The One-Round Multi-player Discrete Voronoi Game on Grids and Trees

Xiaoming Sun[1,2], Yuan Sun[1,2], Zhiyu Xia[1,2], and Jialin Zhang[1,2]

[1] CAS Key Lab of Network Data Science and Technology, Institute of Computing Technology, Chinese Academy of Sciences, Beijing, China
[2] University of Chinese Academy of Sciences, Beijing, China
{sunxiaoming,sunyuan2016,xiazhiyu,zhangjialin}@ict.ac.cn

Abstract. Basing on the two-player Voronoi game introduced by Ahn et al. [1] and the multi-player diffusion game introduced by Alon et al. [2] on grids, we investigate the following *one-round multi-player discrete Voronoi game* on grids and trees. There are n players playing this game on a graph $G = (V, E)$. Each player chooses an initial vertex from the vertex set of the graph and tries to maximize the size of the nearest vertex set. As the main result, we give sufficient conditions for the existence/non-existence of pure Nash equilibrium in 4-player Voronoi game on grids and only a constant gap leaves unknown. We further consider this game with more than 4 players and construct a family of strategy profiles, which are pure Nash equilibrium on sufficiently narrow graphs. Besides, we investigate the game with 3 players on trees and design a linear time/space algorithm to decide the existence of a pure Nash equilibrium.

Keywords: Game theory · Nash equilibrium · Location game · Graph theory.

1 Introduction

1.1 Background

Consider the following scene: Several investors plan to set up laundries in a city and each of them is permitted to manage only one. There are some residents in the city whose addresses have been obtained by the investors. Residents in the city would only choose the nearest laundry and if the nearest ones of some residence are not unique, he/she will choose one of them randomly. In this game, the investors try to attract more customers by locating their laundry at an advantageous position and the payoff of each investor is the number of residents who are certain to choose his/her laundry. Our goal is to determine whether there exists a stable profile in which no investor would change his/her choice to improve the payoff.

In previous works, there are two ways to model this problem:

Voronoi game. The competitive facility location game is a well-studied topic in game theory. Ahn et al. introduced the Voronoi diagram to characterize players' payoff and proposed the Voronoi game [1]. In the Voronoi game, two players alternately locate their facilities. After locating, each player will control the area closer to one of the facilities. The aim of the two players is to maximize the area controlled by them, respectively. Cheong et al. then modified this model and introduced the *one-round Voronoi game* [10]. In their version, the first player locates n facilities first. Then, the second player locates n facilities. Their locating is only allowed to perform in one round. Dürr et al. introduced an interesting multi-player version on graphs, where each player can only choice one facility [11]. A small modification in the model of Dürr et al. is that a "tie" vertex, who has k nearest facilities, contributes $1/k$ to the payoff of the k players.

Diffusion game. A similar model is raised by Alon et al. [2] which describes the following competitive process with k players on graph $G = (V, E)$. Let player-i's influenced set $I_i = \emptyset$ for all $i \in [k]$, at the beginning. The game contains several rounds. In round-0, each player-i chooses a vertex $v_i \in V$ as the initial vertex simultaneously, and v_i is gathered by I_i. In round-$(t+1)$ for $t \geq 0$, if some vertex v is not gathered by any I_j for all $j \in [k]$ until the round-t but has a neighbor gathered in I_i, then v is gathered by I_i in this round. However, if some v is gathered by more than one set in any round, including round-0, v will be deleted from G and all influenced set after this round. The process iterates until each I_i is invariant after some round and the payoff of each player-i is $|I_i|$.

We investigate a one-round discrete version of Voronoi games with k players on a given graph G denoted by $\Gamma(G, k)$, which is also a simplified version of diffusion games. Note that in diffusion games, the duplicate vertices, i.e., the vertices gathered by more than one player, are deleted immediately after each round. In our model, the duplicate vertices will be deleted at the end of the game. The game is also in the Voronoi style, i.e., the owner of each vertex could be determined by which initial vertex is the nearest one from it. We describe the one-round discrete Voronoi game in an equivalent but more natural way. First, each player chooses a vertex simultaneously in a given graph as the *initial vertex*. The payoff of each player is the number of vertices, which have smaller distance to this player's initial vertex than to all the others. Note that the only difference between this model with the model of Dürr et al. is that "tie" vertices will contribute nothing to the payoff of any player.

Remark. It is easy to see that a diffusion game and a one-round Voronoi game are equivalent on paths, circles and trees. But in the general case, they could be totally different. Figure 1 is an example.

Fig. 1. The results of these games are different in a grid graph with 2 players. The left one is the result of the diffusion game, while the right one is the result of our model.

In game theory, the concept Nash equilibrium, named after John Forbes Nash Jr., takes a central position. In a Nash equilibrium, no player can improve the payoff by change the choice unilaterally. John Nash proved every game with finite number of players and a finite strategy space has a mixed-strategy Nash equilibrium in his famous paper [13]. If we just allow pure strategies, in which each player can only make a deterministic choice, the existence of a Nash equilibrium cannot be ensured. In this paper, we investigate this existence in one-round discrete Voronoi games on grids and trees.

1.2 Our Results

In this paper, we consider the one-round Voronoi game on girds or trees. We firstly summerize our results for grids.

1. In $\Gamma(\mathrm{Grid}_{n\times m}, k)$ with $k \leq 2$, there always exists a pure Nash equilibrium;
2. In $\Gamma(\mathrm{Grid}_{n\times m}, 3)$, there exists no pure Nash equilibrium;
3. For $\Gamma(\mathrm{Grid}_{n\times m}, 4)$, we prove the following theorem which almost completely characterizes the existence of a pure Nash equilibrium.

Theorem 1. *In $\Gamma(\mathrm{Grid}_{n\times m}, 4)$ $(n \geq m)$ where n and m are sufficiently large, if $4|n$ and m is odd, there exists a pure Nash profile if $m \leq n/2 + 2\lfloor\sqrt{n}\rfloor + 1$, and does not exist if $m \geq n/2 + 2\lfloor\sqrt{n}\rfloor + 5$. Otherwise, there exists a pure Nash profile if $m \leq n/2$, and does not exist if $m > n/2 + 6$.*

For general k, we show that there always exists a pure Nash profile for sufficiently narrow grids when $k \neq 3$.

Theorem 2. *For any $k \in \mathbb{Z}^+$ $(k \neq 3)$ and sufficiently large integers m, n, there exists a pure Nash equilibrium in $\Gamma(\mathrm{Grid}_{n\times m}, k)$ if $m \leq n/\lceil k/2\rceil$.*

We then consider the case where the graph is a tree. The case with two players on a tree is solved in [14]. In this paper we solve the case with 3 players:

Theorem 3. *In $\Gamma(T, 3)$ where T is a tree with size n, there exists a pure Nash equilibrium if and only if there exists a vertex v such that $n - st(i_1, v) - st(i_2, v) \geq \max\{st(j_1, v), st(j_2, v)\}$ and $st(i_2, v) \geq st(j_1, v)$, where i_1 and i_2 are the children of v with maximum and second maximum $st(\cdot, v)$, and j_k is the child of i_k with maximum $st(\cdot, v)$ $(k = 1, 2)$. Here, $st(v_1, v_2)$ represents the number of vertices in the subtree with root v_1, when the tree is rebuilt as a rooted tree with root v_2.*

Furthermore, if such a vertex v exists, then (i_1, v, i_2) is a Nash equilibrium, and we can design an algorithm to determine whether a pure Nash equilibrium exists in $O(n)$ time/space.

1.3 Related works

Ahn et al. proved that the first player has a winning strategy in Voronoi games on a circle or a line segment [1]. Teramoto et al. showed it is NP-hard to decide whether the second player can win in a 2-player discrete Voronoi game on graphs [15]. Durr and Thang studied a one-round, multi-player version and proved it is also NP-hard to decide the existence of a pure Nash equilibrium in a discrete Voronoi game on a given graph [11]. Besides, several works focus on a one-round 2-player discrete version and try to compute a winning strategy for each player efficiently [3,4,5,6,7,8].

For diffusion games, it is also proved as a NP-hard problem to decide whether there exists a pure Nash equilibrium with multi-player on general networks [12]. Roshanbin et al. studied this game with 2 players [14] and showed the existence or nonexistence of a pure Nash equilibrium in paths, cycles, trees and grids. Especially, on a tree with 2 players, there is always a Nash equilibrium under the diffusion game model. Bulteau et al. proved a pure Nash equilibrium does not exist on $\mathrm{Grid}_{n\times m}$ where $n, m \leq 5$ with 3 players and got some extra conclusions about the existence of pure Nash equilibrium in several special graph classes [9].

1.4 Organization

In section 2, we define some useful notations and concepts used in the latter sections. In section 3, we give a pure Nash profile construction for sufficiently narrow grids. In section 4, we discuss the case when there are at most 4 players on grid graphs. In section 5 we discuss the case where there are 3 players on trees. Due to the space limitation, we leave the most proof details in the appendix.

2 Preliminaries

2.1 General definitions

For notations in the set theory, we denote $[n, m]$ as the set $\{n, n+1, \ldots, m\}$, and $[n]$ as $[1, n]$ for integers n, m $(n \leq m)$.

For notations in the graph theory, An (undirected) *graph* is an ordered pair $G = (V, E)$ where V is a set of *vertices* and E is a set of *edges*. We say $v_1, \ldots, v_\ell$ is a *path* if $\forall i$ $(v_i, v_{i+1}) \in E(G)$. We say vertices $v, w \in V(G)$ are connected if there exists a path between v and w in G. For convenience of discussions, we denote that $v(G) = |V|$ and $e(G) = |E|$ for a graph $G = (V, E)$.

We indicate a *grid* with a set of integer pairs. Define $\mathrm{Grid}_{n\times m} := (V, E)$ with the vertex set $V = [n] \times [m]$ and an edge set E. For vertices $v_1, v_2 \in V$, L_1 norm distance between them is defined as $\|v_1 - v_2\|_1 := |x_1 - x_2| + |y_1 - y_2|$, where $v_1 = (x_1, y_1)$ and $v_2 = (x_2, y_2)$. Then the edge set E of $\mathrm{Grid}_{n\times m}$ is

$$\{(v_1, v_2) \mid \|v_1 - v_2\|_1 = 1\}.$$

A graph $T = (V, E)$ is called a *tree* if $v(T) = e(T) + 1$ and every 2 vertices $v, w \in V(T)$ are connected. We also define a function $st(w, v)$ $(w, v \in V(T))$, as the number of vertices in the subtree rooted at vertex w when T is rebuilt as a rooted tree with root v.

A one-round multi-player Voronoi game, denoted by $\Gamma(G,k)$, processes on the undirected graph G with k players. Each player, player-i for example, can choose one vertex v_i from G. v_i is called *initial vertex* of player-i and we also say player-i *takes* v_i. Player-i will influence the vertices in

$$I_i := \{v \in V(G) \mid \forall j \in [k] \backslash \{i\} (\|v - v_i\|_1 < \|v - v_j\|_1)\},$$

where v_j is the player-j's initial vertex. In this game the payoff of player-i is $U_i := |I_i|$. The aim of each player is to maximize the payoff in the game, respectively. Obviously, the vertex set influenced by a specific player totally depends on every player's initial vertex. So, a strategy profile is defined as a tuple of vertices $p := (v_1, v_2, \dots, v_k)$ where v_i is player-i's initial vertex. The influenced vertex set and payoff of player-i in some strategy profile p are donated as $I_i(p)$ and $U_i(p)$. When a strategy profile p is implied in context, we use tuple (v_i, v_{-i}) to emphasize the player-i's choice where v_{-i} represents the others' choices. We say a strategy profile v is pure Nash equilibrium if

$$\forall i \in [k]\ \forall v_i' \in V(G)\ (U_i(v_i, v_{-i}) \geq U_i(v_i', v_{-i})).$$

Namely, no player can improve the payoff by moving the initial vertex in non-cooperative cases.

2.2 Notations in grid graphs

For grid graphs, we say a vertex v_i' is an *improving* of the original initial vertex v_i if and only if $U_i(v_i', v_{-i}) > U_i(v_i, v_{-i})$ holds. For any player, an improving of its initial vertex cannot be found in a pure Nash equilibrium profile. We then introduce several useful notations which are convenient to group vertices with similar properties.

Definition 1. *In* $\mathrm{Grid}_{n\times m}$ *and for* $i,j \in \{-1,1\}$, *define* $\mathrm{Part}_{i,j} : [n] \times [m] \to 2^{[n]\times[m]}$ *as*

$$\begin{aligned}
\mathrm{Part}_{-1,-1}(x,y) &:= [x] \times [y],\\
\mathrm{Part}_{-1,1}(x,y) &:= [x] \times [y,m],\\
\mathrm{Part}_{1,-1}(x,y) &:= [x,n] \times [y],\\
\mathrm{Part}_{1,1}(x,y) &:= [x,n] \times [y,m].
\end{aligned}$$

When we move some player's initial vertex from $v = (x,y)$ to $v' = (x+1, y+1)$, it is clear that vertices in $\mathrm{Part}_{1,-1}(x+1,y) \cup \mathrm{Part}_{-1,1}(x,y+1)$ keep the invariant distance to the initial vertex, the distances from the vertices in $\mathrm{Part}_{1,1}(v')$ decrease by 2, and the distances from the vertices in $\mathrm{Part}_{-1,-1}(v)$ increase by 2. So, it is convenient to introduce Definition 1 to locate the vertices with similar properties in our following proofs.

Fig. 2. Definition 1&2

Definition 2. *In* $\mathrm{Grid}_{n\times m}$ *and for* $(i,j) \in \{(0,1),(0,-1),(1,0),(-1,0)\}$, *define* $\mathrm{Part}_{i,j} : [n]\times[m] \to 2^{[n]\times[m]}$ *as*

$$\begin{aligned}
\mathrm{Part}_{1,0}(x,y) &:= \{(x',y') \in [n]\times[m] \mid x'+y' > x+y \wedge x'-y' > x-y\},\\
\mathrm{Part}_{-1,0}(x,y) &:= \{(x',y') \in [n]\times[m] \mid x'+y' < x+y \wedge x'-y' < x-y\},\\
\mathrm{Part}_{0,1}(x,y) &:= \{(x',y') \in [n]\times[m] \mid x'+y' > x+y \wedge x'-y' < x-y\},\\
\mathrm{Part}_{0,-1}(x,y) &:= \{(x',y') \in [n]\times[m] \mid x'+y' < x+y \wedge x'-y' > x-y\}.
\end{aligned}$$

We consider the "slashes" crossing some given vertex $v = (x,y)$, a vertex set shaped as $\{v' = (x',y') \in [n]\times[m] \mid |x-x'| = |y-y'|\}$ frequently, since a player's influenced set will change notably while an initial vertex is spanning a slash.

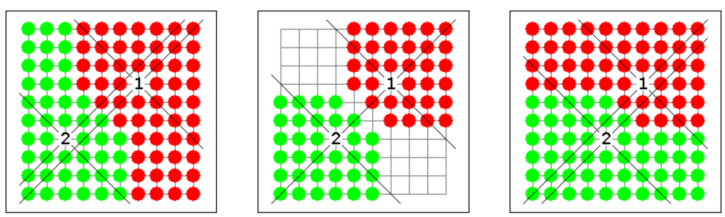

Fig. 3. v_2 moves from (3,4) to (4,4), then, to (5,4) while v_1 is located at (7,7).

The vertices in a grid are divided into several parts by the slashes. When an initial vertex moves in the inner of a single part, the change of its payoff is easy to calculate. Thus, the key point is to analyze the change caused by an initial vertex's moving between different parts.

We also define a new concept called *controller*, which is an initial vertex taking one of the most 'special' positions in the profile.

Definition 3. *An initial vertex* v_i *controls the others, if and only if one of the following conditions holds for all* $j \in [k]\setminus\{i\}$*:*

$$(x_i \le x_j \wedge y_i \le y_j),\ (x_i \le x_j \wedge y_i \ge y_j),\ (x_i \ge x_j \wedge y_i \le y_j),\ (x_i \ge x_j \wedge y_i \ge y_j)$$

The definitions of strictly controlling are risen by Bulteau et al. [9]. We extend their definition by allowing that the others are in the same row or column of a controller. The controller will always appear in our construction.

3 Voronoi game on the narrow grids

In this section, we discuss the case in which k players ($k \neq 3$) take part in a Voronoi game on $\mathrm{Grid}_{n\times m}$. As the main result, a family of pure Nash equilibrium constructions will be given for narrow grids in Theorem 2.

Bulteau et al. study multi-player diffusion games on graphs, which is equivalent to Voronoi games on some special kinds of graph classes, such as paths, circles and trees. Bulteau et al. construct a profile on paths and prove that a pure Nash equilibrium always exists in a game on Path_n, a path with n vertices, with k players, excepted the case where $k = 3$ and $n \geq 6$:

Theorem 4 (Bulteau et al.). *For any $k \in \mathbb{N}$ and any $n \in \mathbb{N}$, there is always a Nash equilibrium for k players on Path_n, except for $k = 3$ and $n \geq 6$.*

In detail, when the number of players k is even, they set the initial vertices of a Nash equilibrium $\{v_i\}_{1\leq i\leq k}$ as:

$$v_i := \begin{cases} \lfloor \frac{n}{k} \rfloor \cdot i + \min\{i, n \bmod k\} & \text{, if } i \text{ is odd} \\ v_{i-1} + 1, & \text{if } i \text{ is even.} \end{cases}$$

For the case with odd number players, Bulteau et al. reduce it to the even case. By constructing the pure Nash-equilibria $(v'_1, \ldots, v'_{k+1})$ for P_{n+1}, they get a pure Nash-equilibria, $(v_1, \ldots, v_k) := (v'_1, \ldots, v'_{k-2}, v'_k - 1, '_{k+1} - 1)$, on Path_n.

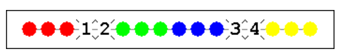

Fig. 4. A pure Nash equilibrium profile for $\Gamma(\mathrm{Path}_{16}, 4)$ constructed by Bulteau et al.

We find that such construction on path sometimes works on grids. Note we might treat a grid as a path when the ratio of the width to the height is sufficiently large. Furthermore, we increase the height of $\mathrm{Grid}_{n\times 1}$ and keep the pure Nash equilibrium profile in some way until there exists an improving breaking the balance.

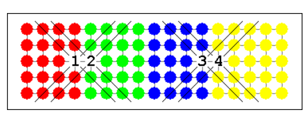

Fig. 5. The constructed profile in $\Gamma(\mathrm{Grid}_{17\times 5}, 4)$.

We construct a family of pure Nash equilibrium profiles. For the odd m, we just embed the construction by Bulteau et al. into the middle row (i.e., the $(m+1)/2$-th row). Whereas, for the even m we would also embed it into one of the two middle rows (i.e., the $m/2$-th or $(m+2)/2$-th row). But in the latter case, we can only promise the pure Nash-property when $(n+k \bmod 2) \bmod (k+k \bmod 2) = 0$ holds. To fix it, we Stretch or Compress a pure Nash equilibrium profile without breaking the pure Nash-property. More precisely, a group of discussions will be given to describe such a process to construct a pure Nash equilibrium profile v, where the routines COMPRESS and STRETCH are given in Algorithm 1:

Case 1: If k is even, consider the parity of m.

Case 2.1: If m is odd, construct v as:

$$v_i := \begin{cases} (\lfloor \frac{n}{k} \rfloor \cdot i + \min\{i, n \bmod k\}, \frac{m+1}{2}), & \text{if } i \text{ is odd;} \\ (x_{i-1} + 1, \frac{m+1}{2}), & \text{if } i \text{ is even.} \end{cases}$$

Case 2.2: If m is even, define $r := n \bmod k$.

Case 2.2.1: In the case where $r = 0$, construct a pure Nash equilibrium profile v for $\Gamma(\mathrm{Grid}_{n\times(m+1)}, k)$.

Case 2.2.2: Otherwise, stretch or compress the profile in the following way:

Case 2.2.2.1: If $r \le k/2$ holds, construct a pure Nash equilibrium v' for $\Gamma(\mathrm{Grid}_{(n-r)\times m}, k)$ and construct v by STRETCH$(n-r, m, k, v', r)$.

Case 2.2.2.2: Otherwise, construct a pure Nash profile v' for $\Gamma(\mathrm{Grid}_{(n+k-r)\times m}, k)$. Then, construct v by COMPRESS$(n + k - r, m, k, v', k - r)$.

Case 2: If k is odd but $k \neq 3$, construct a pure Nash equilibrium profile v' for $\Gamma(\mathrm{Grid}_{(n+1)\times m}, k + 1)$. Then, construct $\{v_i\}_{1\le i<k}$ as

$$v_i := \begin{cases} v'_i & \text{if } i \le k-2 \\ (x_{i-1} - 1, y_{i-1}) & \text{otherwise} \end{cases}$$

For convenience, we use a superscript to represent the profile. Namely, $v_i = (x_i^v, y_i^v)$ means that player-i's initial vertex is located at (x_i^v, y_i^v) in profile v.

Algorithm 1 Stretch or compress a profile

function STRETCH(n, m, k, v', i)
  return STRETCH-OR-COMPRESS$(n, m, k, v', i, 1)$

function COMPRESS(n, m, k, v', i)
  return STRETCH-OR-COMPRESS$(n, m, k, v', i, -1)$

function STRETCH-OR-COMPRESS(n, m, k, v', i, soc)
  for $j \leftarrow 1$ to $k/2$ **do**
    $a \leftarrow soc \cdot \min\{i, j - 1\}$
    $v_{2j-1} \leftarrow (x_{2j-1}^{v'} + a, y_{2j-1}^{v'} + [a \text{ is odd}])$
    $v_{2j} \leftarrow (x_{2j}^{v'} + a, y_{2j}^{v'} + [a \text{ is odd}])$
  return v

For our constructed strategy profile, the positions of the initial vertices are specific. Thus, the payoff of each players can be calculated, and the payoff of their changes can be predicted, wherever the changes are located. For any constructed profile we check all the changes of all the players. In fact, it is an expatiatory work to check that our construction is a pure Nash equilibrium with the tools provided in Appendix. To convince readers that the constructed profile is pure Nash, we show that it works in a simple case where k, m are even and $k|n$.

For v_1, there exists no improving. To see this, consider a change $v'_1 = (x'_1, y'_1)$. Obviously, we can assume that $y'_1 \le y_1$. If $v'_1 \in \mathrm{Part}_{-1,0}(x_2+1, y_2)$, the position with the maximized payoff is (x_1, y_1). If $v'_1 \in \mathrm{Part}_{0,-1}(x_2, y_2+1)$, the position (x_2, y_2-1) is with the maximized payoff nm/k, which cannot be an improving. If $v'_1 \in \mathrm{Part}_{0,-1}(v_i) \cap \mathrm{Part}_{0,-1}(v_{i+1})$ where i is odd, (x_i, y_i-2) or $(x_{i+1}, y_{i+1}-2)$ can be with the maximized payoff, which is no more than $(2n/k+1)(m/2-1)$. If $v'_1 \in \mathrm{Part}_{1,0}(x_i-1, y_i) \cap \mathrm{Part}_{-1,0}(x_{i+1}+1, y_{i+1})$ where i is even, the maximized payoff is touched only if $y'_1 = y_1$. Thus, it cannot be an improving, since v is a pure Nash profile on the sub-path $[n] \times \{m/2+1\}$. If $v'_1 \in \mathrm{Part}_{0,-1}(v_i) \cap \mathrm{Part}_{0,-1}(v_{i+1})$ where i is even, the payoff cannot be more than $nm/2k + o(nm)$. By similar argument, we can also prove that there exists no improving to v_i for all $i \in [2, k]$.

4 Voronoi game on grids within 4 players

Note that there is always a pure Nash equilibrium in $\mathit{\Gamma}(\mathrm{Grid}_{n,m}, 2)$: if the grid has more than one centroid, v_1 and v_2 take a pair of adjacent centroids; Otherwise, v_1 takes the centroid and v_2 an adjacent vertex of v_1. It is easy to verify the pure Nash-property.

For the case $\mathit{\Gamma}(\mathrm{Grid}_{n\times m}, 3)$, we prove the following conclusion whose proof is in in Appendix B:

Theorem 5. *For any sufficiently large integer n, m, there exists no pure Nash equilibrium in $\mathit{\Gamma}(\mathrm{Grid}_{n\times m}, 3)$.*

Next, we discuss a more challenging case $\mathit{\Gamma}(\mathrm{Grid}_{n\times m}, 4)$. In the first step, we will prove there must exist a controller in a pure Nash equilibrium profile.

Lemma 1. *In $\mathit{\Gamma}(\mathrm{Grid}_{n\times m}, 4)$ where n, m are sufficiently large, a profile is not pure Nash equilibrium if no initial vertex controls the others.*

If there does not exist a controller, the leftmost, rightmost, uppermost and bottommost initial vertices must be distinct. Lemma 1 is proved by classifying the cases according to the relationship between the position of the leftmost initial vertex v_3 and the rightmost one v_4.

Our construction contains a pair of adjacent initial vertices. In the second step, we prove this sub-structure is necessary in any pure Nash equilibrium profiles.

Lemma 2. *In $\mathit{\Gamma}(\mathrm{Grid}_{n\times m}, 4)$ where n, m are sufficiently large, the profile is not pure Nash equilibrium, if and only if the players controls the others and the distance between any 2 players' initial vertices is not shorter than 2.*

In the third step, we show that for any pure Nash equilibrium, the 4 initial vertices should be divided into 2 vertex-pairs and the initial vertices in each pair are adjacent.

Lemma 3. *In $\Gamma(\mathrm{Grid}_{n\times m}, 4)$ where n, m are sufficiently large, the profile is not pure Nash equilibrium if $\|v_1 - v_2\|_1 = 1$ and $\|v_3 - v_4\|_1 > 1$.*

As what we discuss in the previous section, the payoff of some player is easy to analyze if the corresponding initial vertex does not span the "slashes" of others. So, we divide all the vertices in the grid into 4 areas by the "slashes" of v_1 and v_2 and classify the cases by the position of v_3 and v_4.

In Lemma 4, we prove the profile like our construction form automatically to avoid an improving. And we show the bound beyond which our construction is not pure Nash equilibrium. In another word, we give a condition in which a pure Nash profile does not exist.

Lemma 4. *In $\Gamma(\mathrm{Grid}_{n\times m}, 4)$ where n and m are sufficiently large and $\|v_1 - v_2\|_1 = 1$ and $\|v_3 - v_4\|_1 = 1$ hold, in the case where $4|n$ and m is odd, there does not exist a pure Nash profile if $m \geq n/2 + 2\lfloor\sqrt{n}\rfloor + 5$ holds. In the other cases, there does not exist a pure Nash profile if $m > n/2 + 6$.*

With Lemma 1, Lemma 2, Lemma 3 and Lemma 4, we can characterize the non-existence condition of pure Nash equilibrium for this game. Meanwhile, we can construct a pure Nash-profile if the grid is sufficiently narrow according to Theorem 2. Particularly, we consider the case where $4|n$ and m is odd.

Lemma 5. *In $\Gamma(\mathrm{Grid}_{n\times m}, 4)$ where n, m are sufficiently large, $4|n$ and m is odd, there exists a pure Nash profile if $m \leq n/2 + 2\lfloor\sqrt{n}\rfloor + 1$ holds.*

In Lemma 5, we show that the necessary condition can be loosen. Combining all the negative results, the pure Nash construction and the analysis to the special case with a loosen condition in this section, Theorem 1 can be proved.

The proofs of all the lemmas in this section is in Appendix A.

5 Voronoi game on trees with 3 players

In this section, we discuss the Voronoi game on trees among 3 players. For the limit of pages, we only show the algorithm induced by Theorem 3.

The main idea towards Theorem 3 is that if there exists a Nash equilibrium on a tree with 3 players, then their positions must induce a Path_3 on the tree (See Appendix C.1). Based on this conclusion, it is easy to verify the group of conditions in Theorem 3 is sufficient and necessary. More information of the proof is in Appendix C.2.

This theorem can induce an efficient algorithm as a judgement. See details in Algorithm 2, where each $val_{v,i}(v \in V(T), i = 1, 2)$ represents the value of $st(w, v)$, where w is a neighbor of v with the i-th maximum $st(\cdot, v)$, and each $lab_{v,i}(v \in V(T), i = 1, 2)$ records the vertex w. If there is a Nash equilibrium, the algorithm will return a triple of three different vertices as a solution, otherwise it will return -1.

The correctness of this algorithm is promised by Theorem 3. Now let us analyze its performance. Let $n = v(T)$. The main space cost of this algorithm is the tree structure and the arrays $\{val_{v,i}\}$ and $\{lab_{v,i}\}$, so it is $O(n)$. For the time cost, it is easy to see that the two arrays $\{val_{v,i}\}$ and $\{lab_{v,i}\}$ can be computed by a dynamic programming on the tree T within $O(n)$ time. Combine the $O(n)$ time cost by Algorithm 2, the total time cost is $O(n)$ too.

Algorithm 2 Nash equilibrium on a tree T with 3 players

```
for v ∈ V(T) do
    sti1 ← val_{v,1}
    sti2 ← val_{v,2}
    if lab_{lab_{v,1},1} = v then
        stj1 ← val_{lab_{v,1},2}
    else
        stj1 ← val_{lab_{v,1},1}
    if label_{lab_{v,2},1} = v then
        stj2 ← val_{lab_{v,2},2}
    else
        stj2 ← val_{lab_{v,2},1}
    if (v(T) − sti1 − sti2 ≥ max{stj1, stj2}) ∧ (sti2 ≥ stj1) then
        return (lab_{v,1}, v, lab_{v,2})
return −1
```

6 Conclusion

In this paper, we consider one-round multi-player Voronoi games on grids and trees. First, we answer such a question: is there a pure Nash equilibrium in a one-round k-player Voronoi game processing on $\text{Grid}_{n\times m}$ $(n \geq m)$? For the game on sufficiently large grids within 4 players, we provide almost complete characterization of the existence of a pure Nash equilibrium. Second, for the game on a tree with 3 players, we give a sufficient-necessary condition for the existence of a pure Nash equilibrium, as well as a linear time/space algorithm to check the condition.

For the case with k player $(k > 4)$ on a grid, we raise a method to construct a pure Nash equilibrium profile if the grid sufficiently narrow. Furthermore, we conjecture the condition is also necessary.

Conjecture 1. For a given integer k and any sufficiently large integers n, m which satisfy $n \geq m > n/\lceil k/2 \rceil + o(n)$, there does not exist a pure Nash equilibrium in $\Gamma(\text{Grid}_{n\times m}, k)$.

We say a profile is non-trivial if in the profile, no obvious improving can be detected by the tools lemmas, i.e, Lemma 6, Lemma 7, Lemma 8 and Lemma 9, stated in Appendix A. The number of non-trivial profiles is limited, especially when k is small. Actually, the approach to prove the non-existence for $k = 4$ is to check each non-trivial profile with some tricks to simplify such checking process. Thus, we believe the approach also works for the larger but constant k.

For the case of tree, it is worth thinking how to determine whether there exists a pure Nash equilibrium with 4 players. It is also interesting to determine whether there is a Nash equilibrium with multi-players on other graph classes.

References

1. Ahn, H.K., Cheng, S.W., Cheong, O., Golin, M., Van Oostrum, R.: Competitive facility location: the voronoi game. Theoretical Computer Science **310**(1-3), 457–467 (2004) [pp. 1, 2, 4]
2. Alon, N., Feldman, M., Procaccia, A.D., Tennenholtz, M.: A note on competitive diffusion through social networks. Information Processing Letters **110**(6), 221–225 (2010) [pp. 1, 2]
3. Banik, A., Bhattacharya, B.B., Das, S.: Optimal strategies for the one-round discrete voronoi game on a line. Journal of Combinatorial Optimization **26**(4), 655–669 (2013) [p. 4]
4. Banik, A., Bhattacharya, B.B., Das, S., Das, S.: Two-round discrete voronoi game along a line. In: Frontiers in Algorithmics and Algorithmic Aspects in Information and Management, pp. 210–220. Springer (2013) [p. 4]
5. Banik, A., Bhattacharya, B.B., Das, S., Mukherjee, S.: The discrete voronoi game in r2. Computational Geometry **63**, 53–62 (2017) [p. 4]
6. Banik, A., De Carufel, J.L., Maheshwari, A., Smid, M.: Discrete voronoi games and -nets, in two and three dimensions. Computational Geometry **55**, 41–58 (2016) [p. 4]
7. de Berg, M., Kisfaludi-Bak, S., Mehr, M.: On one-round discrete voronoi games. arXiv preprint arXiv:1902.09234 (2019) [p. 4]
8. Berg, M.D., Gudmundsson, J., Mehr, M.: Faster algorithms for computing plurality points. ACM Transactions on Algorithms (TALG) **14**(3), 36 (2018) [p. 4]
9. Bulteau, L., Froese, V., Talmon, N.: Multi-player diffusion games on graph classes. In: International Conference on Theory and Applications of Models of Computation. pp. 200–211. Springer (2015) [pp. 4, 6]
10. Cheong, O., Har-Peled, S., Linial, N., Matousek, J.: The one-round voronoi game. Discrete & Computational Geometry **31**(1), 125–138 (2004) [p. 2]
11. Dürr, C., Thang, N.K.: Nash equilibria in voronoi games on graphs. In: European Symposium on Algorithms. pp. 17–28. Springer (2007) [pp. 2, 4]
12. Etesami, S.R., Basar, T.: Complexity of equilibrium in diffusion games on social networks. In: American Control Conference (ACC), 2014. pp. 2065–2070. IEEE (2014) [p. 4]
13. Nash, J.: Non-cooperative games. Annals of mathematics pp. 286–295 (1951) [p. 3]
14. Roshanbin, E.: The competitive diffusion game in classes of graphs. In: International Conference on Algorithmic Applications in Management. pp. 275–287. Springer (2014) [pp. 3, 4]
15. Teramoto, S., Demaine, E.D., Uehara, R.: Voronoi game on graphs and its complexity. In: 2006 IEEE Symposium on Computational Intelligence and Games. pp. 265–271. IEEE (2006) [p. 4]

Appendix

A The proof of Theorem 2

First, we investigate several necessary conditions for a pure Nash profile: Lemma 6, Lemma 7 and Lemma 8. They will be used as tools to find an obvious improving rapidly and to certify such profile is not a pure Nash. Intuitively, we can perceive if an initial vertex v_1 moves close to another initial vertex v_2, it will snatch some vertices from I_2. If some change makes the initial vertex closer to all the other initial vertex, it may be an improving.

Lemma 6. *For some player occupying $v_i = (x_i, y_i)$, $v_i' = (x_i + 1, y_i + 1)$ is an improving of v_i if $x_j + y_j > x_i' + y_i'$ holds for all $j \in [k]\backslash\{i\}$.*

Lemma 6 can be generalized due to the symmetry of grids as well as Lemma 7. Namely, after reflecting coordinates along a dimension or rotating the grid by 90 degrees, these lemmas also hold. In the following proofs, we omit the statement about such generalization.

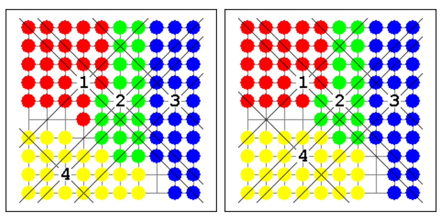

Fig. 6. v_4 moves from (3,2) to (4,3), and the payoff increases by 4.

Proof. Obviously, after moving initial vertex to v_i', the distance between v_i' and the vertices in $\text{Part}_{-1,1}(x_i, y_i+1)$ and $\text{Part}_{1,-1}(x_i+1, y_i)$ cannot be longer than before. Thus, the player-i's payoff will not reduce in these vertex sets. For all vertices $p = (x, y)$ in $\text{Part}_{-1,-1}(x_i, y_i)$ and other initial vertices $v_j = (x_j, y_j)$, we claim that $x_j \geq x$ or $y_j \geq y$ (otherwise, v_j will be in $\text{Part}_{-1,-1}(x_i, y_i)$ and $x_j + y_j > x_i' + y_i'$ cannot hold). Thus,

$$
\begin{aligned}
\|v_i' - p\|_1 &= x_i' + y_i' - x - y \\
&< x_j + y_j - x - y = \|v_j - p\|_1 && \text{when } x_j \geq x \text{ and } y_j \geq y \\
\|v_i' - p\|_1 &= x_i' + y_i' - x - y \\
&< x_j + y_j + (y - y_j) - x - y = x_j - x \\
&< x_j - y_j - x + y = \|v_j - p\|_1 && \text{when } x_j \geq x \text{ and } y_j < y \\
\|v_i' - p\|_1 &= x_i' + y_i' - x - y \\
&< x_j + (x - x_j) + y_j - x - y = y_j - y \\
&< -x_j + y_j + x - y = \|v_j - p\|_1 && \text{when } x_j < x \text{ and } y_j \leq y
\end{aligned}
$$

hold, which shows $\mathrm{Part}_{-1,-1}(x_i, y_i) \subseteq I_i(v'_i, v_{-i})$. Considering the vertices in $\mathrm{Part}_{1,1}(v'_i)$, their distance to v'_i must be shorter and we can claim player-i's payoff in $\mathrm{Part}_{1,1}(v'_i)$ cannot decrease and $I_i \subseteq I_i(v'_i, v_{-i})$ holds. Pick an initial vertex v_j in the other initial vertices, for which $\forall l \in [k]\backslash\{i\} \|v_i - v_j\|_1 \leq \|v_i - v_l\|_1$ holds. Without loss of generality, assume v_j is located in $\mathrm{Part}_{1,0}(x_i, y_i+1)$. For a vertex $p = (x, y) = (x'_i + \left\lceil \frac{|y_j - y'_i| + x_j - x'_i}{2} \right\rceil - 1, y'_i)$, it is clear that v is in $\mathrm{Part}_{1,1}(v'_i)$ and $x \leq x_j$ holds, and

$$
\begin{aligned}
&\|v - v'_i\|_1 = \left\lceil \frac{|y_j - y'_i| + x_j - x'_i}{2} \right\rceil - 1 \\
<&\|v - v_j\|_1 = x_j - x'_i - \left\lceil \frac{|y_j - y'_i| + x_j - x'_i}{2} \right\rceil + 1 + |y_j - y'_i| \\
=&\|v - v_j\|_1 = \left\lfloor \frac{|y_j - y'_i| + x_j - x'_i}{2} \right\rfloor + 1 \\
\leq&\|v - v_i\|_1 = \left\lceil \frac{|y_j - y'_i| + x_j - x'_i}{2} \right\rceil + 1
\end{aligned}
$$

can be deduced. Due to the way that v_j is picked, we cannot find an initial vertex v_l which keeps $\|v - v_l\|_1 \leq \|v'_i\|_1$. As the result, p is influenced by v'_i but not v_i, which means the change v'_i get the extra payoff than before and v'_i is an improving of v_i. □

Furthermore, we study how player-i's payoff changes when v_i moves along the orientation of the slash in this lemma. As the main conclusion, whether the payoff increases or decreases depends on the distance from v_i to the below and the left boundaries of the grid.

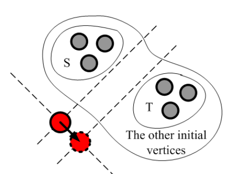

Fig. 7. The change in Lemma 7. The red vertex represents v_i, which is moving along the slash orientation.

Lemma 7. *For some player occupying $v_i = (x_i, y_i)$, $v'_i = (x_i - 1, y_i + 1)$ will be an improving of v_i, if $\{v_j \mid j \in [k]\backslash\{i\}\}$ can be divided into 2 disjoint and non-empty sets S, T satisfying $S \subseteq \mathrm{Part}_{0,-1}(v'_i)$, $T \subseteq \mathrm{Part}_{1,0}(v_i)$ and $x_i - y_i > 1$.*

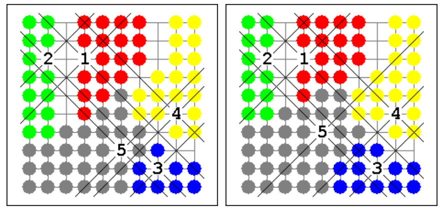

Fig. 8. v_5 moves from (6,3) to (5,4), and its payoff increases by 2.

Proof. The distances between player-i's initial vertex and the vertices in

$$\mathrm{Part}_{-1,-1}(x_i - 1, y_i) \cup \mathrm{Part}_{1,1}(x, y_i + 1)$$

do not change, thus, the payoff in the vertex set will also not change. According to Lemma 6, the payoff will increase in $\mathrm{Part}_{-1,1}(v_i')$.

Considering the vertex set $S_r = \{r\} \times [y_i, m]$ ($r \in [x_i - 1]$), that some vertex (r, y') in S_r is influenced by v_i means $r \times [y_i, y'] \subseteq I_i$. Assume a vertex (r, y'') in $r \times [y_1, y']$ and $(r, y'') \notin I_i$, which mean there is some initial vertex v_j makes

$$\begin{aligned}\|(r, y') - v_i\|_1 =& \|(r, y') - (r, y'')\|_1 + \|(r, y'') - v_i\|_1 \geq \\ & \|(r, y') - (r, y'')\|_1 + \|(r, y'') - v_j\|_1 = \|(r, y') - v_j\|_1\end{aligned}$$

hold and it conflicts with $(r, y') \in I_i$.

Notice $\forall j \in [k]\backslash\{i\}(x_j + y_j > x_i + y_i)$ and S is a non-empty set, (r, y_1) must belong to I_i and (r, m) cannot belong to I_i, thus, the vertex $p = (r, \sup\{y \in [y_1, m] | (r, y) \in I_i\} + 1)$ exists. Consider this vertex $p = (x_p, y_p)$, it is not influenced by v_i but $p' = (x_p, y_p - 1)$ is, so,

$$\begin{aligned}&\exists j \in [k]\backslash\{i\}(\|v_j - p\|_1 \leq \|v_i - p\|_1) \quad \text{and} \\ &\forall j \in [k]\backslash\{i\}(\|v_j - p'\|_1 > \|v_i - p'\|_1) \Rightarrow \\ &\qquad (\|v_j - p\|_1 + 1 \geq \|v_j - p'\|_1 > \|v_i - p\|_1 - 1) \Rightarrow \\ &\qquad (\|v_j - p\|_1 + 1 \geq \|v_i - p\|_1)\end{aligned}$$

hold, by which we can claim $p \in I_i(v_i', v_{-i})$ but $p \notin I_i$.

Let $v_j = (x_j, y_j)$ is the nearest initial vertex from p excepted v_i. Assume there is a vertex $p'' = (r, y_p'')$ in $\{r\} \times [y_p + 2, m]$ which satisfies $\|p'' - v_i\|_1 \leq \|p'' - v_j\|_1 + 1$. Then $\|(r, y_j) - v_i\|_1 \leq \|(r, y_j) - v_j\|_1 + 1$ holds if $y_p'' \geq y_j$, but that is impossible since

$$\begin{aligned}&v_j \in S \Rightarrow y_j - x_j > y_i - x_i + 2 \quad \text{and} \\ &\|(r, y_j) - v_i\|_1 = x_i - r + y_j - y_i > x_j - r + 2 = \|(r, y_j) - v_j\|_1 + 2.\end{aligned}$$

If $y_p'' < y_j$, it is easy to deduce that the vertex $(r, y_p'' - 1)$ belongs to I_i, which conflicts with $y_p'' \in [y_p + 2, m]$. Now, we can claim the player-i's payoff in S_r increase by 1 where $r \in [x_i - 1]$, as well as the player-i's payoff in $\mathrm{Part}_{-1,1}(v_i')$ increase by $x_i - 1$ after moving player-i's initial vertex to v_i'.

If we image v_i as v_i''s change, for the same reason, its payoff in $\mathrm{Part}_{1,-1}(v_i)$ increase by y_i, thus, in the reverse processing the same number of vertices loss. $\square$

The following lemma shows the profile in which all the initial vertices are bounded by a small block should not be pure Nash equilibrium.

Lemma 8. *In $\Gamma(\mathrm{Grid}_{n\times m}, k)$ where $k \geq 3$ and n, m are sufficiently large, the profile is not pure Nash equilibrium if for some constant c there exists no distinct $i, j \in [k]$ such that $x_i - x_j > c$ or $y_i - y_j > c$.*

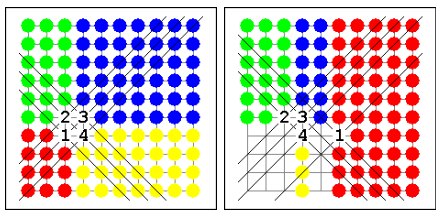

Fig. 9. The initial vertices gather in a 1-size block, (6,4) is an improving of v_1.

Proof. Consider two points $p' = (\min\{x_i|i \in [k]\} - c - 1, \min\{y_i|i \in [k]\})$ and $p'' = (\max\{x_i|i \in [k]\} + c + 1, \min\{y_i|i \in [k]\})$. First we show at least one of p' and p'' must be on the grid. If not, then $\min\{x_i|i \in [k]\} - c - 1 < 1$ and $\max\{x_i|i \in [k]\} + c + 1 > n$, which turns out that

$$n < \max\{x_i|i \in [k]\} + c + 1 \leq \min\{x_i|i \in [k]\} + 2c + 1 < 3c + 3.$$

This result conflicts with that n is sufficiently large. If exact one of these 2 vertices, suppose p'' exists but p' w.l.o.g. Thus, $\min\{x_i|i \in [k]\} - c - 1 < 1$ holds. We will prove $p'' = (x'', y'')$ is an improving of some player's initial vertex. For any initial vertex v_j, $p'' \in \mathrm{Part}_{1,0}(v_j)$ because

$$\begin{aligned} x_j + y_j &\leq \max\{x_i|i \in [k]\} + \max\{y_i|i \in [k]\} \\ &< (\max\{x_i|i \in [k]\} + c + 1) + \min\{y_i|i \in [k]\} \quad \text{and} \\ x_j - y_j &\leq \max\{x_i|i \in [k]\} - \min\{y_i|i \in [k]\} \\ &< (\max\{x_i|i \in [k]\} + c + 1) - \min\{y_i|i \in [k]\}. \end{aligned}$$

Then, for vertex $p = (x_p, y_p)$ in $\mathrm{Part}_{1,1} \cup \mathrm{Part}_{1,-1}(p'')$, we get

$$\begin{aligned} \|p - v_j\|_1 &= \|p - (x'', y_j)\|_1 + \|(x'', y_j) - v_j\|_1 \\ &= \|p - (x'', y_j)\|_1 + |x'' - x_j| < \|p - (x'', y_j)\|_1 + |y_j - y''| \\ &= \|p - (x'', y_j)\|_1 + \|(x'', y_j) - p''\|_1 \leq \|p - p''\|_1 \end{aligned}$$

which means $\mathrm{Part}_{1,1} \cup \mathrm{Part}_{1,-1}(p'') \subseteq I_i(p'', v_{-i})$ for $i \in [k]$. Let v_i is the initial vertex of player-i whose payoff is the least,

$$\begin{aligned} U_i(p'', v_{-i}) \geq m(n - x'' + 1) &= m(n - (\max\{x_i | i \in [k]\} + c + 1) + 1) \\ &\geq m(n - (\min\{x_i | i \in [k]\} + 2c + 1) + 1) \\ &> m(n - (3c + 3)) > \left\lfloor \frac{mn}{k} \right\rfloor \geq U_i \end{aligned}$$

holds when $k \geq 3$. For the same reason, if both p' and p'' exist, we can get

$$\begin{aligned} U_i(p', v_{-i}) + U_i(p'', v_{-i}) &\geq m(n - \max\{x_i | i \in [k]\} - c) + m(\min\{x_i | i \in [k]\} - c - 1) \\ &= m(n - (\max\{x_i | i \in [k]\} - \min\{x_i | i \in [k]\}) - 2c - 1) \\ &\geq m(n - 3c - 1) > 2 \left\lfloor \frac{mn}{k} \right\rfloor \geq 2U_i \end{aligned}$$

hold when $k \geq 3$ and n, m are sufficiently large. Thus, p' or p'' is an improving of v_i. □

The following lemma excludes a non-Nash case, where the 4 players are divided into 2 pairs, initial vertices in each pair are constantly close and all initial vertex is constantly close to a slash.

Lemma 9. *In $\Gamma(\mathrm{Grid}_{n \times m}, 4)$ where n, m are sufficiently large, a profile v is not a pure Nash where $\|v_1 - v_2\|_1$, $\|v_3 - v_4\|_1$ and $|(x_1 + y_1) - (x_3 + y_3)|$ are upper bounded by a constant.*

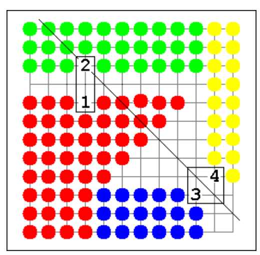

Fig. 10. An example in the case discussed in Lemma 9.

Proof. Assume v is a pure Nash. Note that $\|v_2 - v_3\|_1 \geq \omega(1)$ holds since Lemma 8. W.l.o.g, assume $x_2 \leq x_3$. We claim $m - y_1 \leq m/4 + O(1)$. Otherwise, the initial vertex with the minimal payoff, v_i for example, can be improved by moving to v_i' where $v_i' = (x_i', y_i')$ is in $\bigcup_{j \in [4]} \mathrm{Part}_{0,1}(v_j)$ with y_i' minimized. Because of our assumption, it is easy to show that $y_i' - y_1 = O(1)$. Thus, $U_i(v_i', v_{-i}) > nm/4 > U_i(v_i, v_{-i})$, due to the definition of v_i. For the similar reason, $y_3 \leq m/4 + O(1)$, $x_1 \leq n/4 + O(1)$ and $n - x_3 \leq n/4 + O(1)$ should be satisfied. Thus, either there exists $v_i' := (n/2 - O(1), m/2 + O(1))$ in

$$\mathrm{Part}_{0,-1}(v_1) \cap \mathrm{Part}_{0,-1}(v_2) \cap \mathrm{Part}_{-1,0}(v_3) \cap \mathrm{Part}_{-1,0}(v_4),$$

or $v_i'' := (n/2 + O(1), m/2 - O(1))$ in

$$\mathrm{Part}_{1,0}(v_1) \cap \mathrm{Part}_{1,0}(v_2) \cap \mathrm{Part}_{0,1}(v_3) \cap \mathrm{Part}_{0,1}(v_4).$$

W.l.o.g, the profile is in the first case. Thus, the initial vertex with the minimal payoff, v_i for example, can be improved since $U_i(v', v_{-i}) \geq 3nm/8 - o(nm)$. □

Lemma 10 (Restate Lemma 1). *In $\Gamma(\mathrm{Grid}_{n\times m}, 4)$ where n, m are sufficiently large, a profile is not pure Nash equilibrium if no initial vertex controls the others.*

Proof of Lemma 1. Since there exists no controller, the leftmost, rightmost, bottommost and uppermost initial vertices are distinct. Suppose the leftmost and the rightmost ones are v_3 and v_4 respectively. We claim $v_4 \in \mathrm{Part}_{1,0}(v_3)$. Otherwise, assume $x_3 + y_3 \geq x_4 + y_4$ without loss of generality. Note that $x_3 < x_2 < x_4$ and $y_2 > y_3$ holds, where v_2 is the uppermost initial vertex. According to Lemma 6, $v_2 = (x_2, y_2)$ can improve it payoff by moving to $(x_2 - 1, y_2 - 1)$ unless v_2 is located at $(x_3 + 1, y_3 + 1)$. If $v_2 = (x_3 + 1, y_3 + 1)$, it easy to see that $v_2' = (x_3 + 1, y_3)$ will be an improving of v_2. Thus, the inequalities

$$\begin{aligned} &|(y_1 + x_1) - (y_3 + x_3)| \leq 2, \quad |(y_2 + x_2) - (y_4 + x_4)| \leq 2, \\ &|(y_1 - x_1) - (y_4 - x_4)| \leq 2, \quad |(y_2 - x_2) - (y_3 - x_3)| \leq 2 \end{aligned}$$

hold. Otherwise, there exists an improving according the Lemma 6.

Then, we claim the distance between each pair of the initial vertices cannot be a constant. W.l.o.g, assume $x_1 \leq x_2$ holds and toward a contradiction, assume $x_1 - x_3 \leq O(1)$. (Note that the assumption about the distance between v_3, v_4 or v_1, v_2 will violate Lemma 8.) Thus, $|(y_3 - x_3) - (y_4 - x_4)|$, $\|v_1 - v_3\|_1$ and $\|v_2 - v_4\|_1$ are upper bounded by a constant, which conflicts with Lemma 9 Note that the feasibility of several changes in the proof is ensured by such dispersiveness and sometimes, by the specific discussions.

Case 1: The profile v cannot be a pure Nash where

$$y_2 - x_2 \geq y_3 - x_3 \text{ and } y_1 + x_1 \leq y_3 + x_3$$

hold. W.l.o.g, assume $x_1 \leq x_2$ holds. Consider the changes

$$v_3' = (x_3 + 2, y_3 - 2) \text{ and } v_3'' = (x_3 - 2, y_3 + 2)$$

where the feasibility of the change v_3'' will be discussed later. Their payoff can be calculated as following:

$$\begin{aligned} &U_3(v_3', v_{-3}) \geq U_3 + 2(x_2 - x_3) - 2x_3 - O(1) \text{ and} \\ &U_3(v_3'', v_{-3}) \geq U_3 + (x_3 - 2)(m - y_2 + 1) - 2(x_2 - x_3) - O(1) \end{aligned}$$

Thus, to avoid improving, $U_3(v_3', v_{-3}) + U_3(v_3'', v_{-3}) \leq 2U_3$ should be satisfied, which implies either $x_3 = O(1)$ or $m - y_2 \leq 1$. $x_3 = O(1)$ is impossible since the change $v_3' = (x_3 + 1, y_3 + 1)$ whose payoff is $U_3(v_3', v_{-3}) \geq$

$U_3 + x_2 - x_3 - O(1)$. Whereas, $x_2 - x_3 = O(1)$ conflicts with Lemma 8. Note that $x_3 = \omega(1)$ also ensures that v_3'' is feasible. Furthermore, $m - y_2 \leq 1$ is also impossible since the changes $v_2' = (x_2-2, y_2-2)$ and $v_2'' = (x_2+2, y_2-2)$. If $x_4 + y_4 \geq x_2 + y_2$ holds, the payoff of v_2' is

$$U_2(v_2', v_{-2}) \geq U_2 + 2x_2 - O(1).$$

Thus, $x_3 - x_2 = O(1)$ should be satisfied to avoid improving, which conflicts with Lemma 8 however. If $x_4 + y_4 < x_2 + y_2$ holds, the payoffs are

$$U_2(v_2', v_{-2}) \geq U_2 + 2x_2 - 2(n - x_4) - O(1) \text{ and}$$
$$U_2(v_2'', v_{-2}) \geq U_2 + 2(n - x_2) - 2x_3 - O(1).$$

To avoid improving, $x_4 - x_3 = O(1)$ holds, which conflicts with Lemma 8.

Case 2: The profile v cannot be a pure Nash where

$$v_1 \in \mathrm{Part}_{0,-1}(v_3) \cap \mathrm{Part}_{-1,0}(v_4) \text{ and}$$
$$v_2 \in \mathrm{Part}_{1,0}(v_3) \cap \mathrm{Part}_{0,1}(v_4)$$

hold. Consider the changes

$$v_1' = (x_1 - 1, y_1 + 1),$$
$$v_2' = (x_2 + 1, y_2 - 1),$$
$$v_3' = (x_3 + 1, y_3 + 1) \text{ and}$$
$$v_4' = (x_4 - 1, y_4 - 1).$$

According the Lemma 7,

$$y_1 \geq x_1 - 1,$$
$$x_3 \geq m - y_3,$$
$$m - y_2 + 1 \geq n - x_2 \text{ and}$$
$$n - x_4 + 1 \geq y_4 - 1$$

should be satisfied to avoid improving. Thus, $(y_4 - y_1) + (y_2 - y_3) + (x_1 - x_3) + (x_4 - x_2) \leq 4$ holds, which conflicts with Lemma 8.

Case 3: For the other cases, the profile can always be reduced to **Case 1** or **Case 2** by rotating and mirror-flipping.

□

Lemma 11 (Restate Lemma 2). *In $\Gamma(\mathrm{Grid}_{n\times m}, 4)$ where n, m are sufficiently large, the profile is not pure Nash equilibrium, if and only if the players controls the others and the distance between any 2 players' initial vertices is not shorter than 2.*

Proof of Lemma 2. Before proving the lemma, we also need to divide the profiles into some cases according to the relative position between the control vertex and the others. We will show that improving for some initial vertex can always be found and as a result she pure Nash equilibrium profile does not exist in any case. Without loss of generality, let v_1 be the initial vertex which controls other vertices.

Case 1: If other players' initial vertices are far way from v_1. (Precisely, $\forall j \in \{2,3,4\}$ $\|v_j - v_1\|_2 > 2$ holds.) $v_1' = (x_1+1, y_1+1)$ is an improving of v_1 for player-1 because of Lemma 6.

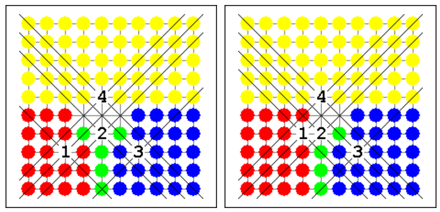

Fig. 11. v_1 improve its payoff by moving to (x_1+1, y_1+1).

Case 2.1: Otherwise, some vertex in $\{(x_1+2, y_1), (x_1+1, y_1+1), (x_1, y_1+2)\}$ would be occupied by v_2 (the initial vertex of player-2) without loss of generality. If (x_1+1, y_1+1) is occupied, the proof of the second case in Lemma 2 can be applied in this case and $v_1' = (x_1+1, y_1)$ is an improving of v_1.

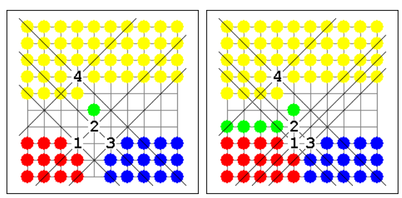

Fig. 12. v_1 improve its payoff by moving to (x_1+1, y_1).

Case 2.2: If (x_1+1, y_1+1) is not occupied, (x_1+2, y_1) or (x_1, y_1+2) must be taken to differ it from Case 1. When the 2 vertices are both taken (player-2 takes $v_2 = (x_1+1, y_1)$ and player-3 takes $v_3 = (x_1, y_1+1)$ without loss of generality), the payoff of player-1 in $\mathrm{Part}_{1,1}(x_1+1, y_1+1) \cup \mathrm{Part}_{1,-1}(x_1+1, y_1) \cup \mathrm{Part}_{-1,1}(x_1, y_1+1)$ is zero because

$$\begin{aligned}\forall v \in \mathrm{Part}_{1,1}(v')(\|v - v_1\|_1 =& \|v - v'\|_1 + \|v' - v_1\|_1 \\ =& \|v - v'\|_1 + \|v' - v_2\|_1 \le \|v - v_2\|_1)\end{aligned}$$

holds where $v' = (x_1+1, y_1+1)$,

$$\begin{aligned}\forall v \in \mathrm{Part}_{1,-1}(v'')(\|v - v_1\|_1 =& \|v - v''\|_1 + \|v'' - v_1\|_1 \\ =& \|v - v''\|_1 + \|v'' - v_2\|_1 \le \|v - v_2\|_1)\end{aligned}$$

holds where $v'' = (x_1+1, y_1)$ and similar conclusion is clear for $\mathrm{Part}_{-1,1}(x_1, y_1+1)$. Since (x_1+1, y_1) and (x_1, y_1+1) are not occupied, we can get $x_1 + y_1 + 1 < x_j + y_j$ for any other player-j with initial vertex

$v_j = (x_j, y_j)$. Note v_1 is the control vertex, Thus, $x_1 \leq x_j$ and $y_1 \leq y_j$ hold. Because of

$$\begin{aligned}\forall v \in \text{Part}_{-1,-1}(v_1)(\|v - v_1'\|_1 = \|v - v_1\|_1 + 1 \\ = x_1 - x_v + y_1 - y_v + 1 < x_j - x_v + y_j - y_v = \|v_j - v\|_1)\end{aligned}$$

player-1 influences all of the vertex in $\text{Part}_{-1,-1}(v_1)$ even if we change his initial vertex to $v_1' = (x+1, y_1)$. But player-1 take (x_1+1, y_1), a vertex in $\text{Part}_{1,-1}(x_1+1, y_1)$ in which player-1 have no influenced vertex before. So, we can claim that $v_1' = (x_1+1, y_1)$ is an improving of v_1.

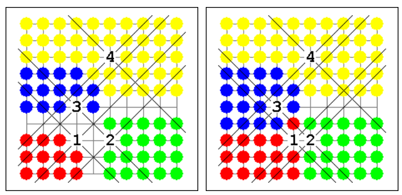

Fig. 13. v_1 improve its payoff by moving to (x_1+1, y_1).

Case 2.3: If only one vertex in $\{(x_1+2, y_1), (x_1, y_1+2)\}$ is taken (without loss of generality, assuming (x_1+2, y_1) is taken by player-2), another important vertex (x_1+1, y_1+2) should be considered. If (x_1+1, y_1+2) is not occupied, we can claim $v_1' = (x_1, y_1+1)$ is an improving of v_1. Obviously, player-1's payoff in $\text{Part}_{-1,1}(v_1') \cup \text{Part}_{1,1}(v_1')$ could not reduce since we move player-1's initial vertex closer to these vertices in the vertex set. And $\text{Part}_{0,0}(v_1) \subseteq I_i$ holds because v_1 controls the others. For the same reason as **Case 2.2**, $\text{Part}_{1,1}(x_1+1, y_1+1) \cap I_1 = \emptyset$, but (x_1+1, y_1+1) belong to $I_i(v_1', v_{-1})$.

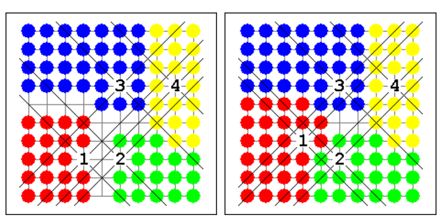

Fig. 14. v_1 improve its payoff by moving to (x_1, y_1+1).

Case 2.4: The last case, in which the vertices (x_1+2, y_1) and (x_1+1, y_1+2) are taken by v_2 and v_3, should be analyzed now. The only remaining initial vertex v_4 cannot be in $\text{Part}_{1,1}(x_1+3, y_1+3)$, otherwise v_4 is the strict control vertex. According to Lemma 6, if we do not want an improving of v_4, its feasible positions will be quite limited which include $(x_1, y_1+3), (x_1+1, y_1+3), (x_1+$

$2, y_1+3), (x_1+3, y_1+2), (x_1+3, y_1+1), (x_1+4, y_1+1)$ and (x_1+4, y_1). Each of them cannot form a pure Nash equilibrium profile because of Lemma 8.

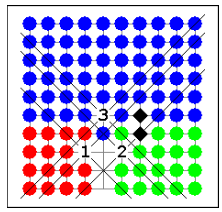

Fig. 15. v_4 can only be located at (x_1+3, y_1+1) or (x_1+3, y_1+2).

□

Lemma 12 (Restate Lemma 3). *In* $\Gamma(\mathrm{Grid}_{n\times m}, 4)$ *where* n, m *are sufficiently large, the profile is not pure Nash equilibrium if* $\|v_1 - v_2\|_1 = 1$ *and* $\|v_3 - v_4\|_1 > 1$.

Proof of Lemma 3. Without loss of generality, we can fix the positions of v_1 and v_2 at (x_1, y_1) and (x_1+1, y_1) and divide $V(\mathrm{Grid}_{n\times m})$ into

$$
\begin{aligned}
P_1 &= \mathrm{Part}_{0,-1}(v_1) \cup \mathrm{Part}_{0,-1}(v_2)\\
P_2 &= \mathrm{Part}_{1,0}(v_1) \backslash \{v_2\}\\
P_3 &= \mathrm{Part}_{0,1}(v_1) \cup \mathrm{Part}_{0,1}(v_2)\\
P_4 &= \mathrm{Part}_{-1,0}(v_2) \backslash \{v_1\}.
\end{aligned}
$$

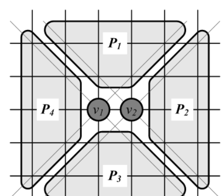

Fig. 16. The grid is divided into 4 parts.

Now, we will debate upon each case separately in which v_3 and v_4 is in different part in grid.

Case 1: If both v_3 and v_4 is in P_1, the profile cannot be pure Nash equilibrium. If $x_3+y_3 \not\equiv x_4+y_4 \pmod 2$ holds, there exists an improving for v_3 or v_4, unless $\|v_3 - v_4\|_1 = 1$, according to Lemma 6. Otherwise, the only case we need to consider is that $y_3 = y_4$ and $x_4 - x_3 = 2$. W.l.o.g, assume $x_3 + y_3 \equiv x_1 + y_1 \pmod 2$ holds. Thus, it can improve v_3 by moving it to $(x_3, y_3 - 1)$. The similar argument also shows v_3 and v_4 cannot be both in P_2, P_3, P_4.

Case 2: If $v_3 \in P_2$ and $v_4 \in P_4$ hold, v_3 and v_4 will be improved unless $\|v_1 - v_3\|_1, \|v_1 - v_4\|_1 = O(1)$ due to Lemma 6, which conflicts with Lemma 8. This argument also is adequate for the case where $v_3 \in P_1$ and $v_4 \in P_3$ hold.

Case 3: The profile v where $v_3 \in P_4$ and $v_4 \in P_1$ hold cannot be a pure Nash. According to Lemma 6, $\|v_3 - v_4\|_1 = O(1)$, $x_1 + y_1 - (x_3 + y_3) \leq 2$ and $x_4 + y_4 - (x_2 + y_2) \leq 2$ hold. Thus, there exists a contradiction due to Lemma 9.

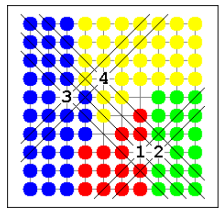

Fig. 17. A sample profile in Case 3

□

Lemma 13 (Restate Lemma 4). *In $\Gamma(\mathrm{Grid}_{n\times m}, 4)$ where n and m are sufficiently large and $\|v_1 - v_2\|_1 = 1$ and $\|v_3 - v_4\|_1 = 1$ hold, in the case where $4|n$ and m is odd, there does not exist a pure Nash profile if $m \geq n/2 + 2\lfloor\sqrt{n}\rfloor + 5$ holds. In the other cases, there does not exist a pure Nash profile if $m > n/2 + 6$.*

Proof of Lemma 4. In the most cases, it is not pure Nash equilibrium, if $m > n/2 + 6$ holds. We will discuss them in detail.

Case 1: If $x_2 = x_1 + 1$ and $y_4 = y_3 + 1$ hold, the profile cannot be a pure Nash if $m > n/2 + 6$. W.l.o.g, assume the $v_2 \in \mathrm{Part}_{-1,0}(x_3 + 1, y_3) \cap \mathrm{Part}_{-1,0}(x_4 + 1, y_4)$ holds. Note that $|(x_2 + y_2) - (x_3 + y_3)|, |(y_2 - x_2) - (y_3 - x_3)| \geq \omega(1)$, since Lemma 9. Besides, $\|v_2 - v_3\|_1 \geq \omega(1)$ should be satisfied since Lemma 8. Next, $n - x_4 \leq n/4 + 1$ holds, unless the initial with the minimal payoff v_i can be improved by moving to $v'_i = (x_4 + 2, y_4)$ where $U_i(v'_i, v_{-i}) \geq (n - x_4 - 1)m$. Then, consider the changes for v_3 and v_4, $v'_3 = (x_3 - 1, y_3 - 1)$ and $v'_4 = (x_4 - 1, y_4 + 1)$ where $U_3(v'_3, v_{-3}) = U_3 + y_3 - 1 - (n - x_3 + 1)$ and $U_4(v'_4, v_{-4}) = U_4 + (m - y_4) - (n - x_4 + 1)$. To avoid improving, $m \leq n/2 + 6$ should be satisfied, which conflicts with our assumption.

Case 2: If $y_2 = y_1 + 1$ and $y_4 = y_3 + 1$ hold, the profile cannot be a pure Nash if $m > n/2 + 6$. W.l.o.g, assume the $v_2 \in \mathrm{Part}_{-1,0}(x_3 + 1, y_3) \cap \mathrm{Part}_{-1,0}(x_4 + 1, y_4)$ holds. The argument in Case 1 is also adequate for this case.

Case 3: If $x_2 = x_1 + 1$, $x_4 = x_3 + 1$ and $v_2 \in \mathrm{Part}_{-1,0}(v_3)$ hold, it should be discussed carefully since this case includes our constructions in Section 3. Note that $|(x_2 + y_2) - (x_3 + y_3)|, |(y_2 - x_2) - (y_3 - x_3)|, \|v_2 - v_3\| \geq \omega(1)$ according to Lemma 8 and Lemma 9. First, we claim $|y_2 - y_3| < 2$ holds. If not, we can assume $y_2 - y_3 \geq 2$. Thus, $m - y_2 + y_3 \leq m - 2$ holds. W.l.o.g, suppose $m - y_2 + 1 \leq (m - 1)/2$. Consider the change $v'_2 = (x_2 + 1, y_2 - 1)$, where $U_2(v'_2, v_{-2}) = U_2 + (y_2 - 1) - (m - y_2 + 1)$. The new payoff is more than the original one. Actually, we can learn more from it, that $|y_i - m/2| < 1$ should satisfied to avoid such improving in a pure Nash profile for all $i \in [4]$.

Case 3.1 For even m, let v_i be the initial vertex with the minimize payoff. If $i \in \{1, 4\}$, suppose $i = 1$ and $y_1 = m/2 + 1$ w.l.o.g. Consider the change $v_1' = (x_2, y_2 - 1)$. It is not so hard to verify that $U_1(v_1', v_{-1}) \geq U_1$. Thus, v is a pure Nash only if the new profile in a pure Nash. Note that $x_1 \leq n/4+1$, otherwise, $v_i'' = (x_1 - 1, y_1)$ will improve the poorest initial vertex. Whereas, $x_1' \geq y_1' - 1$ should be satisfied according to Lemma 7, which conflicts with $m > n/2 + 6$. If $i \in \{2, 3\}$, the same argument also works.

Case 3.2 If m is odd, $y_i = (m + 1)/2$ holds for all $i \in [4]$. Note that v is a pure Nash only if it is a pure Nash on the sub-path $\{(m + 1)/2\} \times [n]$. Thus, all the non-trivial case is corresponding to a pure Nash profile on Path_n.

Case 3.2.1: If $n \not\equiv 0 \pmod 4$, the profile is not a pure Nash if $m > n/2 + 6$. Recall $\Gamma(\mathrm{Path}_n, 4)$. It is easy to see that for a pure Nash profile, the payoff of each initial vertex is no less than $\lfloor n/4 \rfloor m$. The minimal one can touch the lower bound and the maximal one is no more than $\lceil n/4 \rceil m$.

Case 3.2.1.1 If $U_1 = \lfloor n/4 \rfloor m$ and $U_2 = \lceil n/4 \rceil m$, consider the change $v_1' = (x_2, y_2 - 1)$ where $U_1(v_1', v_{-1}) \geq (2\lfloor n/4 \rfloor + 1)(m - 1)/2$. It is an improving if $m > n/2 + 6$.

Case 3.2.1.2 If $U_1 = \lceil n/4 \rceil m$ and $U_2 = \lfloor n/4 \rfloor m$, consider the change $v_2' = (x_1, y_1 - 1)$ where $U_2(v_2', v_{-2}) \geq (2\lfloor n/4 \rfloor + 1)(m - 1)/2$. It is an improving if $m > n/2 + 6$.

Case 3.2.1.3 If $U_1 = U_2 = \lfloor n/4 \rfloor m$ and $x_2 \equiv x_3 \pmod 2$, consider the change $v_1' = (x_2, y_2 - 1)$ where $U_1(v_1', v_{-1}) \geq (2\lfloor n/4 \rfloor + 1)(m - 1)/2$. It is an improving if $m > n/2 + 6$.

Case 3.2.1.3 The other cases can be reduced to these 3 cases.

Case 3.2.2: If $n \equiv 0 \pmod 4$, there exists the unique pure Nash profile in $\Gamma(\mathrm{Path}_n, 4)$. Thus, the profile is not a pure Nash unless

$$v_1 = \left(\frac{n}{4}, \frac{m+1}{2}\right), v_2 = (x_1{+}1, y_1), v_3 = \left(\frac{3n}{4}, y_1\right) \text{ and } v_4 = (x_3{+}1, y_1).$$

Then, considering the vertex

$$v_1' = \left(x_1 + \left\lfloor \frac{m-1}{4} - \frac{n}{8} + \frac{1}{2} \right\rfloor, y_1 - \left\lfloor \frac{m-1}{4} - \frac{n}{8} + \frac{1}{2} \right\rfloor\right)$$

as v_1's change, we can calculate its payoff,

$$U_1(v_1', v_{-1}) = \left\lfloor \frac{1}{16}m^2 + \frac{3}{16}mn - \frac{3}{8}m + \frac{1}{64}n^2 - \frac{1}{16}n + \frac{9}{16} \right\rfloor$$

and it is more than U_1 when $m \geq n/2 + 2\left\lceil \sqrt{n} \right\rceil + 5$ holds.

□

Lemma 14 (Restate Lemma 5). *In $\Gamma(\mathrm{Grid}_{n \times m}, 4)$ where n, m are sufficiently large, $4|n$ and m is odd, there exists a pure Nash profile if $m \leq n/2 + 2\lfloor \sqrt{n} \rfloor + 1$ holds.*

Proof. We will check whether there is any improving for v_1 first. Donate $v_1' = (x_1', y_1')$ as a change of v_1. Without loss of generality, we assume $y_1' \leq y_1$. Note that the positions $(x_1, y_1), (x_1, y_1 - 1)$ and $(x_2, y_2 - 1)$ lead to the maximized payoff according to Lemma 6 if $x_1' \leq x_1$. Furthermore, it is easy to see $U_1((x_1, y_1 - 1), v_{-1}) \leq U_1(v_1, v_{-1})$ and $U_1((x_2, y_2 - 1), v_{-1}) = n(m-1)/4 \leq U_1(v_1, v_{-1})$. Thus, we can assume $x_1' > x_1$. If $v_1' = (x_1', y_1')$ is in $\mathrm{Part}_{1,0}(v_1) \cap \mathrm{Part}_{-1,0}(x_4 + 1, y_4)$ excepted $(n/2 + 1, (m+1)/2 - n/4)$

$$\begin{aligned} U_1((x_1', y_1'), v_{-1}) \leq & \left(\left\lceil \frac{x_1' - x_2 - y_1' + y_2}{2} \right\rceil + \left\lceil \frac{x_3 - x_1' - y_1' + y_2}{2} \right\rceil - 1 \right) \cdot m \\ & + (y_1' - y_2) \cdot (m - (y_1' - y_2)) \\ = & (n-1)m/4 - (y_1' - y_2)^2, \end{aligned}$$

and $U_1((n/2 + 1, (m+1)/2 - n/4), v_{-1}) = nm/4 - n^2/16 - (m-1)/2$. For the positions in $\mathrm{Part}_{0,-1}(x_2, y_2 + 1) \cap \mathrm{Part}_{0,-1}(v_3)$, $v_1' = (n/2 + 1, (m+1)/2 - n/4 - 1)$ maximizes the payoff where $U_1(v_1', v_{-1}) = nm/2 - 3n^2/16 - 3n/4 \leq nm/4$ with sufficiently large n, m if the condition holds. For the positions in $\mathrm{Part}_{0,-1}(v_2) \cap \mathrm{Part}_{-1,0}(v_4)$, the maximized-payoff can be touched only when $y_1' + x_1' = y_1 + x_1$ according to Lemma 6. Thus, with $r := x_1' - x_2$ is

$$U_1(v_1', v_{-1}) = nm/4 + rm/2 - rn/4 - n/4 - r^2 - 3r/2$$

which is no more than $nm/4$ when $m \leq n/2 + 2\lfloor \sqrt{n} \rfloor + 1$ holds.

An argument similar to the one for v_1 shows that the payoff of player-2 in the constructed profile can also not be improved. Due to the symmetry of the profile, the pure Nash property can be shown. □

B Proof of Theorem 5

Theorem 6 (Restate Theorem 5). *For any sufficiently large integer n, m, there exists no pure Nash equilibrium in $\Gamma(\mathrm{Grid}_{n \times m}, 3)$.*

Proof. By the similar argument in the proof of Lemma 2, it is easy to prove that there exists a pure Nash in $\Gamma(\mathrm{Grid}_{n \times m}, 3)$ only if, without loss of generality, $\|v_1 - v_2\|_1 = 1$ holds. Without of loss generality, assume $x_3 \leq x_1$, $x_3 \leq x_2$, $y_3 \leq y_1$ and $y_3 \leq y_2$ hold. If $\|v_1 - v_3\| \geq 100$, $x_1 + y_1 - x_3 - y_3 > 50$ and $x_2 + y_2 - x_3 - y_3 > 50$ hold, which implies the profile cannot be a pure Nash according to Lemma 6. Otherwise, the same result can also be indicated due to Lemma 8. □

C Arguments in Section 5

C.1 A lemma used in the proof of Theorem 3

Lemma 15. *In the 3-player Voronoi game on a tree T, if there exists a pure Nash equilibrium, then these three vertices induce a Path_3 on T.*

Proof. Suppose v_1, v_2, v_3 is a pure Nash equilibrium on the tree T, where each player i chooses vertex i $(i = 1, 2, 3)$. Then there are four cases at all:

Case 1: v_3 is on the path between v_1 and v_2. If v_1 is not a neighbor of v_3, then there exists a vertex $v' \notin \{v_1, v_2, v_3\}$ and v' is on the path between v_1 and v_3. It is easy to check that for player 1, (v', v_2, v_3) is better than (v_1, v_2, v_3), which is contradicted to Nash equilibrium. If v_2 is not a neighbor of v_3, then there exists a vertex $v'' \notin \{v_1, v_2, v_3\}$ and v'' is on the path between v_2 and v_3. It is easy to check that for player 2, (v_1, v'', v_3) is better than (v_1, v_2, v_3), which is contradicted to Nash equilibrium. So, both v_1 and v_2 must be neighbors of v_3, and $(v_1, v_3), (v_3, v_2)$ induce a Path_3.

Case 2: v_1 is on the path between v_3 and v_2. This case is solved by the same approach of Case 1.

Case 3: v_2 is on the path between v_1 and v_3. This case is solved by the same approach of Case 1.

Case 4: There exists a vertex $v \notin \{v_1, v_2, v_3\}$ such that v is on the path between v_2 and v_3, and v_2 and v_1. Since T is a tree and the Case 1,2,3 are all unsatisfied, the vertex v must be exist and unique, and it is also on the path between v_1 and v_3. It is easy to check that for player 1, (v, v_2, v_3) is better than (v_1, v_2, v_3), which is contradicted to Nash equilibrium. So, this case is not a pure Nash equilibrium.

Combining all these cases, we can prove this lemma. □

C.2 Proof of Theorem 3

Theorem 7 (Restate Theorem 3). *In $\Gamma(T, 3)$ where T is a tree with size n, there exists a pure Nash profile if and only if there exists a vertex v such that $n - st(i_1, v) - st(i_2, v) \geq \max\{st(j_1, v), st(j_2, v)\}$ and $st(i_2, v) \geq st(j_1, v)$, where i_1 and i_2 are the children of v with maximum and second maximum $st(\cdot, v)$, and j_k is the child of i_k with maximum $st(\cdot, v)$ $(k = 1, 2)$.*

Proof. "$\Longrightarrow$": If there exists a pure Nash equilibrium v_1, v_2, v_3 on the tree T, then because of Lemma 15, v_1, v_2, v_3 induce a Path_3 on the tree T. Suppose v_2 is the degree-2 vertex in this Path_3. Let i_1, i_2 are the children of v_2 with maximum and second maximum $st(\cdot, v_2)$, and j_k is the child of i_k with maximum $st(\cdot, v_2)$ $(k = 1, 2)$.

First, we show that $\{v_1, v_3\} = \{i_1, i_2\}$. This is obvious, since v_1 and v_3 are neighbors of v_2, in the rooted tree with root v_2 they will be children of v_2. To get most profit they must choose the vertices among all children of v_2 with maximum and second maximum $st(\cdot, v_2)$.

Now we can verify all conditions:

- Condition 1 holds because if not, then player 2 can move from v_2 to j_1 to get profit $st(j_1, v_2)$, which is better than $n - st(i_1, v_2) - st(i_2, v_2)$, the profit player 2 can get on vertex v_2;
- Condition 2 holds for the same reason;

– Condition 3 holds because if not, then player 3 can move from i_2 to j_1 to get profit $st(j_1, v_2)$, which is better than $st(i_2, v_2)$, the profit player 3 can get on vertex i_2.

"$\Longleftarrow$": Suppose such a vertex v exists, then we will prove (i_1, v, i_2) is a Nash equilibrium:

1. For player 1, the definition of i_1 makes it meaningless to move from i_1 to any other child of v. Obviously, it is also useless to move position from i_1 to other vertices. So, player 1 will not move;
2. For player 2, condition 1 and 2 promise that it is useless to move position from v to any vertex in the subtree rooted at i_1 or i_2. Obviously, it is also useless to move position from v to other vertices. So, player 2 will not move;
3. For player 3, condition 3 promises that it is useless to move position from i_2 to any vertex in the subtree rooted at i_1. The definition of i_2 makes it meaningless to move from i_2 to any other child of v except for i_1. Obviously, it is also useless to move position from i_2 to other vertices. So, player 3 will not move.

Thus, we finish this proof. □