The One-Round Multi-player Discrete Voronoi Game on Grids and Trees

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Abstract

Basing on the two-player Voronoi game introduced by Ahn et al. [1] and the multi-player diffusion game introduced by Alon et al. [2], we investigate the following *one-round multi-player discrete Voronoi game* on grids and trees. There are n players playing this game on a graph $G = (V, E)$. Each player chooses an initial vertex from the vertex set of the graph and tries to maximize the size of the nearest vertex set. As the main result, we give sufficient conditions for the existence/non-existence of a pure-strategy Nash equilibrium in 4-player games on grids and only a constant gap leaves unknown. We further consider this game with more than 4 players and construct a family of strategy profiles, which are pure-strategy Nash equilibria on sufficiently narrow graphs. Besides, we investigate the game with 3 players on trees and design a linear time/space algorithm to decide the existence of a pure-strategy Nash equilibrium.

Keywords: Game theory, Nash equilibrium, Location game, Graph theory.

1 Introduction

Consider the following scene: Several investors plan to set up laundries in a city and each of them is permitted to manage only one. There are some residents in the city whose addresses have been obtained by the investors. Residents in the city would only choose the nearest laundry and if the nearest ones of some residence are not unique, he/she will choose one of them arbitrarily. In this game, the investors try to attract more customers by locating their laundry at an advantageous position and the payoff of each investor is the number of residents who are certain to choose the laundry. Our goal is to determine whether there exists a stable profile in which no investor would change his/her choice to improve the payoff.

The competitive facility location game is a well-studied topic in game theory. Ahn et al. introduced the Voronoi diagram to characterize players' payoffs and proposed the Voronoi game [1]. In the Voronoi game, two players alternately locate their facilities. After locating, each player will control the area close to one of the facilities. The aim of the two players is to maximize the area controlled by them, respectively. Cheong et al. then modified this model and introduced the *one-round Voronoi game* [9]. In their version, a player locates n facilities first, and the other player locates n facilities then. Their locating is only allowed to perform in one round. Dürr et al. introduced an interesting multi-player version on graphs, where each player can only choice one facility [11]. A small modification in the model of Dürr et al. is that a "tie" vertex, who has k nearest facilities, contributes $1/k$ to the payoffs of the k players.

A similar model is raised by Alon et al. [2] which describes the following competitive process with k players on graph $G = (V, E)$. Let player-i's influenced set $I_i = \emptyset$ for all $i \in [k]$, in the beginning. The game contains several rounds. In round-0, each player-i chooses a vertex $v_i \in V$ as the initial vertex simultaneously, and v_i is gathered by I_i. In round-$(t+1)$ for $t \geq 0$, if some vertex v is not gathered by any I_j for all $j \in [k]$ until the round-t but has a neighbor gathered in I_i, then v is gathered by I_i in this round. However, if some v is gathered by more than one set in any round, including round-0, v will be deleted from G and all influenced set after this round. The process iterates until each I_i is invariant after some round and the payoff of each player-i is $|I_i|$.

We investigate *one-round discrete version of Voronoi games* with k players on a given graph G denoted by $\Gamma(G,k)$. Note that in diffusion games, the duplicate vertices, i.e., the vertices gathered by more than one player, are deleted immediately after each round. In our model, the duplicate vertices will be deleted at the end of the game. The game is also in the Voronoi style, i.e., the owner of each vertex is determined by which initial vertex is the nearest one from it. We describe the one-round discrete Voronoi game in an equivalent but more natural way. First, each player chooses a vertex simultaneously in a given graph as the *initial vertex*. The payoff of each player is the number of vertices which have smaller distance to this player's initial vertex than to all the others. See Section 1.3 for the formal definition of the game. Note that the only difference between this model with the model of Dürr et al. is that "tie" vertices will contribute nothing to the payoff of any player.

Remark It is easy to see that a diffusion game and a one-round Voronoi game are equivalent on paths, circles and trees. But in the general case, they could be totally different, as following example.

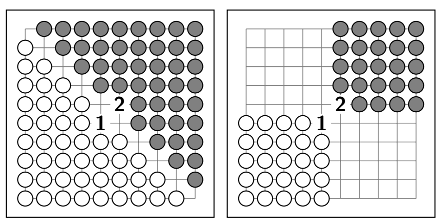

Figure 1: The results of these games are different in a grid graph with 2 players. The left one is the result of the diffusion game, while the right one is the result of our model.

In game theory, the concept Nash equilibrium, named after John Forbes Nash equilibrium Jr., takes a central position. In a Nash equilibrium, no player can improve the payoff by change the choice unilaterally. John Nash equilibrium proved every game with finite number of players and a finite strategy space has a mixed-strategy Nash equilibrium in his famous paper [13]. If we just allow pure strategies, in which each player can only make a deterministic choice, the existence of a Nash equilibrium cannot be ensured. In this paper, a Nash equilibrium refers to a pure-strategy Nash equilibrium by default. In this paper, we investigate the existence of a Nash equilibrium in one-round discrete Voronoi games on a given graph.

Unfortunately, such a property is hard to test, since given a graph G and an integer k, judging the existence in $\Gamma(G,k)$ is NP-hard[1]. Nevertheless, the existence of a Nash equilibrium in the game on some important graph classes is is worth examining. In this work, we select two non-trivial cases, the game on grids and trees. The grid graph, or the two-dimensional lattice graph, is an natural embedded structure of Euclidean space $\mathbb{R}^2$, utilized to approximate a space equipped L_1

[1] An NP-hardness result of diffusion games is given in [12]. The reduction in this papers also works for one-round discrete Voronoi games.

distance, describe practical senses such as city residents and some physical molecular structure in theoretical computer science. The tree graph is one of the most basic graph classes, defined as an acyclic connected graph. See Section 1.3 for formal definitions of these graph classes.

Organization To draw the full picture of this topic, we give a brief survey of the results about Voronoi games and diffusion games in Section 1.1. Then we list our main results in Section 1.2 and define some useful notations and our game model formally in Section 1.3. We consider the game on grids in Section 2 and give a Nash equilibrium construction for sufficiently narrow grids in Section 2.1. We discuss the case when there are at most 4 players on grid graphs in Section 2.3. Finally, we discuss the game with 3 players on trees in Section 3.

1.1 Related results

Ahn et al. proved that the first player has a winning strategy in Voronoi games on a circle or a line segment [1]. Teramoto et al. showed it is NP-hard to decide whether the second player can win in a 2-player discrete Voronoi game on graphs [15]. Durr and Thang studied a one-round, multi-player version and proved it is also NP-hard to decide the existence of a Nash equilibrium in a discrete Voronoi game on a given graph [11]. Besides, several works focus on a one-round 2-player discrete version and try to compute a winning strategy for each player efficiently [3, 4, 5, 6, 10, 7].

For diffusion games, it is also proved as a NP-hard problem to decide whether there exists a Nash equilibrium with multi-player on general networks [12]. Roshanbin et al. studied this game with 2 players [14] and showed the existence or nonexistence of a Nash equilibrium in paths, cycles, trees and grids. Especially, on a tree with 2 players, there is always a Nash equilibrium under the diffusion game model. Bulteau et al. proved a Nash equilibrium does not exist on $\mathrm{Grid}_{n\times m}$ where $n, m \leq 5$ with 3 players and got some extra conclusions about the existence of Nash equilibrium in several special graph classes [8].

1.2 Our Results

First, we show that there always exists a Nash equilibrium for sufficiently narrow grids when $k \neq 3$ by a family of constructions.

Theorem 1. *For any $k \in \mathbb{Z}^+$ ($k \neq 3$) and integers n, m ($n \geq k$), there exists a Nash equilibrium in $\Gamma(\mathrm{Grid}_{n\times m}, k)$ if $m \leq n/\lceil k/2 \rceil$.*

Furthermore, we attempt to characterize the existence of a Nash equilibrium for small k. The results are enumerated as following:

1. In $\Gamma(\mathrm{Grid}_{n\times m}, k)$ with $k \leq 2$, there always exists a Nash equilibrium;
2. In $\Gamma(\mathrm{Grid}_{n\times m}, 3)$, there exists no Nash equilibrium for sufficiently large n, m;
3. For $\Gamma(\mathrm{Grid}_{n\times m}, 4)$, we prove the following theorem which almost completely characterizes the existence of a Nash equilibrium as following theorem.

Remark. In this paper, that some lemma or theorem holds for "sufficiently large n and m" means there exists an universal constant c, which is not with respect to any quantity, such that the lemma or theorem holds for all n, m larger than c. For convenience, we also using "big-O" notations in some assertions. That means if the corresponding value is not more than a constant c, there exists another constant c' with respect to c such that this assertion holds for all n, m larger than c'.

Theorem 2. *In $\Gamma(\mathrm{Grid}_{n\times m}, 4)$ $(n \geq m)$ where n and m are sufficiently large, if $4 \mid n$ and m is odd, there exists a Nash equilibrium if $m \leq n/2+2\lfloor\sqrt{n}\rfloor+1$, and does not exist if $m \geq n/2+2\lfloor\sqrt{n}\rfloor+5$. Otherwise, there exists a Nash equilibrium if $m \leq n/2$, and does not exist if $m > n/2+6$.*

Then, we consider the game a tree. The case with two players is solved in [14]. In this paper we investigate the case with 3 players:

Theorem 3. *In $\Gamma(T, 3)$ where T is a tree with size n, there exists a Nash equilibrium if and only if there exists a vertex v such that $n - st(i_1, v) - st(i_2, v) \geq \max\{st(j_1, v), st(j_2, v)\}$ and $st(i_2, v) \geq st(j_1, v)$, where i_1 and i_2 are the children of v with maximum and second maximum $st(\cdot, v)$, and j_k is the child of i_k with maximum $st(\cdot, v)$ $(k = 1, 2)$. Here, $st(v_1, v_2)$ represents the number of vertices in the subtree with root v_1, when the tree is rebuilt as a rooted tree with root v_2.*

Furthermore, if such a vertex v exists, then (i_1, v, i_2) is a Nash equilibrium, and we can design an algorithm to determine whether a Nash equilibrium exists in $O(n)$ time/space.

1.3 Preliminaries

Denote $[n, m]$ as the set $\{n, n+1, \ldots, m\}$, and $[n]$ as $[1, n]$ for integers n, m $(n \leq m)$. An (undirected) *graph* is an ordered pair $G = (V, E)$ where V is a set of *vertices* and E is a set of *edges*. We say $u_1, \ldots, u_\ell$ is a *path* if $\forall i$ $(u_i, u_{i+1}) \in E(G)$. We say vertices $v, w \in V(G)$ are connected if there exists a path between v and w in G. For convenience of discussions, we denote that $v(G) = |V|$ and $e(G) = |E|$ for a graph $G = (V, E)$.

We indicate a *grid* with a set of integer pairs. Define $\mathrm{Grid}_{n\times m} := (V, E)$ with the vertex set $V = [n] \times [m]$ and an edge set E. For vertices $u, v \in V$, L_1 norm distance between them is defined as $\|u - v\|_1 := |x_u - x_v| + |y_u - y_v|$, where $v = (x_v, y_v)$ and $u = (x_u, y_u)$. Then the edge set E of $\mathrm{Grid}_{n\times m}$ is $\{(u, v) \mid \|u - v\|_1 = 1\}$. Note that for all $u, v \in V(\mathrm{Grid}_{n\times m})$, $\mathrm{dist}(u, v) = \|u - v\|_1$ holds.

A graph $T = (V, E)$ is called a *tree* if $v(T) = e(T) + 1$ and every 2 vertices $v, w \in V(T)$ are connected. We also define a function $st(w, v)$ $(w, v \in V(T))$, as the number of vertices in the subtree rooted at vertex w when T is rebuilt as a rooted tree with root v.

A one-round multi-player Voronoi game, denoted by $\Gamma(G, k)$, processes on the undirected graph G with k players. Each player, player-i for example, can choose one vertex v_i from G. v_i is called *initial vertex* of player-i and we also say player-i *takes* v_i. Player-i will influence the vertices in

$$I_i := \{v \in V(G) \mid \forall j \in [k]\backslash\{i\} : \mathrm{dist}_G(v, v_i) < \mathrm{dist}_G(v, v_j)\},$$

where v_j is the player-j's initial vertex and $\mathrm{dist}(u, v)$ is the length of the shortest path between vertices u and v. In this game the payoff of player-i is $U_i := |I_i|$. The aim of each player is to maximize the payoff in the game, respectively. Obviously, the vertex set influenced by a specific player totally depends on every player's initial vertex. So, a strategy profile is defined as a tuple of vertices $p := (v_1, v_2, \ldots, v_k)$ where v_i is player-i's initial vertex. The influenced vertex set and payoff of player-i in some strategy profile p are donated as $I_i(p)$ and $U_i(p)$. When a strategy profile p is implied in context, we use tuple (v_i, v_{-i}) to emphasize the player-i's choice where v_{-i} represents the others' choices. We say a strategy profile v is a Nash equilibrium if

$$\forall i \in [k]\ \forall v_i' \in V(G) : U_i(v_i, v_{-i}) \geq U_i(v_i', v_{-i}).$$

Namely, no player can improve the payoff by moving the initial vertex in non-cooperative cases.

2 Voronoi games on grids

2.1 Voronoi game on the narrow grids

In this section, we discuss the case in which k players ($k \neq 3$) take part in a Voronoi game on $\mathrm{Grid}_{n\times m}$. As the main result, a family of Nash equilibria will be given for narrow grids in Theorem 1.

Bulteau et al. construct a profile on paths and prove that a Nash equilibrium always exists in a game on Path_n, a path with n vertices, with k players, except the case where $k = 3$ and $n \geq 6$. When the number of players k is even, they set the initial vertices $\{v_i\}_{1\leq i\leq k}$ as:

$$v_i := \begin{cases} \lfloor \frac{n}{k} \rfloor \cdot i + \min\{i, n \bmod k\} & \text{if } i \text{ is odd,} \\ v_{i-1} + 1, & \text{if } i \text{ is even.} \end{cases}$$

For the case with odd number players, Bulteau et al. reduce it to the even case. By constructing a Nash equilibrium $(v'_1, \dots, v'_{k+1})$ for P_{n+1}, they get a Nash equilibrium, $(v_1, \dots, v_k) := (v'_1, \dots, v'_{k-2}, v'_k - 1, '_{k+1} - 1)$ on Path_n.

Note we might treat a grid as a path when the ratio of the width to the height is sufficiently large. Furthermore, we increase the height of $\mathrm{Grid}_{n\times 1}$ and keep the Nash equilibrium in some way until there exists an improvement breaking the balance.

Figure 2: Basing on the construction of Bulteau et al., construct a Nash equilibrium in $\Gamma(\mathrm{Grid}_{17\times 5}, 4)$.

We construct a family of Nash equilibria. For the odd m, we just embed the construction by Bulteau et al. into the middle row (i.e., the $(m+1)/2$-th row). Whereas, for the even m we would also embed it into one of the two middle rows (i.e., the $m/2$-th or $(m+2)/2$-th row). But in the latter case, we can only promise the Nash-property when $(n + k \bmod 2) \bmod (k + k \bmod 2) = 0$. To fix it, we Stretch or Compress a Nash equilibrium without breaking the Nash-property. More precisely, a group of discussions will be given to describe such a process to construct a Nash equilibrium v, where the routines Compress and Stretch are given in Algorithm 1:

Case 1: If k is even, consider the parity of m.

Case 1.1: If m is odd, construct v as:

$$v_i := \begin{cases} (\lfloor \frac{n}{k} \rfloor \cdot i + \min\{i, n \bmod k\}, \frac{m+1}{2}), & \text{if } i \text{ is odd;} \\ (x_{i-1} + 1, \frac{m+1}{2}), & \text{if } i \text{ is even.} \end{cases}$$

Case 1.2: If m is even, define $r := n \bmod k$.

Case 1.2.1: In the case where $r = 0$, construct a Nash equilibrium v for $\Gamma(\mathrm{Grid}_{n\times(m+1)}, k)$.

Case 1.2.2: Otherwise, stretch or compress the profile in the following way:

Case 1.2.2.1: If $r \leq k/2$, construct a Nash equilibrium v' for $\Gamma(\mathrm{Grid}_{(n-r)\times m}, k)$ and construct v by $\textsc{Stretch}(n - r, m, k, v', r)$.

Case 1.2.2.2: Otherwise, construct a Nash equilibrium v' for $\Gamma(\mathrm{Grid}_{(n+k-r)\times m}, k)$. Then, construct v by $\textsc{Compress}(n + k - r, m, k, v', k - r)$.

Case 2: If k is odd but $k \neq 3$, construct a Nash equilibrium v' for $\Gamma(\mathrm{Grid}_{(n+1)\times m}, k+1)$. Then, construct $\{v_i\}_{1\leq i<k}$ as

$$v_i := \begin{cases} v'_i & \text{if } i \leq k-2 \\ (x_{i-1}-1, y_{i-1}) & \text{otherwise} \end{cases}$$

Algorithm 1 Stretch or compress a profile

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function $\textsc{Stretch}(n, m, k, v', i)$
  return $\textsc{Stretch-or-Compress}(n, m, k, v', i, 1)$

function $\textsc{Compress}(n, m, k, v', i)$
  return $\textsc{Stretch-or-Compress}(n, m, k, v', i, -1)$

function $\textsc{Stretch-or-Compress}(n, m, k, v', i, soc)$
  for $j \leftarrow 1$ to $k/2$ **do**
    $a \leftarrow soc \cdot \min\{i, j-1\}$
    $v_{2j-1} \leftarrow (x'_{2j-1} + a, y'_{2j-1} - [a \text{ is odd}])$
    $v_{2j} \leftarrow (x'_{2j} + a, y'_{2j} - [a \text{ is odd}])$
  return v

For our constructed strategy profile, the positions of the initial vertices are specific. Thus, the payoff of each players can be calculated, and the payoffs of their changes can be predicted, wherever the changes are located. For any constructed profile we check all the changes of all the players. In fact, it is a lengthy and tedious work to check that our construction is a Nash equilibrium, which is omitted in this paper. Nevertheless, to convince readers that our construction works, we provide a proof for a special case where $k = 3$, m is even and $n \bmod 6 = 4$, in Appendix, which can be expanded to other cases.

2.2 Sufficient conditions to assert a profile is not a Nash equilibrium

In this section, we give 4 sufficient conditions to assert a profile is not a Nash equilibrium, Lemma 1, Lemma 2, Lemma 3 and Lemma 4, where are used as tools to detect an improvement and certify a profile is not a Nash equilibrium. For convenience, we define the following notations:

- In $\mathrm{Grid}_{n\times m}$, for $i, j \in \{-1, 1\}$ define $\mathrm{Part}_{i,j} : [n] \times [m] \to 2^{[n]\times[m]}$ as

$$\begin{aligned}
\mathrm{Part}_{-1,-1}(x, y) &:= [x] \times [y], \\
\mathrm{Part}_{-1,1}(x, y) &:= [x] \times [y, m], \\
\mathrm{Part}_{1,-1}(x, y) &:= [x, n] \times [y], \\
\mathrm{Part}_{1,1}(x, y) &:= [x, n] \times [y, m].
\end{aligned}$$

- In $\mathrm{Grid}_{n\times m}$, for $(i, j) \in \{(0, 1), (0, -1), (1, 0), (-1, 0)\}$, define $\mathrm{Part}_{i,j} : [n] \times [m] \to 2^{[n]\times[m]}$ as

$$\begin{aligned}
\mathrm{Part}_{1,0}(x, y) &:= \{(x', y') \in [n] \times [m] \mid x' + y' > x + y \wedge x' - y' > x - y\}, \\
\mathrm{Part}_{-1,0}(x, y) &:= \{(x', y') \in [n] \times [m] \mid x' + y' < x + y \wedge x' - y' < x - y\}, \\
\mathrm{Part}_{0,1}(x, y) &:= \{(x', y') \in [n] \times [m] \mid x' + y' > x + y \wedge x' - y' < x - y\}, \\
\mathrm{Part}_{0,-1}(x, y) &:= \{(x', y') \in [n] \times [m] \mid x' + y' < x + y \wedge x' - y' > x - y\}.
\end{aligned}$$

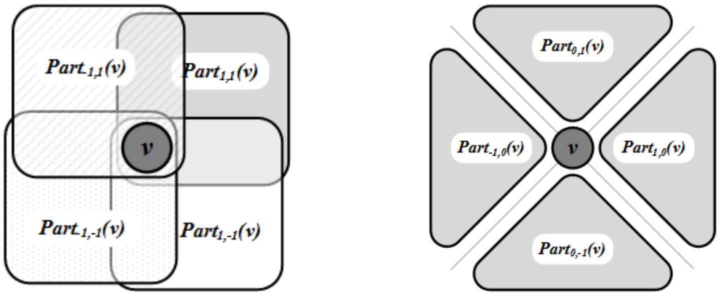

Figure 3: Definitions of Part($\cdot$)

Intuitively, we can perceive if an initial vertex v_1 moves close to another initial vertex v_2, it will snatch some vertices from I_2. If some change makes the initial vertex closer to all the other initial vertex, it may be an improvement. The first lemma is a formal statement of such idea.

Lemma 1. *$v'_i = (x_i + 1, y_i + 1)$[2] is an improvement of v_i if $x_j + y_j > x'_i + y'_i$ for all $j \in [k] \backslash \{i\}$.*

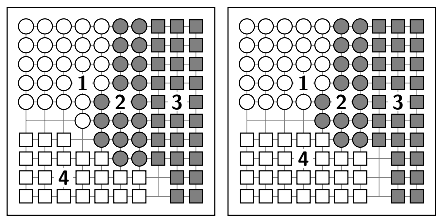

Figure 4: v_4 moves from (3,2) to (4,3), and the payoff increases by 4.

Proof. Obviously, after moving initial vertex to v'_i, the distance between v'_i and the vertices in $\text{Part}_{-1,1}(x_i, y_i + 1)$ and $\text{Part}_{1,-1}(x_i + 1, y_i)$ cannot be longer than before. Thus, the player-i's payoff will not reduce in these vertex sets. For all vertices $p = (x, y)$ in $\text{Part}_{-1,-1}(x_i, y_i)$ and other initial vertices v_j, we claim that $x_j \geq x$ or $y_j \geq y$ (otherwise, v_j will be in $\text{Part}_{-1,-1}(x_i, y_i)$ and $x_j + y_j > x'_i + y'_i$ cannot hold). Thus,

$$
\begin{aligned}
\|v'_i - p\|_1 &= x'_i + y'_i - x - y \\
&< x_j + y_j - x - y = \|v_j - p\|_1 && \text{when } x_j \geq x \text{ and } y_j \geq y \\
\|v'_i - p\|_1 &= x'_i + y'_i - x - y \\
&< x_j + y_j + (y - y_j) - x - y = x_j - x \\
&< x_j - y_j - x + y = \|v_j - p\|_1 && \text{when } x_j \geq x \text{ and } y_j < y \\
\|v'_i - p\|_1 &= x'_i + y'_i - x - y \\
&< x_j + (x - x_j) + y_j - x - y = y_j - y \\
&< -x_j + y_j + x - y = \|v_j - p\|_1 && \text{when } x_j < x \text{ and } y_j \geq y
\end{aligned}
$$

hold, which shows $\text{Part}_{-1,-1}(x_i, y_i) \subseteq I_i(v'_i, v_{-i})$. Considering the vertices in $\text{Part}_{1,1}(v'_i)$, their distance to v'_i must be shorter and we can claim player-i's payoff in $\text{Part}_{1,1}(v'_i)$ cannot decrease and $I_i \subseteq I_i(v'_i, v_{-i})$. Pick an initial vertex v_j in the other initial vertices, for which $\forall l \in [k] \backslash \{i\} \|v_i -$

[2] In this paper, given an profile v, define $(x_i, y_i) := v_i$ for all index i. In addition, $(x'_i, y'_i) := v'_i$ and (x''_i, y''_i) are defined similarly. These definition will be omitted. Generally, v'_i and v''_i represent changes of initial vertex v_i.

$v_j\|_1 \leq \|v_i - v_l\|_1$. Without loss of generality, assume v_j is located in $\mathrm{Part}_{1,0}(x_i, y_i+1)$. For a vertex $p = (x,y) = (x_i' + \left\lceil \frac{|y_j - y_i'| + x_j - x_i'}{2} \right\rceil - 1, y_i')$, it is clear that v is in $\mathrm{Part}_{1,1}(v_i')$ and $x \leq x_j$, and

$$\begin{aligned}
&\|v - v_i'\|_1 = \left\lceil \frac{|y_j - y_i'| + x_j - x_i'}{2} \right\rceil - 1 \\
<&\|v - v_j\|_1 = x_j - x_i' - \left\lceil \frac{|y_j - y_i'| + x_j - x_i'}{2} \right\rceil + 1 + |y_j - y_i'| \\
=&\|v - v_j\|_1 = \left\lfloor \frac{|y_j - y_i'| + x_j - x_i'}{2} \right\rfloor + 1 \\
\leq&\|v - v_i\|_1 = \left\lceil \frac{|y_j - y_i'| + x_j - x_i'}{2} \right\rceil + 1
\end{aligned}$$

can be deduced. Due to the way that v_j is picked, we cannot find an initial vertex v_l which keeps $\|v - v_l\|_1 \leq \|v_i'\|_1$. As the result, p is influenced by v_i' but not v_i, which means the change v_i' get the extra payoff than before and v_i' is an improvement of v_i. $\square$

Lemma 1 can be generalized due to the symmetry of grids as well as Lemma 3. Namely, after reflecting coordinates along a dimension or rotating the grid by 90 degrees, these lemmas also hold. In the following proofs, we omit the statement about such generalization.

The second lemma shows the profile in which all the initial vertices are bounded by a small block cannot be a Nash equilibrium.

Lemma 2. *In $\Gamma(\mathrm{Grid}_{n\times m}, k)$ where $k \geq 3$ and $\max\{n, m\} > 3c + 2$ for some c, a profile is not a Nash equilibrium if $|x_i - x_j|, |y_i - y_j| \leq c$ for all distinct $i, j \in [k]$.*

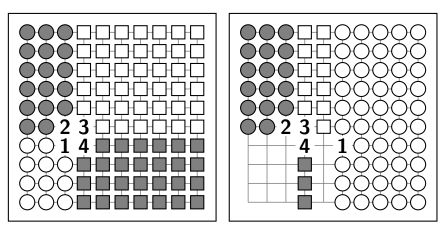

Figure 5: The initial vertices gather in a 1-size block, (6,4) is an improvement of v_1.

Proof. Assume $n \geq m$ w.l.o.g. Consider two points $p' = (\min\{x_i | i \in [k]\} - c - 1, \min\{y_i | i \in [k]\})$ and $p'' = (\max\{x_i | i \in [k]\} + c + 1, \min\{y_i | i \in [k]\})$. First we show at least one of p' and p'' must be on the grid. If not, then $\min\{x_i | i \in [k]\} - c - 1 < 1$ and $\max\{x_i | i \in [k]\} + c + 1 > n$, which turns out that $n < \max\{x_i | i \in [k]\} + c + 1 \leq \min\{x_i | i \in [k]\} + 2c + 1 < 3c + 3$. This conflicts with that $n > 3c + 2$. If exact one of these 2 vertices is on the grid, suppose p'' exists but p' w.l.o.g. Thus, $\min\{x_i | i \in [k]\} - c - 1 < 1$. We will prove $p'' = (x'', y'')$ is an improvement of some player's initial vertex. For any initial vertex v_j, $p'' \in \mathrm{Part}_{1,0}(v_j)$ because

$$\begin{aligned}
x_j + y_j &\leq \max\{x_i | i \in [k]\} + \max\{y_i | i \in [k]\} \\
&< (\max\{x_i | i \in [k]\} + c + 1) + \min\{y_i | i \in [k]\} \quad \text{and} \\
x_j - y_j &\leq \max\{x_i | i \in [k]\} - \min\{y_i | i \in [k]\} \\
&< (\max\{x_i | i \in [k]\} + c + 1) - \min\{y_i | i \in [k]\}.
\end{aligned}$$

Then, for vertex $p = (x_p, y_p)$ in $\text{Part}_{1,1} \cup \text{Part}_{1,-1}(p'')$, we get

$$\begin{aligned}
\|p - v_j\|_1 &= \|p - (x'', y_j)\|_1 + \|(x'', y_j) - v_j\|_1 \\
&= \|p - (x'', y_j)\|_1 + |x'' - x_j| < \|p - (x'', y_j)\|_1 + |y_j - y''| \\
&= \|p - (x'', y_j)\|_1 + \|(x'', y_j) - p''\|_1 \leq \|p - p''\|_1
\end{aligned}$$

which means $\text{Part}_{1,1} \cup \text{Part}_{1,-1}(p'') \subseteq I_i(p'', v_{-i})$ for $i \in [k]$. Let v_i is the initial vertex of player-i whose payoff is the least,

$$\begin{aligned}
U_i(p'', v_{-i}) \geq m(n - x'' + 1) &= m(n - (\max\{x_i | i \in [k]\} + c + 1) + 1) \\
&\geq m(n - (\min\{x_i | i \in [k]\} + 2c + 1) + 1) \\
&> m(n - (3c + 3)) > \left\lfloor \frac{mn}{k} \right\rfloor \geq U_i
\end{aligned}$$

when $k \geq 3$. For the same reason, if both p' and p'' exist, we can get

$$\begin{aligned}
U_i(p', v_{-i}) + U_i(p'', v_{-i}) &\geq m(n - \max\{x_i | i \in [k]\} - c) + m(\min\{x_i | i \in [k]\} - c - 1) \\
&= m(n - (\max\{x_i | i \in [k]\} - \min\{x_i | i \in [k]\}) - 2c - 1) \\
&\geq m(n - 3c - 1) > 2 \left\lfloor \frac{mn}{k} \right\rfloor \geq 2U_i
\end{aligned}$$

hold when $k \geq 3$ and $n > 3c + 2$. Thus, p' or p'' is an improvement of v_i. $\square$

Furthermore, we study how player-i's payoff changes when v_i moves along the orientation of the slash in this lemma. As the main conclusion, whether the payoff increases or decreases depends on the distance from v_i to the below and the left boundaries of the grid.

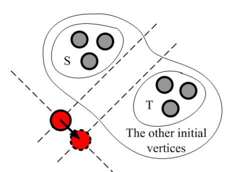

Figure 6: The change in Lemma 3. The red vertex represents v_i, which is moving along the slash orientation.

Lemma 3. *$v_i' = (x_i - 1, y_i + 1)$ will be an improvement of v_i, if $\{v_j \mid j \in [k] \backslash \{i\}\}$ can be divided into 2 disjoint and non-empty sets S, T satisfying $S \subseteq \text{Part}_{0,-1}(v_i')$, $T \subseteq \text{Part}_{1,0}(v_i)$ and $x_i - y_i > 1$.*

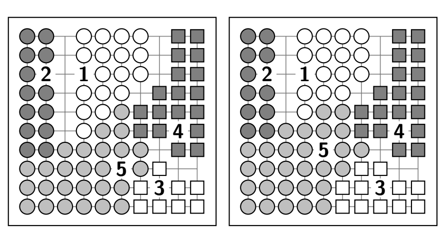

Figure 7: v_5 moves from (6,3) to (5,4), and its payoff increases by 2.

Proof. The distances between player-i's initial vertex and the vertices in

$$\text{Part}_{-1,-1}(x_i - 1, y_i) \cup \text{Part}_{1,1}(x, y_i + 1)$$

do not change. Thus, the payoff in the vertex set will also not change. According to Lemma 1, the payoff will increase in $\text{Part}_{-1,1}(v_i')$.

Considering the vertex set $S_r = \{r\} \times [y_i, m]$ ($r \in [x_i - 1]$), in which some vertex (r, y') is influenced by v_i means $r \times [y_i, y'] \subseteq I_i$. (Assume a vertex (r, y'') in $r \times [y_i, y']$ and $(r, y'') \notin I_i$, which means some initial vertex v_j makes

$$\begin{aligned}\|(r,y') - v_i\|_1 =& \|(r,y') - (r,y'')\|_1 + \|(r,y'') - v_i\|_1 \geq \\ & \|(r,y') - (r,y'')\|_1 + \|(r,y'') - v_j\|_1 = \|(r,y') - v_j\|_1\end{aligned}$$

hold and it conflicts with $(r, y') \in I_i$.)

Notice $\forall j \in [k] \backslash \{i\} (x_j + y_j > x_i + y_i)$ and S is a non-empty set, (r, y_1) must belong to I_i and (r, m) cannot belong to I_i. Thus, the vertex $p = (r, \sup\{y \in [y_1, m] | (r, y) \in I_i\} + 1)$ exists. Consider this vertex $p = (x_p, y_p)$, it is not influenced by v_i but $p' = (x_p, y_p - 1)$ is. So,

$$\begin{aligned}&\exists j \in [k] \backslash \{i\} (\|v_j - p\|_1 \leq \|v_i - p\|_1) \quad \text{and} \\ &\forall j \in [k] \backslash \{i\} (\|v_j - p'\|_1 > \|v_i - p'\|_1) \Rightarrow \\ &\qquad (\|v_j - p\|_1 + 1 \geq \|v_j - p'\|_1 > \|v_i - p\|_1 - 1) \Rightarrow \\ &\qquad (\|v_j - p\|_1 + 1 \geq \|v_i - p\|_1)\end{aligned}$$

hold, by which we can claim $p \in I_i(v_i', v_{-i})$ but $p \notin I_i$.

Let $v_j = (x_j, y_j)$ is the nearest initial vertex from p excepted v_i. Assume there is a vertex $p'' = (r, y_p'')$ in $\{r\} \times [y_p + 2, m]$ which satisfies $\|p'' - v_i\|_1 \leq \|p'' - v_j\|_1 + 1$. Then $\|(r, y_j) - v_i\|_1 \leq \|(r, y_j) - v_j\|_1 + 1$ if $y_p'' \geq y_j$, but that is impossible since

$$\begin{aligned}&v_j \in S \Rightarrow y_j - x_j > y_i - x_i + 2 \quad \text{and} \\ &\|(r, y_j) - v_i\|_1 = x_i - r + y_j - y_i > x_j - r + 2 = \|(r, y_j) - v_j\|_1 + 2.\end{aligned}$$

If $y_p'' < y_j$, it is easy to deduce that the vertex $(r, y_p'' - 1)$ belongs to I_i, which conflicts with $y_p'' \in [y_p + 2, m]$. Now, we can claim the player-i's payoff in S_r increase by 1 where $r \in [x_i - 1]$, as well as the player-i's payoff in $\text{Part}_{-1,1}(v_i')$ increase by $x_i - 1$ after moving player-i's initial vertex to v_i'.

If we image v_i as v_i''s change, for the same reason, its payoff in $\text{Part}_{1,-1}(v_i)$ increase by y_i, thus, in the reverse processing the same number of vertices loss. □

The forth lemma excludes another non-Nash equilibrium case, where the 4 players are divided into 2 pairs, initial vertices in each pair are constantly close and all initial vertex is constantly close to a slash.

Lemma 4. *In* $\Gamma(\text{Grid}_{n\times m}, 4)$ *where* $n > 64c + 8, m > 48c + 36$ *for some integer* c*, profile* v *is not a Nash equilibrium if* $\max\{\|v_1 - v_2\|_1, \|v_3 - v_4\|_1, |(x_1 + y_1) - (x_3 + y_3)|\} \leq c$.

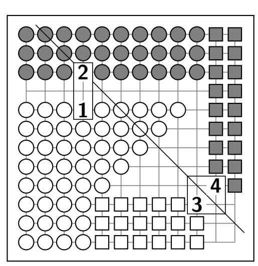

Figure 8: An example in the case discussed in Lemma 4.

Proof. Assume v is a Nash equilibrium. Note that $\|v_2 - v_3\|_1 \geq \omega(1)$ since Lemma 2. W.l.o.g, assume $x_2 \leq x_3$. We claim $m - y_1 \leq m/4 + 2c$. Otherwise, the initial vertex with the minimum payoff, v_i for example, can be improved by moving to v_i' where v_i' is in $\bigcup_{j\in[4]} \mathrm{Part}_{0,1}(v_j)$ with y_i' minimized. Because of our assumption, it is easy to show that $y_i' - y_1 \leq 2c+1$. Thus, $U_i(v_i', v_{-i}) > nm/4 > U_i(v_i, v_{-i})$, if $m - y_1 > m/4+2c$. For the similar reason, $y_3 \leq m/4+2c+1$, $x_1 \leq n/4+2c+1$ and $n - x_3 \leq n/4+2c$ should be satisfied. Consider vertices $v_5' := (\lceil (n+1)/2\rceil + c, \lceil (m+1)/2\rceil + c)$ and $v_5'' := (\lfloor (n+1)/2\rfloor - c, \lfloor (m+1)/2\rfloor - c)$. Note that either $x_5'' + y_5'' < x_i + y_i$ for all $i \in [4]$, or $x_5' + y_5' > x_i + y_i$ for all $i \in [4]$, since $|(x_i + y_i) - (x_j + y_j)| \leq 3c$ for all distinct $i, j \in [4]$. In the first case, add a new initial vertex at v_5'' and the payoff is $U_5(v_5'', v)$. Obviously, v_5'' will occupy all vertex in $\mathrm{Part}_{-1,-1}(x_5'' - 1, y_5'')$. Furthermore, v_5'' occupy at least almost a half of vertices in rectangle $T := [x_5'', \min\{x_3, x_4\}] \times [\max\{y_3, y_4\}, y_5'']$, or formally, $|I_5(v_5'', v) \cap T| \geq (|T| - \min\{w, h\})/2$ where w, h are the width and height of T. Thus, we have

$$\begin{aligned} U_5(v_5'', v) \geq& (\lfloor (n+1)/2\rfloor - c - 1) \cdot (\lfloor (m+1)/2\rfloor - c) + (|T| - \min\{w, h\})/2 \\ \geq& 9mn/32 - (c + 1/8)n - (3c/4 + 9/16)m + (5c^2 + 3c) \end{aligned}$$

which will exceed $nm/4$ when $m > 64c + 8$ and $n > 48c + 36$. Thus, there exists some v_i with $U_i(v) \leq nm/4$. $U_i(v_5'', v_{-i}) \geq U_5(v_5'', v) > nm/4$ implies that v is not a Nash equilibrium. It conflicts with our assumption. The second case is symmetric with the first one, which leads to the same result. □

2.3 Voronoi game on grids within 4 players

Naturally, it is a good start point to consider the cases with a few players. In this section, we show the existence of a Nash equilibrium in $\Gamma(\mathrm{Grid}_{n\times m}, k)$ where $k \leq 4$.

In the first non-trivial case $\Gamma(\mathrm{Grid}_{n\times m}, 2)$, there is always a Nash equilibrium in $\Gamma(\mathrm{Grid}_{n,m}, 2)$. If the grid has more than one centroid, v_1 and v_2 take a pair of adjacent centroids; Otherwise, v_1 takes the centroid and v_2 an adjacent vertex of v_1. It is easy to verify the Nash-property.

For the case $\Gamma(\mathrm{Grid}_{n\times m}, 3)$, the situation turns to be a bit more complicated. We say an initial vertex v_i is a *controller*, if and only if one of the following conditions

$$\begin{aligned} x_i \leq x_j \wedge y_i \leq y_j \quad \forall j \neq i, \\ x_i \leq x_j \wedge y_i \geq y_j \quad \forall j \neq i, \\ x_i \geq x_j \wedge y_i \leq y_j \quad \forall j \neq i, \\ x_i \geq x_j \wedge y_i \geq y_j \quad \forall j \neq i. \end{aligned}$$

It is not hard to verify that there always exists a controller in the game $\Gamma(\mathrm{Grid}_{n\times m}, 3)$, which leads to the following theorem.

Theorem 4. *For any sufficiently large integer n, m, there exists no Nash equilibrium in* $\Gamma(\mathrm{Grid}_{n\times m}, 3)$.

Proof. Suppose there exists a Nash equilibrium $v = \{v_1, v_2, v_3\}$ in $\Gamma(\mathrm{Grid}_{n\times m}, 3)$. First, Lemma 7, which will be proved later, says there must be initial vertices v_1, v_2 satisfying $\|v_1 - v_2\| = 1$. Then it is easy to verify that v_3 is a controller wherever it lies on the grid. Without loss of generality, suppose $x_3 \leq x_1$, $x_3 \leq x_2$, $y_3 \leq y_1$ and $y_3 \leq y_2$. Now consider $\|v_1 - v_3\|$. If it is large (for example, no less than 100), Lemma 1 will give a contradiction; Otherwise, Lemma 2 will lead to a contradiction. □

Next, we discuss a much more challenging case $\Gamma(\mathrm{Grid}_{n\times m}, 4)$. As the first step, we claim a controller is necessary in a Nash equilibrium.

Lemma 5. *In $\Gamma(\mathrm{Grid}_{n\times m}, 4)$ where n, m are sufficiently large, a profile is not a Nash equilibrium unless there is no controller.*

Proof. Assume v is a Nash equilibrium toward a contradiction. Since there exists no controller, the leftmost, rightmost, bottommost and uppermost initial vertices are distinct. Suppose the leftmost, rightmost, uppermost and bottommost ones are v_3, v_4, v_2 and v_1 respectively. We claim $v_4 \in \mathrm{Part}_{1,0}(v_3)$. Otherwise, assume $x_3 + y_3 \geq x_4 + y_4$ without loss of generality. Note that $x_3 < x_2 < x_4$ and $y_2 > y_3$, where v_2 is the uppermost initial vertex. According to Lemma 1, v_2 can improve the payoff by moving to $(x_2 - 1, y_2 - 1)$ unless v_2 is located at $(x_3 + 1, y_3 + 1)$. If $v_2 = (x_3 + 1, y_3 + 1)$, it easy to see that $v_2' = (x_3 + 1, y_3)$ will be an improvement of v_2.

Furthermore, the inequalities

$$
\begin{aligned}
&|(y_1 + x_1) - (y_3 + x_3)| \leq 2, \quad |(y_2 + x_2) - (y_4 + x_4)| \leq 2, \\
&|(y_1 - x_1) - (y_4 - x_4)| \leq 2, \quad |(y_2 - x_2) - (y_3 - x_3)| \leq 2
\end{aligned}
$$

hold. Otherwise, there exists an improvement according the Lemma 1. In addition, $\|v_3 - v_1\| = O(1)$ implies $\|v_2 - v_4\| = O(1)$. In this case, combining $|(y_2 - x_2) - (y_3 - x_3)| = O(1)$, Lemma 4 would say v is not a Nash equilibrium. For the same reason, $\|v_3 - v_2\|, \|v_4 - v_1\|$ and $\|v_4 - v_2\|$ are $\omega(1)$.

Case 1: The profile v cannot be a Nash equilibrium where

$$y_2 - x_2 \geq y_3 - x_3 \text{ and } y_1 + x_1 \leq y_3 + x_3$$

hold. W.l.o.g, assume $x_1 \leq x_2$. Consider the changes

$$v_3' = (x_3 + 2, y_3 - 2) \text{ and } v_3'' = (x_3 - 2, y_3 + 2)$$

where the feasibility of the change v_3'' will be discussed later. Their payoffs can be calculated as following:

$$
\begin{aligned}
&U_3(v_3', v_{-3}) \geq U_3 + 2(x_2 - x_3) - 2x_3 - O(1) \\
&U_3(v_3'', v_{-3}) \geq U_3 + (x_3 - 2)(m - y_2 + 1) - 2(x_2 - x_3) - O(1)
\end{aligned}
$$

Thus, to avoid improvement, $U_3(v_3', v_{-3}) + U_3(v_3'', v_{-3}) \leq 2U_3$ should be satisfied, which implies either $x_3 = O(1)$ or $m - y_2 \leq 1$. If $x_3 = O(1)$, $x_2 - x_3 = O(1)$ since $U_3 \geq U_3(v_3', v_{-3}) \geq U_3 + 2(x_2 - x_3) - 2x_3 - O(1)$, which conflicts with $\|v_2 - v_3\|_1 = \omega(1)$. Note that $x_3 = \omega(1)$ also ensures that v_3'' is in this grid.

If, $m - y_2 \leq 1$, there is also an improvement. Consider changes $v_2' = (x_2 - 2, y_2 - 2)$ and $v_2'' = (x_2 + 2, y_2 - 2)$. If $x_4 + y_4 \geq x_2 + y_2$, the payoff of v_2' is $U_2(v_2', v_{-2}) \geq U_2 + 2(x_2 - x_3) - O(1)$. $U_2(v_2', v_{-2}) \leq U_2$ implies $x_2 - x_3 = O(1)$ and v is not a Nash equilibrium. So, $x_4 + y_4 < x_2 + y_2$, and $y_2 - x_2 < y_3 - x_3$ for similar reason. Then, the following lower bounds for the payoffs of v_2'' and v_3'' can be given:

$$
\begin{aligned}
&U_2(v_2', v_{-2}) \geq U_2 + 2x_2 - 2(n - x_4) - O(1) \\
&U_2(v_2'', v_{-2}) \geq U_2 + 2(n - x_2) - 2x_3 - O(1)
\end{aligned}
$$

To avoid improvement, $x_4 - x_3 = O(1)$ should be satisfied, which conflicts with Lemma 2.

Case 2: The profile v cannot be a Nash equilibrium where

$$v_1 \in \mathrm{Part}_{0,-1}(v_3) \cap \mathrm{Part}_{-1,0}(v_4) \text{ and}$$
$$v_2 \in \mathrm{Part}_{1,0}(v_3) \cap \mathrm{Part}_{0,1}(v_4)$$

hold. Consider the changes $v_1' = (x_1 - 1, y_1 + 1)$, $v_2' = (x_2 + 1, y_2 - 1)$, $v_3' = (x_3 + 1, y_3 + 1)$ and $v_4' = (x_4 - 1, y_4 - 1)$. According the Lemma 3, $y_1 \geq x_1 - 1$, $x_3 \geq m - y_3$, $m - y_2 + 1 \geq n - x_2$ and $n - x_4 + 1 \geq y_4 - 1$ should be satisfied to avoid an improvement. Thus, $(y_4 - y_1) + (y_2 - y_3) + (x_1 - x_3) + (x_4 - x_2) \leq 4$ holds, which conflicts with Lemma 2.

Case 3: For the other cases, the profile can always be reduced to **Case 1** or **Case 2** by rotating and mirror-flipping.

□

Second, we claim that in a Nash equilibrium, there always be a pair of adjacent initial vertices.

Lemma 6. *In $\Gamma(\mathrm{Grid}_{n\times m}, 4)$ where n, m are sufficiently large, a profile is not a Nash equilibrium, if there exists a controller and the distance between any 2 players' initial vertices is not shorter than 2.*

Proof. Before proving the lemma, we pick a controller and examine the relative position between the controller and the others. We will show that improvement for some initial vertex can always be found. Without loss of generality, let v_1 be a controller.

Case 1: If other players' initial vertices are far way from v_1. (Precisely, $\forall j \in \{2, 3, 4\}$ $\|v_j - v_1\|_2 > 2$.) $v_1' = (x_1 + 1, y_1 + 1)$ is an improvement of v_1 for player-1 due to Lemma 1.

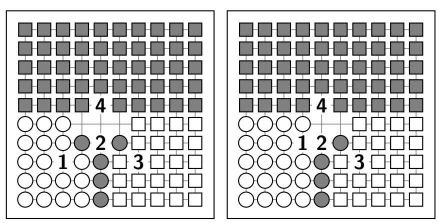

Figure 9: v_1 improve its payoff by moving to $(x_1 + 1, y_1 + 1)$.

Case 2.1: Otherwise, some vertex in $\{(x_1 + 2, y_1), (x_1 + 1, y_1 + 1), (x_1, y_1 + 2)\}$ would be occupied by v_2 without loss of generality. If $(x_1 + 1, y_1 + 1)$ is occupied, the proof of the second case in Lemma 6 can be applied in this case and $v_1' = (x_1 + 1, y_1)$ is an improvement of v_1.

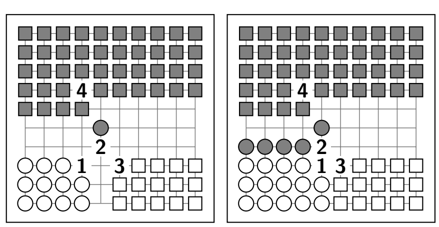

Figure 10: v_1 improve its payoff by moving to $(x_1 + 1, y_1)$.

Case 2.2: If (x_1+1, y_1+1) is not occupied, (x_1+2, y_1) or (x_1, y_1+2) must be taken to differ it from Case 1. When the 2 vertices are both taken (player-2 takes $v_2 = (x_1+1, y_1)$ and player-3 takes $v_3 = (x_1, y_1+1)$ without loss of generality), the payoff of player-1 in $\mathrm{Part}_{1,1}(x_1+1, y_1+1) \cup \mathrm{Part}_{1,-1}(x_1+1, y_1) \cup \mathrm{Part}_{-1,1}(x_1, y_1+1)$ is zero because for all $v \in \mathrm{Part}_{1,1}(v')$,

$$\|v - v_1\|_1 = \|v - v'\|_1 + \|v' - v_1\|_1 = \|v - v'\|_1 + \|v' - v_2\|_1 \leq \|v - v_2\|_1$$

where $v' = (x_1+1, y_1+1)$, for all $v \in \mathrm{Part}_{1,-1}(v'')$,

$$\|v - v_1\|_1 = \|v - v''\|_1 + \|v'' - v_1\|_1 = \|v - v''\|_1 + \|v'' - v_2\|_1 \leq \|v - v_2\|_1$$

where $v'' = (x_1+1, y_1)$ and similar conclusion is clear for $\mathrm{Part}_{-1,1}(x_1, y_1+1)$. Since (x_1+1, y_1) and (x_1, y_1+1) are not occupied, we can get $x_1 + y_1 + 1 < x_j + y_j$ for any other player-j. Note v_1 is a controller. Thus, $x_1 \leq x_j$ and $y_1 \leq y_j$ hold. Because for all $v \in \mathrm{Part}_{-1,-1}(v_1)$,

$$\|v - v_1'\|_1 = \|v - v_1\|_1 + 1 = x_1 - x_v + y_1 - y_v + 1 < x_j - x_v + y_j - y_v = \|v_j - v\|_1,$$

player-1 influences all of the vertex in $\mathrm{Part}_{-1,-1}(v_1)$ even if we change his initial vertex to $(x+1, y_1)$. But player-1 take (x_1+1, y_1), a vertex in $\mathrm{Part}_{1,-1}(x_1+1, y_1)$ in which player-1 have no influenced vertex before. So, we can claim that (x_1+1, y_1) is an improvement of v_1.

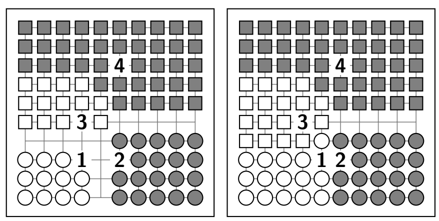

Figure 11: v_1 improve its payoff by moving to (x_1+1, y_1).

Case 2.3: If only one vertex in $\{(x_1+2, y_1), (x_1, y_1+2)\}$ is taken (without loss of generality, assuming (x_1+2, y_1) is taken by player-2), another important vertex (x_1+1, y_1+2) should be considered. If (x_1+1, y_1+2) is not occupied, we can claim $v_1' = (x_1, y_1+1)$ is an improvement of v_1. Obviously, player-1's payoff in $\mathrm{Part}_{-1,1}(v_1') \cup \mathrm{Part}_{1,1}(v_1')$ could not reduce since we move player-1's initial vertex closer to these vertices in the vertex set. And $\mathrm{Part}_{0,0}(v_1) \subseteq I_i$ because v_1 is a controller. For the same reason as **Case 2.2**, $\mathrm{Part}_{1,1}(x_1+1, y_1+1) \cap I_1 = \emptyset$, but (x_1+1, y_1+1) belong to $I_i(v_1', v_{-1})$.

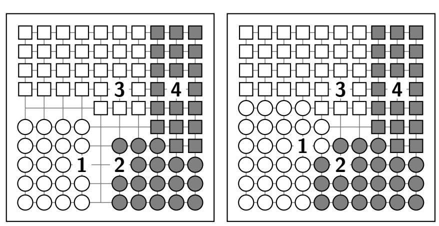

Figure 12: v_1 improve its payoff by moving to (x_1, y_1+1).

Case 2.4: In the last case, (x_1+2, y_1) and (x_1+1, y_1+2) are taken by v_2 and v_3. It easy to see that either v_4 can be improved by getting closer to the others due to Lemma 1, or all the initial vertices are blocked in a constant block, which implies the profile is not a Nash equilibrium according to Lemma 2.

□

Note that we do not really care about the number of players in the proof of Lemma 6. When there are only 3 players, the proof also works obviously. So, we have the following lemma.

Lemma 7. *In* $\Gamma(\mathrm{Grid}_{n\times m}, 3)$ *where* n, m *are sufficiently large, a profile is not a Nash equilibrium, if there exists a controller and the distance between any 2 players' initial vertices is not shorter than 2.*

W.l.o.g, assume v_1 and v_2 are adjacent, which is ensured by Lemma 6. In the third step, we show v_3 and v_4 must be adjacent if v is a Nash equilibrium.

Lemma 8. *In* $\Gamma(\mathrm{Grid}_{n\times m}, 4)$ *where* n, m *are sufficiently large, a profile is not a Nash equilibrium if* $\|v_1 - v_2\|_1 = 1$ *and* $\|v_3 - v_4\|_1 > 1$.

Proof. Without loss of generality, we can fix the positions of v_1 and v_2 at (x_1, y_1) and $(x_1 + 1, y_1)$ and divide $V(\mathrm{Grid}_{n\times m})$ into

$$
\begin{aligned}
P_1 &= \mathrm{Part}_{0,-1}(v_1) \cup \mathrm{Part}_{0,-1}(v_2)\\
P_2 &= \mathrm{Part}_{1,0}(v_1) \backslash \{v_2\}\\
P_3 &= \mathrm{Part}_{0,1}(v_1) \cup \mathrm{Part}_{0,1}(v_2)\\
P_4 &= \mathrm{Part}_{-1,0}(v_2) \backslash \{v_1\}.
\end{aligned}
$$

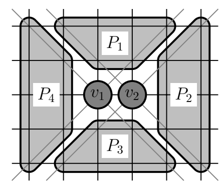

Figure 13: The grid is divided into 4 parts.

Now, we will debate upon each case separately in which v_3 and v_4 is in different part in grid.

Case 1: If both v_3 and v_4 is in P_1, the profile cannot be Nash equilibrium. If $x_3 + y_3 \not\equiv x_4 + y_4 \pmod 2$, there exists an improvement for v_3 or v_4, unless $\|v_3 - v_4\|_1 = 1$, according to Lemma 1. Otherwise, the only case we need to consider is that $y_3 = y_4$ and $x_4 - x_3 = 2$. W.l.o.g, assume $x_3 + y_3 \equiv x_1 + y_1 \pmod 2$. Thus, it can improve v_3 by moving it to $(x_3, y_3 - 1)$. The similar argument also shows v_3 and v_4 cannot be both in P_2, P_3, P_4.

Case 2: If $v_3 \in P_2$ and $v_4 \in P_4$ hold, v_3 and v_4 will be improved unless $\|v_1 - v_3\|_1, \|v_1 - v_4\|_1 = O(1)$ due to Lemma 1, which conflicts with Lemma 2. This argument also is adequate for the case where $v_3 \in P_1$ and $v_4 \in P_3$ hold.

Case 3: The profile v where $v_3 \in P_4$ and $v_4 \in P_1$ hold cannot be a Nash. According to Lemma 1, $\|v_3 - v_4\|_1 = O(1)$, $x_1 + y_1 - (x_3 + y_3) \le 2$ and $x_4 + y_4 - (x_2 + y_2) \le 2$ hold. Thus, there exists a contradiction due to Lemma 4.

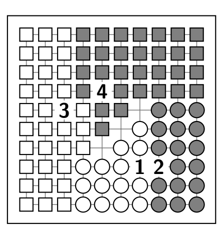

Figure 14: A sample profile in Case 3

□

Thus, the only remaining case is that the 4 initial vertices are divided into 2 neighbor pair. The structure is simple enough such that we can analyze all the case. Finally, we get the following two lemmas Lemma 9 and Lemma 10, whose combination is Theorem 2.

Lemma 9. *In $\Gamma(\text{Grid}_{n\times m}, 4)$ where n and m are sufficiently large and $\|v_1 - v_2\|_1 = 1$ and $\|v_3 - v_4\|_1 = 1$ hold, in the case where $4 \mid n$ and m is odd, there does not exist a Nash equilibrium if $m \geq n/2 + 2\lfloor\sqrt{n}\rfloor + 5$. In the other cases, there does not exist a Nash equilibrium if $m > n/2 + 6$.*

Proof. In the most cases, it is not a Nash equilibrium, if $m > n/2 + 6$. We will discuss them in detail.

Case 1: If $x_2 = x_1 + 1$ and $y_4 = y_3 + 1$ hold, the profile cannot be a Nash equilibrium if $m > n/2 + 6$. W.l.o.g, assume the $v_2 \in \text{Part}_{-1,0}(x_3 + 1, y_3) \cap \text{Part}_{-1,0}(x_4 + 1, y_4)$. Note that $|(x_2+y_2)-(x_3+y_3)|, |(y_2-x_2)-(y_3-x_3)| \geq \omega(1)$, since Lemma 4. Besides, $\|v_2-v_3\|_1 \geq \omega(1)$ should be satisfied since Lemma 2. Next, $n-x_4 \leq n/4+1$, unless the initial with the minimum payoff v_i can be improved by moving to $v_i' = (x_4 + 2, y_4)$ where $U_i(v_i', v_{-i}) \geq (n - x_4 - 1)m$. Then, consider the changes for v_3 and v_4, $v_3' = (x_3 - 1, y_3 - 1)$ and $v_4' = (x_4 - 1, y_4 + 1)$ where $U_3(v_3', v_{-3}) = U_3 + y_3 - 1 - (n - x_3 + 1)$ and $U_4(v_4', v_{-4}) = U_4 + (m - y_4) - (n - x_4 + 1)$. To avoid improvement, $m \leq n/2 + 6$ should be satisfied, which conflicts with our assumption.

Case 2: If $y_2 = y_1 + 1$ and $y_4 = y_3 + 1$ hold, the profile cannot be a Nash equilibrium if $m > n/2+6$. W.l.o.g, assume that $v_2 \in \text{Part}_{-1,0}(x_3+1, y_3) \cap \text{Part}_{-1,0}(x_4+1, y_4)$. The argument in Case 1 is also adequate for this case.

Case 3: If $x_2 = x_1 + 1$, $x_4 = x_3 + 1$ and $v_2 \in \text{Part}_{-1,0}(v_3)$ hold, it should be discussed carefully since this case includes our constructions in Section 3. Note that $|(x_2 + y_2) - (x_3 + y_3)|, |(y_2 - x_2) - (y_3 - x_3)|, \|v_2 - v_3\| \geq \omega(1)$ according to Lemma 2 and Lemma 4. First, we claim $|y_2 - y_3| < 2$. If not, we can assume $y_2 - y_3 \geq 2$. Thus, $m - y_2 + y_3 \leq m - 2$ holds. W.l.o.g, suppose $m - y_2 + 1 \leq (m - 1)/2$. Consider the change $v_2' = (x_2 + 1, y_2 - 1)$, where $U_2(v_2', v_{-2}) = U_2 + (y_2 - 1) - (m - y_2 + 1)$. The new payoff is more than the original one. Actually, we can learn more from it, that $|y_i - m/2| < 1$ should satisfied to avoid such improvement in a Nash equilibrium for all $i \in [4]$.

Case 3.1 For even m, let v_i be the initial vertex with the minimum payoff. If $i \in \{1, 4\}$, suppose $i = 1$ and $y_1 = m/2 + 1$ w.l.o.g. Consider the change $v_1' = (x_2, y_2 - 1)$. It is not so hard to verify that $U_1(v_1', v_{-1}) \geq U_1$. Thus, v is a Nash equilibrium only if the new profile in a Nash. Note that $x_1 \leq n/4 + 1$, otherwise, $v_i'' = (x_1 - 1, y_1)$ will improve the poorest initial vertex. Whereas, $x_1' \geq y_1' - 1$ should be satisfied according to Lemma 3, which conflicts with $m > n/2 + 6$. If $i \in \{2, 3\}$, the same argument also works.

Case 3.2 If m is odd, $y_i = (m + 1)/2$ for all $i \in [4]$. Note that v is a Nash equilibrium only if it is a Nash equilibrium on the sub-path $\{(m + 1)/2\} \times [n]$. Thus, all the non-trivial case is corresponding to a Nash equilibrium on Path_n.

Case 3.2.1: If $n \not\equiv 0 \pmod 4$, the profile is not a Nash equilibrium if $m > n/2 + 6$. Recall $\Gamma(\text{Path}_n, 4)$. It is easy to see that for a Nash equilibrium, the payoff of each initial vertex is no less than $\lfloor n/4\rfloor m$. The minimum one can touch the lower bound and the maximum one is no more than $\lceil n/4\rceil m$.

Case 3.2.1.1 If $U_1 = \lfloor n/4\rfloor m$ and $U_2 = \lceil n/4\rceil m$, consider the change $v_1' = (x_2, y_2 - 1)$ where $U_1(v_1', v_{-1}) \geq (2\lfloor n/4\rfloor + 1)(m - 1)/2$. It is an improvement if $m > n/2 + 6$.

Case 3.2.1.2 If $U_1 = \lceil n/4\rceil m$ and $U_2 = \lfloor n/4\rfloor m$, consider the change $v_2' = (x_1, y_1 - 1)$ where $U_2(v_2', v_{-2}) \geq (2\lfloor n/4\rfloor + 1)(m - 1)/2$. It is an improvement if $m > n/2 + 6$.

Case 3.2.1.3 If $U_1 = U_2 = \lfloor n/4 \rfloor m$ and $x_2 \equiv x_3 \pmod 2$, consider the change $v_1' = (x_2, y_2 - 1)$ where $U_1(v_1', v_{-1}) \geq (2\lfloor n/4 \rfloor + 1)(m-1)/2$. It is an improvement if $m > n/2 + 6$.

Case 3.2.1.3 The other cases can be reduced to these 3 cases.

Case 3.2.2: If $n \equiv 0 \pmod 4$, there exists the unique Nash equilibrium in $\Gamma(\mathrm{Path}_n, 4)$. Thus, the profile is not a Nash equilibrium unless

$$v_1 = \left(\frac{n}{4}, \frac{m+1}{2}\right), v_2 = (x_1 + 1, y_1), v_3 = \left(\frac{3n}{4}, y_1\right) \text{ and } v_4 = (x_3 + 1, y_1).$$

Then, considering the vertex

$$v_1' = \left(x_1 + \left\lfloor \frac{m-1}{4} - \frac{n}{8} + \frac{1}{2} \right\rfloor, y_1 - \left\lfloor \frac{m-1}{4} - \frac{n}{8} + \frac{1}{2} \right\rfloor\right)$$

as v_1's change, we can calculate its payoff,

$$U_1(v_1', v_{-1}) = \left\lfloor \frac{1}{16}m^2 + \frac{3}{16}mn - \frac{3}{8}m + \frac{1}{64}n^2 - \frac{1}{16}n + \frac{9}{16} \right\rfloor$$

and it is more than U_1 when $m \geq n/2 + 2\lceil \sqrt{n} \rceil + 5$.

□

Lemma 10. *In $\Gamma(\mathrm{Grid}_{n \times m}, 4)$ where n, m are sufficiently large, $4 \mid n$ and m is odd, there exists a Nash equilibrium if $m \leq n/2 + 2\lfloor \sqrt{n} \rfloor + 1$.*

Proof. We will check whether there is any improvement for v_1 first. Donate v_1' as a change of v_1. Without loss of generality, we assume $y_1' \leq y_1$. Note that the positions $(x_1, y_1), (x_1, y_1 - 1)$ and $(x_2, y_2 - 1)$ lead to the maximized payoffs according to Lemma 1 if $x_1' \leq x_1$. Furthermore, it is easy to see $U_1((x_1, y_1 - 1), v_{-1}) \leq U_1(v_1, v_{-1})$ and $U_1((x_2, y_2 - 1), v_{-1}) = n(m-1)/4 \leq U_1(v_1, v_{-1})$. Thus, we can assume $x_1' > x_1$. If v_1' is in $\mathrm{Part}_{1,0}(v_1) \cap \mathrm{Part}_{-1,0}(x_4 + 1, y_4)$ excepted $(n/2+1, (m+1)/2 - n/4)$

$$\begin{aligned} U_1((x_1', y_1'), v_{-1}) \leq & \left(\left\lceil \frac{x_1' - x_2 - y_1' + y_2}{2} \right\rceil + \left\lceil \frac{x_3 - x_1' - y_1' + y_2}{2} \right\rceil - 1 \right) \cdot m \\ & + (y_1' - y_2) \cdot (m - (y_1' - y_2)) \\ = & (n-1)m/4 - (y_1' - y_2)^2, \end{aligned}$$

and $U_1((n/2 + 1, (m+1)/2 - n/4), v_{-1}) = nm/4 - n^2/16 - (m-1)/2$. For the positions in $\mathrm{Part}_{0,-1}(x_2, y_2 + 1) \cap \mathrm{Part}_{0,-1}(v_3)$, $v_1' = (n/2 + 1, (m+1)/2 - n/4 - 1)$ maximizes the payoff where $U_1(v_1', v_{-1}) = nm/2 - 3n^2/16 - 3n/4 \leq nm/4$ with sufficiently large n, m if the condition holds. For the positions in $\mathrm{Part}_{0,-1}(v_2) \cap \mathrm{Part}_{-1,0}(v_4)$, the maximized payoff can be touched only when $y_1' + x_1' = y_1 + x_1$ according to Lemma 1. Thus, with $r := x_1' - x_2$ is

$$U_1(v_1', v_{-1}) = nm/4 + rm/2 - rn/4 - n/4 - r^2 - 3r/2$$

which is no more than $nm/4$ when $m \leq n/2 + 2\lfloor \sqrt{n} \rfloor + 1$.

An argument similar to the one for v_1 shows that the payoff of player-2 in the constructed profile can also not be improved. Due to the symmetry of the profile, the Nash equilibrium property can be shown. □

3 Voronoi games on trees

In this section, we discuss the Voronoi game on trees among 3 players. The main idea towards Theorem 3 is that if there exists a Nash equilibrium on a tree with 3 players, then their positions must induce a Path_3 on the tree.

Lemma 11. *In the 3-player Voronoi game on a tree T, if there exists a Nash equilibrium, then these three vertices induce a* Path_3 *on T.*

Proof. Suppose v_1, v_2, v_3 is a Nash equilibrium on the tree T, where each player i chooses vertex i $(i = 1, 2, 3)$. Then there are four cases at all:

Case 1: v_3 is on the path between v_1 and v_2. If v_1 is not a neighbor of v_3, then there exists a vertex $v' \notin \{v_1, v_2, v_3\}$ and v' is on the path between v_1 and v_3. It is easy to check that for player 1, (v', v_2, v_3) is better than (v_1, v_2, v_3), which is contradicted to Nash equilibrium. If v_2 is not a neighbor of v_3, then there exists a vertex $v'' \notin \{v_1, v_2, v_3\}$ and v'' is on the path between v_2 and v_3. It is easy to check that for player 2, (v_1, v'', v_3) is better than (v_1, v_2, v_3), which is contradicted to Nash equilibrium. So, both v_1 and v_2 must be neighbors of v_3, and $(v_1, v_3), (v_3, v_2)$ induce a Path_3.

Case 2: v_1 is on the path between v_3 and v_2. This case is solved by the same approach of Case 1.

Case 3: v_2 is on the path between v_1 and v_3. This case is solved by the same approach of Case 1.

Case 4: There exists a vertex $v \notin \{v_1, v_2, v_3\}$ such that v is on the path between v_2 and v_3, and v_2 and v_1. Since T is a tree and the Case 1,2,3 are all unsatisfied, the vertex v must be exist and unique, and it is also on the path between v_1 and v_3. It is easy to check that for player 1, (v, v_2, v_3) is better than (v_1, v_2, v_3), which is contradicted to Nash equilibrium. So, this case is not a Nash equilibrium.

Combining all these cases, we can prove this lemma. □

Based on this conclusion, it is easy to verify the group of conditions in Theorem 3 is sufficient and necessary.

Proof of Theorem 3. "$\Longrightarrow$": If there exists a Nash equilibrium v_1, v_2, v_3 on the tree T, then because of Lemma 11, v_1, v_2, v_3 induce a Path_3 on the tree T. Suppose v_2 is the degree-2 vertex in this Path_3. Let i_1, i_2 are the children of v_2 with maximum and second maximum $st(\cdot, v_2)$, and j_k is the child of i_k with maximum $st(\cdot, v_2)$ $(k = 1, 2)$.

First, we show that $\{v_1, v_3\} = \{i_1, i_2\}$. This is obvious, since v_1 and v_3 are neighbors of v_2, in the rooted tree with root v_2 they will be children of v_2. To get most profit they must choose the vertices among all children of v_2 with maximum and second maximum $st(\cdot, v_2)$.

Now we can verify all conditions:

- Condition 1 holds because if not, then player 2 can move from v_2 to j_1 to get profit $st(j_1, v_2)$, which is better than $n - st(i_1, v_2) - st(i_2, v_2)$, the profit player 2 can get on vertex v_2;
- Condition 2 holds for the same reason;
- Condition 3 holds because if not, then player 3 can move from i_2 to j_1 to get profit $st(j_1, v_2)$, which is better than $st(i_2, v_2)$, the profit player 3 can get on vertex i_2.

"$\Longleftarrow$": Suppose such a vertex v exists, then we will prove (i_1, v, i_2) is a Nash equilibrium:

1. For player 1, the definition of i_1 makes it meaningless to move from i_1 to any other child of v. Obviously, it is also useless to move position from i_1 to other vertices. So, player 1 will not move;
2. For player 2, condition 1 and 2 promise that it is useless to move position from v to any vertex in the subtree rooted at i_1 or i_2. Obviously, it is also useless to move position from v to other vertices. So, player 2 will not move;
3. For player 3, condition 3 promises that it is useless to move position from i_2 to any vertex in the subtree rooted at i_1. The definition of i_2 makes it meaningless to move from i_2 to any other child of v except for i_1. Obviously, it is also useless to move position from i_2 to other vertices. So, player 3 will not move.

Thus, we finish this proof. $\square$

Theorem 3 can induce an efficient algorithm as a judgment. See details in Algorithm 2, where each $val_{v,i}(v \in V(T), i = 1, 2)$ represents the value of $st(w, v)$, where w is a neighbor of v with the i-th maximum $st(\cdot, v)$, and each $lab_{v,i}(v \in V(T), i = 1, 2)$ records the vertex w. If there is a Nash equilibrium, the algorithm will return a triple of three different vertices as a solution, otherwise it will return -1.

The correctness of this algorithm is promised by Theorem 3. Now let us analyze its performance. Let $n = v(T)$. The main space cost of this algorithm is the tree structure and the arrays $\{val_{v,i}\}$ and $\{lab_{v,i}\}$, so it is $O(n)$. For the time cost, it is easy to see that the two arrays $\{val_{v,i}\}$ and $\{lab_{v,i}\}$ can be computed by a dynamic programming on the tree T within $O(n)$ time. Combine the $O(n)$ time cost by Algorithm 2, the total time cost is $O(n)$ too.

Algorithm 2 Nash equilibrium on a tree T with 3 players

```
for v ∈ V(T) do
    sti1 ← val_{v,1}
    sti2 ← val_{v,2}
    if lab_{lab_{v,1},1} = v then
        stj1 ← val_{lab_{v,1},2}
    else
        stj1 ← val_{lab_{v,1},1}
    if lab_{lab_{v,2},1} = v then
        stj2 ← val_{lab_{v,2},2}
    else
        stj2 ← val_{lab_{v,2},1}
    if (v(T) − sti1 − sti2 ≥ max{stj1, stj2}) ∧ (sti2 ≥ stj1) then
        return (lab_{v,1}, v, lab_{v,2})
return −1
```

4 Conclusion

In this paper, we consider one-round multi-player Voronoi games on grids and trees. First, we answer such a question: is there a Nash equilibrium in a one-round k-player Voronoi game processing on $\text{Grid}_{n\times m}$ $(n \geq m)$? For the game on sufficiently large grids within 4 players, we provide almost complete characterization of the existence of a Nash equilibrium. Second, for the game on a tree with 3 players, we give a sufficient-necessary condition for the existence of a Nash equilibrium, as well as a linear time/space algorithm to check the condition.

For the case with k player $(k > 4)$ on a grid, we raise a method to construct a Nash equilibrium if the grid sufficiently narrow. Furthermore, we conjecture:

Conjecture 1. *For a given integer k and any sufficiently large integers n, m which satisfy $n \geq m > n/\lceil k/2 \rceil + o(n)$, there does not exist a Nash equilibrium in $\Gamma(\text{Grid}_{n\times m}, k)$.*

We say a profile is non-trivial if in the profile, no obvious improvement can be detected by the tools lemmas, i.e, Lemma 1, Lemma 3, Lemma 2 and Lemma 4, stated in Section 2.2. The number of non-trivial profiles is limited, especially when k is constant. Actually, the approach to prove the non-existence for $k = 4$ is to check each non-trivial profile with some tricks to simplify such checking process. So, we believe the approach also works for the larger but constant k.

For the case of tree, it is worth thinking how to determine whether there exists a Nash equilibrium with 4 players. It is also interesting to determine whether there is a Nash equilibrium with multi-players on other graph classes.

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A A special case of Theorem 1

Theorem 5. *There is a Nash equilibrium in* $\Gamma(\mathrm{Grid}_{n\times m}, 6)$ *if* m *is even,* $n \bmod 6 = 4$ *and* $m \le n/3$.

Proof. Let $t := (n-4)/6$ and v is the constructed profile with $v_1 = (t+1, m/2+1), v_2 = (t+2, m/2+1), v_3 = (3t+2, m/2), v_4 = (3t+3, m/2), v_5 = (5t+3, m/2+1)$ and $v_6 = (5t+4, m/2+1)$. It suffices to prove v_1, v_2 and v_3 cannot be improved duo to the symmetry. We calculate the maximum $U_i(v'_i, v_{-i})$ when v'_i is in different areas, for all $i \in [3]$, as following table:

	Area	Best v'_i	Max. payoff	Orig. payoff
v'_1	$\mathrm{Part}_{-1,0}(x_2+1, y_2)$	v_1	$(t+1)m$	$(t+1)m$
	$\mathrm{Part}_{0,-1}(v_2)$	(x_2, y_2-1)	$(2t+1)m/2$	
	$\mathrm{Part}_{0,1}(v_2)$	(x_2, y_2+1)	$(2t+2)(m/2-1)$	
	$\mathrm{Part}_{1,0}(v_1) \cap \mathrm{Part}_{-1,0}(v_4)$	(x_2+1, y_2)	$(t-1/2)m$	
	$\mathrm{Part}_{0,-1}(v_3) \cap \{(x,y) \mid x \le n/2\}$	(x_3, y_3-2)	$(2t+3)(m/2-2)$	
	$\mathrm{Part}_{0,1}(v_3) \cap \{(x,y) \mid x \le n/2\}$	(x_3, y_3+2)	$(2t+1)(m/2-1)$	
v'_2	$\mathrm{Part}_{-1,0}(v_2)$	(x_1-1, y_1)	tm	$(t+1/2)m$
	$\mathrm{Part}_{0,-1}(v_1)$	(x_1, y_1-1)	$(2t+1)m/2$	
	$\mathrm{Part}_{0,1}(v_1)$	(x_1, y_1+1)	$(2t+2)(m/2-1)$	
	$\mathrm{Part}_{1,0}(x_1-1, y_1) \cap \mathrm{Part}_{-1,0}(v_4)$	v_2	$(t+1/2)m$	
	$\mathrm{Part}_{0,-1}(v_3) \cap \{(x,y) \mid x \le n/2\}$	(x_3, y_3-2)	$(2t+3)(m/2-2)$	
	$\mathrm{Part}_{0,1}(v_3) \cap \{(x,y) \mid x \le n/2\}$	(x_3, y_3+2)	$(2t+1)(m/2-1)$	
v'_3	$\mathrm{Part}_{-1,0}(v_2)$	(x_1-1, y_1)	tm	$(t+1/2)m$
	$\mathrm{Part}_{0,-1}(v_1) \cup \mathrm{Part}_{0,-1}(v_2)$	(x_1, y_1-2)	$(2t+2)(m/2-1)$	
	$\mathrm{Part}_{0,1}(v_1) \cup \mathrm{Part}_{0,1}(v_2)$	(x_1, y_1+2)	$(2t+3)(m/2-2)$	
	$\mathrm{Part}_{1,0}(v_1) \cap \mathrm{Part}_{-1,0}(x_4+1, y_4)$	v_3	$(t+1/2)m$	
	$\mathrm{Part}_{0,-1}(v_4) \cap \{(x,y) \mid x \le n/2\}$	(x_3, y_3-2)	$(2t+3)(m/2-2)+t+1$	
	$\mathrm{Part}_{0,1}(v_4) \cap \{(x,y) \mid x \le n/2\}$	(x_3, y_3+2)	$(2t+1)(m/2-1)+t$	

Note that all case not listed in this tables can be reduced to a listed case. When $m < 6$, some best v'_i in this table could be out of the grid. But the Max. payoff is still an upper bound of the changed payoff. If $m \le n/3$, the payoff of any change cannot exceed the original payoff, which means v is an Nash equilibrium. □