

Approximate Fictitious Play for Mean Field Games

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Abstract

The theory of Mean Field Games (MFG) allows characterizing the Nash equilibria of an infinite number of identical players, and provides a convenient and relevant mathematical framework for the study of games with a large number of agents in interaction. Until very recently, the literature only considered Nash equilibria between fully informed players. In this paper, we focus on the realistic setting where agents with no prior information on the game learn their best response policy through repeated experience. We study the convergence to a (possibly approximate) Nash equilibrium of a fictitious play iterative learning scheme where the best response is approximately computed, typically by a reinforcement learning (RL) algorithm. Notably, we show for the first time convergence of model free learning algorithms towards non-stationary MFG equilibria, relying only on classical assumptions on the MFG dynamics. We illustrate our theoretical results with a numerical experiment in continuous action-space setting, where the best response of the iterative fictitious play scheme is computed with a deep RL algorithm.

1 Introduction

We consider the computation of Nash equilibria for systems with a large number of anonymous identical agents in interaction. Such systems encompass the modeling of numerous applications such as traffic jam dynamics, swarm systems, financial market equilibrium, crowd evacuation, smart grid control, web advertising auction, vaccination dynamics, rumor spreading on social media, among others. As each agent is small in comparison to the population size, its own effect on the global system can be neglected. Consequently, the asymptotic limit with infinite population size is highly relevant and corresponds to considering Mean Field Games (MFG), introduced by Lasry and Lions [2006a,b, 2007] and by Huang et al. [2006, 2007]. In a sequential game theory setting, each player needs to take into account its impact on the other players strategy. Studying games with an infinite number of players is easier from this point of view, as the impact of one single player on the others can be neglected. Furthermore, the optimal policy for the asymptotic MFG often provides an approximate Nash equilibrium for the game with a finite number of players [Cardaliaguet, 2013; Bensoussan et al., 2013; Carmona and Delarue, 2018b].

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A solution to a dynamic MFG is determined via the optimal policy of a representative agent in response to the flow of the entire population. A mean field Nash equilibrium arises when the distribution of the best response policies over the population generates the exact same population flow. In the rapidly growing literature on MFG, most papers consider fully informed agents about the game operation scheme and the MF population dynamics. Only a few works consider the realistic case where the agents are learning the game dynamics and rewards as they play, e.g. [Yin et al., 2010; Cardaliaguet and Lehalle, 2018; Hadikhanloo, 2018]. In this paper, we focus on a realistic setting where agents with no prior information learn sequentially as they play.

Similarly, in a Multi-Agent Reinforcement Learning (MARL) setting, all agents are learning simultaneously how to act optimally in a dynamic system. In comparison to the single-agent case, the derivation of efficient learning algorithms in such context is difficult due to the lack of stationarity of the environment, whose dynamics evolves as the population learns [Bu et al., 2008]. Our approach investigates how any single-agent RL algorithm can be applied in a MFG setting, in order to learn a (possibly approximate) Nash equilibrium, via repeated experiences and without any prior knowledge.

For this purpose, our approach relies on a Fictitious Play (FP) iterative learning scheme for repeated games, where each agent calibrates its belief to the empirical frequency of the previously observed population flows. Whenever each agent is able to compute its exact best response to a population flow, FP is proved to reach asymptotically a Nash equilibrium in some (but not all [Shapley, 1964]) classes of games, such as first order monotone MFG [Hadikhanloo, 2018]. We focus here on a realistic setting, where the exact best response remains unattainable and is typically approximated via a single-agent RL algorithm. This induces an approximation error at each iteration, which propagates through the FP learning scheme.

The main contribution of this paper is the rigorous study of the error propagation in Approximate FP algorithms for MFGs, using an innovative line of proof in comparison to the standard two time scale approximation convergence results [Leslie and Collins, 2006; Borkar, 1997]. This allows discussing the convergence to a (possibly approximate) MF Nash equilibrium, when using any standard off-the-shelf single-agent RL algorithm. Especially, our theoretical framework encompasses the convergence of RL algorithms to MFG equilibrium in non stationary settings, which, as far as we know, is new in the literature. The convergence properties are obtained for 1st order monotone MFGs [Gomes et al., 2010] and rely on reasonable and verifiable assumptions on the MFG dynamics. We illustrate our theoretical results with a numerical experiment in a newly addressed continuous state-action environment, where the approximate best response of the iterative FP scheme is computed with a deep RL algorithm.

2 Background

Mean Field Games. MFGs were introduced by Lasry and Lions [2006a,b, 2007] and by Huang et al. [2006, 2007]. An MFG corresponds to the asymptotic limit of a differential game, when the number of agents is infinite. Due to game interactions, the state dynamics and the reward of each agent depend on the states of the other ones, as in crowd motion. Since all agents are assumed to be identical and indistinguishable, individual interactions are irrelevant in the limit and only the distribution of states matters (see [Cardaliaguet, 2013; Bensoussan et al., 2013; Carmona and Delarue, 2018a] for an introduction). In order to emphasize the proximity of MFG with regard to the RL literature, we choose to present our analysis in a discrete time setting.

Finding a Mean Field Nash equilibrium boils down to identifying the equilibrium distribution dynamics of the population and the best response (or optimal policy) of a representative agent to this population mean field flow. Since the number of players is infinite, each agent has an infinitesimal influence on the population

distribution. Yet, since all agents are rational, the state distribution generated by the optimal policy must coincide at equilibrium with the population distribution.

Notations. Let \mathcal{X} and \mathcal{A} be compact convex subsets of \mathbb{R}^ℓ and \mathbb{R}^d respectively, which represent the state and action spaces common to every agent. Let $T > 0$ be a time horizon and let \mathbb{T} denote the time sequence $\{0, 1, \dots, T\}$. We denote by $\mathcal{P}(\mathcal{X})$ the set of probability measures on \mathcal{X} and by $M_T = \mathcal{P}(\mathcal{X})^T$ the set of all possible flows of population state distributions $\mu = (\mu_0, \mu_1, \dots, \mu_T)$. The initial distribution of the population is an atomless measure on \mathcal{X} denoted by μ_0 . For $\mu \in M_T$, μ_t represents the distribution at time t of the state occupation of the entire population.

State dynamics & Mean field population flow. At any time $t \in \mathbb{T}$, each agent belongs to a state $x_t \in \mathcal{X}$ and picks an action $a_t \in \mathcal{A}$. For a sequence of actions $a := (a_t)_{t \in \mathbb{T}}$, the dynamics of x is governed by a Markov Decision Process (MDP) with transition density $p_t(\cdot | x_t, a_t, \mu_t)$ parametrized by the mean field flow $\mu \in M_T$ of the population. This indexation transcribes the interactions with the other agents, through their state distribution μ_t . Typically, the dynamics of x is described by an equation of the form

$$x_{t+1} = x_t + b(x_t, a_t, \mu_t) + \epsilon_{t+1}, \quad (1)$$

where $b : \mathcal{X} \times \mathcal{A} \times \mathcal{P}(\mathcal{X}) \rightarrow \mathcal{X}$ is a drift function and $(\epsilon_t)_{t \geq 1}$ is a source of noise. We stress that the mean field term μ_t , is the whole population distribution and not just the average state.

We denote by Π_T the set of policies (or controls) $\pi : \mathbb{T} \times \mathcal{X} \rightarrow \mathcal{A}$ which are feedback in the state: at time t , an agent using policy π while in state x_t plays the action $a_t = \pi(t, x_t)$. The process x controlled by $\pi \in \Pi_T$ is denoted x^π .

Agent's reward scheme. An infinitesimal agent starting at time 0 in state $x_0 \in \mathcal{X}$ chooses a policy $\pi \in \Pi_T$ in order to maximize the following discounted expected sum of rewards:

$$J(x_0, \pi, \mu) := \mathbb{E} \left[\sum_{t=0}^{T-1} \gamma^t r(x_t^\pi, \mu_t, a_t) \right], \quad (2)$$

while interacting with the population MF flow $\mu \in M_T$. At time t , the agent's rewards are impacted by μ_t , which represents the aggregate state distribution of all the other agents (*i.e.* of the whole population). Since the agents are anonymous, only the MF distribution flow μ of the states matters.

As μ_0 denotes the state distribution at time 0, the average reward for a typical agent is given by

$$\mathbb{J}(\pi, \mu) := \mathbb{E}_{x_0 \sim \mu_0} [J(x_0, \pi, \mu)], \quad (3)$$

when this agent uses policy $\pi \in \Pi_T$, while the mean field population flow is μ .

Definition 1 (Best response). *A policy π^* maximizing $\mathbb{J}(\cdot, \mu)$ is called a best response of the representative agent to the MF population dynamic flow μ .*

MF Nash Equilibrium. While interacting, the agents may or may not reach a Nash equilibrium, whose definition, based on the previous best response policy characterization, is:

Definition 2 (Mean Field Nash equilibrium). *A pair (π^*, μ^*) consisting of a policy and a MF population distribution flow is called MF Nash equilibrium if it satisfies*

→ **agent rationality:** π^* is a best response to μ^* ;

→ **population consistency:** for all $t \in \mathbb{T}$, μ_t^* is the distribution of x_t^* , starting with distribution μ_0 and controlled by π^* .

Namely, if the mean field population flow is μ^* , the policy π^* is optimal, and if all the agents play according to π^* , the induced mean field population flow coincides with μ^* . Hereby, (π^*, μ^*) identifies to an MF Nash equilibrium.

Observe that reaching an MF Nash equilibrium requires the computation of exact best response policy, which can be difficult in practice. We are concerned with the design of an iterative learning scheme, where the available best response is partially accurate and typically approximated by RL through repeated experiences. E.g., this realistic situation arises when agents are repeatedly optimizing their daily driving trajectories, without any prior information on the traffic jam flow dynamic rules.

3 Fictitious Play Algorithms for MFG

Fictitious play [Robinson, 1951] is an iterative learning scheme for repeated games, where each agent calibrates its belief to the empirical frequency of previously observed strategies of other agents, and plays optimally according to its beliefs. This constitutes its best response. Even in simple two-player games, the convergence of FP to a Nash equilibrium is not guaranteed [Shapley, 1964]. However, the convergence of FP has recently been proved for some classes of Mean Field Games [Hadikhanloo, 2018, 2017; Cardaliaguet and Hadikhanloo, 2017].

Yet, in most cases, agents do not have access to the exact best response policy π^* but use an approximate version of it instead, in the spirit of [Leslie and Collins, 2006; Pérolat et al., 2018]. At each iteration step n , the agent has only access to an approximate version $\hat{\pi}^{n+1}$ of the best response $\pi^{*,n+1}$ to the mean field flow $\bar{\mu}^{(n)}$, as presented in Algorithm 1. For sake of clarity, we also report the exact FP algorithm in the Appendix. So at each iteration step n , the learning scheme induces an additional error ℓ_n defined as

$$\ell_n := \mathbb{E}_{x_0 \sim \mu_0} [J(x_0, \pi^{*,(n+1)}, \bar{\mu}^{(n)}) - J(x_0, \hat{\pi}^{(n+1)}, \bar{\mu}^{(n)})] \geq 0, \quad n \in \mathbb{N}.$$

Observe that ℓ_n identifies to the expected loss over the entire population at step n , when using the approximate policy $\hat{\pi}^{(n+1)}$ instead of the exact one $\pi^{*,(n+1)}$. In Sec 4 below, we quantify the error propagation of ℓ_n and clarify the convergence properties of Algorithm 1. The specific setting where the approximate optimal policy is computed using single-agent RL algorithms is discussed in Sec 4.3.

Approximate Nash equilibrium At each step n , we denote by $\bar{\pi}^{(n)}$ the agent belief on the aggregate population policy, defined as an equally randomized version of all previous approximate best responses $(\hat{\pi}^{(k)})_{k=1, \dots, n}$: for each $t \in \mathbb{T}$ and $x \in \mathcal{X}$, $\bar{\pi}^{(n)}(t, x) \in \mathcal{P}(\mathcal{A})$ is the probability distribution on the set of actions \mathcal{A} according to which the player picks uniformly at random an element of $\{\hat{\pi}^{(1)}(t, x), \dots, \hat{\pi}^{(n)}(t, x)\}$. With a slight abuse of notation, we write

$$J(x_0, \bar{\pi}^{(n)}, \bar{\mu}^{(n)}) := \frac{1}{n} \sum_{k=1}^n J(x_0, \hat{\pi}^{(k)}, \bar{\mu}^{(n)}), \quad n \in \mathbb{N}.$$

By construction, $\bar{\mu}^{(n)}$ identifies to the population MF flow induced by the policy $\bar{\pi}^{(n)}$. In order to assess the quality of $(\bar{\mu}^{(n)}, \bar{\pi}^{(n)})$ as an (approximate) MF Nash equilibrium, we introduce

$$e_n := \mathbb{E}_{x_0 \sim \mu_0} [J(x_0, \pi^{*,(n+1)}, \bar{\mu}^{(n)}) - J(x_0, \bar{\pi}^{(n)}, \bar{\mu}^{(n)})] \geq 0, \quad n \in \mathbb{N}.$$

Algorithm 1: Approximate Fictitious Play for MFG

Data: An initial distribution μ_0 ; an initial flow of distributions $\bar{\mu}^{(0)}$; a number of steps N .

Result: A flow of distribution μ and a policy π .

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1 begin
2   for  $n = 0, 1, \dots, N - 1$  do
3     Compute  $\hat{\pi}^{(n+1)}$ , the approximate response policy against  $\bar{\mu}^{(n)}$ . Compute  $\hat{\mu}^{(n+1)}$ , the mean field
       flow associated to  $\hat{\pi}^{(n+1)}$  starting with  $\hat{\mu}_0^{(n+1)} = \mu_0$ 
4     Update  $\bar{\mu}^{(n+1)} = \frac{n}{n+1}\bar{\mu}^{(n)} + \frac{1}{n+1}\hat{\mu}^{(n+1)}$ 
5   return  $(\bar{\mu}^{(N)}, \hat{\pi}^{(N)})$ 

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The error e_n quantifies at iteration n the expected gain for a typical agent, when shifting its belief $\bar{\pi}^{(n)}$ to the exact best response $\pi^{*,(n+1)}$, while interacting with the MF population flow $\bar{\mu}^{(n)}$. After n iterations in Algorithm 1, e_n measures the quality of $(\bar{\mu}^{(n)}, \bar{\pi}^{(n)})$ as a MF Nash equilibrium. For the sake of clarification, let us introduce a weaker notion of MF Nash equilibrium, borrowed from [Carmona, 2004].

Definition 3 (Approximate MF Nash equilibrium). *For $\epsilon > 0$ and $\delta \in (0, 1)$, a pair $(\pi_{\epsilon, \delta}^*, \mu_{\epsilon, \delta}^*)$ consisting of a policy and a population distribution flow is called an (ϵ, δ) -MF Nash equilibrium if*

$$\mu_0 \left(\{x_0 \in \text{supp}(\mu_0) ; J(x_0, \pi_{\epsilon, \delta}^*, \mu_{\epsilon, \delta}^*) \geq J(x_0, \pi', \mu_{\epsilon, \delta}^*) - \epsilon, \forall \pi' \in \Pi_T\} \right) \geq 1 - \delta,$$

and $\mu_{\epsilon, \delta}^*$ coincides with the MF distribution flow starting from μ_0 , when every agent uses policy $\pi_{\epsilon, \delta}^*$.

An (ϵ, δ) -MF Nash equilibrium is interpreted as a weak equilibrium which is ϵ -optimal for at least a fraction $(1 - \delta)$ of the population. We are now able to clarify how e_n characterizes the quality of $(\bar{\mu}^{(n)}, \bar{\pi}^{(n)})$ as an MF Nash equilibrium.

Theorem 4. *If $0 \leq e_n \leq \epsilon^2$, then $(\bar{\mu}^{(n)}, \bar{\pi}^{(n)})$ is an (ϵ, ϵ) -MF Nash equilibrium. Furthermore, if e_n goes to 0 as n goes to infinity, $\bar{\mu}^{(n)}$ converges to an MF Nash equilibrium.*

Proof. Assume $0 \leq e_n \leq \epsilon^2$. Let us introduce $\varphi(x_0) := J(x_0, \pi^{*,(n+1)}, \bar{\mu}^{(n)}) - J(x_0, \bar{\pi}^{(n)}, \bar{\mu}^{(n)})$, which is non-negative since $\pi^{*,(n+1)}$ is the best response to the MF flow $\bar{\mu}^{(n)}$.

Using Markov's inequality and the bound on e_n we obtain

$$\mu_0(\{x_0 \in \mathcal{X} : \varphi(x_0) \geq \epsilon\}) = \mathbb{P}_{x_0 \sim \mu_0}[\varphi(x_0) \geq \epsilon] \leq \frac{\mathbb{E}_{x_0 \sim \mu_0}[\varphi(x_0)]}{\epsilon} = \frac{e_n}{\epsilon} \leq \frac{\epsilon^2}{\epsilon} = \epsilon.$$

Collecting the terms and using the definition of φ , we deduce

$$\mu_0 \left(\left\{ x_0 \in \mathcal{X} : J(x_0, \pi^{*,(n+1)}, \bar{\mu}^{(n)}) \leq J(x_0, \bar{\pi}^{(n)}, \bar{\mu}^{(n)}) + \epsilon \right\} \right) \geq 1 - \epsilon,$$

so that $(\bar{\pi}^{(n)}, \bar{\mu}^{(n)})$ is an (ϵ, ϵ) -MF Nash equilibrium.

The second part of the theorem is a direct consequence of the first one. □

4 Error propagation & Nash equilibrium approximation for 1st order MFG

Since $(e_n)_n$ identifies to a relevant quality measure of Algorithm 1 after n iterations, we now evaluate how the individual learning errors $(\ell_k)_{0 \leq k \leq n}$ aggregate over e_n . For the sake of simplicity, we focus our discussion on 1st order mean field games, *i.e.* without source of noise in the dynamics. This allows us to build our reasoning on the analysis of Hadikhanloo [2017] and to avoid a restriction to potential games. Nevertheless, since Fictitious Play has also been studied for potential second order MFGs [Cardaliaguet and Hadikhanloo, 2017], we think that similar results should hold in that setting as well.

4.1 First order mean field game

The state $(x_t)_t$ evolves in \mathbb{R}^d with dynamics (1), where we take $b(x, a, \mu) = a$, and $(\epsilon_t)_t = 0$. In other words, each agent controls exactly its state variation between two time steps and does not endure any noise. While interacting with a MF flow $\mu \in M_T$, each agent intends to maximize the classical reward scheme given by (2) with a running reward at time t of the form:

$$r(x_t^\pi, \mu_t, a_t) \mapsto \tilde{r}(x_t^\pi, a_t) + \bar{r}(x_t^\pi, \mu_t), \quad (4)$$

where the extra \bar{r} captures the impact of the other agents' positions. In Sec. 5, we provide in particular an example where \bar{r} models an appeal for non-crowded regions. This type of conditions translates into the so-called Lasry-Lions monotonicity condition [Lasry and Lions, 2007] which ensures uniqueness of MF Nash equilibrium. More precisely, existence and uniqueness of solution to the 1st order MFG of interest holds under the following assumptions.

Assumption 1. *For some constant C , the reward functions \tilde{r} and \bar{r} satisfy:*

- *For every $x \in \mathcal{X}$, the map $\tilde{r}(x, \cdot)$ is twice differentiable and $\frac{1}{C}I_d \leq D_{aa}\tilde{r}(x, \cdot) \leq CI_d$,*
- *The function \bar{r} is continuous on $\mathcal{X} \times \mathcal{P}(\mathcal{X})$ and $\bar{r}(\cdot, m)$ is \mathcal{C}^1 on \mathcal{X} ,*
- *$\|\tilde{r}(\cdot, \cdot)\|_\infty + \|\bar{r}(\cdot, \cdot)\|_\infty \leq C$,*
- *The Lasry-Lions monotonicity condition holds:*

$$\int_{\mathcal{X}} [\bar{r}(\cdot, m_1) - \bar{r}(\cdot, m_2)] d[m_1 - m_2] < 0, \quad \text{for all } m_1, m_2 \in \mathcal{P}(\mathcal{X}). \quad (5)$$

4.2 Error propagation in the Fictitious Play algorithm

We first investigate how the learning error $(\ell_n)_n$ propagates through FP in general, while Sec 4.3 focuses on the specific case where the best response is approximated via RL.

The main feature of FP is the quick stabilization of the sequence of beliefs $(\bar{\mu}^{(n)})_n$.

Lemma 5. *Under Assumption 1, the fictitious play MF flow $\bar{\mu}^{(n)}$ satisfies:*

$$d_1(\bar{\mu}^{(n)}, \bar{\mu}^{(n+1)}) \leq \frac{C}{n}, \quad n \in \mathbb{N}, \quad \text{for some constant } C > 0,$$

where d_1 is the Wasserstein distance. The proof follows from a straightforward adaptation of the arguments used in [Hadikhanloo, 2018, Lemma 3.3.2] to our setting. It is provided in the Appendix for sake of completeness, together with the definition of the Wasserstein distance d_1 .

As the sequence of beliefs $\bar{\mu}_n$ stabilizes, the impact of recent learning errors (ℓ_n) reduces and we are in position to quantify the global error e_n of the algorithm after n iteration steps. This is the main result of the paper, whose line of proof reported in the Appendix interestingly differs from the more classical two-time scale approximation argumentation [Borkar, 1997].

Theorem 6. *Under Assumption 1, the Nash equilibrium quality $(e_n)_n$ satisfies both estimates:*

$$0 \leq e_n \leq \frac{C_1}{n} + \frac{C_1}{n} \sum_{i=1}^n d_1(\mu^{*,(i+1)}, \hat{\mu}^{(i+1)}) + \frac{1}{n} \sum_{i=1}^n \ell_i, \quad n \in \mathbb{N}, \quad (6)$$

$$0 \leq e_n \leq \ell_n + \frac{C_2}{n} + \frac{C_2}{n} \sum_{i=1}^n d_1(\hat{\mu}^{(i+1)}, \hat{\mu}^{(i+2)}) + \sum_{i=1}^n \frac{i+1}{n} \ell_i, \quad n \in \mathbb{N}, \quad (7)$$

for some constants C_1 and C_2 .

Bound (6) indicates a nice averaging aggregation of the learning errors $(\ell_n)_n$, but requires a strong additional control on the Wasserstein distance between the MF flows generated by both approximate and exact best responses. Such estimate is readily available for the numerical approximation of convex stochastic control problems [Kushner and Dupuis, 2013] but less classical in the RL literature, as discussed in Sec 4.3. When such an estimate is not available, Bound (7) provides a slower $n\ell_n$ convergence rate, up to a weak d_1 -regularity of the approximate best response in terms of the mean field flow $\bar{\mu}^{(n)}$, recall Lemma 5. Such a weak estimate is for example highly classical in the setting of convex stochastic control problems with Lipschitz rewards [Fleming and Rishel, 2012; Kushner and Dupuis, 2013]. At finite distance and asymptotically, the following corollary sums up these properties in terms of MF Nash equilibrium.

Corollary 7. *If $\ell_n + C_1 d_1(\mu^{*,(n)}, \hat{\mu}^{(n)})$ or $n\ell_n + C_2 d_1(\hat{\mu}^{(n+1)}, \hat{\mu}^{(n)})$ is bounded by $\epsilon^2/2$, $(\bar{\mu}_n, \bar{\pi}_n)_n$ is an (ϵ, ϵ) -MF Nash equilibrium, for n large enough.*

In a similar fashion, we can conclude on the general asymptotic convergence of Algorithm 1 to the unique MF-Nash equilibrium, before discussing the specific implications for RL best response approximation schemes.

Corollary 8. *The approximate FP algorithm converges to the unique MF Nash equilibrium whenever one of the following two conditions holds:*

1. *The approximate best response update procedure $\bar{\mu}^{(n)} \mapsto \hat{\mu}^{(n+1)}$ is continuous in d_1 , and $n\ell_n$ converges to 0, as n goes to ∞ ;*
2. *The learning and policy approximation errors ℓ_n and $(d_1(\mu^{*,(n)}, \hat{\mu}^{(n)}))_n$ converge to 0.*

4.3 Discussion on the convergence for Best Response RL approximation

The result in Theorem 6 is general and relies on standard assumptions of MFGs. It also relies on a good enough control of the approximation error of the best response at each iteration. Here, we discuss to what extent existing theoretical results for RL algorithms allow satisfying this assumption.

As stated in Corollary 8, in order for the approximate FP to converge to the exact MF Nash equilibrium, the approximate best response should converge quickly enough to the best one, depending on the number

of iterations. From an RL perspective, this would require being able to compute the approximate optimal policy to an arbitrary precision, with high probability. As far as we know, such a result is possible only when an exact representation of any value function is possible, that is, in the tabular setting which imposes finite state and action spaces. Notably, convergence and rate of convergence of Q-learning-like algorithms have been studied in the literature, see e.g. [Szepesvári, 1998; Kearns and Singh, 1999; Even-Dar and Mansour, 2003; Azar et al., 2011]. For example, the speedy Q-learning algorithm requires $\mathcal{O}(\ln(K)/(\epsilon^2(1-\gamma)^4))$ steps to learn an ϵ -optimal state-action value function with high probability, with K the number of state-action couples. According to Corollary 8, if the error is in $\mathcal{O}(n^{-\alpha})$ with $\alpha > 1$ (and if we have continuity in d_1), then the scheme converges to the Nash equilibrium. This suggests using $\mathcal{O}(\ln(K)n^{2\alpha}/(1-\gamma)^4)$ steps for the RL agent at iteration n . Yet, this kind of results does not provide guarantees on the continuity in d_1 .

According to Corollary 7, bounding the learning errors and the distance between two iterates of the distribution is sufficient to reach an approximate Nash equilibrium. As approximate FP can be seen as repeated RL problems, RL (or approximate dynamic programming) can be seen as repeated supervised learning problems, and the propagation of errors from supervised to RL is a well studied field, see e.g. [Farahmand et al., 2010; Scherrer et al., 2015]. Basically, if the supervised learning steps are bounded by some $\mathcal{O}(\epsilon)$, then the learning error of the RL algorithm is bounded by $\mathcal{O}(C\epsilon/(1-\gamma)^2)$, where C is a so-called concentrability coefficient, measuring the mismatch between some measures. In principle, we could then propagate the learning error of the supervised learning part up to the FP error, through the RL error. However, these results do not provide any guarantees on the proximity between the estimated optimal policy and the actual one (which would be a sufficient condition for the proximity between population distributions); it only provides a guarantee on the distance between their respective returns. This is due to the fact that in RL, the optimal value function is unique, but not the optimal policy. A perspective would be to consider regularized MDPs [Geist et al., 2019], where the optimal policy is unique (and greediness is Lipschitz). Yet, this would come at the cost of a bias in the Nash equilibria. The approach in [Guo et al., 2019] somehow builds partially on this idea.

5 Numerical illustration

As an illustration, we consider a stylized MFG model with congestion in the spirit of Alnulla et al. [2017]. This application should be seen as a proof of concept showing that the method described above can be applied beyond the framework used for our theoretical results. We compute a model free approximation of the MFG solution combining Algorithm 1 with Deep Deterministic Policy Gradient (DDPG) [Lillicrap et al., 2016]. As far as we know, this is the first numerical illustration of model free deep RL Algorithm for Mean Field Games with continuous states and actions.

Environment Each agent has a position x located on the torus $\mathbf{T} = [0, 1]$ with periodic boundary conditions, whose dynamics is governed by $x_{t+1} = x_t + a_t\Delta_t + \sqrt{\Delta_t}\epsilon_t$, $t = 0, 1, \dots$, where Δ_t is the time step of the continuous time process. It receives the per-step reward

$$r(x_t, \mu_t, a_t) = \tilde{r}(x_t) - \frac{1}{2}|a_t|^2 - \log(\mu_t) ,$$

where the last term motivates agents to avoid congestion, *i.e.* the proximity to a region with a large population density. In the continuous time setting with no discounting, a direct PDE argument provides the ergodic

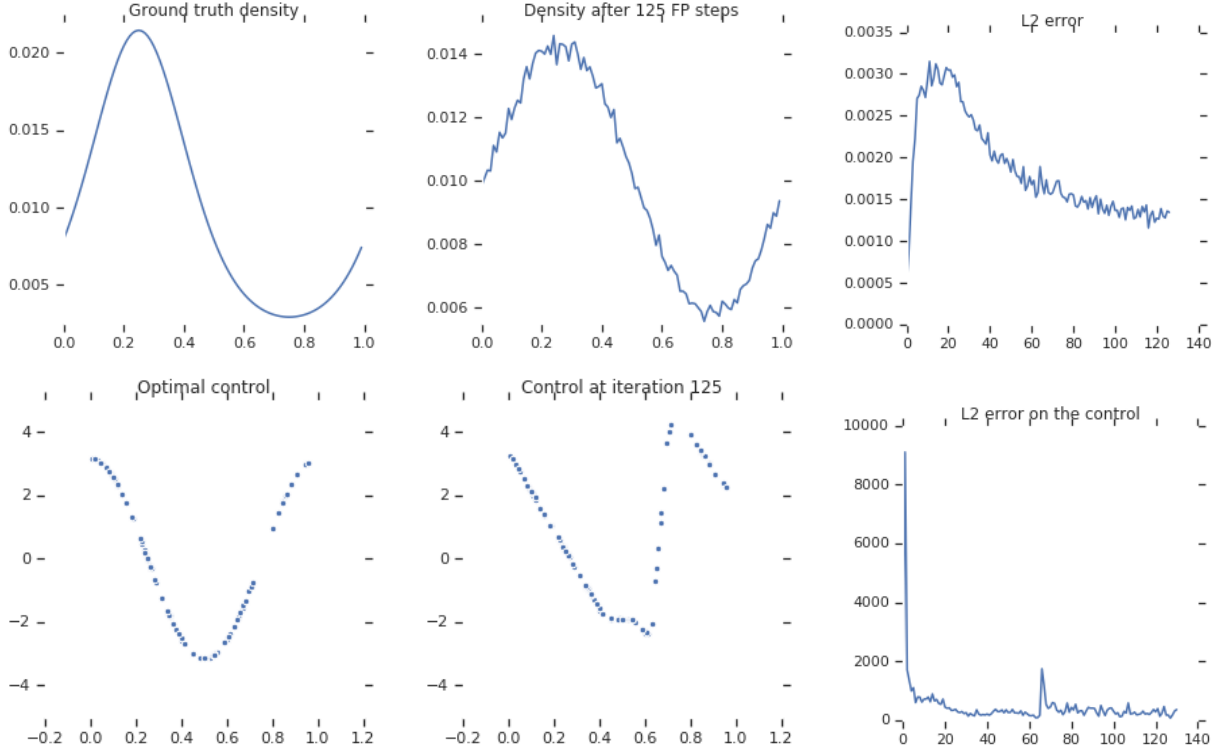


Figure 1: First row from left to right: 1) the continuous time explicit solution for $\gamma = 1$, 2) the discrete time learned distribution of the policy with $\gamma = 0.95$ and $\Delta_t = 0.05$ and 3) the L_2 error between the explicit solution and the density for each fictitious play iterations. Second row from left to right: 1) the optimal control 2) the learned control at iteration 125, and 3) the L_2 error between the control and the optimal control and for each fictitious play iterations on a trajectory.

solution (a^*, μ^*) in closed form [Almulla et al., 2017]

$$a^* : x \mapsto \pi \cos(2\pi x) \quad \text{and} \quad \mu^* : x \mapsto \frac{e^{2\sin(2\pi x)}}{\int_{\mathbf{T}} e^{2\sin(2\pi y)} dy}, \quad (8)$$

when the geographic reward is of the form $\tilde{r}(x_t) = 2\pi^2 \sin(2\pi x_t) - 2\pi^2 \cos(2\pi x_t)^2 + 2\sin(2\pi x_t)$. This closed form solution offers a nice benchmark for our experiments.

Implemented Algorithm Model free FP for MFGs takes a somehow similar approach as Lanctot et al. [2017] in the sense that we estimate the best response using a model free RL algorithm (namely DDPG). However we do not maintain those best responses as in [Lanctot et al., 2017] but rather learn the population MF flow of the distribution of the representative agents. The implemented algorithm and the subroutines used are described in the Appendix. The best response approximation through DDPG and the estimation of the population MF are left in the Algorithm. We ran 30000 trajectories of DDPG with a trajectory length of 300. The noise used for the for exploration is a normal noise centered around 0 and of variance 0.02 and we used Adam optimizers with a starting learning rate of 0.001 and $\tau = 0.01$. At each iteration of FP, we added 3000 trajectories of length 1000 to the replay buffer. Finally, we estimated the density using 100 classes and

doing 30000 steps of Adam (with an initial learning rate of 0.001).

Algorithm 2: Fictitious Play for continuous state and action Mean Field Games

Data: Number of FP iterations N_{FP} , an empty buffer of trajectories R and a number of trajectories to collect,
Result: a density $\bar{\mu}^{(N_{FP})}$ and a strategy $\hat{\pi}^{(N_{FP})}$.

```

1 begin
2   for  $n = 1, \dots, N_{FP}$  do
3     Initialize replay buffer  $R$ , for  $t = 0, 1, \dots, T - 1$  do
4       Compute a best response  $\hat{\pi}^{(n)}$  using DDPG,
5       Collect trajectories of the strategy  $\hat{\pi}^{(n)}$  and add them to the buffer  $R$ ,
6       Learn the population mean field flow  $\bar{\mu}^{(n)}$  by Monte Carlo samples on  $R$ 
7   return  $\bar{\mu}^{(N_{FP})}$  and  $\hat{\pi}^{(N_{FP})}$ 

```

Results. Figure 1 presents the learned equilibrium computed for $\gamma = 0.95$, $N = 90$ and uniform initial distribution, as well as the continuous time closed form ergodic solution for $\gamma = 1$, see (8). We emphasize that the variation in γ together with the discrete/continuous time difference setting implies that the theoretical solutions to both problems are close but do not exactly coincide. We keep this benchmark since no ergodic closed form solution is available for $\gamma \neq 1$. As observed on Figure 1, both ergodic explicit and learned distributions and controls are close. As expected, the density of players is larger around the point of maximum of the reward but the distribution is not highly concentrated due to the logarithmic penalty encoding aversion for congested regions. More precisely, Figure 1 indicates that the L_2 errors between the distributions and the controls decrease with the number n of iterations. The convergence of control distributions echoes to the discussion on error propagation in Section 4.2. This example clearly illustrates the numerical convergence of the Deep RL fictitious play mean field algorithm.

6 Conclusion, related work and future research

We presented a rigorous convergence analysis of model free FP learning algorithm for the resolution of MFG, in particular when the best response is approximated using single agent RL algorithms. Our setting encompasses for the first time the consideration of non-stationary MFG and relies on reasonable verifiable assumptions on the MFG. The convergence is illustrated for the first time by numerical experiments in a continuous (state,action) setting, based on deep RL algorithm.

The related literature is as follows. Recently, Guo et al. [2019] and Subramanian and Mahajan [2019] respectively introduced Q -learning and policy gradient model free RL algorithms for solving MFGs. However, their studies restrict to a stationary setting and their convergence results hold under assumptions that are hard to verify in practice. Yang et al. [2018] focused on a MF interaction through the empirical action average. Numerical illustrations provided in these papers are in a finite state-action setting. As for model-based learning algorithms, Yin et al. [2010] studied a MF oscillator game while Hu [2019] proposed a decentralized deep FP learning architecture for large MARL, whose convergence holds on linear quadratic MFG examples with explicit solution and small maturity.

We showed how the convergence of model free iterative FP algorithm reduces to the error analysis of each RL iteration step, as the convergence of RL algorithm reduces to the aggregation of repeated supervised

learning approximation errors. This allows to study the convergence of MARL algorithms using single RL approximation properties in this context. Besides, our analysis suggests a must faster convergence rate, whenever the best response approximation quality can be controlled in Wasserstein distance. This kind of estimate is currently not available in RL convergence literature, although highly classical in the literature of numerical approximation of stochastic control problems. This observation enlights relevant directions for future research.

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Appendix

This Appendix regroups the technical proofs related to the error propagation bounds on the Approximate Fictitious Play algorithm detailed in Theorem 6. Many arguments reported here are inspired by the results presented in [Hadikhanloo, 2018] for the exact fictitious play algorithm.

We follow the notations of Section 4. In particular, we recall that \mathcal{X} denotes the set of accessible states before time T . Since we take $b(x, a, \mu) = a$ and the set \mathcal{A} is compact, $\mathcal{X}^T = \mathcal{X} \times \dots \times \mathcal{X}$ is compact too and in particular it is a bounded subset of $\mathbb{R}^{T \times d}$.

In order to measure the proximity between MF population flows, we denote by d_1 , the 1- Wasserstein distance defined (using Kantorovitch-Rubinstein duality) as:

$$d_1(\mu, \mu') = \sup_{f \in Lip_1(\mathcal{X}^T, \mathbb{R})} \sum_{t \in \mathbb{T}} \int_{\mathcal{X}} h(x) d(\mu_t - \mu'_t)(x), \quad \text{for all } \mu, \mu' \in M_T,$$

where $Lip_1(\mathcal{X}^T, \mathbb{R})$ is the set of 1-Lipschitz continuous function from \mathcal{X}^T to \mathbb{R} .

A. Stability of the FP mean field flow $(\bar{\mu}^{(n)})_n$

Let us first provide the proof of Lemma 5 which ensures the closedness in d_1 of two consecutive elements of the Mean field flow learning sequence $\bar{\mu}^{(n)}$. Let first recall from the definition of $\bar{\mu}^{(n)}$ that we have:

$$\bar{\mu}^{(n+1)} - \bar{\mu}^{(n)} = \frac{1}{n} \left[\hat{\mu}^{(n+1)} - \bar{\mu}^{(n+1)} \right] = \frac{1}{n+1} \left[\hat{\mu}^{(n+1)} - \bar{\mu}^{(n)} \right], \quad n \in \mathbb{N}. \quad (9)$$

Proof of Lemma 5. Let $h \in Lip_1(\mathcal{X}^T, \mathbb{R})$. We recall that \mathcal{X}^T is bounded and pick $\mathbf{x}_0 \in \mathcal{X}^T$. Then, using (9) together with the definition of $\bar{\mu}^{(n)}$, we compute

$$\begin{aligned} \left| \int_{\mathcal{X}^T} h(\mathbf{x}) d(\bar{\mu}^{(n+1)} - \bar{\mu}^{(n)})(\mathbf{x}) \right| &= \frac{1}{n+1} \left| \int_{\mathcal{X}^T} h(\mathbf{x}) d(\hat{\mu}^{(n)} - \bar{\mu}^{(n)})(\mathbf{x}) \right| \\ &= \frac{1}{n+1} \left| \int_{\mathcal{X}^T} (h(\mathbf{x}) - h(\mathbf{x}_0)) d(\hat{\mu}^{(n)} - \bar{\mu}^{(n)})(\mathbf{x}) \right| \\ &\leq \frac{1}{n+1} \left(\int_{\mathcal{X}^T} \|\mathbf{x} - \mathbf{x}_0\| d\hat{\mu}^{(n)}(\mathbf{x}) + \int_{\mathcal{X}^T} \|\mathbf{x} - \mathbf{x}_0\| d\bar{\mu}^{(n)}(\mathbf{x}) \right) \\ &\leq \frac{C}{n+1}, \end{aligned}$$

since \mathcal{X}^T is bounded. This result being valid for any $h \in Lip_1(\mathcal{X}^T, \mathbb{R})$, we obtain

$$d_1(\bar{\mu}^{(n+1)}, \bar{\mu}^{(n)}) \leq \frac{C}{n+1}, \quad n \in \mathbb{N}.$$

□

B. Propagation error estimates

This section is dedicated to the rigorous derivation of the bounds (7) and (6) presented in Theorem 4.

We first recall the following useful result, see e.g. [Hadikhanloo, 2018, Lemma 3.3.1].

Lemma 9. Let $(\varphi_n)_n$ and $(\lambda_n)_n$ be two sequences of real numbers such that

$$(n+1)\varphi_{n+1} - n\varphi_n \leq \lambda_n, \quad n \in \mathbb{N}.$$

Then, we have the estimate:

$$\varphi_n \leq \frac{\varphi_0}{n} + \frac{1}{n} \sum_{i=1}^n \lambda_i, \quad n \in \mathbb{N}.$$

For ease of notation, we introduce $J_n := J(\cdot, \bar{\mu}^{(n)})$ and $J_{n+1} := J(\cdot, \bar{\mu}^{(n+1)})$ (which are functions defined over \mathcal{X}^T), for all $n \geq 0$. More precisely, for $x \in \mathcal{X}^T$,

$$J_n(x) := J(x_0, (x_{t+1} - x_t)_{t=0, \dots, T-1}, \bar{\mu}^{(n)}).$$

Observe that this definition is accurate by the definition of the first order MFG setting presented in Section 4.1 and because there is a bijection between process trajectory and the combination of initial position and policy.

Proof of estimate (7) in Theorem 6. We adapt the arguments in the proof of [Hadikhanloo, 2018, Theorem 3.3.1] to our setting with approximate best responses.

Let introduce the approximate learning error \hat{e}_n defined by

$$\hat{e}_n := \mathbb{E}_{x_0 \sim \mu_0} [J(x_0, \hat{\pi}^{(n+1)}, \bar{\mu}^{(n)}) - J(x_0, \bar{\pi}^{(n)}, \bar{\mu}^{(n)})] \geq 0, \quad n \in \mathbb{N},$$

so that $e_n = \ell_n + \hat{e}_n$. In order to control e_n , we will focus our analysis on \hat{e}_n and compute

$$\begin{aligned} & (n+1)\hat{e}_{n+1} - n\hat{e}_n \\ &= (n+1) \int_{\mathcal{X}^T} J_{n+1} d(\hat{\mu}^{(n+2)} - \bar{\mu}^{(n+1)}) - n \int_{\mathcal{X}^T} J_n d(\hat{\mu}^{(n+1)} - \bar{\mu}^{(n)}) \\ &= (n+1) \int_{\mathcal{X}^T} J_{n+1} d(\hat{\mu}^{(n+1)} - \bar{\mu}^{(n+1)}) - n \int_{\mathcal{X}^T} J_n d(\hat{\mu}^{(n+1)} - \bar{\mu}^{(n)}) \\ &\quad + (n+1) \int_{\mathcal{X}^T} J_{n+1} d(\hat{\mu}^{(n+2)} - \hat{\mu}^{(n+1)}) \\ &= n(n+1) \int_{\mathcal{X}^T} (J_{n+1} - J_n) d(\bar{\mu}^{(n+1)} - \bar{\mu}^{(n)}) + (n+1) \int_{\mathcal{X}^T} J_{n+1} d(\hat{\mu}^{(n+2)} - \hat{\mu}^{(n+1)}), \end{aligned}$$

where the last equality follows from (9).

Thanks to Assumption 1, the monotonicity of the reward function implies directly

$$(n+1)\hat{e}_{n+1} - n\hat{e}_n \leq (n+1) \int_{\mathcal{X}^T} J_{n+1} d(\hat{\mu}^{(n+2)} - \hat{\mu}^{(n+1)}). \quad (10)$$

By definition of J_n together with the first order MFG dynamics, we have the expression

$$J_n(x) = \sum_{t=0}^{T-1} \gamma^t \left[\tilde{r}(x_t, x_{t+1} - x_t) + \bar{r}(x_t, \bar{\mu}_t^{(n)}) \right].$$

Moreover, using Assumption 1 and the compactness of \mathcal{X} , we deduce as in [Hadikhanloo, 2018, Lemma 3.5.2], the existence of a constant C such that for all $\mathbf{x}, \mathbf{x}' \in \mathcal{X}^T$,

$$|J_{n+1}(\mathbf{x}) - J_n(\mathbf{x}) - J_{n+1}(\mathbf{x}') + J_n(\mathbf{x}')| \leq C \|\mathbf{x} - \mathbf{x}'\|_\infty d_1(\bar{\mu}^{(n+1)}, \bar{\mu}^{(n)}). \quad (11)$$

This property of the reward function together with Lemma 5 indicate that $J_{n+1} - J_n$ is C/n -Lipschitz, so that

$$\begin{aligned} (n+1)\hat{e}_{n+1} - n\hat{e}_n &\leq (n+1) \int_{\mathcal{X}^T} J_n d(\hat{\mu}^{(n+2)} - \hat{\mu}^{(n+1)}) + Cd_1(\hat{\mu}^{(n+2)}, \hat{\mu}^{(n+1)}) \\ &\leq (n+1) \int_{\mathcal{X}^T} J_n d(\mu^{*,(n+1)} - \hat{\mu}^{(n+1)}) + Cd_1(\hat{\mu}^{(n+2)}, \hat{\mu}^{(n+1)}), \end{aligned}$$

where, in the last inequality, we used the fact that $\pi^{*,(n+1)}$ is the best response with respect to $\bar{\mu}^{(n)}$ and hence

$$\int_{\mathcal{X}^T} J_n d\mu^{*,(n+1)} \geq \int_{\mathcal{X}^T} J_n d\hat{\mu}^{(n+2)}. \quad (12)$$

Indeed, by optimality of $\pi^{*,(n+1)}$, we have

$$\int_{\mathcal{X}^T} J_n d\mu^{*,(n+1)} = \mathbb{J}(\pi^{*,(n+1)}, \bar{\mu}^{(n)}) \geq \mathbb{J}(\hat{\pi}^{(n+2)}, \bar{\mu}^{(n)}).$$

By definition of the learning error ℓ_n in (3), we deduce

$$(n+1)\hat{e}_{n+1} - n\hat{e}_n \leq (n+1)\ell_n + Cd_1(\hat{\mu}^{(n+2)}, \hat{\mu}^{(n+1)}).$$

By Lemma 9 applied to

$$\varphi_n = \hat{e}_n, \quad \text{and} \quad \lambda_n = (n+1)\ell_n + Cd_1(\hat{\mu}^{(n+2)}, \hat{\mu}^{(n+1)}),$$

we derive the estimate

$$\hat{e}_n \leq \ell_n + \frac{\hat{e}_0}{n} + \frac{1}{n} \sum_{i=1}^n (i+1)\ell_i + \frac{1}{n} \sum_{i=1}^n d_1(\hat{\mu}^{(i+2)}, \hat{\mu}^{(i+1)}).$$

Combining this estimate with the relation $e_n = \ell_n + \hat{e}_n$ provides (7). \square

We are now in position to turn to the proof of the remaining estimate (6).

Proof of estimate (6) in Theorem 6. Combining the relation $e_n = \ell_n + \hat{e}_n$ together with estimate (10), we compute

$$\begin{aligned} &(n+1)e_{n+1} - ne_n \\ &\leq (n+1) \int_{\mathcal{X}^T} J_{n+1} d(\mu^{*,(n+2)} - \hat{\mu}^{(n+1)}) - n \int_{\mathcal{X}^T} J_n d(\mu^{*,(n+1)} - \hat{\mu}^{(n+1)}) \\ &= (n+1) \int_{\mathcal{X}^T} J_{n+1} d(\mu^{*,(n+2)} - \mu^{*,(n+1)}) + \int_{\mathcal{X}^T} [(n+1)J_{n+1} - nJ_n] d(\mu^{*,(n+1)} - \hat{\mu}^{(n+1)}) \\ &= (n+1) \int_{\mathcal{X}^T} [J_{n+1} - J_n] d(\mu^{*,(n+2)} - \mu^{*,(n+1)}) + (n+1) \int_{\mathcal{X}^T} J_n d(\mu^{*,(n+2)} - \mu^{*,(n+1)}) \\ &\quad + (n+1) \int_{\mathcal{X}^T} [J_{n+1} - J_n] d(\mu^{*,(n+1)} - \hat{\mu}^{(n+1)}) + \int_{\mathcal{X}^T} J_n d[\mu^{*,(n+1)} - \hat{\mu}^{(n+1)}]. \end{aligned}$$

Hence, the Lipschitz property (11) together with Lemma 5 implies

$$(n+1)e_{n+1} - ne_n \leq Cd_1(\mu^{*,(n+2)}, \mu^{*,(n+1)}) + Cd_1(\mu^{*,(n+1)}, \hat{\mu}^{(n+1)}) + \ell_n ,$$

where we also used that $\mu^{*,(n+1)}$ is the mean field flow induced by the best response to the population distribution $\bar{\mu}^{(n)}$ so that

$$\int_{\mathcal{X}^T} J_n d\mu^{*,(n+1)} \geq \int_{\mathcal{X}^T} J_n d\hat{\mu}^{(n+2)}.$$

Finally, Lemma 5 together with the continuity of the best response in the first order MFG (which stems from Assumption 1, see e.g. [Hadikhanloo, 2018, Remark 3.5.3]) ensure that $d_1(\mu^{*,(n+2)}, \mu^{*,(n+1)})$ converges to zero as n goes to infinity. Finally, applying Lemma 9 to

$$\varphi_n = e_n, \quad \text{and} \quad \lambda_n = \ell_n + Cd_1(\mu^{*,(n+1)}, \hat{\mu}^{(n+1)}) + \frac{C}{n}$$

directly concludes the proof. □

C. Exact Mean Field Fictitious Play Algorithm

Algorithm 3: Exact Fictitious Play for MFG

Data: An initial distribution μ_0 ; an initial mean field flow $\bar{\mu}^{*,(0)}$; a number of steps N .

Result: A mean field flow μ and a policy π .

```
1 begin
2   for  $n = 0, 1, \dots, N - 1$  do
3     Compute  $\pi^{*,(n+1)}$ , the best response policy against  $\bar{\mu}^{*,(n)}$  (see Definition 1)
4     Update  $\bar{\mu}^{*,(n+1)} = \frac{n}{n+1}\bar{\mu}^{*,(n)} + \frac{1}{n+1}\mu^{*,(n+1)}$ 
5   return  $(\bar{\mu}^{*,(N)}, \pi^{*,(N)})$ 
```

D. Deep Deterministic Policy Gradient

Algorithm 4: DDPG

Data: Randomly initialize critic network $Q(x, a|\theta^Q)$ and actor network $\pi(x|\theta^\pi)$ with weights θ^Q and θ^π and initialize target network Q' and actor network π' with weights $\theta^{Q'} \leftarrow \theta^Q$ and $\theta^{\pi'} \leftarrow \theta^\pi$.

Result: a policy π .

```

1 begin
2   for  $episode = 0, 1, \dots, M - 1$  do
3     Initialize replay buffer  $R$ , for  $t = 0, 1, \dots, T - 1$  do
4       Select an action  $a_t = \pi(x|\theta^\pi) + \mathcal{N}_t$  according to the current policy and exploration noise,
5       Execute action  $a_t$  and observe reward  $r_t$  and new state  $x_{t+1}$ ,
6       Store transition  $(x_t, a_t, r_t, x_{t+1})$  in  $R$ 
7       Sample a random minibatch of  $N$  transitions  $(x_i, a_i, r_i, x_{i+1})$  from  $R$ 
8       Set  $y_i = r_i + \gamma Q'(x_{i+1}, \pi'(x_{i+1}|\theta^{\pi'}))$ 
9       Update critic by minimizing the loss:  $L = \frac{1}{N} \sum_i (y_i - Q(x_i, a_i|\theta^Q))^2$ ,
10      Update the actor policy using the sampled policy gradient:
11         $\nabla_{\theta^\pi} J \simeq \frac{1}{N} \sum_i \nabla_a Q(x, a|\theta^Q)|_{x=x_i, a=\pi(x_i)} \nabla_{\theta^\pi} \pi(x|\theta^\pi)|_{x=x_1}$ 
12      Update target networks:
        
$$\theta^{Q'} \leftarrow \tau \theta^Q + (1 - \tau) \theta^{Q'}$$

        
$$\theta^{\pi'} \leftarrow \tau \theta^\pi + (1 - \tau) \theta^{\pi'}$$

    return  $\pi$ 
```

Algorithm 5: Density Estimation

Data: State dataset X , a number of iterations N , a number of classes N_{classes} and the parameters θ of the function representing the density.

Result: a policy π .

```
1 begin
2   for  $i = 0, 1, \dots N$  do
3     Sample a random minibatch of  $N$  states  $x_i$  from  $R$ ,
4     Construct the classes  $y_i$  to which  $x_i$  belong to,
5     minimize the loss  $L = \frac{1}{N} \sum_i y_i \log(f_\theta)$ 
6   return  $\pi$ 
```
