$, so that the magnitude of the vector is 1, $|\hat{r}| = 1$.
- This process of taking a vector and turning it into a unit vector is called normalization, or "taking the *norm*".



Gravity

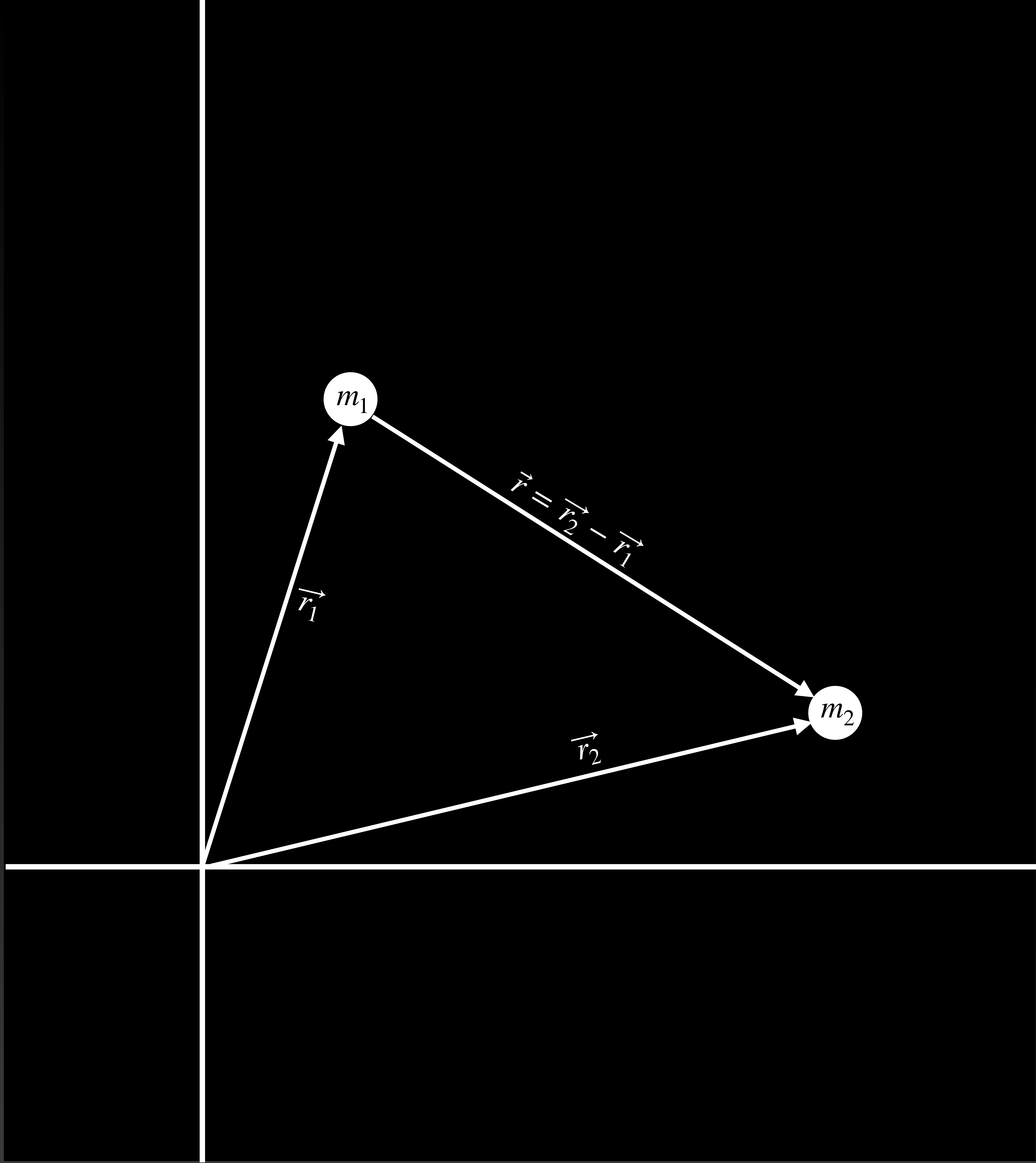
The Two Body Problem

- Every particle attracts every other particle with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.
- $F = -\dfrac{Gm_1m_2}{r^2}$
- Remember, the minus sign indicates that the force is an attractive or restoring force.



Gravity
The Two Body Problem

- With a knowledge of vectors, let us work out how to solve the Two Body Problem.
- We can write Newton's Law of Gravity, this time with vectors.
- $\overrightarrow{F} = -\frac{Gm_1m_2}{r^2}\hat{r}$
- This says that the force $\overrightarrow{F}$ is in the opposite direction of $\hat{r}$.



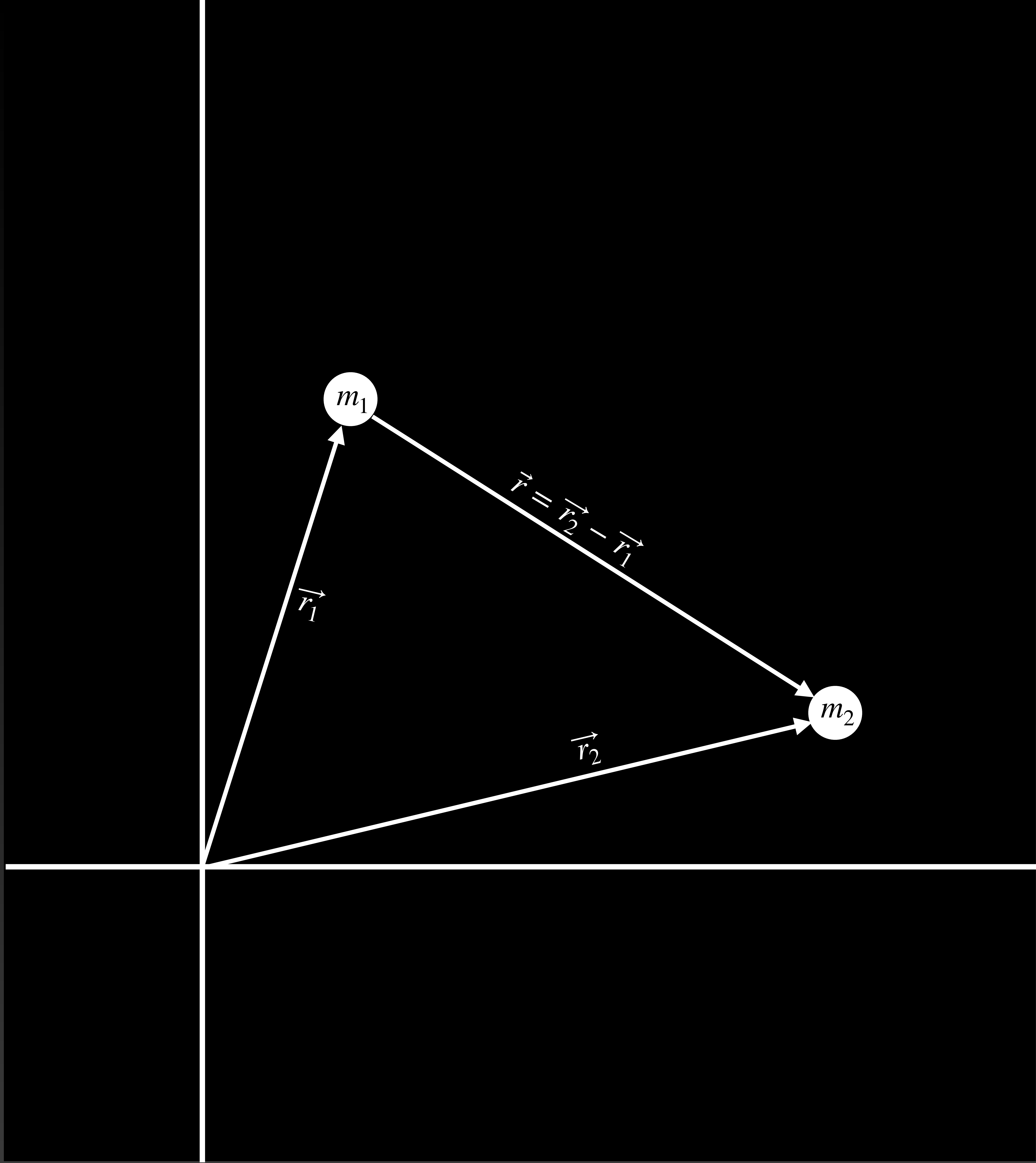
Gravity

The Two Body Problem

- From Newton's 2nd Law we know

  that $\overrightarrow{F} = m\overrightarrow{a}$, or $\overrightarrow{F} = \dfrac{d\overrightarrow{p}}{dt}$

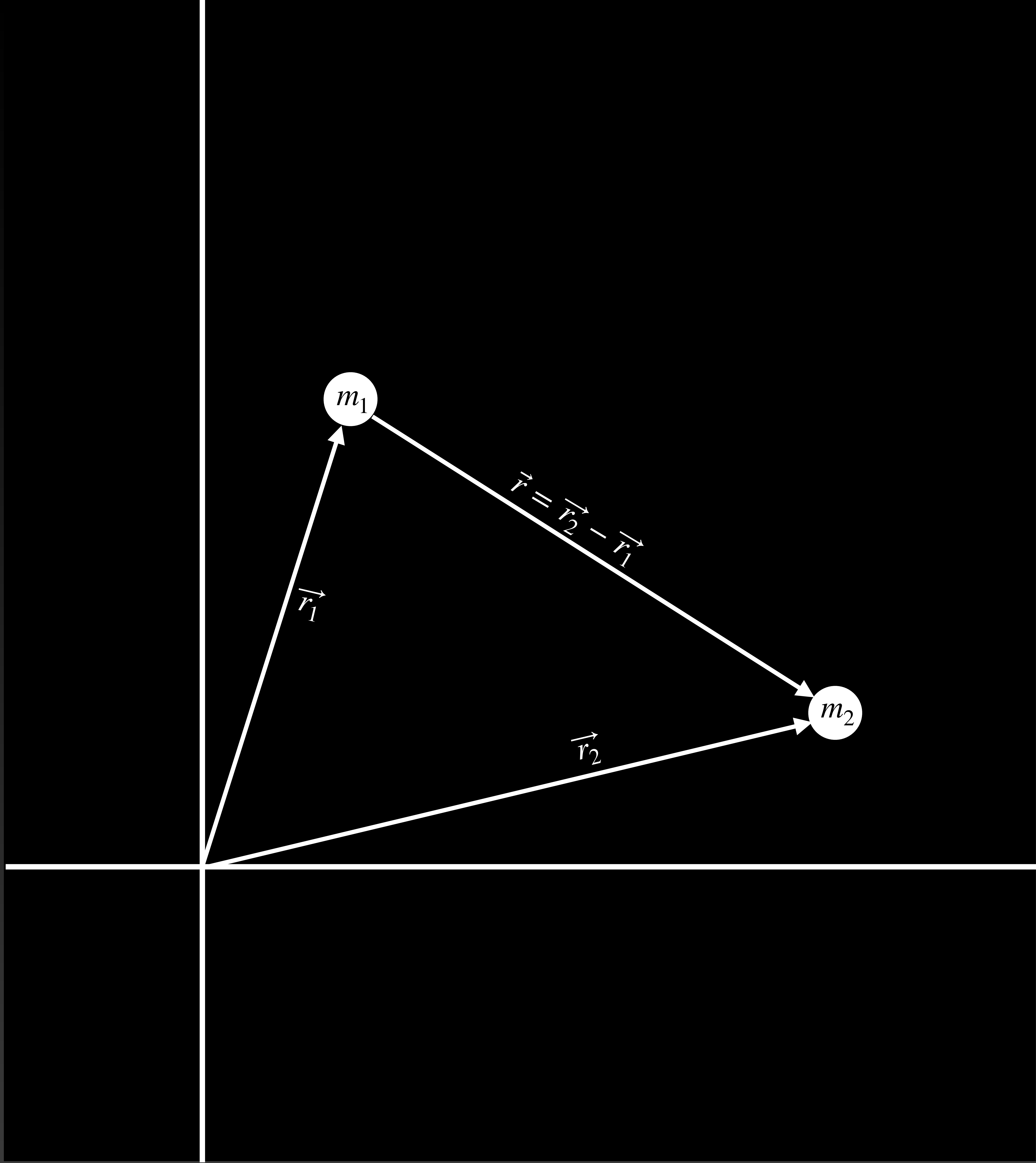
  if we use momentum $\overrightarrow{p} = m\overrightarrow{v}$.



Gravity

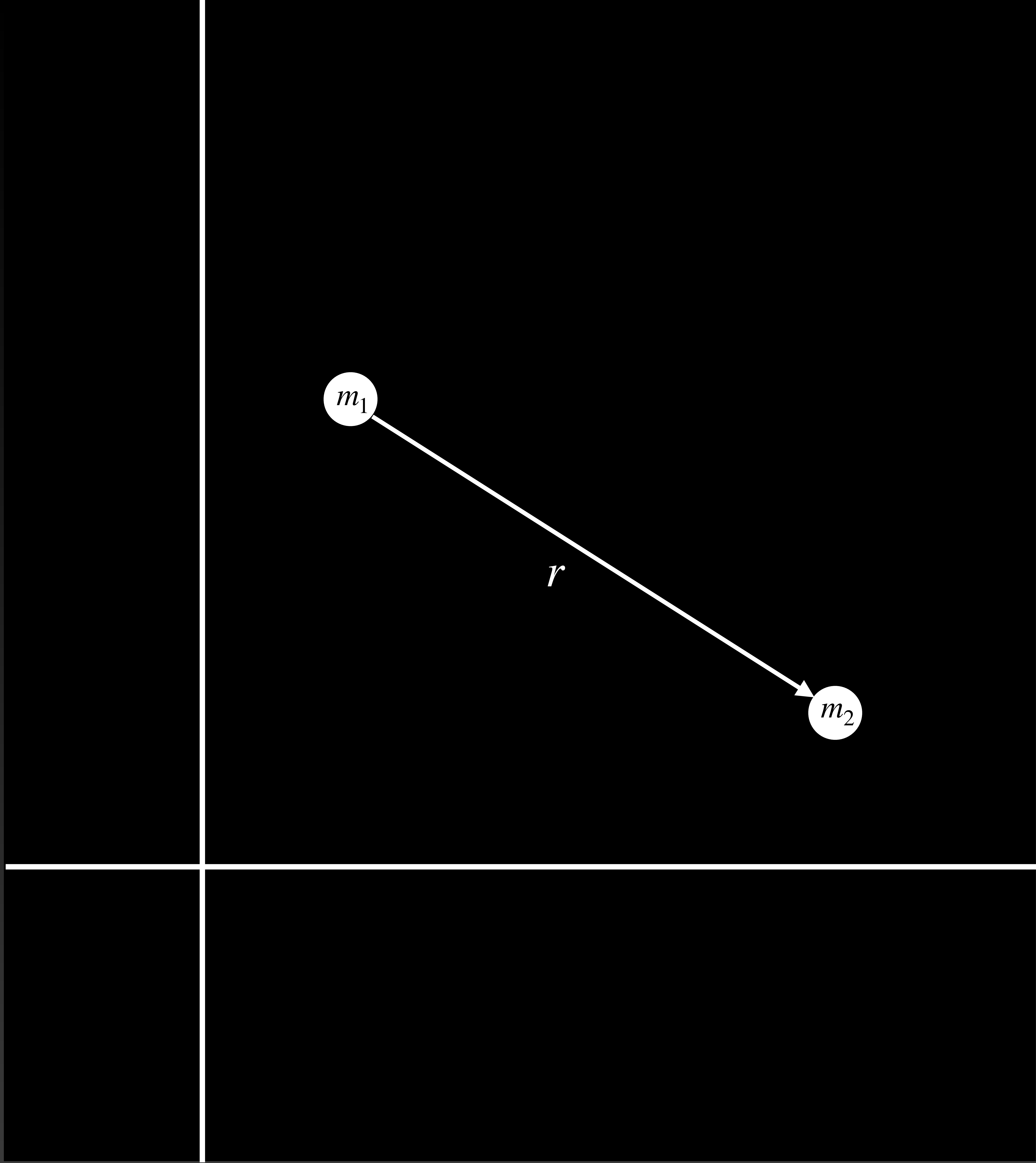
The Two Body Problem

- If we follow the same path as before and rearrange for how the momentum changes due to the force. We have
  - $\dfrac{\Delta \overrightarrow{p}}{\Delta t} = \overrightarrow{F}$
  - $\Delta \overrightarrow{p} = \overrightarrow{F} \Delta t$
  - $\overrightarrow{p}_f - \overrightarrow{p}_i = \overrightarrow{F} \Delta t$
  - $\overrightarrow{p}_f = \overrightarrow{p}_i + \overrightarrow{F} \Delta t$



Gravity
The Two Body Problem

- Using the momentum to update the position and putting it all together for m_1 we have
- $\vec{F}_{12} = -\dfrac{Gm_1m_2}{r_{12}^2}\hat{r}_{12}$
- $\vec{p}_{1_f} = \vec{p}_{1_i} + \vec{F}_{12}\Delta t$
- $\vec{r}_{1_f} = \vec{r}_{1_i} + \dfrac{\vec{p}_{1_f}}{m_1}\Delta t$
- Then do the same for m_2 remembering Newton's 3rd law ($F_{12} = -F_{21}$).



The Two Body Problem
Coding Time!

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- Once again, find the part that says,

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- and code up your solution between the lines.
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The Three Body Problem

Follow Along!

- Go to https://trinket.io/library/trinkets/3b7770e582
- Again, code your solution between the lines,

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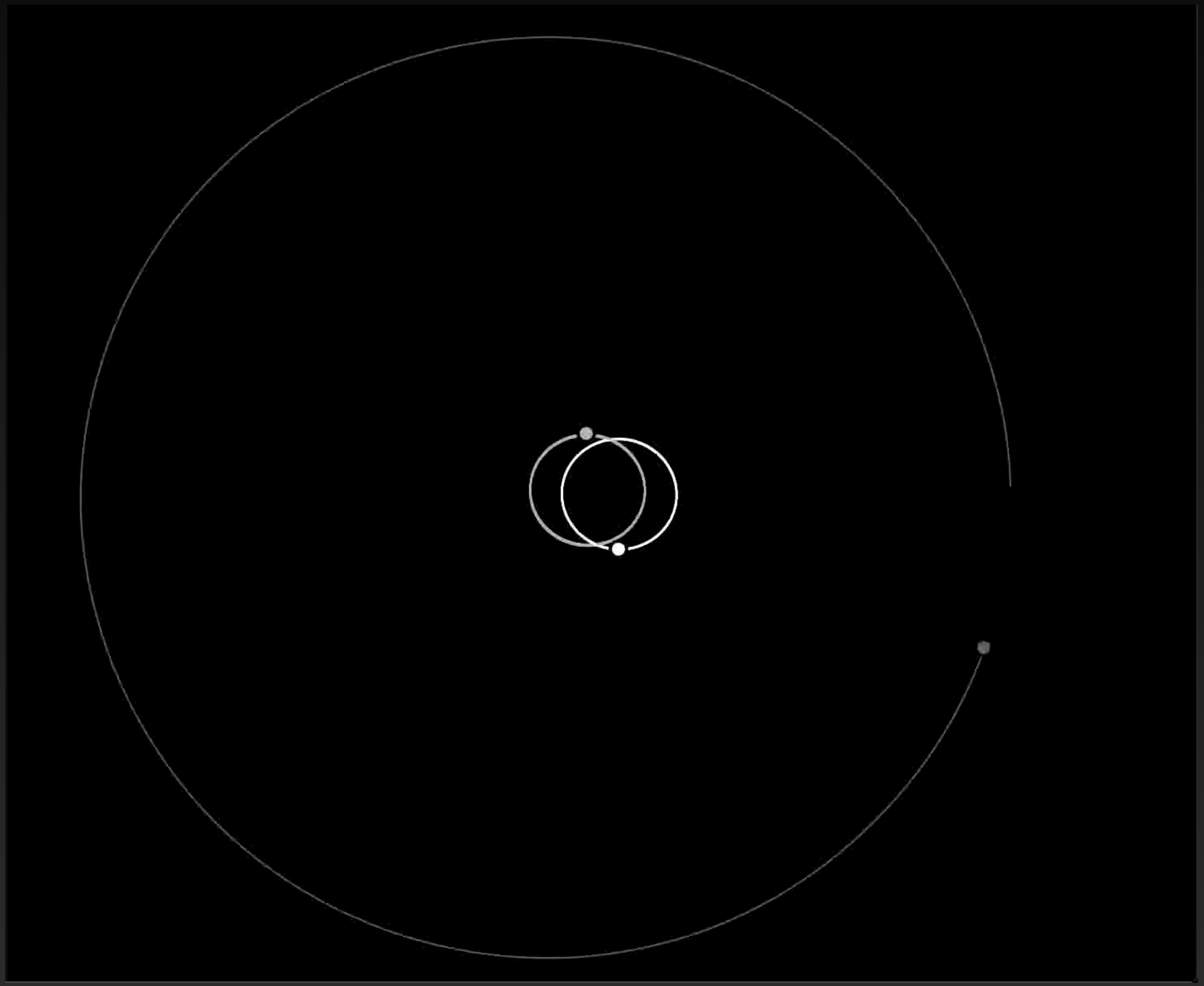


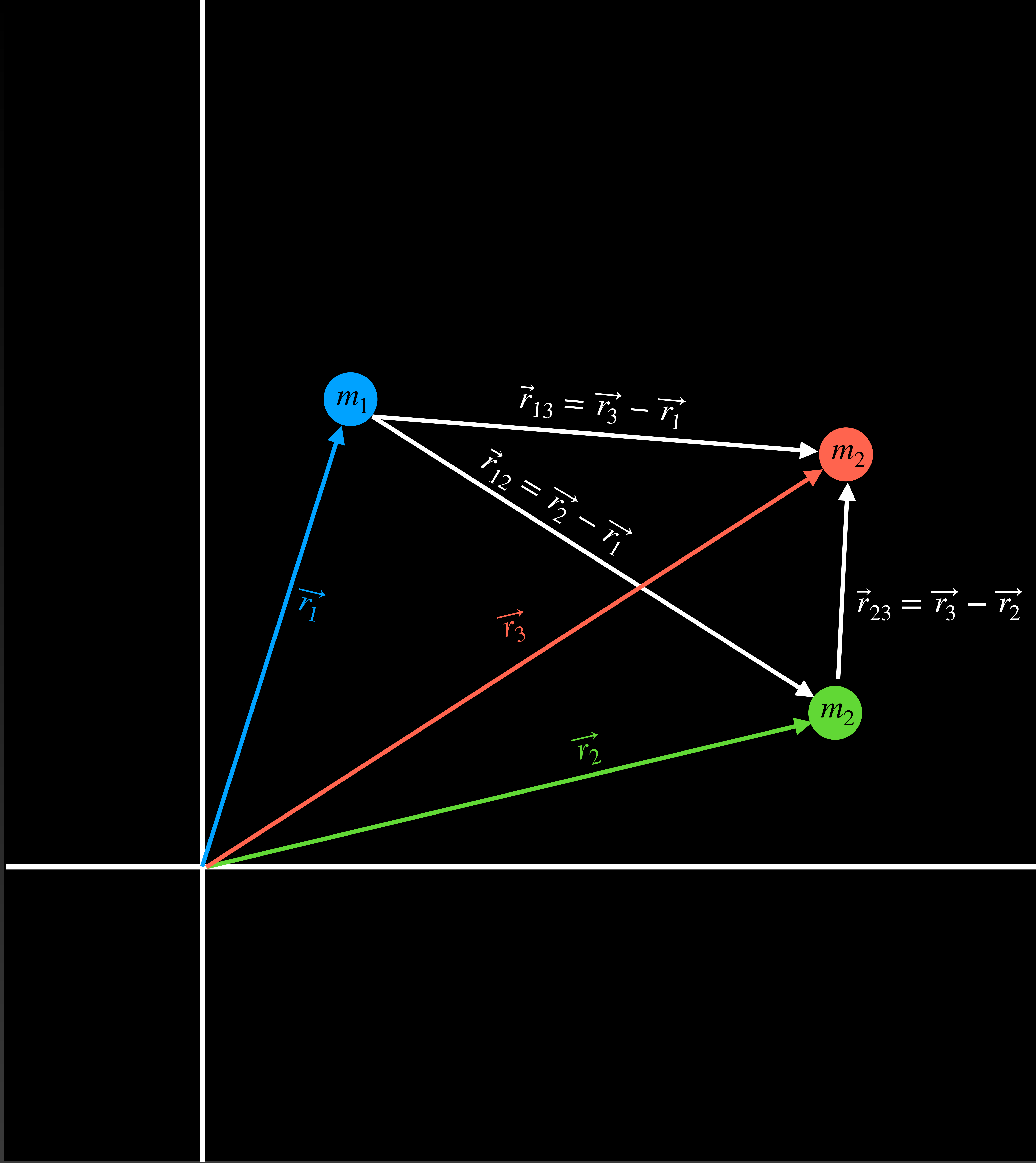
- There are 3 masses this time which means there are 3 pairs of forces.
- The solution can be found at https://trinket.io/library/trinkets/424bbef5df

Gravity

The Three Body Problem

- We follow the same path as before, combining all the forces on m_1 that we have the forces from m_2 and m_3,
- $$\overrightarrow{F}_{12} = -\frac{Gm_1m_2}{r_{12}^2}\hat{r}_{12} \text{ and}$$

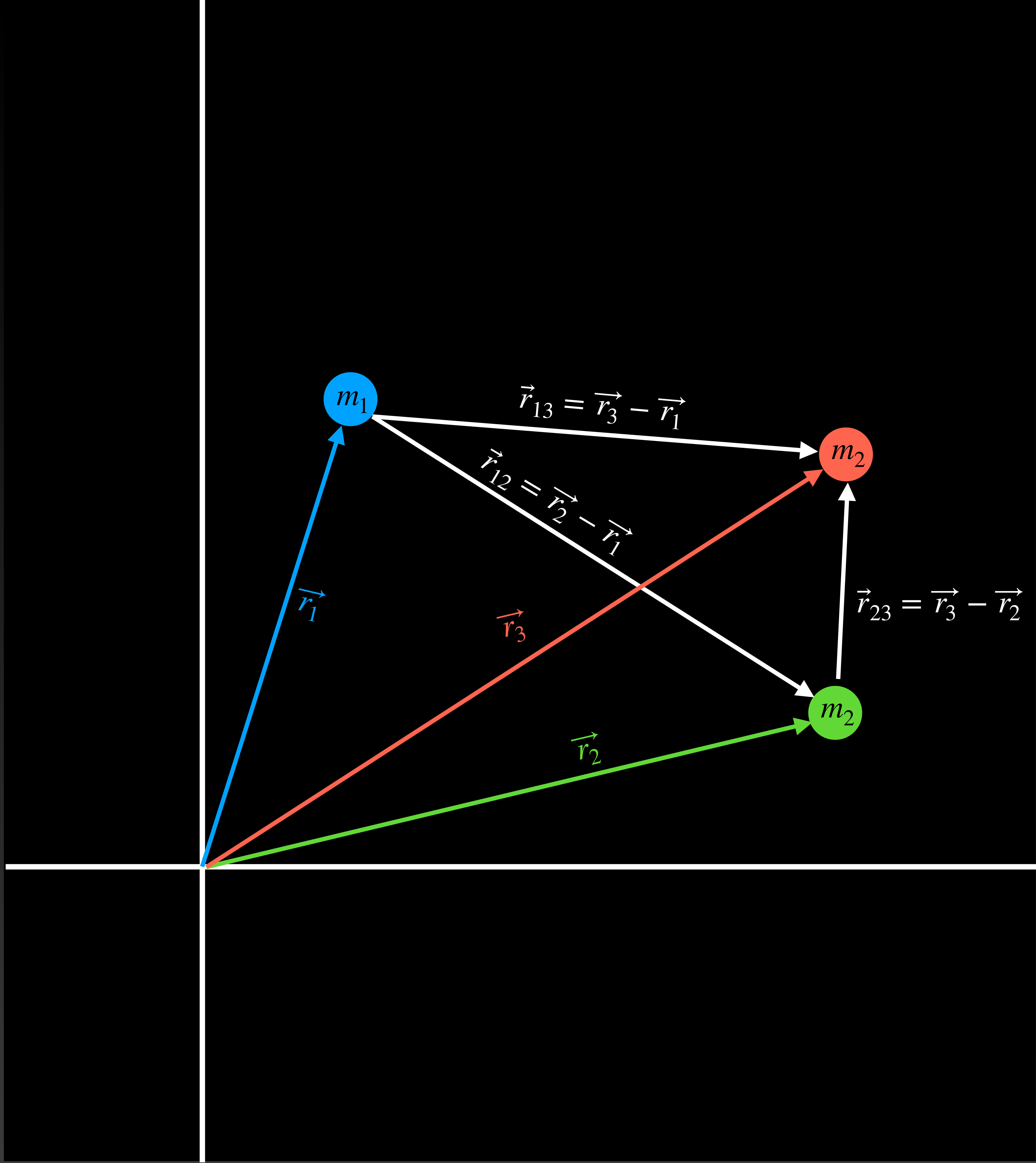
  $$\overrightarrow{F}_{13} = -\frac{Gm_1m_3}{r_{13}^2}\hat{r}_{13}$$
- $$\overrightarrow{p}_{1_f} = \overrightarrow{p}_{1_i} + \left(\overrightarrow{F}_{12} + \overrightarrow{F}_{13}\right)\Delta t$$
- $$\vec{r}_{1_f} = \vec{r}_{1_i} + \frac{\overrightarrow{p}_{1_f}}{m_1}\Delta t$$



Gravity

The Three Body Problem

- Then do the same for m_2 and m_3 remembering Newton's 3rd law ($\overrightarrow{F}_{21} = -\overrightarrow{F}_{12}$ and $\overrightarrow{F}_{31} = -\overrightarrow{F}_{13}$).
- Which is to say that the force of m_1 on m_2 is the opposite to the force of m_1 on m_2.
- But also don't forget the force pair between m_2 and m_3, $\overrightarrow{F}_{32} = -\overrightarrow{F}_{23}$

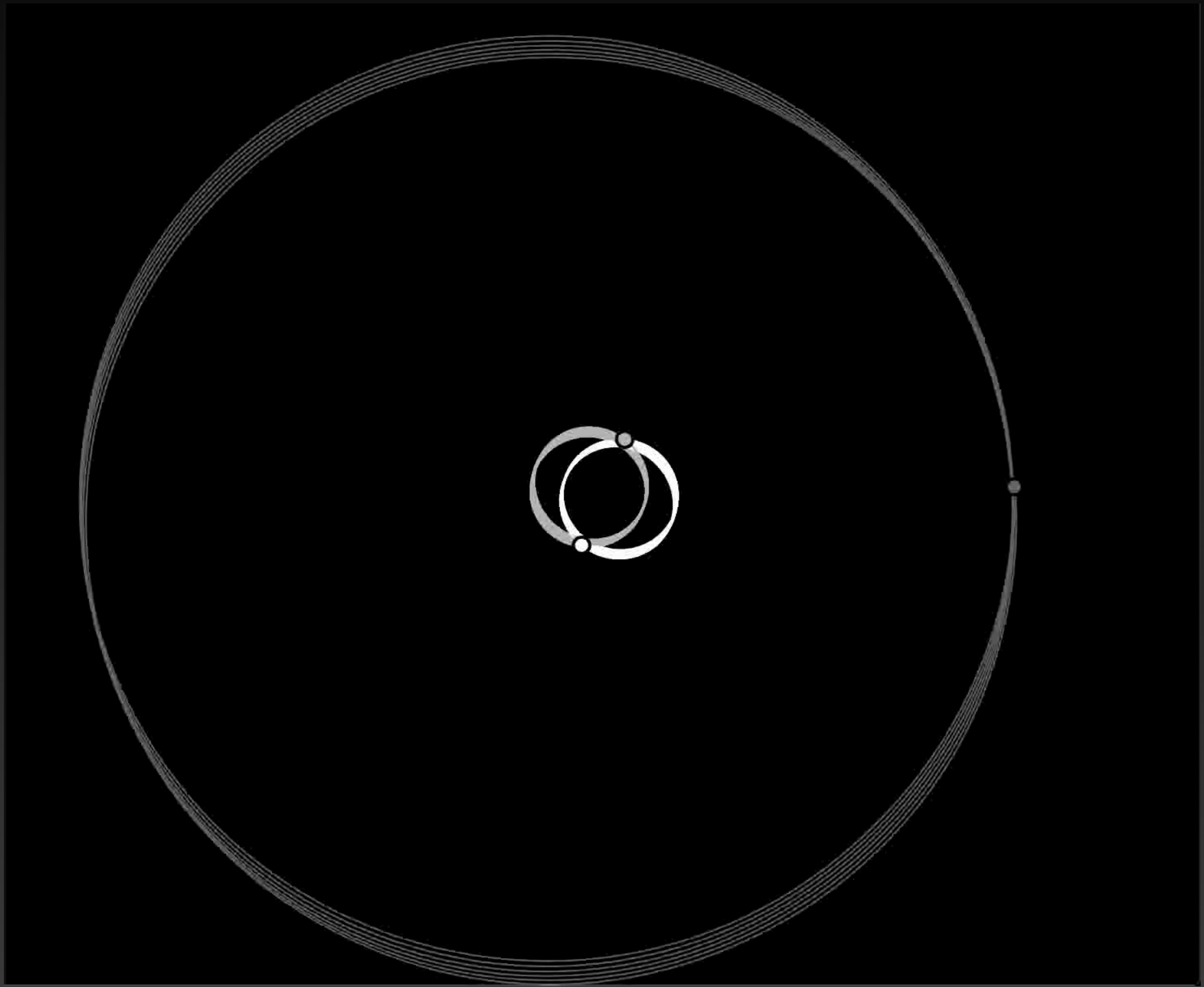


The Three Body Problem

Coding Time!

- Template: https://trinket.io/library/trinkets/3b7770e582

- Solution: https://trinket.io/library/trinkets/424bbef5df

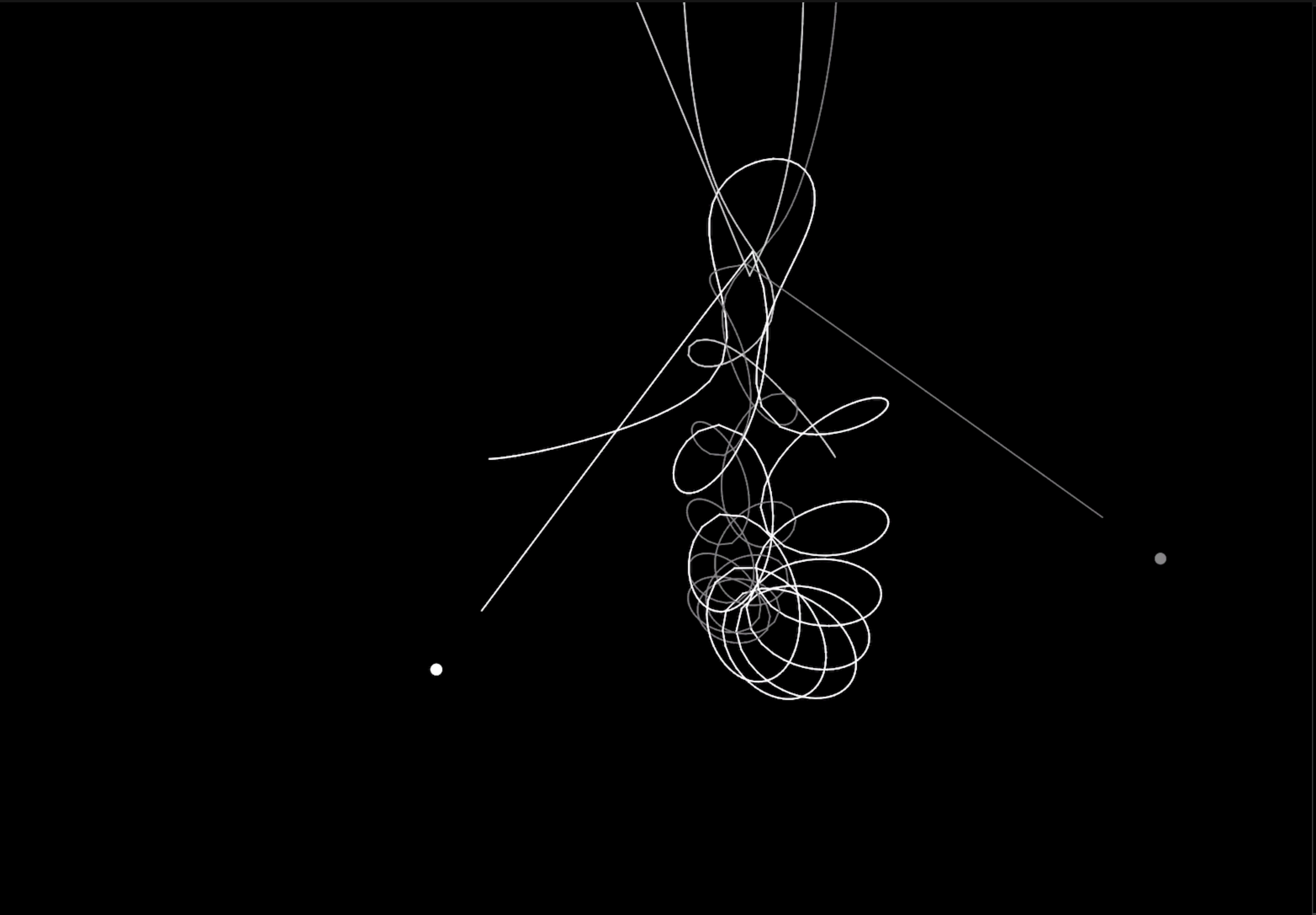


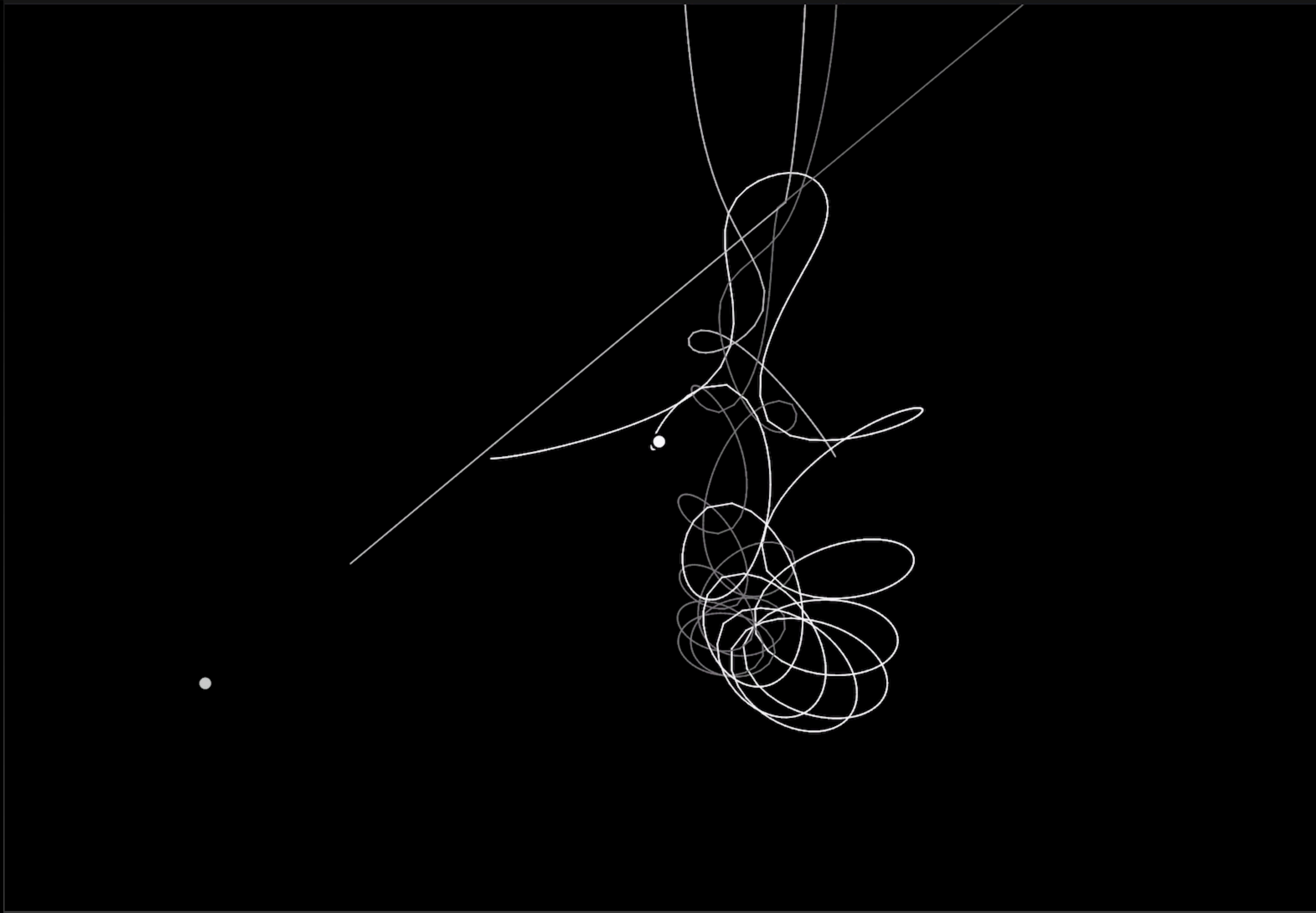
Chaos

- Small changes to the initial conditions can give very different results.
- The motion is determined, but not unpredictable.
- Small differences in initial conditions, such as errors in measurements or rounding errors in numerical computation, can yield widely diverging outcomes.

Chaos

- If we start the three body system with two almost identical initial conditions, after a short time their paths diverge.
- Play around with this using your code.





Thank you

Happy Coding

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