Coding Physics

Gravity and the Three Body Problem

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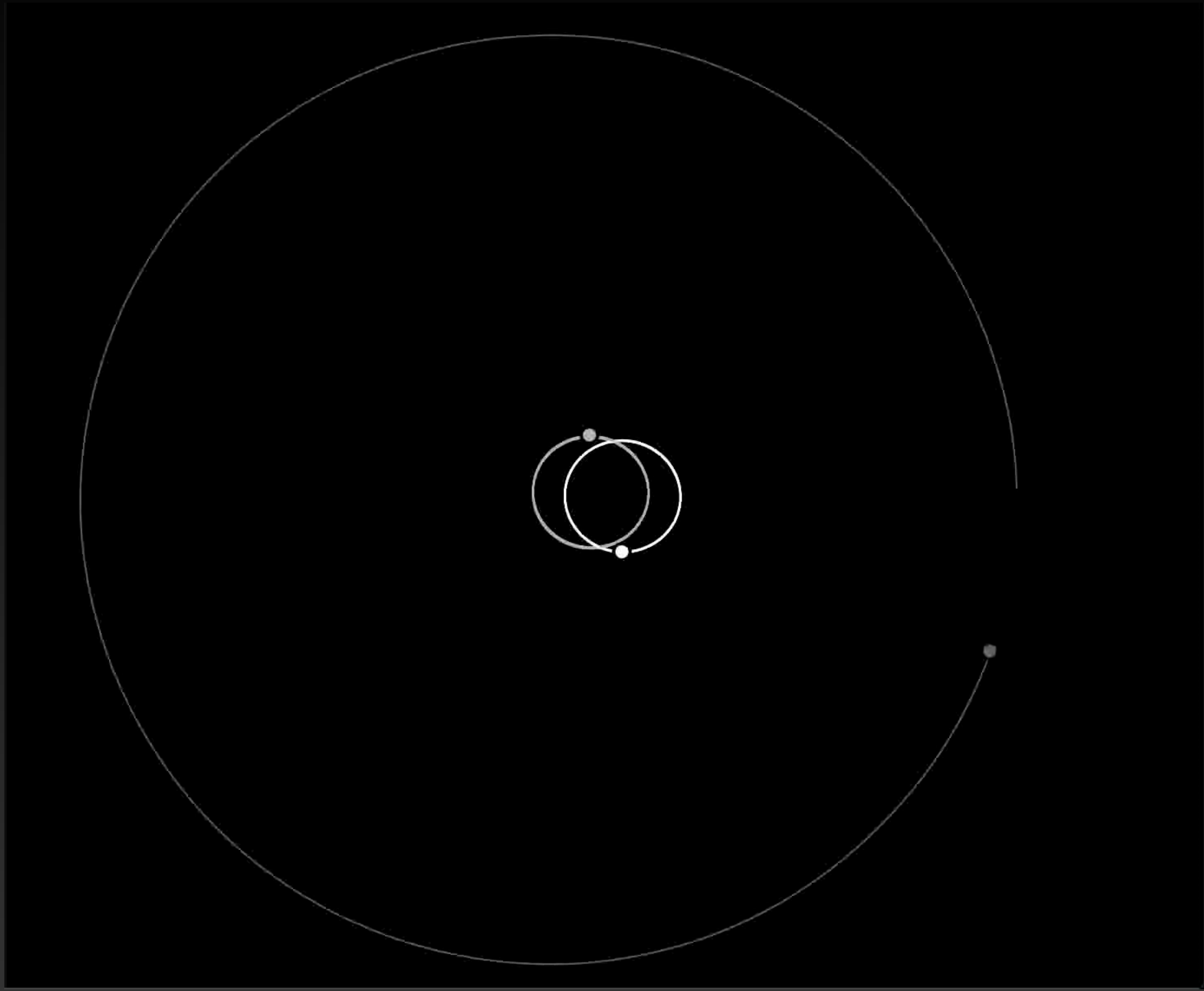
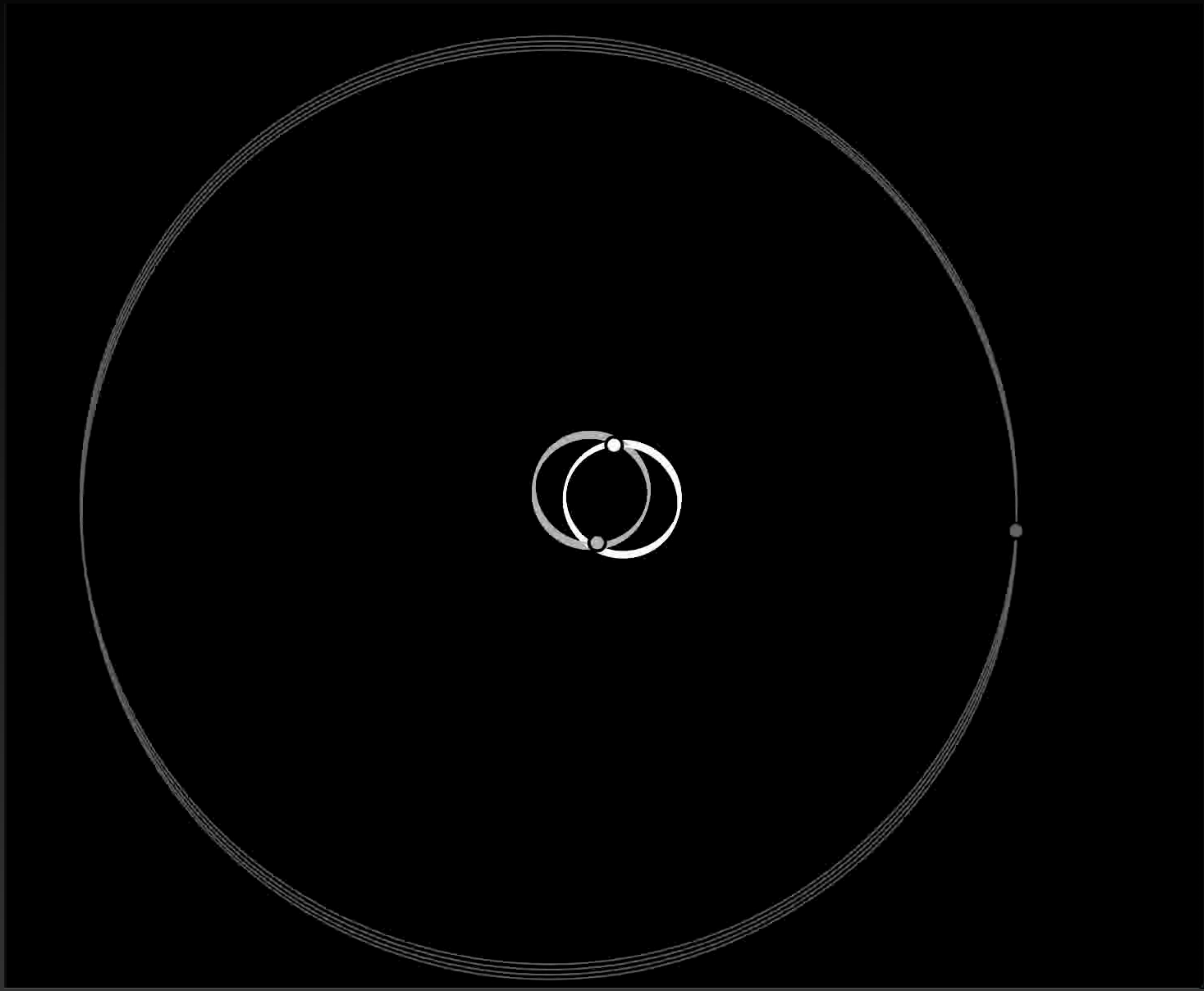
Why should we solve Physics Problems with Computer Code?

- Some experiments cannot be done in a physical lab.
  - Simulating the universe or the planets.
- Some experiments are too costly to practically run thousands of times.
- Machine Learning methods and AI can help us see new patterns



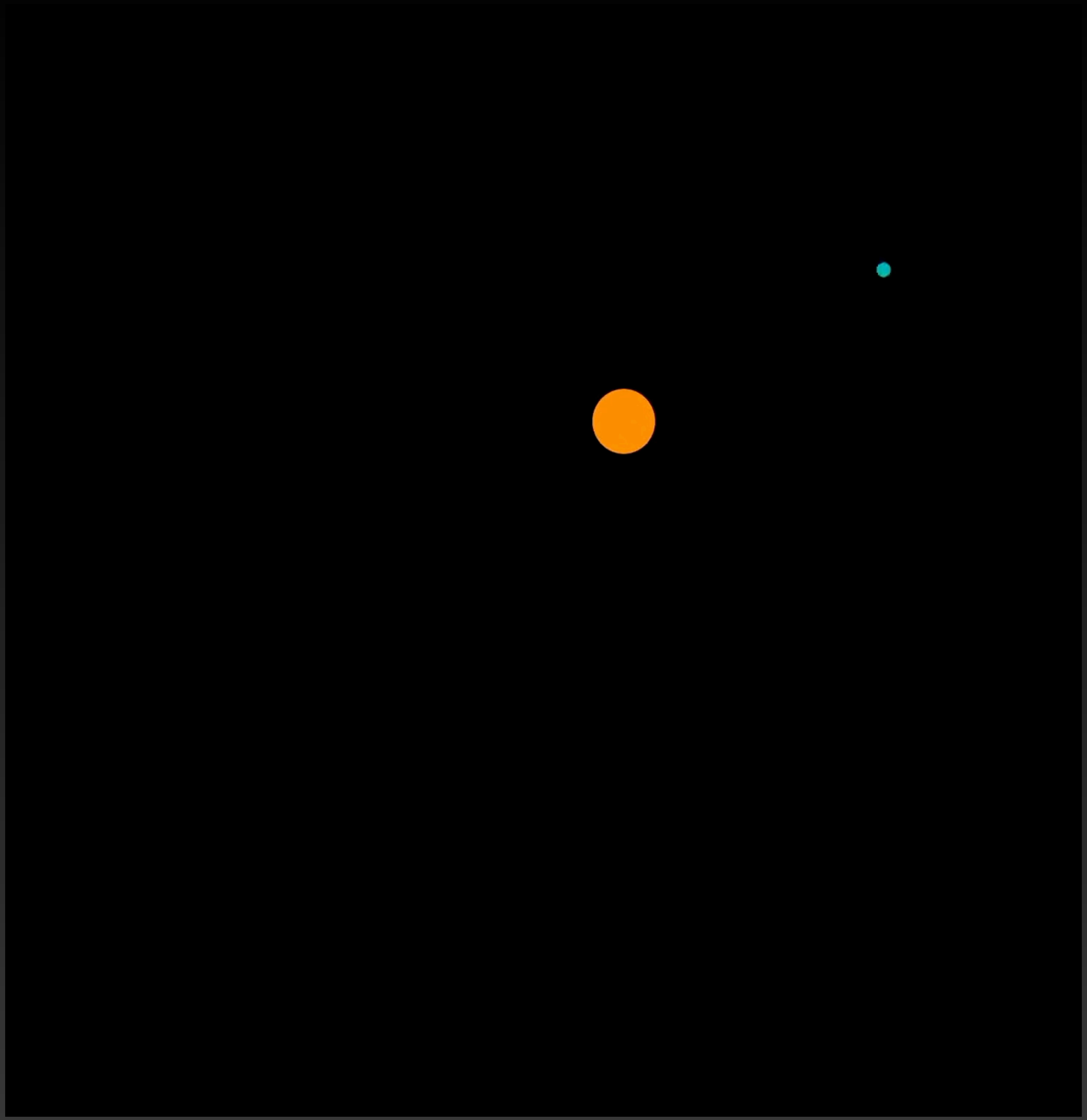
How do we solve Physics Problems with Computer Code?

- First, we need to be able to describe the motion.
- For many problems this can be done with Newton's 2nd law.
    - $F = ma$
- We will do this for the 2 and 3 body problems involving Newton's Law of Gravity.



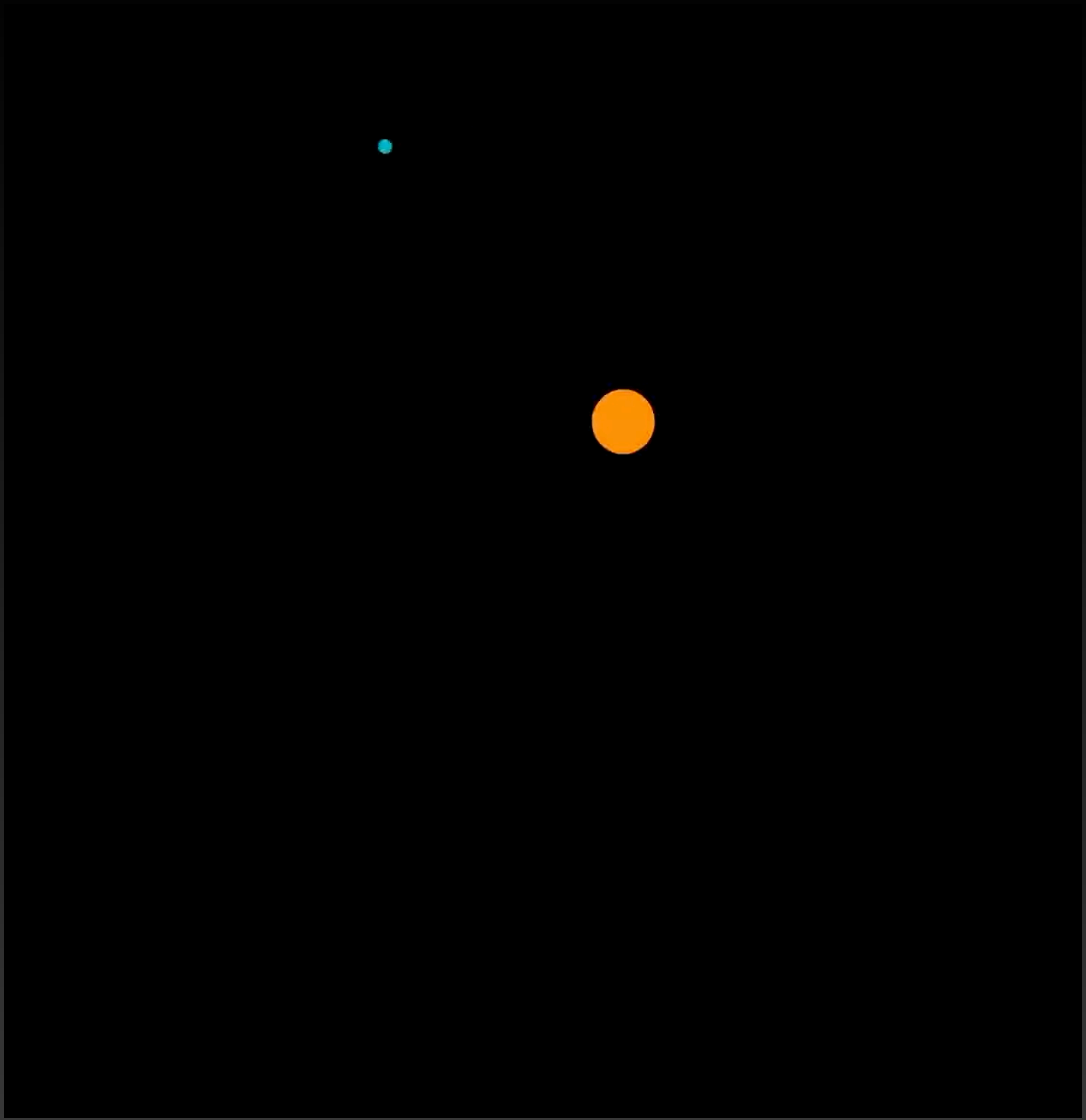
The Two Body Problem

- The two body problem is the only gravitational problem that can be solved analytically, making it useful to study.
- Although, the solution can be extended to some special cases of the three body problem.
- We will not solve the problem analytically, but we will use it as a stepping stone to the three body problem.



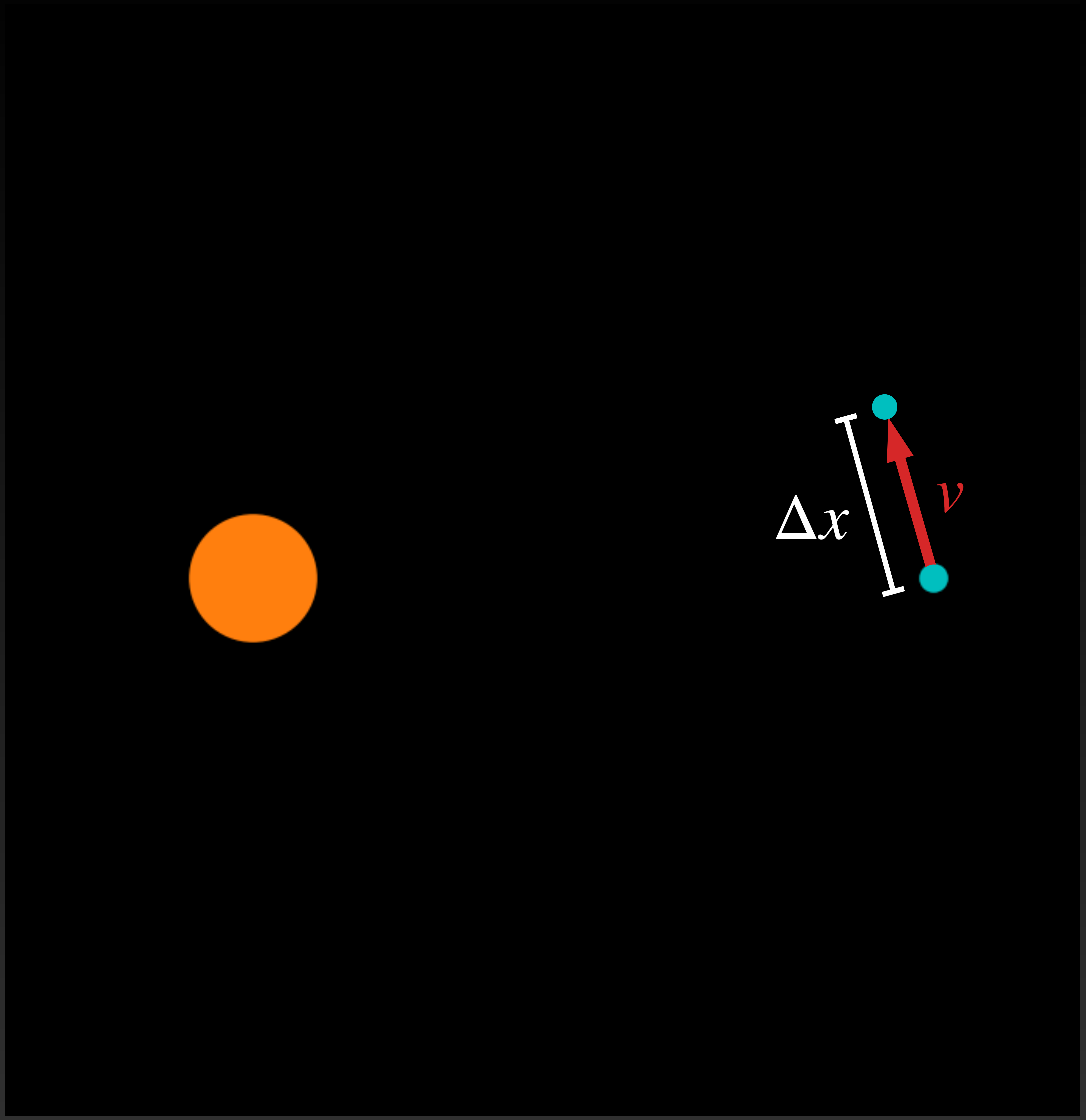
How do we solve Physics Problems with Computer Code?

- If we know the location (position) of an object, how can we know where it will go next?
- We can determine where it will be next if we know the velocity of the object.



How do we solve Physics Problems with Computer Code?

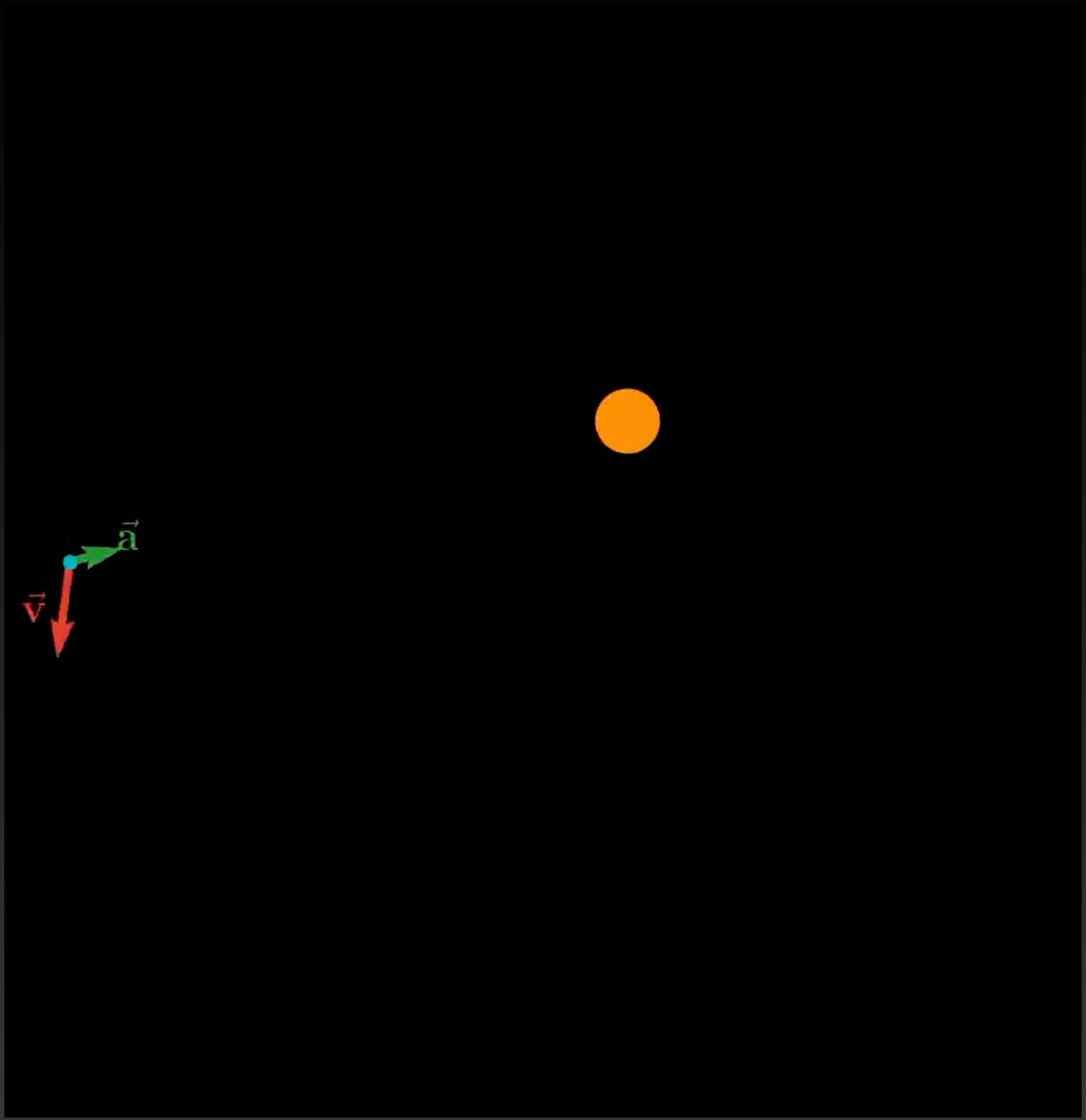
- The position changes as the object moves.
- Velocity is the change in position over the change in time.
  - $v = \frac{x_f - x_i}{t_f - t_i} = \frac{\Delta x}{\Delta t}$
- We can use this information to calculate where it will be next.
  - $\Delta x = v \Delta t$
- But what if the velocity is changing too?



Δ is the Greek letter "Delta"
Often used to mean "change in" or "difference"

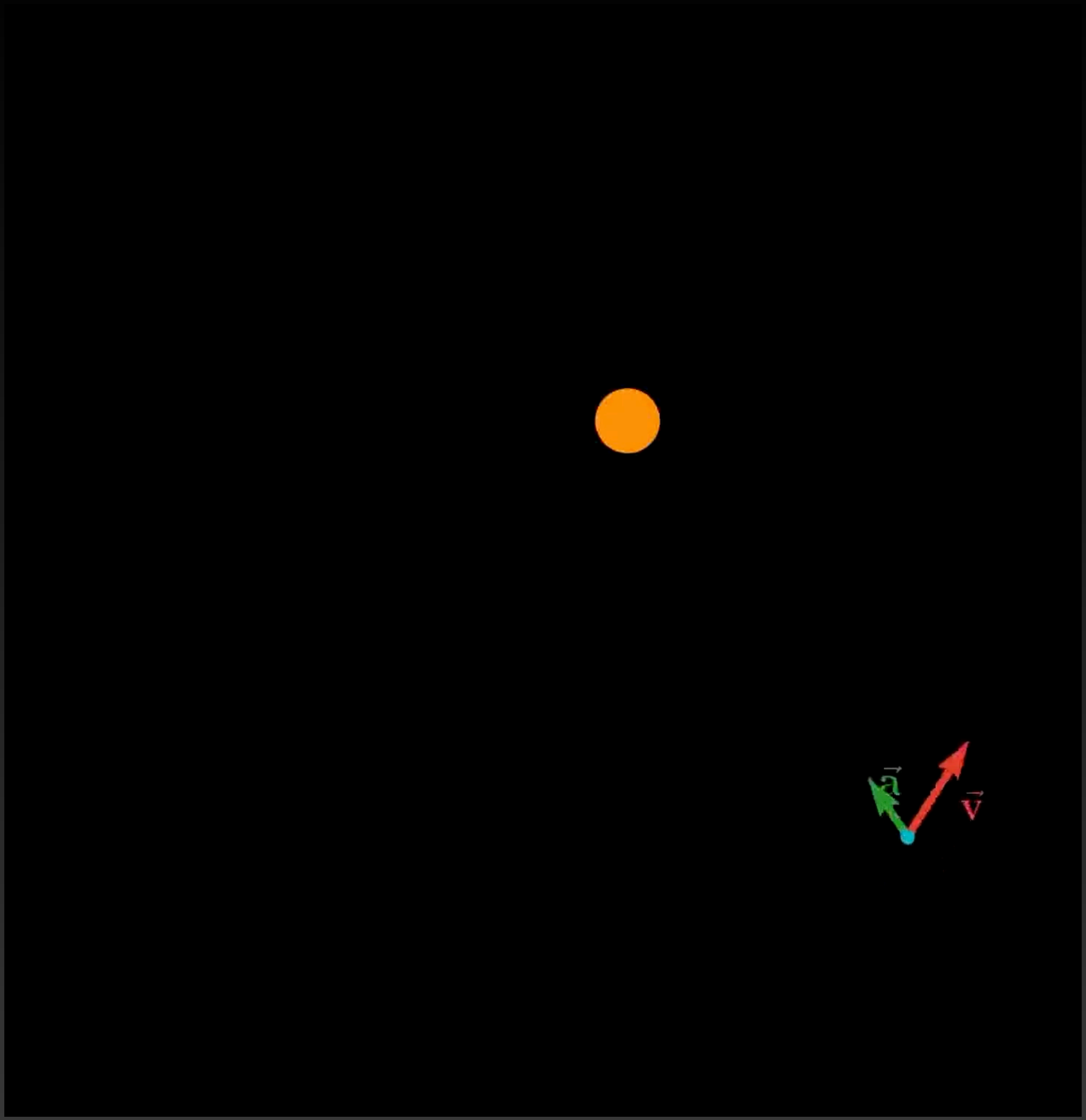
How do we solve Physics Problems with Computer Code?

- If the velocity is also changing, then we can determine how the velocity changes based on the acceleration.
- This is because acceleration is the change in velocity over the change in time.
- $$a = \frac{v_f - v_i}{t_f - t_i} = \frac{\Delta v}{\Delta t}$$



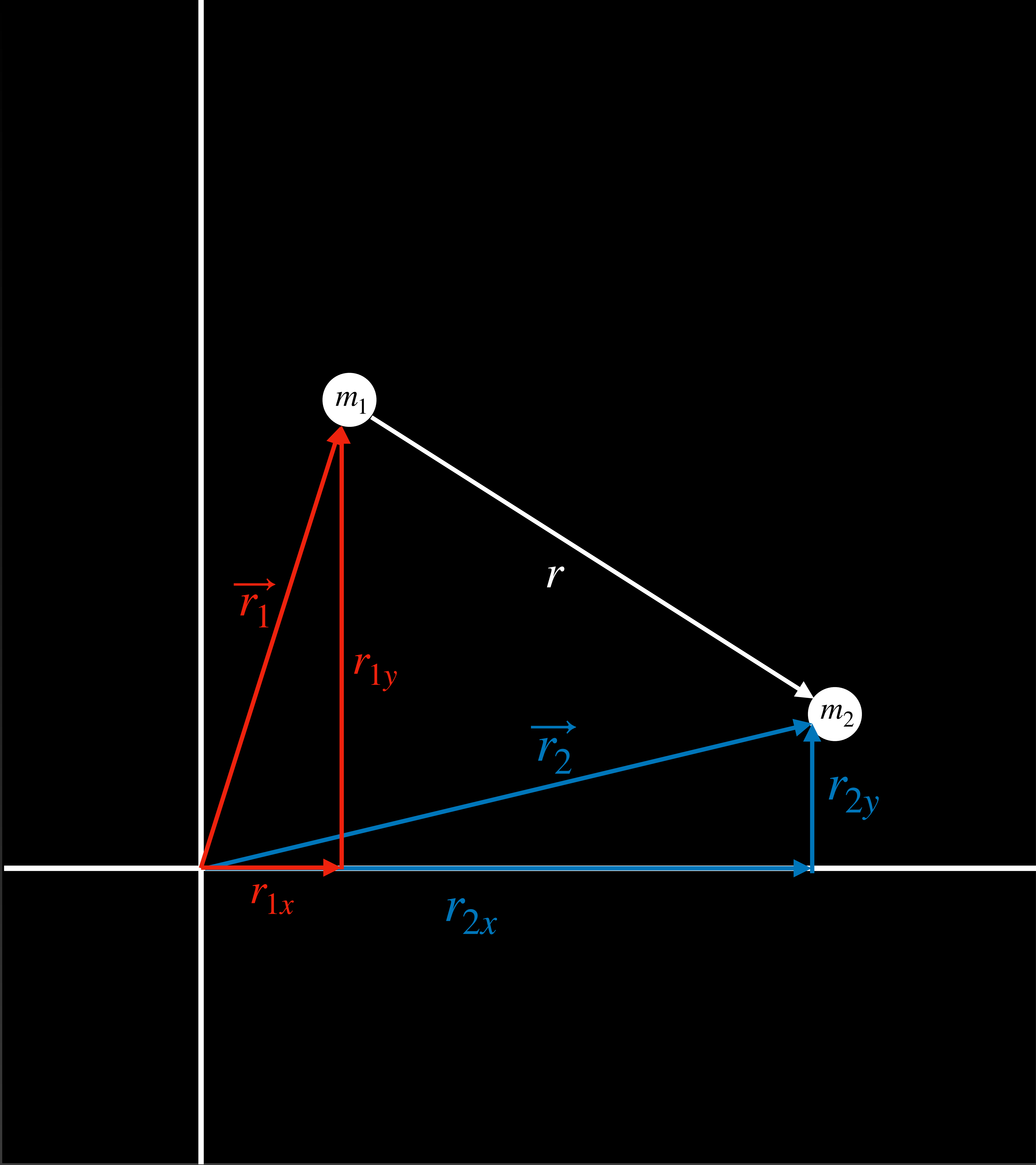
How do we solve Physics Problems with Computer Code?

- But we come full circle.
- The change in position tells us the velocity and the change in velocity tells us the acceleration, but we don't know where the object will be unless we first know the velocity and the acceleration.
- Thus, we use Newton's 2nd Law of motion to first determine the acceleration of the object.
- But first…



Gravity
Vector Review

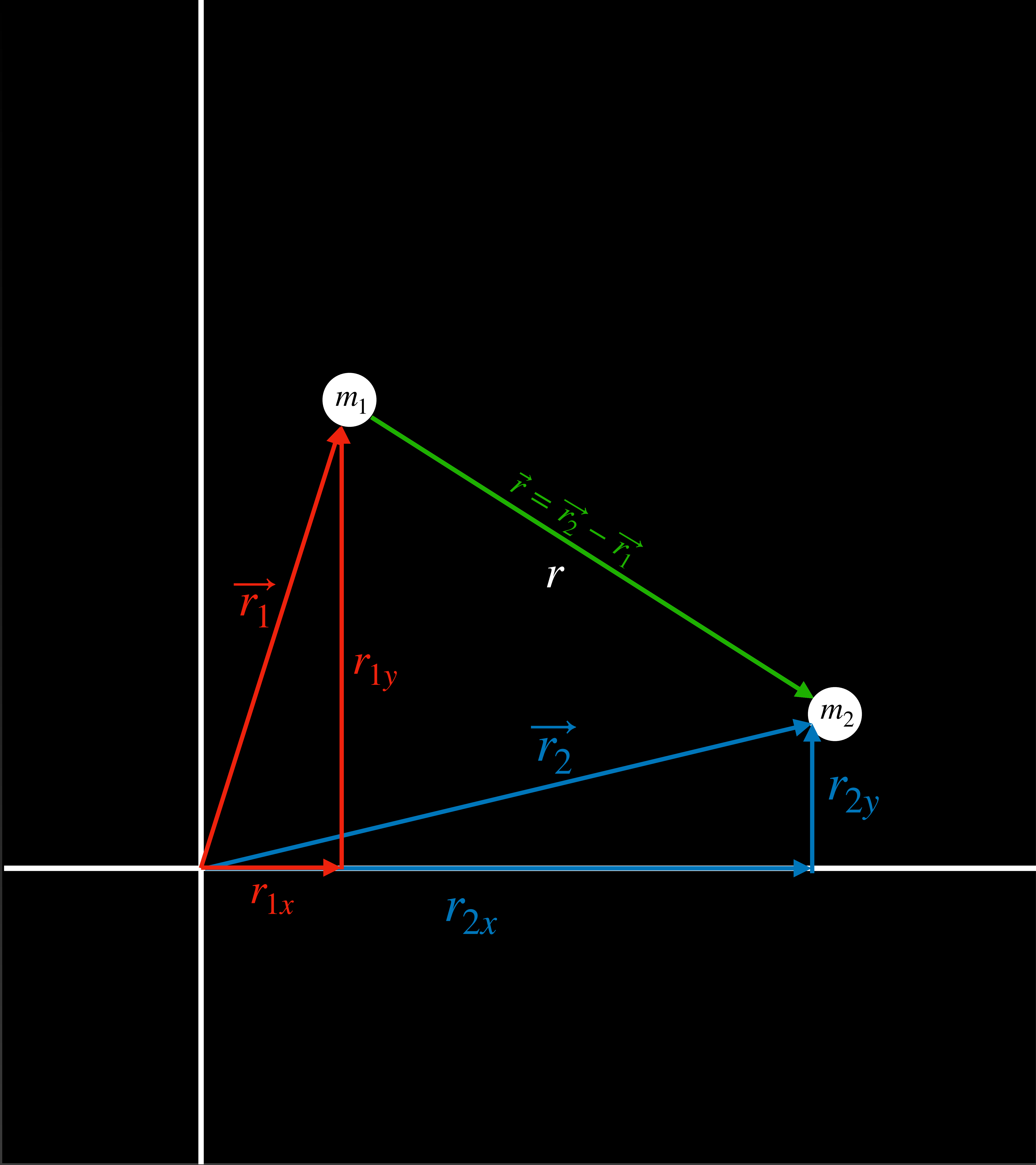
- Solving the Two and Three Body Problems will require vectors.
- Let's do a quick review.
  - If we have vectors $\overrightarrow{r_1}$ and $\overrightarrow{r_2}$ they can be split into their x, y components r_{1x}, r_{1y}, r_{2x}, and r_{2y}.



Gravity

Vector Review

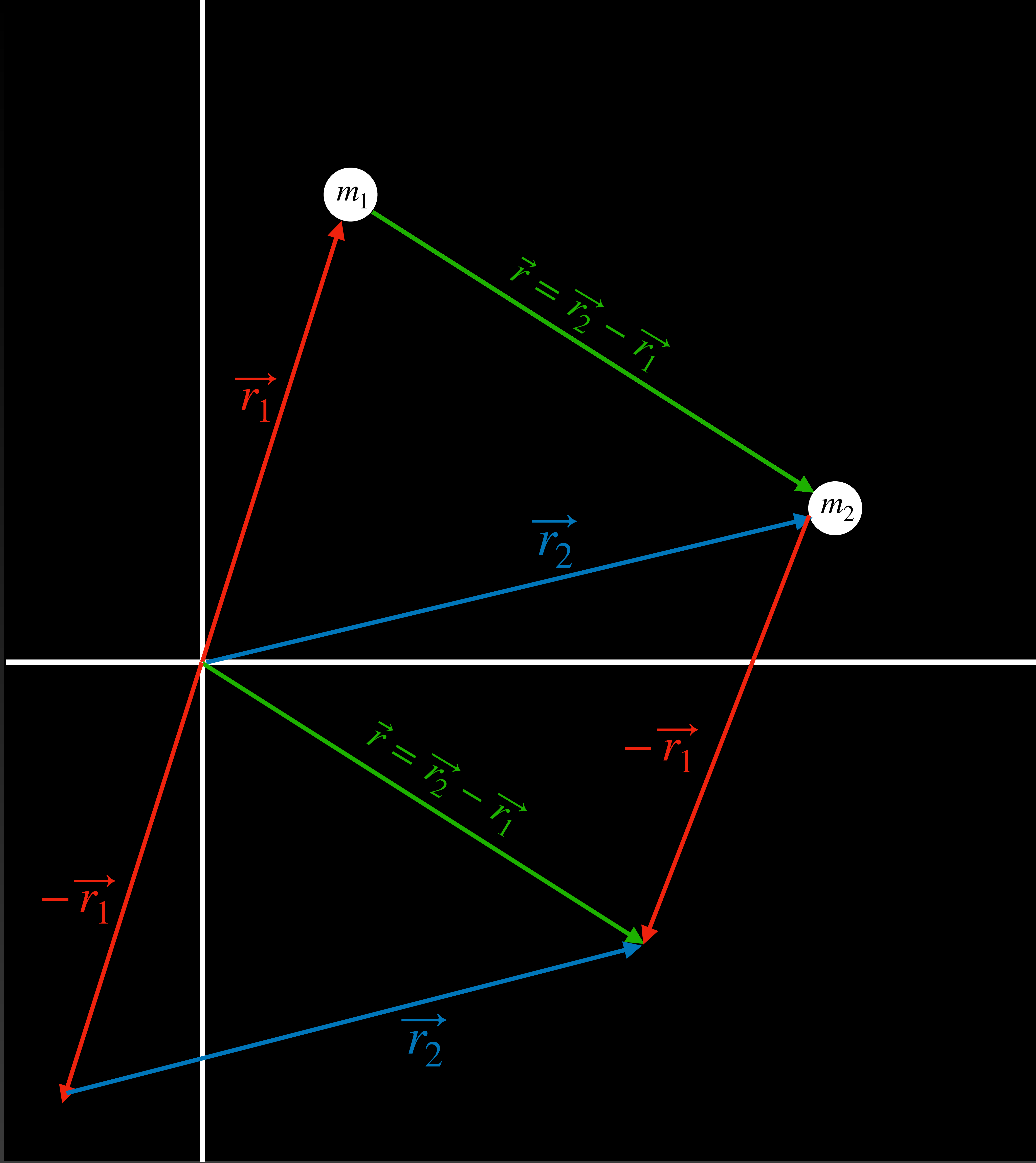
- When we add or subtract vectors like $\vec{r_1}$ and $\vec{r_2}$ the result is the same adding or subtracting their x,y respective components.
- For example, $\vec{r} = \vec{r_2} - \vec{r_1}$, or $r_x = r_{2x} - r_{1x}$ and $r_y = r_{2y} - r_{1y}$.



Gravity

Vector Review

- Vector Addition/Subtraction:
  - place the end of one vector at the tip of the other, $\vec{r} = \vec{r_2} - \vec{r_1}$.

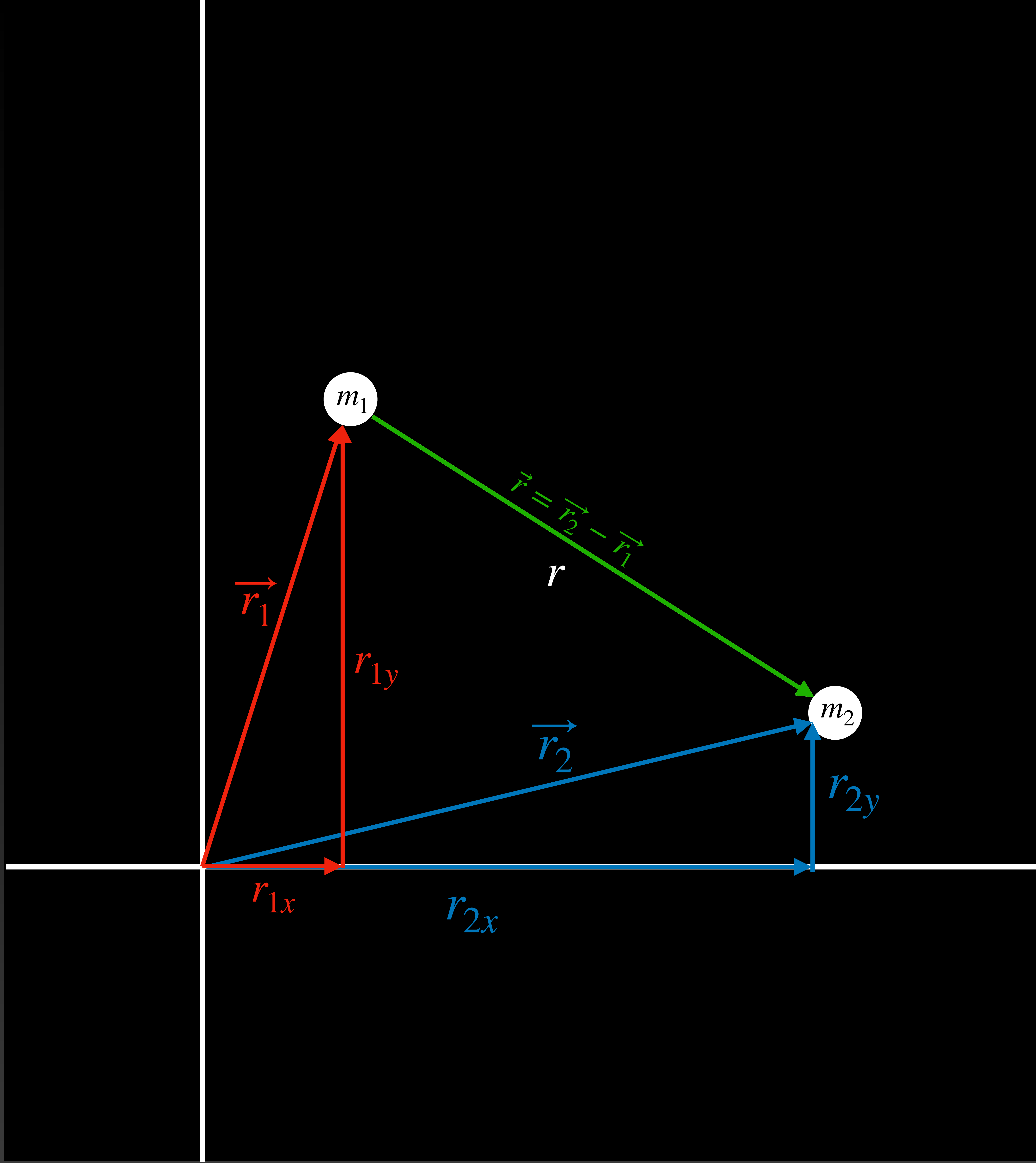


Gravity
Vector Review

- Vectors have a *magnitude* and a *direction*.
- We can obtain the *magnitude* of the vector by using Pythagoras' Theorem.
- For example, the magnitude of $\vec{r_1}$, usually written as $|\vec{r_1}|$ or r_1, is

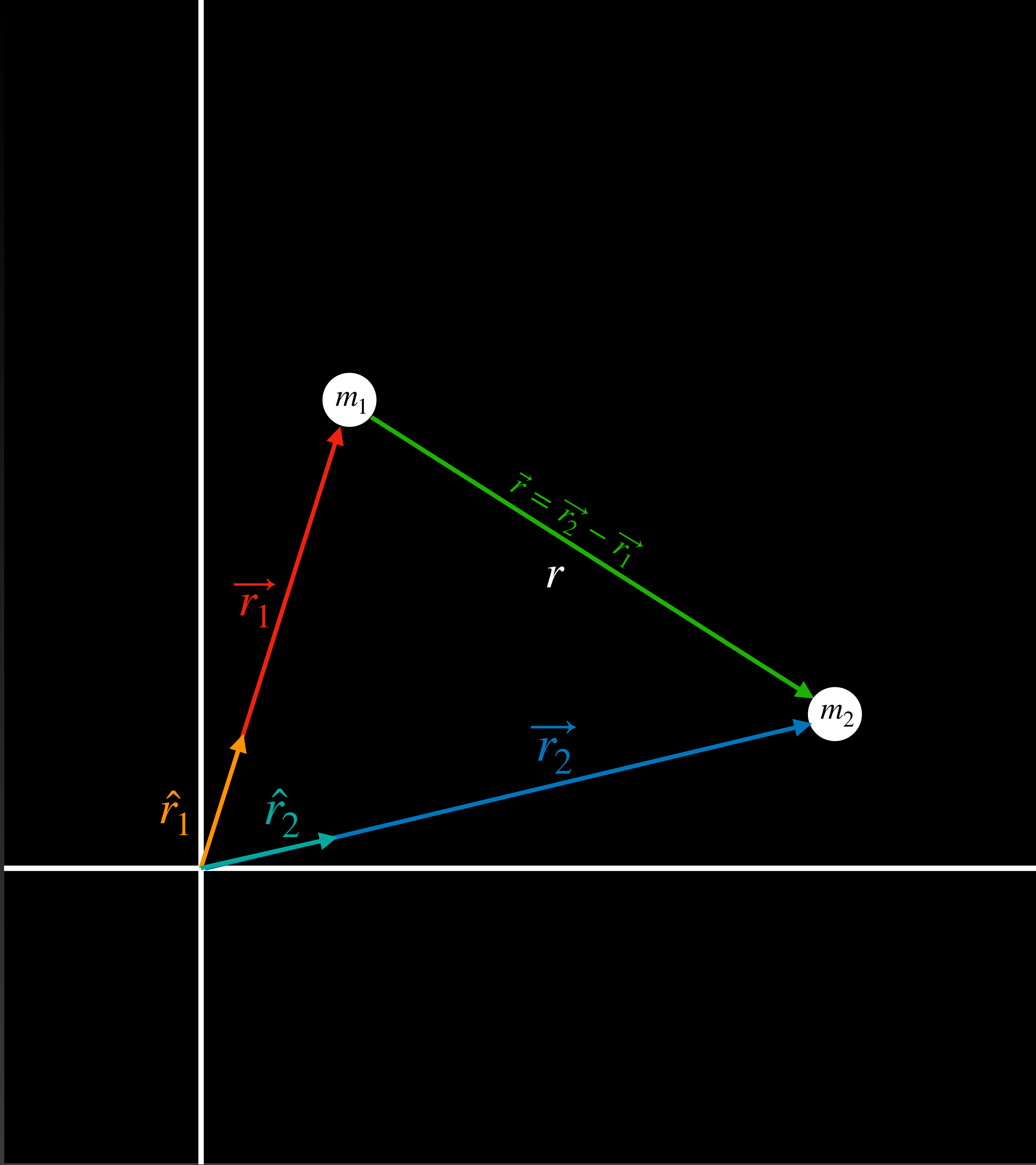
$$\left|\vec{r_1}\right| = r_1 = \sqrt{r_{1x}^2 + r_{1y}^2}$$

- For 3D vectors $r = \sqrt{r_x^2 + r_y^2 + r_z^2}$



Gravity
Vector Review

- We can obtain the *direction* of $\overrightarrow{r_1}$, by calculating its "unit vector".
- A unit vector is a vector divided by its magnitude, $\hat{r}_2 = \dfrac{\overrightarrow{r_2}}{\left|\overrightarrow{r_2}\right|}$, so that the magnitude of the vector is 1, $|\hat{r}| = 1$.
- This process of taking a vector and turning it into a unit vector is called normalization, or "taking the *norm*".



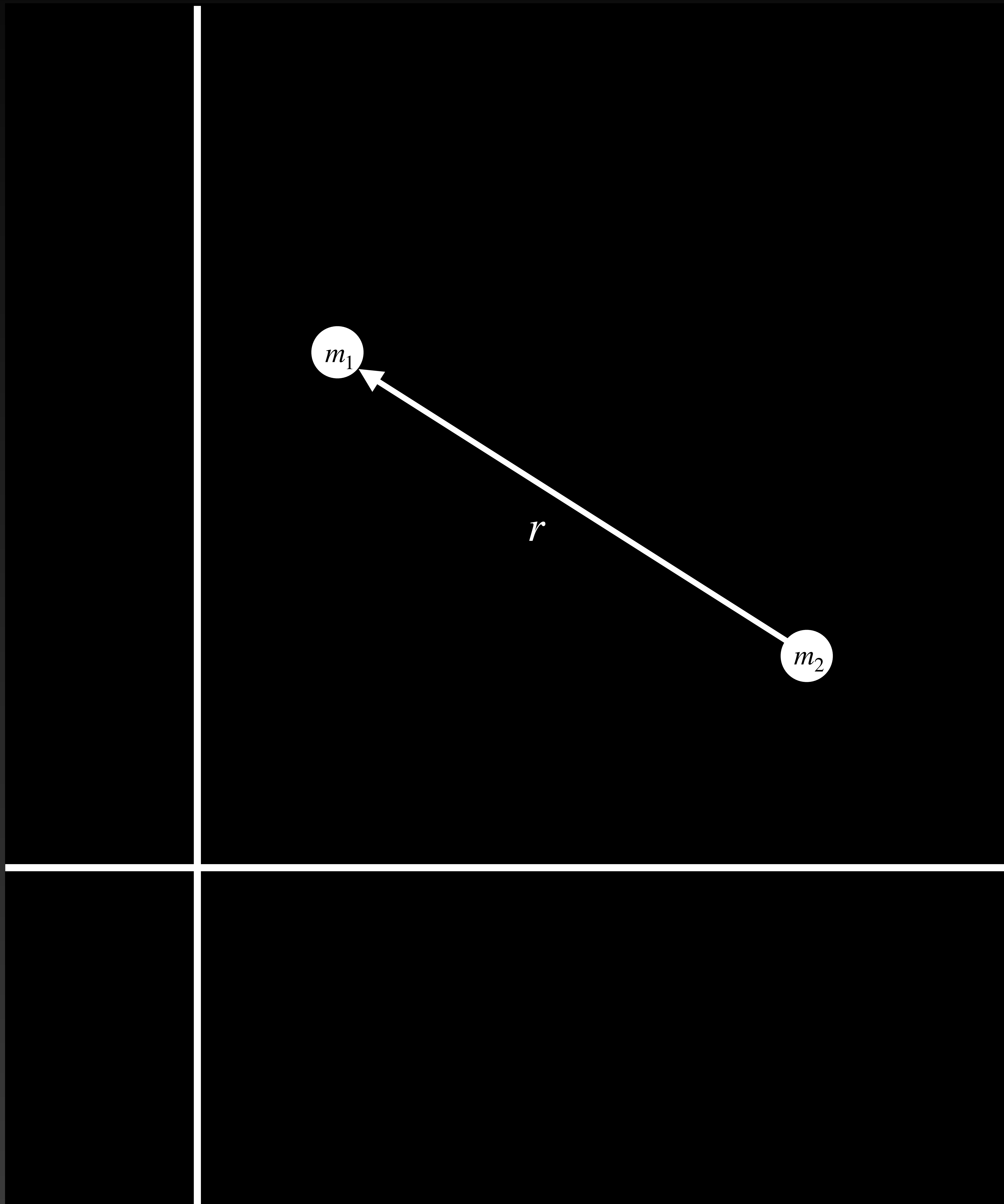
The Two Body Problem

Follow along!

- Go to https://trinket.io/library/trinkets/0b17a8381c
- Don’t worry about understanding how the animations are done right now, all of the code used to animate the pendulum is already there. Just find the part that says,

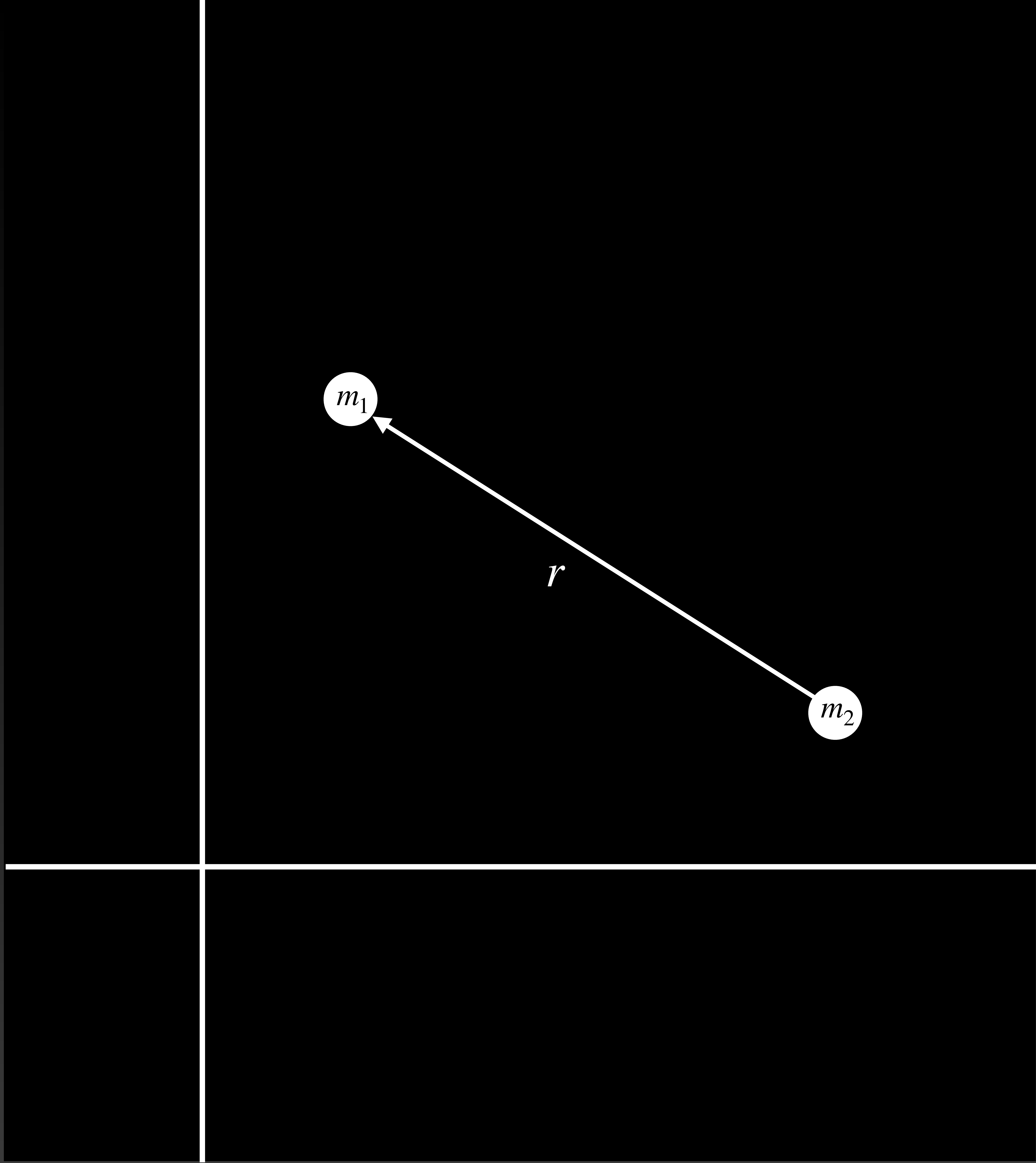
```
#***********************************
      Code Solution Here:
#***********************************


#***********************************
```

- and code up your solution between the lines. When you think you’ve got it, hit the play button at the top of the page and see if it worked! Also, play around with the initial conditions and physical constants



Gravity
The Two Body Problem

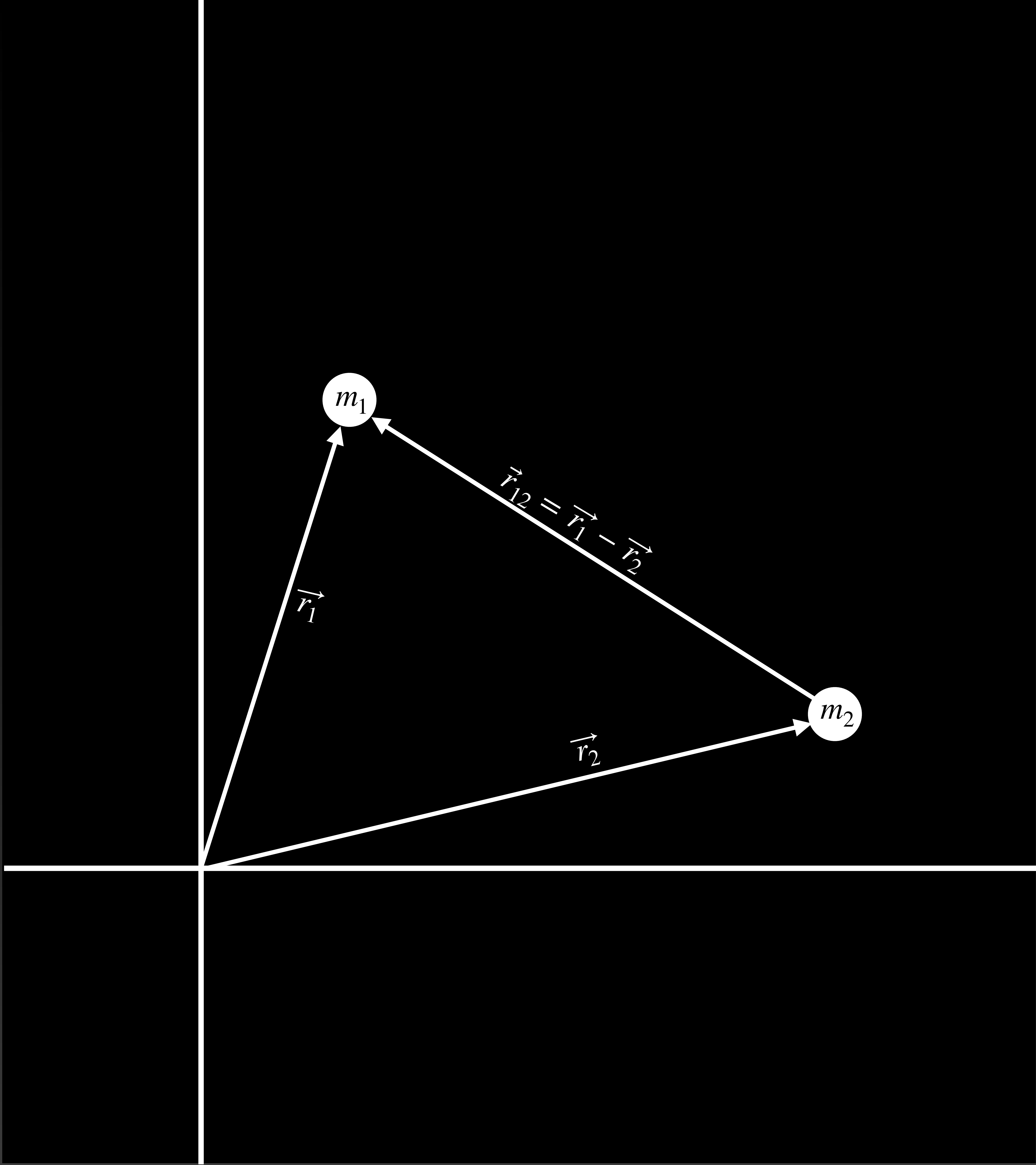
- Every particle attracts every other particle with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.
- $F = -\dfrac{Gm_1m_2}{r^2}$
- Remember, the minus sign indicates that the force is an attractive or restoring force.



Gravity

The Two Body Problem

- With a knowledge of vectors, let us work out how to solve the Two Body Problem.
- We can write Newton's Law of Gravity, this time with vectors.
- $$\overrightarrow{F} = -\frac{Gm_1m_2}{r^2}\hat{r}$$
- This says that the force $\overrightarrow{F}$ is in the opposite direction of $\hat{r}$.

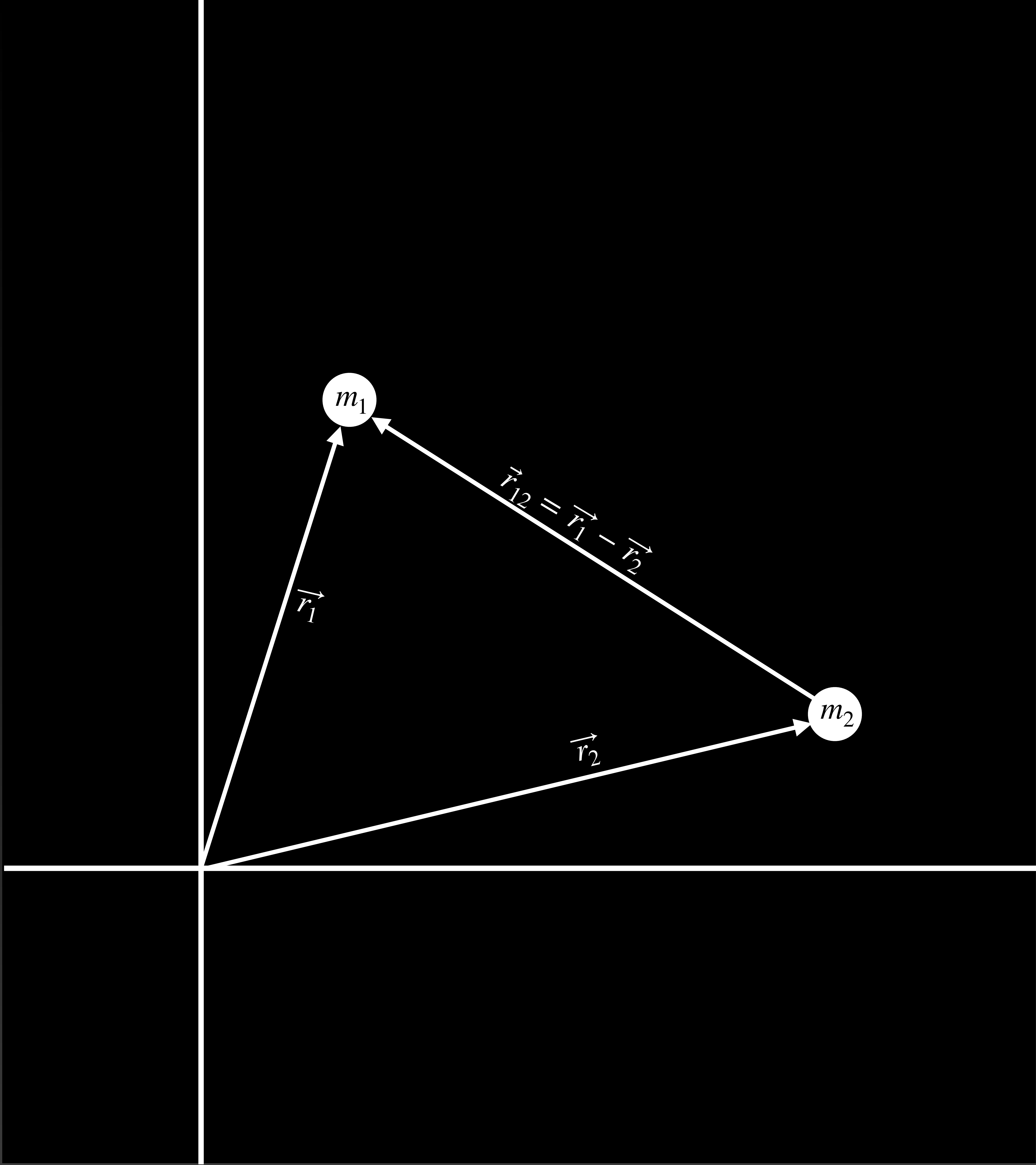


Gravity
The Two Body Problem

- From Newton's 2nd Law we know

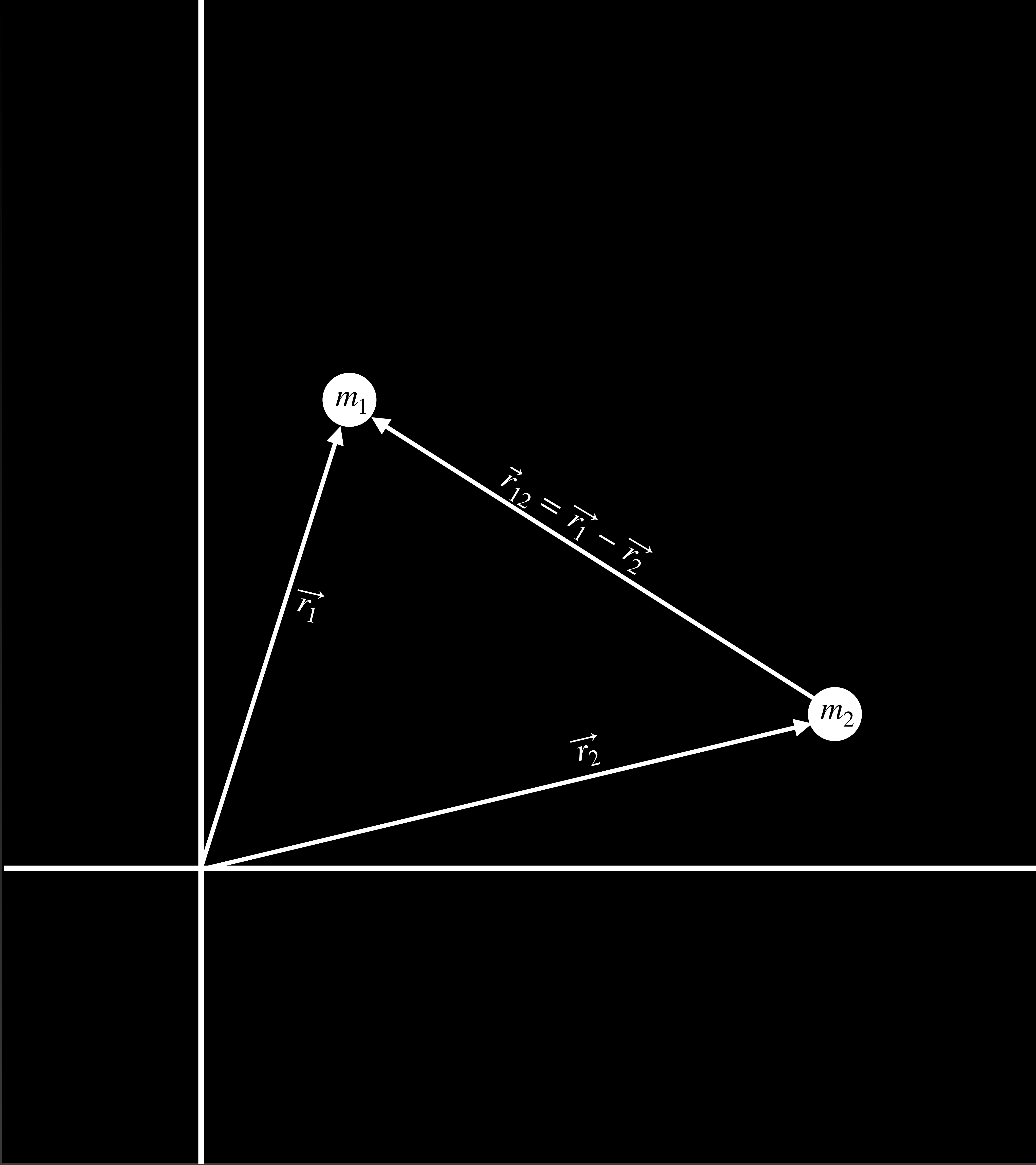
that $\overrightarrow{F} = m\overrightarrow{a}$, or $\overrightarrow{F} = \dfrac{\Delta\overrightarrow{p}}{\Delta t}$

if we use momentum $\overrightarrow{p} = m\overrightarrow{v}$.



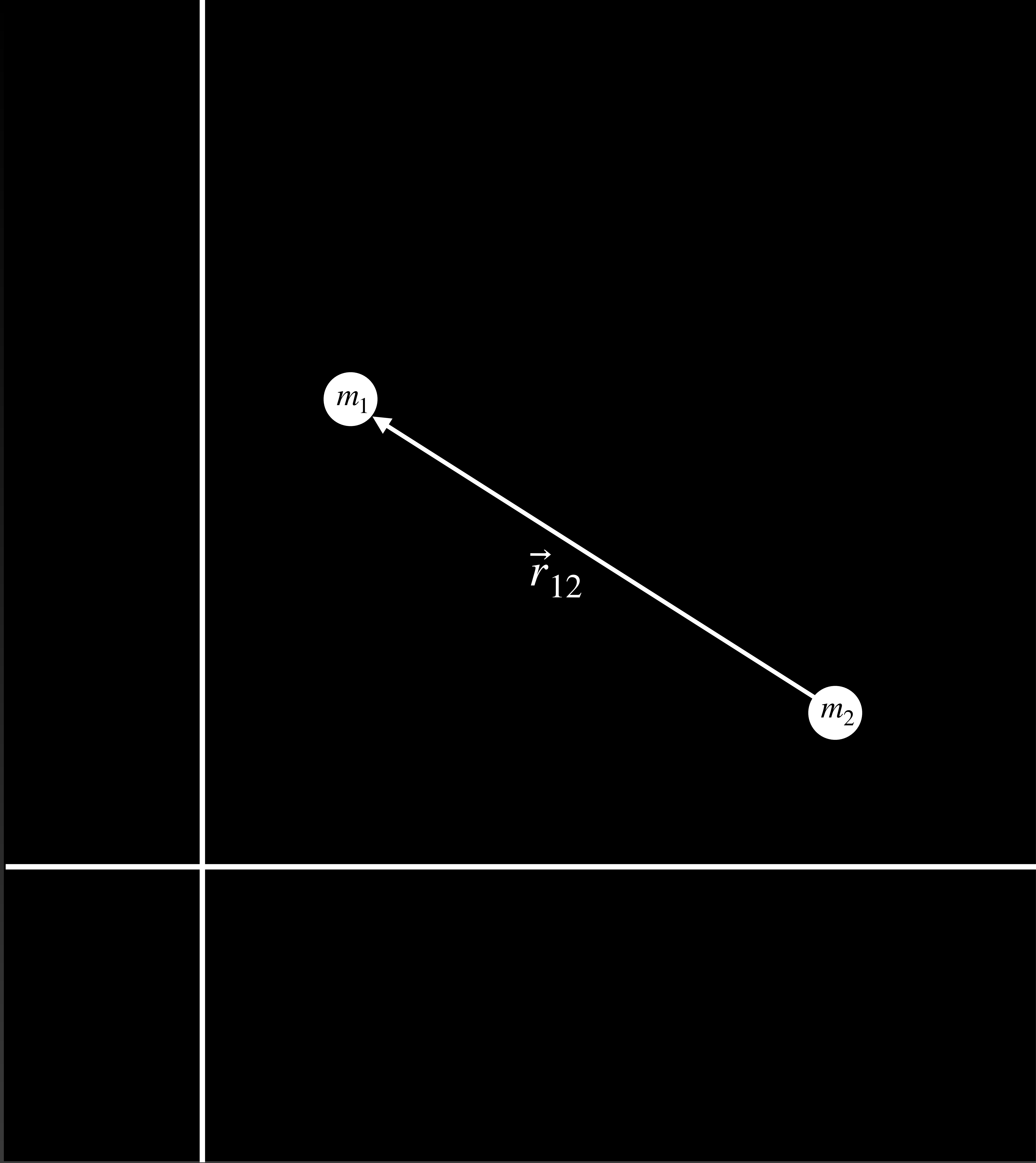
Gravity
The Two Body Problem

- With this relationship between force and momentum we can rearrange for how the momentum changes due to the force. We have
  - $\dfrac{\Delta \overrightarrow{p}}{\Delta t} = \overrightarrow{F}$
  - $\Delta \overrightarrow{p} = \overrightarrow{F} \Delta t$
  - $\overrightarrow{p}_{\text{final}} - \overrightarrow{p}_{\text{initial}} = \overrightarrow{F} \Delta t$
  - $\overrightarrow{p}_{\text{final}} = \overrightarrow{p}_{\text{initial}} + \overrightarrow{F} \Delta t$



Gravity
The Two Body Problem

- Using the momentum to update the position and putting it all together for m_1 we have
- $\overrightarrow{F}_{12} = -\dfrac{Gm_1m_2}{r_{12}^2}\hat{r}_{12}$
- $\overrightarrow{p}_{1_{\text{final}}} = \overrightarrow{p}_{1_{\text{initial}}} + \overrightarrow{F}_{12}\Delta t$
- $\vec{r}_{1_{\text{final}}} = \vec{r}_{1_{\text{initial}}} + \dfrac{\overrightarrow{p}_{1_{\text{final}}}}{m_1}\Delta t$
- Then do the same for m_2 remembering Newton's 3rd law ($\overrightarrow{F}_{12} = -\overrightarrow{F}_{21}$).



The Two Body Problem

Coding Time!

- Go to https://trinket.io/library/trinkets/0b17a8381c
- Once again, find the part that says,

  ```
  ***********************************
          Code Solution Here:
  ***********************************


  ***********************************
  ```

- and code up your solution between the lines.
- The solution can be found at https://trinket.io/library/trinkets/0f411818d9

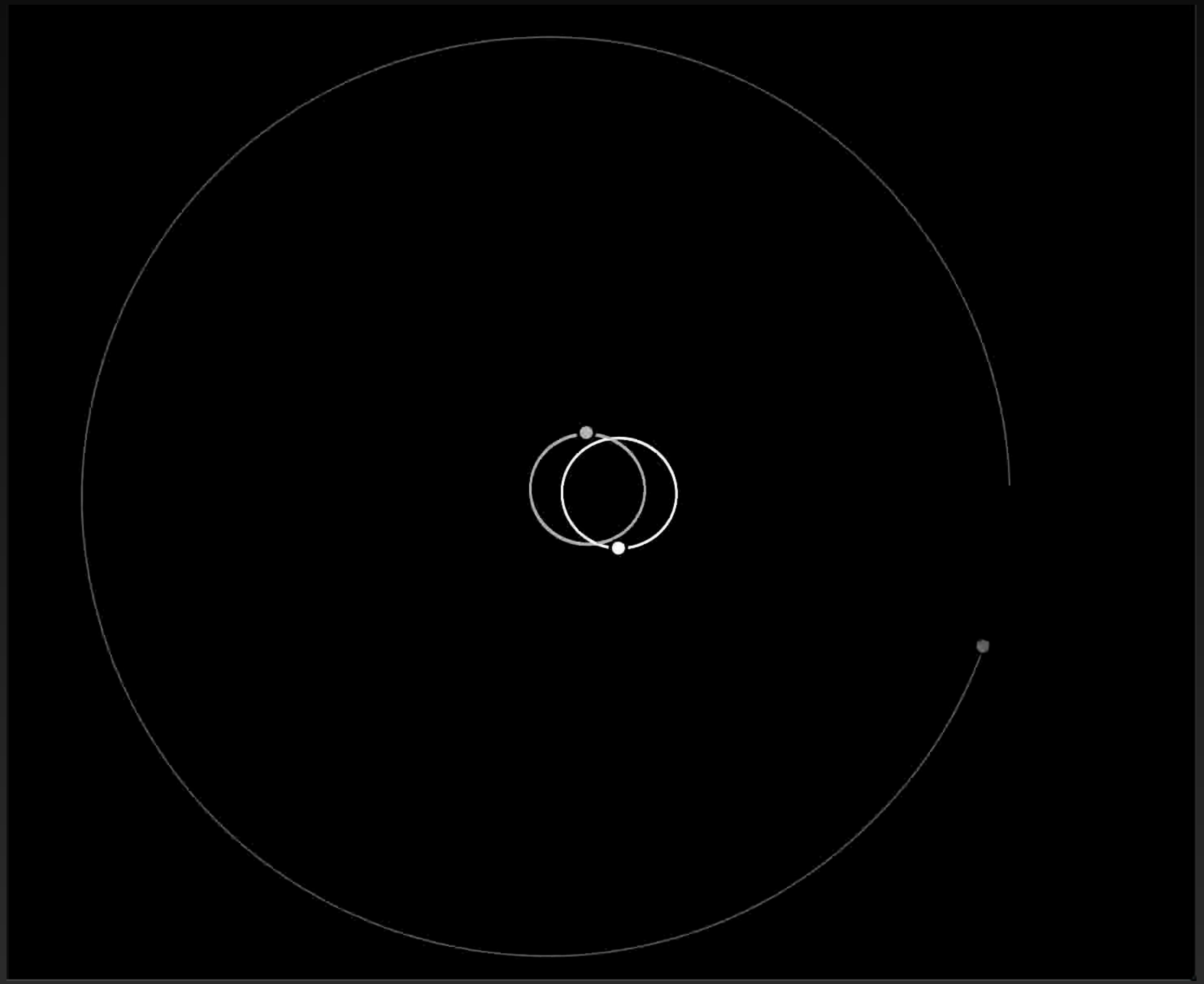
- Remember
  - $\vec{r}_{12} = \vec{r}_1 - \vec{r}_2$
  - $\vec{F}_{12} = -\dfrac{Gm_1m_2}{r_{12}^2}\hat{r}_{12}$
  - $\vec{F}_{21} = -\vec{F}_{12}$
  - $\vec{p}_{1_{\text{final}}} = \vec{p}_{1_{\text{initial}}} + \vec{F}_{12}\Delta t$
  - $\vec{r}_{1_{\text{final}}} = \vec{r}_{1_{\text{initial}}} + \dfrac{\vec{p}_{1_{\text{final}}}}{m_1}\Delta t$

The Three Body Problem

Follow Along!

- Go to https://trinket.io/library/trinkets/3b7770e582
- Again, code your solution between the lines,

```
#*********************************
     Code Solution Here:
#*********************************



#*********************************
```



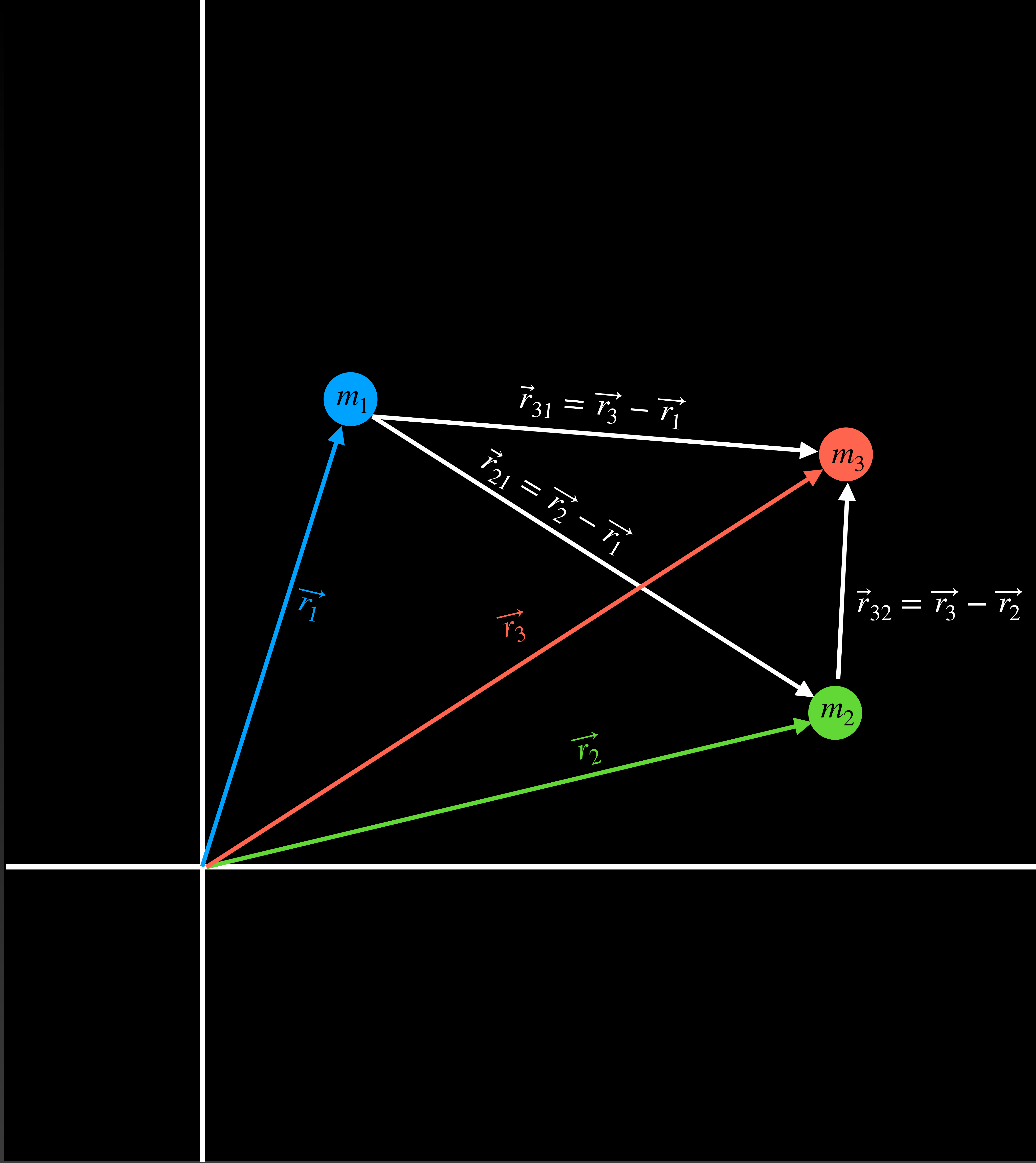
- There are 3 masses this time which means there are 3 pairs of forces.
- The solution can be found at https://trinket.io/library/trinkets/424bbef5df

Gravity

The Three Body Problem

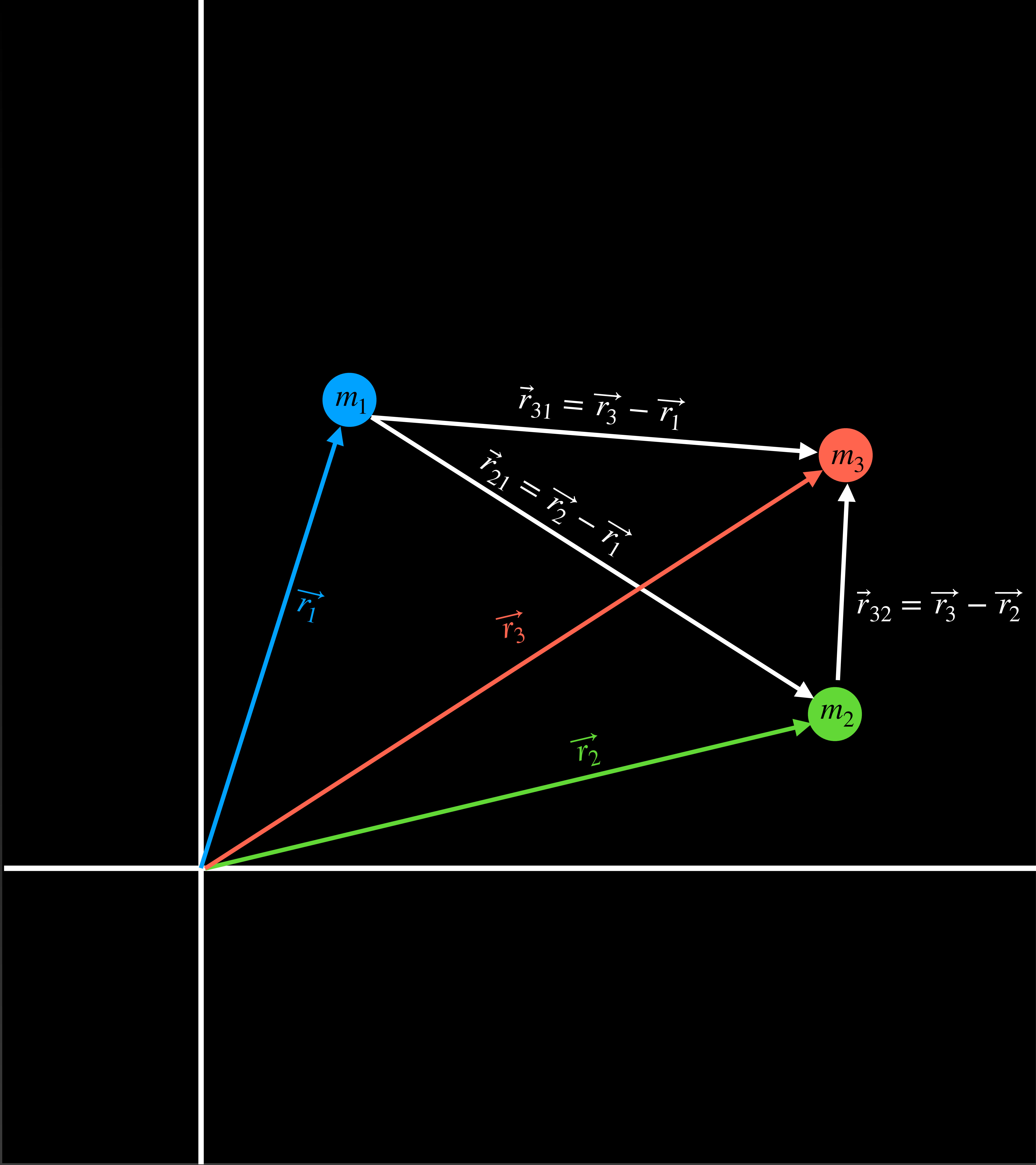
- We follow the same path as before, combining all the forces on m_1 that we have the forces from m_2 and m_3,
- $\overrightarrow{F}_{12} = -\dfrac{Gm_1m_2}{r_{12}^2}\hat{r}_{12}$ and

  $\overrightarrow{F}_{13} = -\dfrac{Gm_1m_3}{r_{13}^2}\hat{r}_{13}$
- $\overrightarrow{p}_{1_f} = \overrightarrow{p}_{1_i} + \left(\overrightarrow{F}_{12} + \overrightarrow{F}_{13}\right)\Delta t$
- $\vec{r}_{1_f} = \vec{r}_{1_i} + \dfrac{\overrightarrow{p}_{1_f}}{m_1}\Delta t$



Gravity
The Three Body Problem

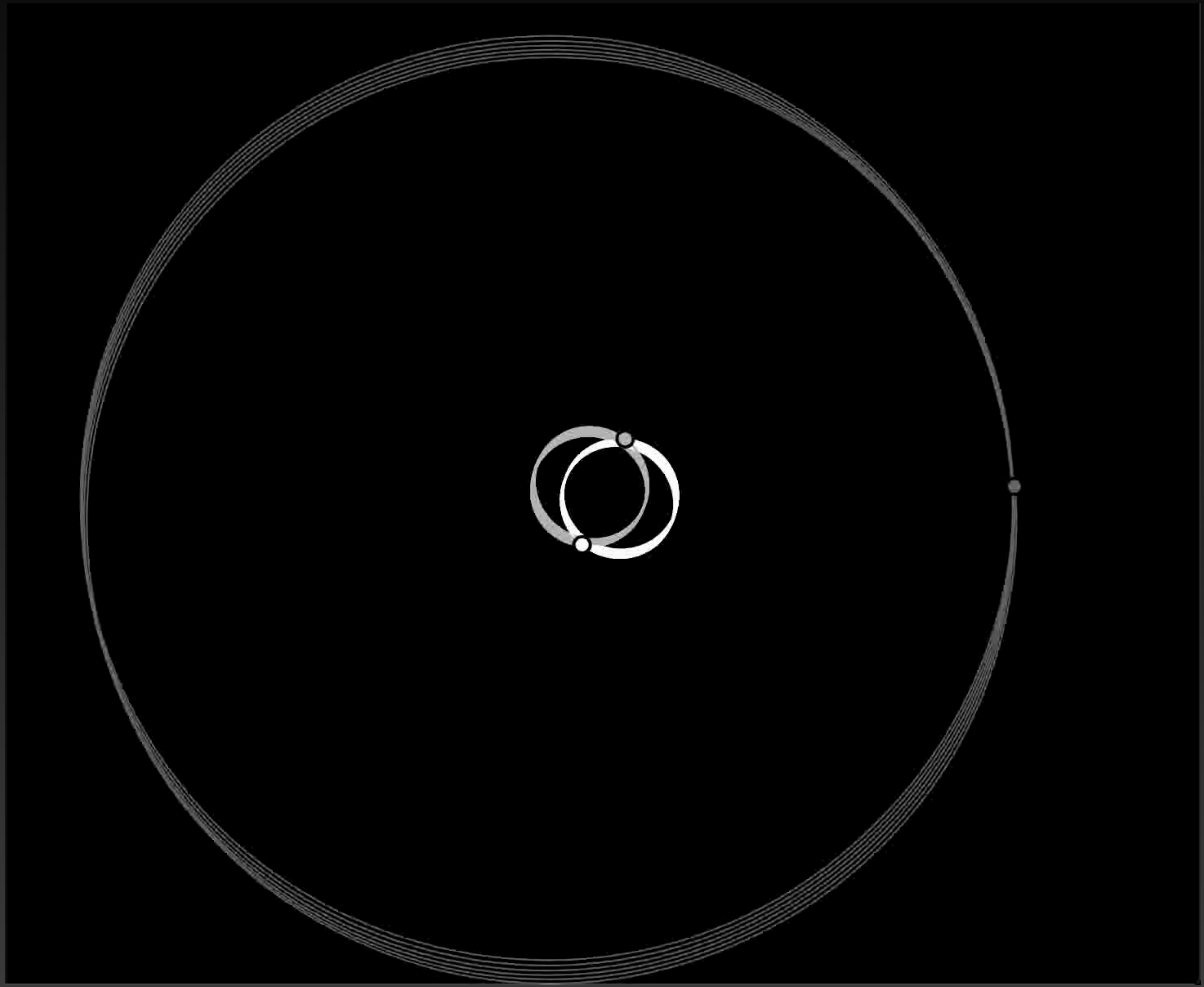
- Then do the same for m_2 and m_3 remembering Newton's 3rd law ($\overrightarrow{F}_{21} = -\overrightarrow{F}_{12}$ and $\overrightarrow{F}_{31} = -\overrightarrow{F}_{13}$).
- Which is to say that the force of m_1 on m_2 is the opposite to the force of m_1 on m_2.
- But also don't forget the force pair between m_2 and m_3, $\overrightarrow{F}_{32} = -\overrightarrow{F}_{23}$



The Three Body Problem
Coding Time!

- Template: https://trinket.io/library/trinkets/3b7770e582

- Solution: https://trinket.io/library/trinkets/424bbef5df



Chaos

- Small changes to the initial conditions can give very different results.
- The motion is determined, but unpredictable.
- Small differences in initial conditions, such as errors in measurements or rounding errors in numerical computation, can yield widely diverging outcomes.

Chaos

- If we start the three body system with two almost identical initial conditions, after a short time their paths diverge.
- Play around with this using your code.

Thank you

Happy Coding

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