Phase Transition in Classical analog of a Model in Synthetic Dimension

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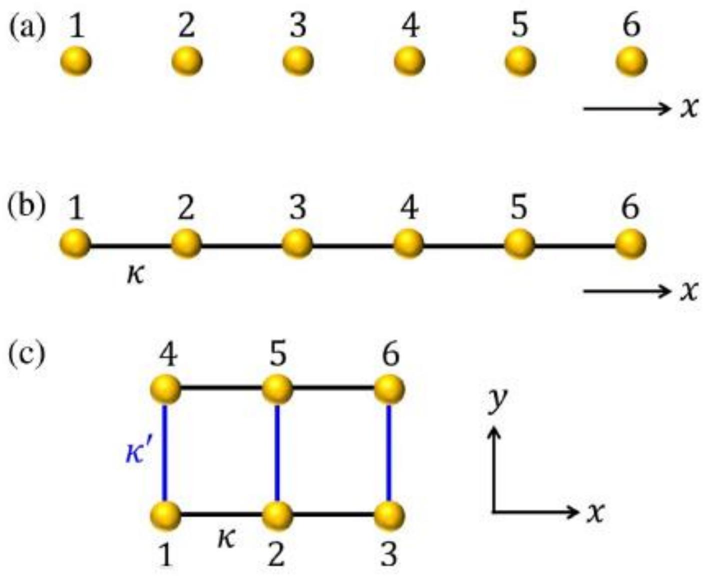
Outline

- What is Synthetic Dimension?
- Phase Transitions in General
- My Model
- Methodology: Classical Monte Carlo
- Results
- Future Plan

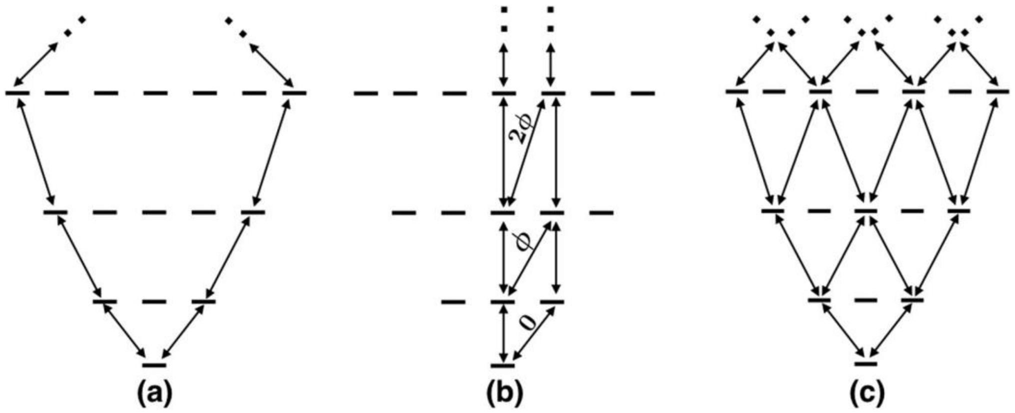
Synthetic Dimensions – What Is It?

- To explore physics in a higher dimensionality then its geometrical dimension
- Dimensionality depends on how the states are coupled



Significance of Synthetic dimension

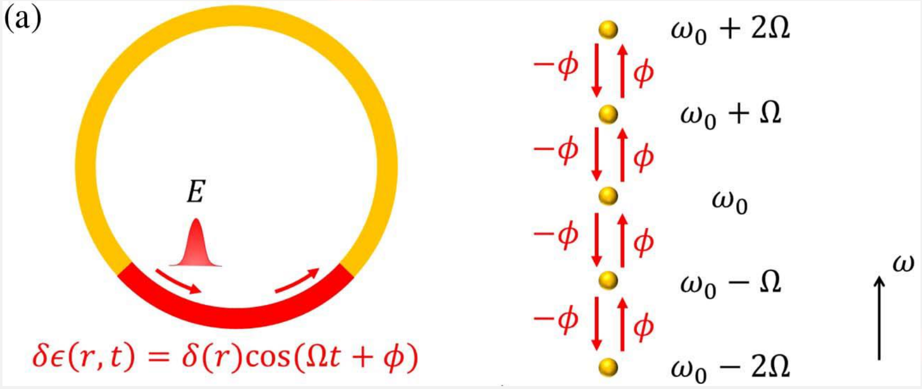
- Explore higher dimensional physics
- Easy to synthesize and control
- Useful in communication
  - Effective gauge potential to control frequency etc. of light
- Useful in experimentally realizing physical systems:
  - Synthetic band structure, including topological ones
  - Create strong correlated matter
  - Engineering topology of the lattice and the boundary condition



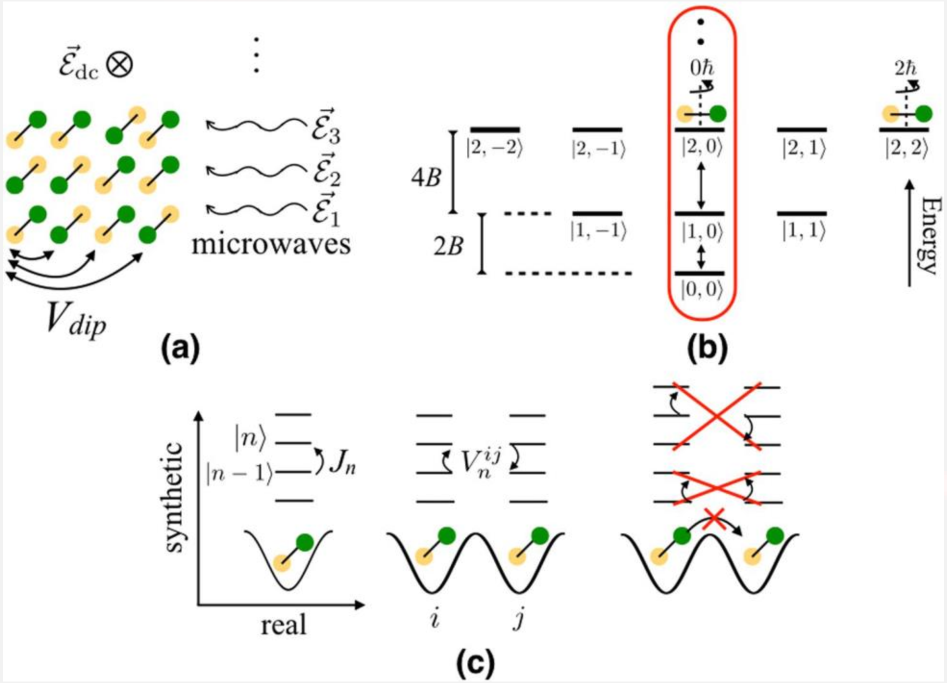
Experimental Realization

Optical Method

- Modulated permittivity ϵ of a ring resonator
- $\nabla \times \nabla \times E + \mu_0 \frac{\partial^2}{\partial t^2}\epsilon_s E = -\mu_0 \frac{\partial^2 P}{\partial t^2}$ $\quad\quad P = \epsilon(t)E$
- Time varying $\epsilon(t)$ induces polarization
- Polarization would induce transition of E to neighboring mode
- Resulting in a nearest neighbor tight binding model



Ultracold polar molecular array in Microwave – the system we study

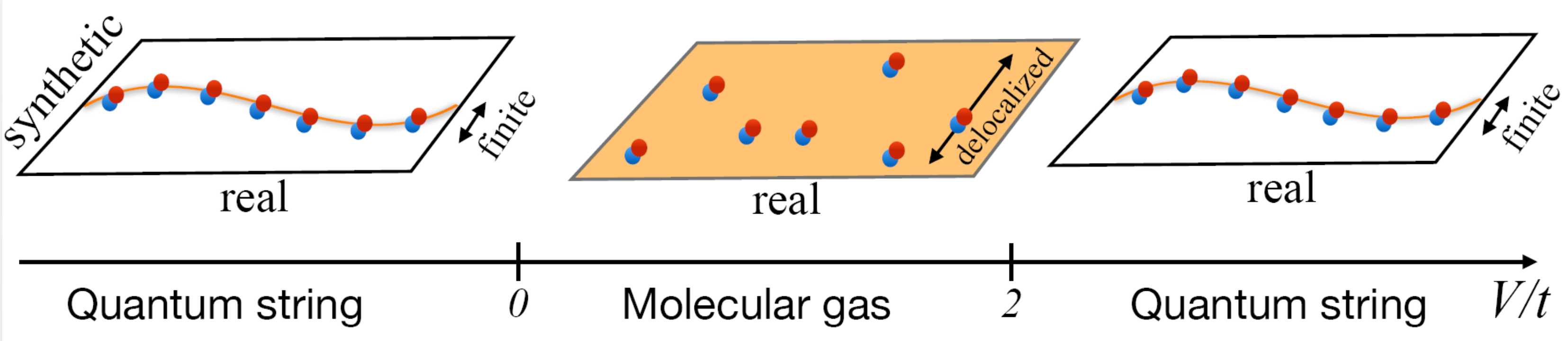


- An array of localized ultracold molecular
- Microwave of specific frequency induces transition between $|n,m\rangle$ and $|n\pm 1,m\rangle$
- Dipole interaction between molecules induce exchange of total angular momentum n
- External field to fix molecule in the $m=0$ state for different total angular momentum n

$$\hat{H} = \sum_j B\hat{N}_j^2 + \sum_j \sum_{n=1}^{N_{\mathrm{rot}}} \hat{\vec{d}}_j \cdot \vec{\mathcal{E}}_n(t) + \sum_{ij} \frac{\hat{\vec{d}}_i \cdot \hat{\vec{d}}_j - 3\left(\hat{\vec{d}}_i \cdot \hat{r}_{ij}\right)\ \left(\hat{\vec{d}}_j \cdot \hat{r}_{ij}\right)}{4\pi\epsilon_0 r_{ij}^3}$$

The Quantum Hamiltonian & String/Sheet Formation

- The Quantum Hamiltonian and where it came from
  - $\hat{H} = -\sum_{nj} J_n \hat{c}^+_{n-1,j}\hat{c}_{n,j} - \sum_{nij} V_n^{ij}\, \hat{c}^+_{n-1,i}\hat{c}_{n,i}\hat{c}^+_{n,j}\hat{c}_{n-1,j} + h.c.$
  - Must have 1 molecule per real-space site, so $\sum_n \hat{c}^+_{n,i}\hat{c}_{n,i} = 1,\ \forall i$
- When V dominates, adjacent sites favor similar total angular momentum n
- String/sheet formation
- Additional real-space hopping term $W \sum_{nj} \hat{c}^+_{n,i}\hat{c}_{n,i}\hat{c}^+_{n,j+1}\hat{c}_{n,j+1}$ from what
  - Controls the width of string/sheet



Classical Analog & Our Model

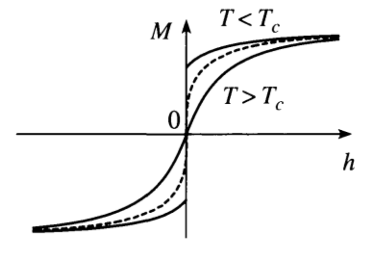
- The Classical Analog
  - To mimic the behavior of the quantum Hamiltonian where it favors the same and neighboring synthetic sites
  - Revised Potts model: $H = J_0 \sum_{\langle i,j \rangle} \delta_{q_i, q_j}$
  - Our Hamiltonian:

$$H = J_0 \sum_{\langle i,j \rangle} \delta_{q_i,q_j} + J_1 \sum_{\langle i,j \rangle} \left(\delta_{q_i+1,q_j} + \delta_{q_{i-1},q_j} \right)$$

    - $q_i = 1 \cdots Q$, Q is length of synthetic dimension
    - Periodic Boundary Condition(PBC) on the synthetic dimension

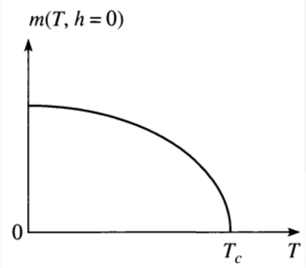
Phase transitions in general

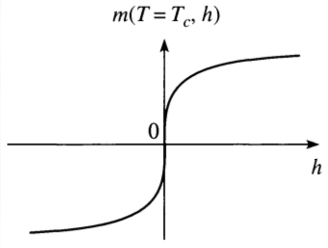
- Happens at thermal dynamic limit $N \to \infty$ only!
- Intuitively, $F = E - TS$, it's a competition between E and S
- Typically Characterized by an order parameter

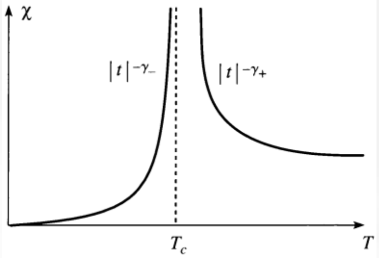


- 1st Order Phase Transition
- 2nd Order Phase Transition
  - Critical exponent ($t = T - T_c$)

$M \sim |t|^{\beta}$ $\quad M \sim h^{\frac{1}{\delta}}$ $\quad \chi \sim |t|^{-\gamma}$ $\quad C_V \sim |t|^{-\alpha}$ $\quad \xi \sim |t|^{-\nu}$ $\quad \langle s(0)s(r)\rangle \sim r^{-d+2-\eta}$







- Typically break(change) symmetry

Monte Carlo Method

- Goal: Sampling the Boltzmann distribution:

$$P(\{x_i\}) = \frac{e^{-\beta E(\{x_i\})}}{Z} \qquad Z = \sum_{\{x_i\}} e^{\beta E(\{x_i\})} \qquad \beta = \frac{1}{T}$$

- Difficulty: Only know the probability ratio between two microstates
- Method: Using a Markov Process to Randomly Generate Samples
  - Starting with a random configuration
  - Transit to a new configuration according to a transition matrix T_{ij}

How and why does it work?

- Detailed Balance Condition: $T_{ji}P_i = T_{ij}P_j$
  - Ensures that P_i is the eigenvector with $\lambda_P = 1$

$$\sum_i T_{ji}P_i = \sum_i T_{ij}P_j = P_j \sum_i T_{ij} = P_j$$

- Transition rule: Stochastic matrix T_{ij}: $T_{ij} > 0$ $\quad\quad \sum_j T_{ij} = 1$
  - Ensures eigenvalues have absolute values less than one: $|\lambda| \leq 1$

- Ergodicity:
  - The sampling process should be able to explore all microstates

- Aperiodicity & Irreducibility

Quick proof of $|\lambda| \leq 1$:

$$\left|\sum_i T_{ji}v_i\right| = |\lambda v_j| = |\lambda||v_j|$$

$$\sum_i T_{ji}|v_i| \geq |\lambda||v_j|$$

$$\sum_j \sum_i T_{ji}|v_i| \geq \sum_j |\lambda||v_j|$$

$$\sum_i |v_i| \geq |\lambda| \sum_i |v_i|$$

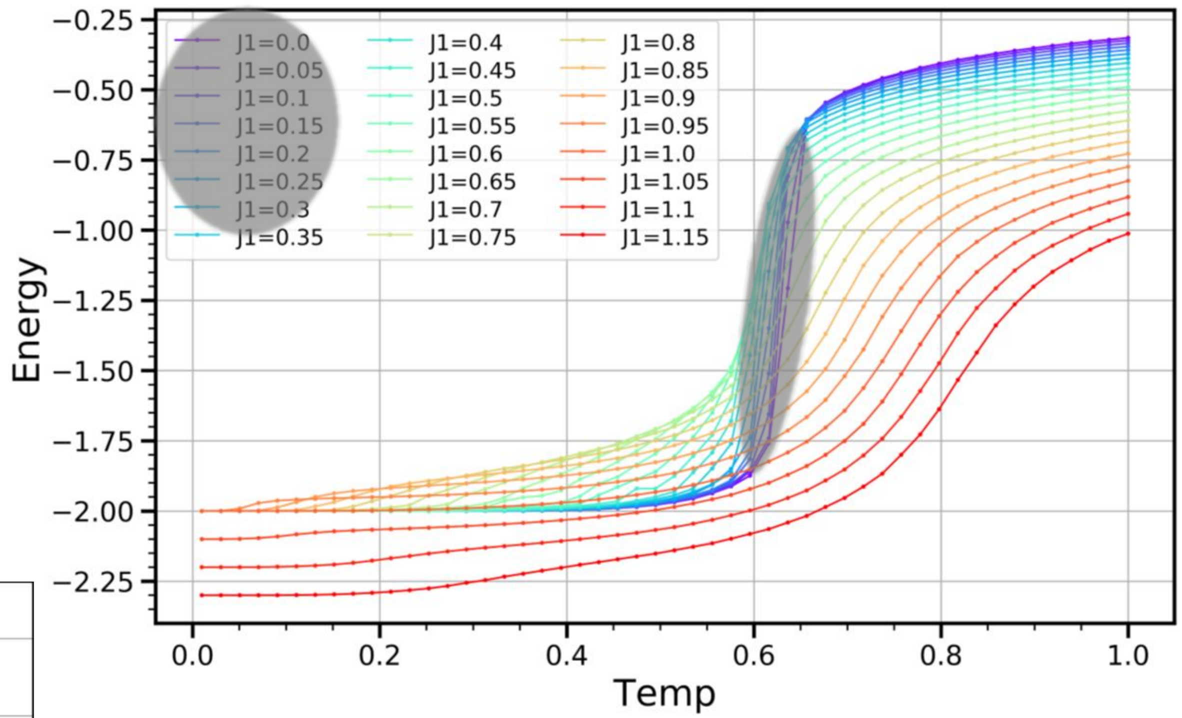
$$1 \geq |\lambda|$$

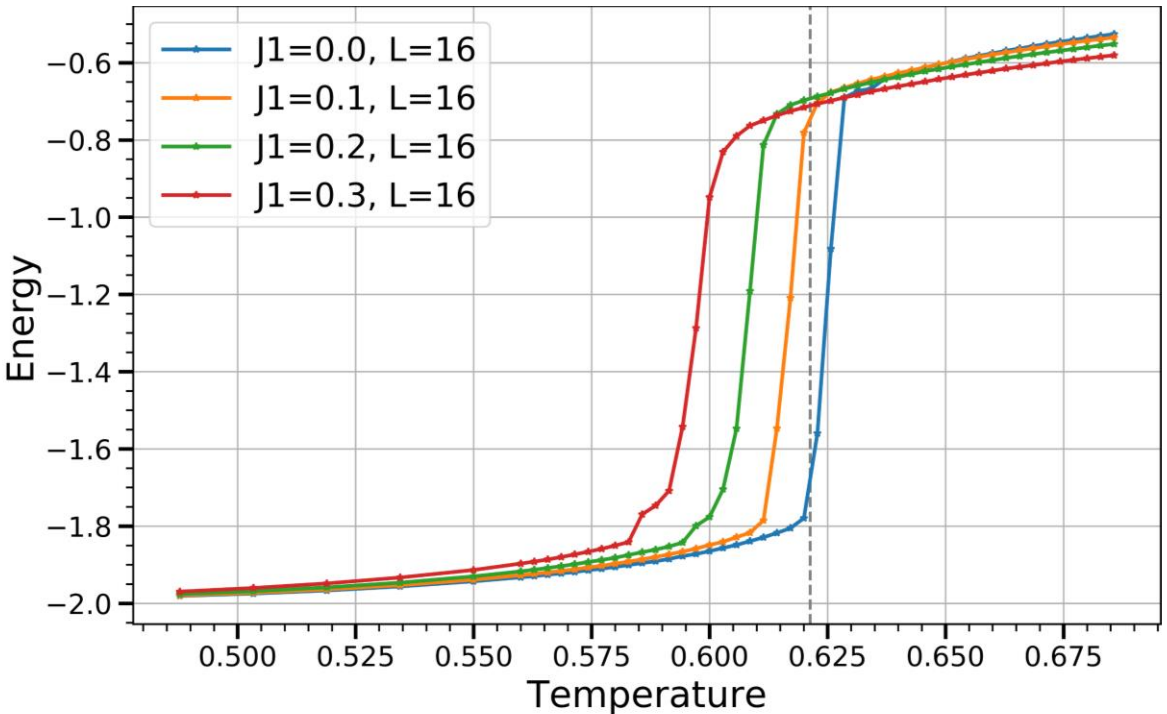
Metropolis Algorithm

- Initialization Process:
  - Choose an arbitrary starting state $\{x_i\}$
  - Choose a proposal function $g(\{x_i\}|\{y_i\})$. To ensure detailed balance, we should make sure $g(\{x_i\}|\{y_i\}) = g(\{y_i\}|\{x_i\})$
- For n steps, do:
  - Suggest a next move with proposal function $g(\{x_i\}|\{y_i\})$
  - Calculate probability ratio between the two states $r = \frac{e^{-\beta E(\{y_i\})}}{e^{-\beta E(\{x_i\})}} = e^{-\beta \Delta E}$
  - Accept the suggested state with the probability $\min\{1, r\}$
- Check for detailed balance:
  - $\frac{P_y}{P_x} = \frac{e^{-\beta(E(\{y_i\})-E(\{x_i\}))}}{1} = \frac{T_{yx}}{T_{xy}}$ (if $P_y > P_x$, also g is symmetric)
  - Similar for when $P_y < P_x$

When J_1 is small

- Single Transition
- 1st order
- Transition temperature decrease slightly as $J1$ increases



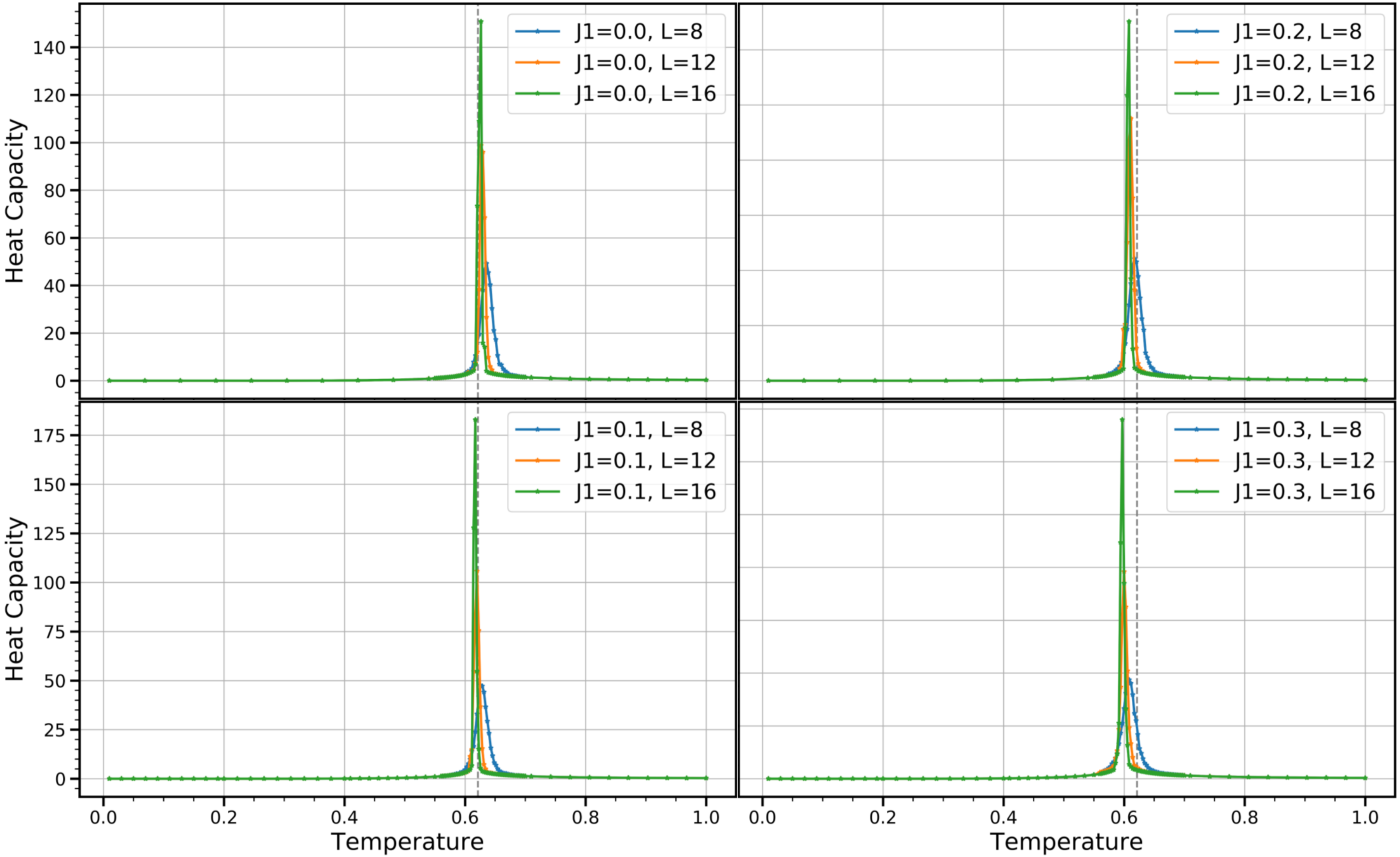


Energy Plot for Small J_1

Potts Model : $H = -J_0 \sum_{\langle i,j \rangle} \delta_{q_i, q_j}$

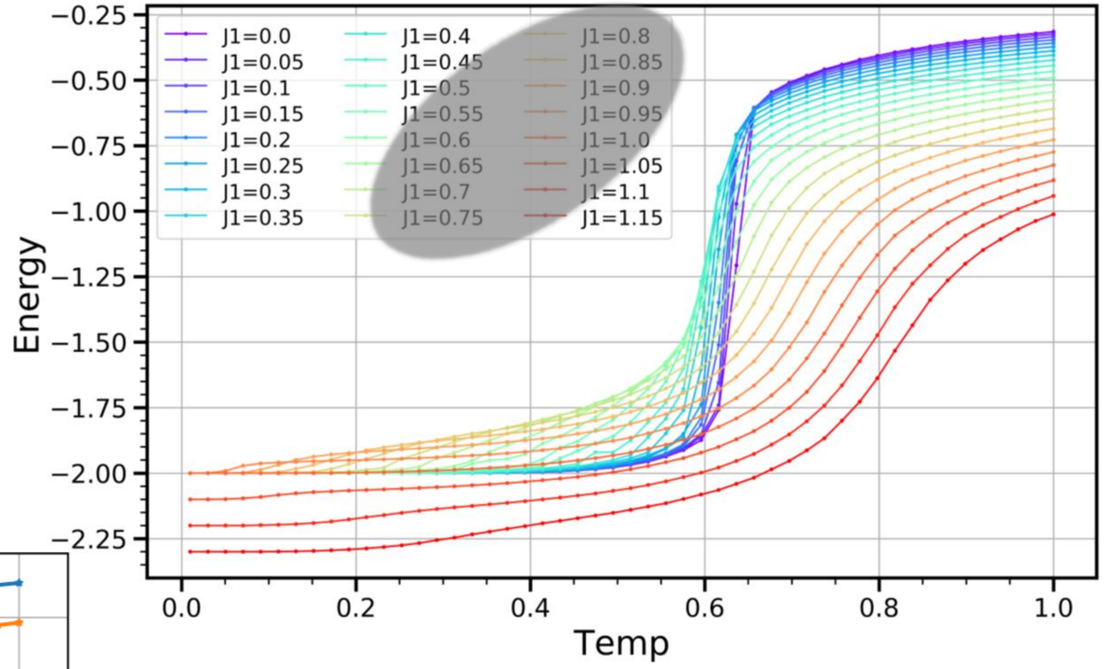
- The model is identical to the Potts model when $J1 = 0$
- We expect similar behavior when $J1$ is small
- Some properties of the 2D Potts model:
  - Generalization of the Ising model($Q = 2$ identical to Ising model)
  - Transition point at $\beta_c = \frac{1}{J_0} \ln(\sqrt{q} + 1)$
  - Different order of transition for different Q values
    - $Q \leq 4$
      - $Q = 3$ is still not rigorously proved
      - $Q = 2$ and $Q = 4$ are identical to the Ising model, exhibit single 2^{nd} order phase transition
    - $Q > 5$
      - Exhibit single 1^{st} order phase transition
- Check result with Potts model

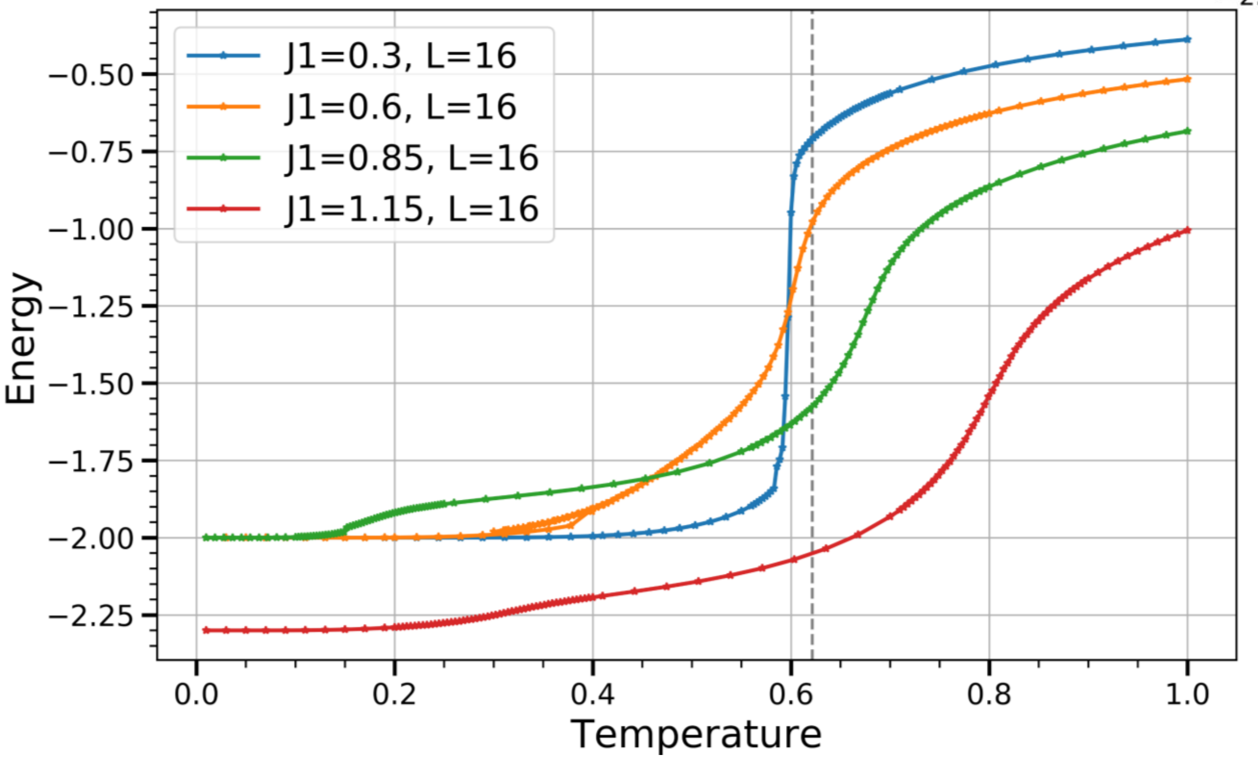
Heat Capacity Plots for Small J_1



As J_1 increases…(show E)

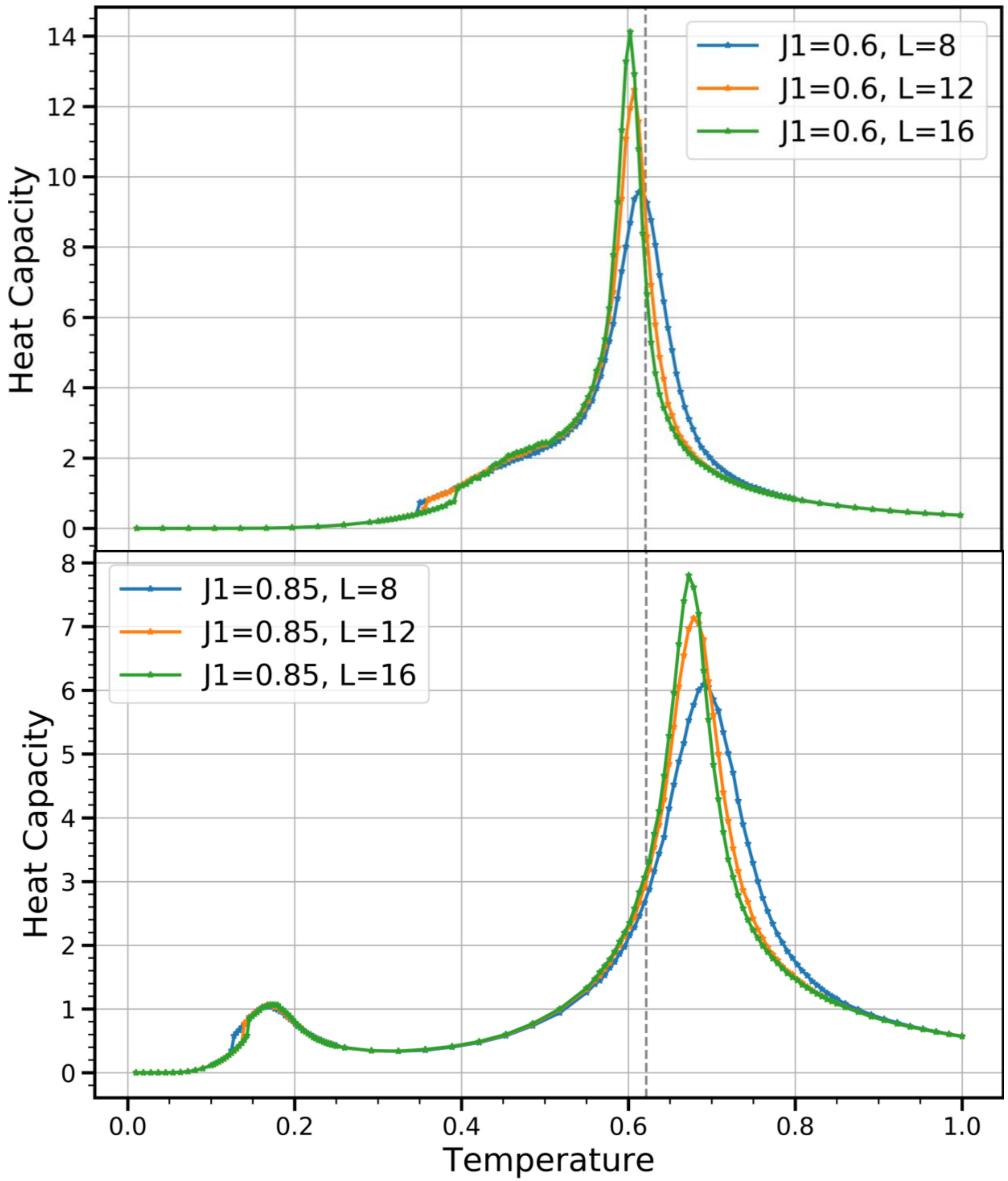
- Starting to show two peaks in heat capacity
- Peak sizes have no difference for different L
- Suggesting two phase transitions





Energy Plot for Larger J_1

Heat Capacity Plots for Large J_1



Binder's ratio crossing

- Want to define a dimensionless quantity
- Introduced by Binder, studying the distribution of blocked spin $P_L(s)$
- Suppose we scale the system by b, $P_L(s)$ should scale like

$$P_{\frac{L_0}{b}}(s, \xi) = b^y P_{L_0}(sb^y, \frac{\xi}{b})$$

- Scale by $b = \frac{L}{L_0}$:

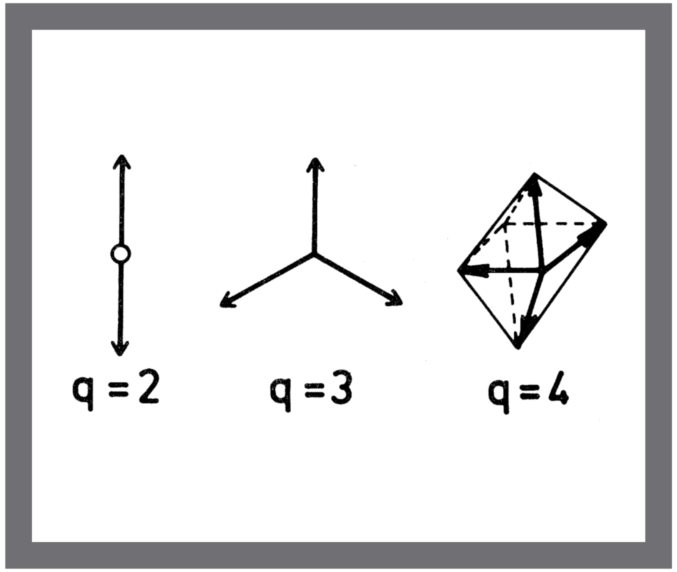
$$P_L(s, \xi) = \left(\frac{L}{L_0}\right)^y P_L\left(s\left(\frac{L}{L_0}\right)^y, \frac{\xi L_0}{L}\right) = L^y \tilde{P}\,(sL^y, \frac{\xi}{L})$$

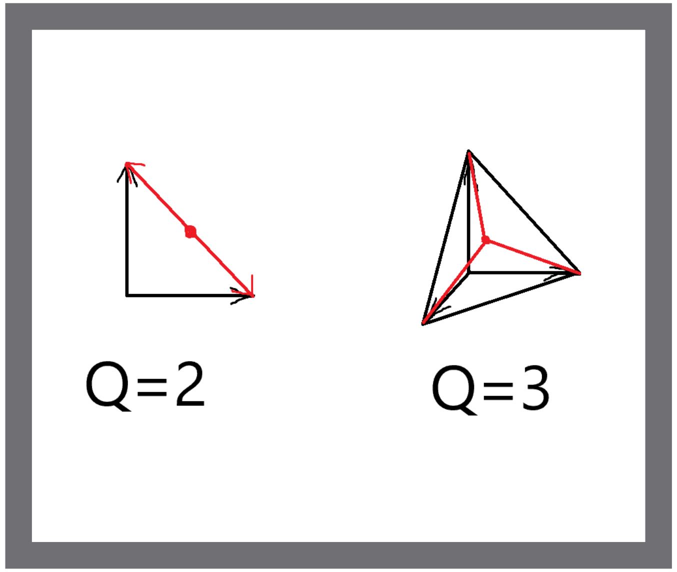
- So the ratio (approaching critical temperature T_c)

$$\frac{\langle s^2 \rangle^2}{\langle s^4 \rangle} = \frac{\left(\int s^2\, P_L(s, \xi) ds\right)^2}{\int s^4\, P_L(s, \xi) ds} \sim \frac{\left(L^{-2y} f_2\left(\frac{\xi}{L}\right)\right)^2}{L^{-4y} f_4\left(\frac{\xi}{L}\right)} = \frac{f_2\left(\frac{\xi}{L}\right)^2}{f_4\left(\frac{\xi}{L}\right)} \to \frac{f_2(\infty)^2}{f_4(\infty)} = const$$

- To make it approach 0 at $T \to \infty$, for e.g. Ising model, define

$$B = 1 - \frac{\langle M^4 \rangle}{3\langle M^2 \rangle^2}$$





Defining Binder's Ratio in Our Model

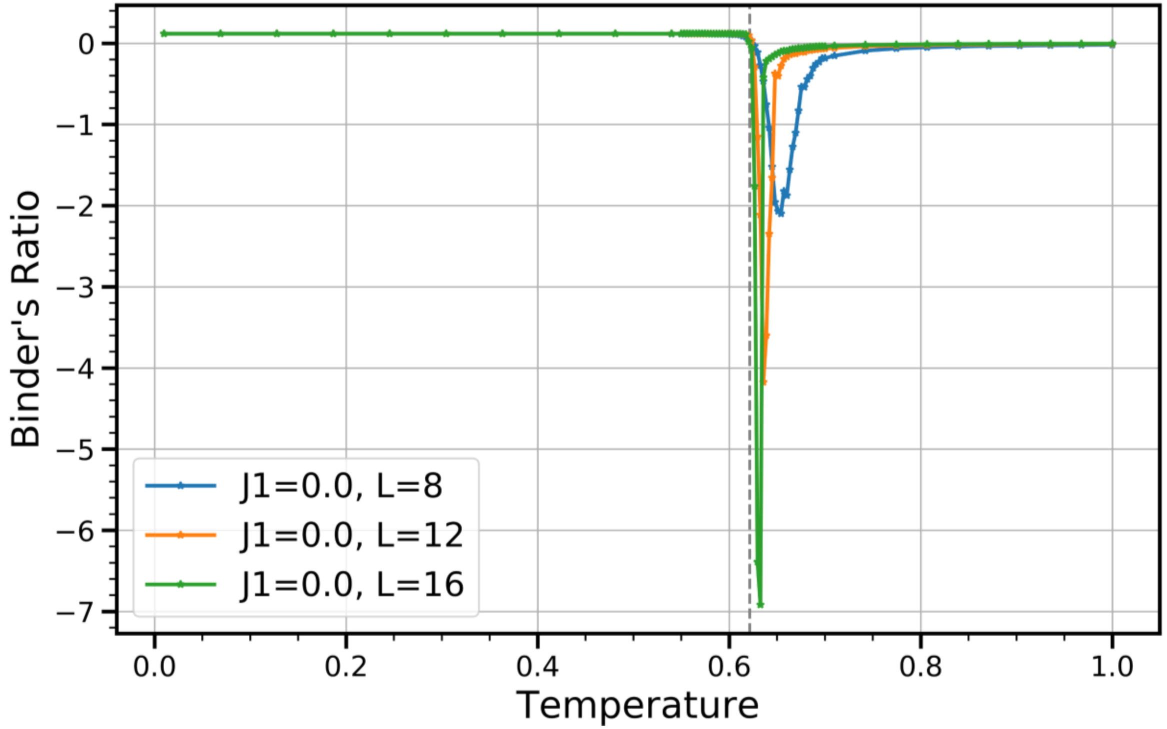
- Q dimensions in total
- $\boldsymbol{M} = \sum_i \frac{N_i}{N} \hat{\boldsymbol{e}}_{\boldsymbol{i}}$ for a microstate
- $\hat{\boldsymbol{e}}_{\boldsymbol{i}}, i = 1 \cdots Q$ forms a $Q-1$ dimensional hyper-tetrahedron
- Definition: $B = 1 - \frac{\langle M^4 \rangle}{\left(1+\frac{2}{Q-1}\right)\langle M^2 \rangle^2}$
- $B \to 0$ in high T limit
- Correspond to the traditional definition

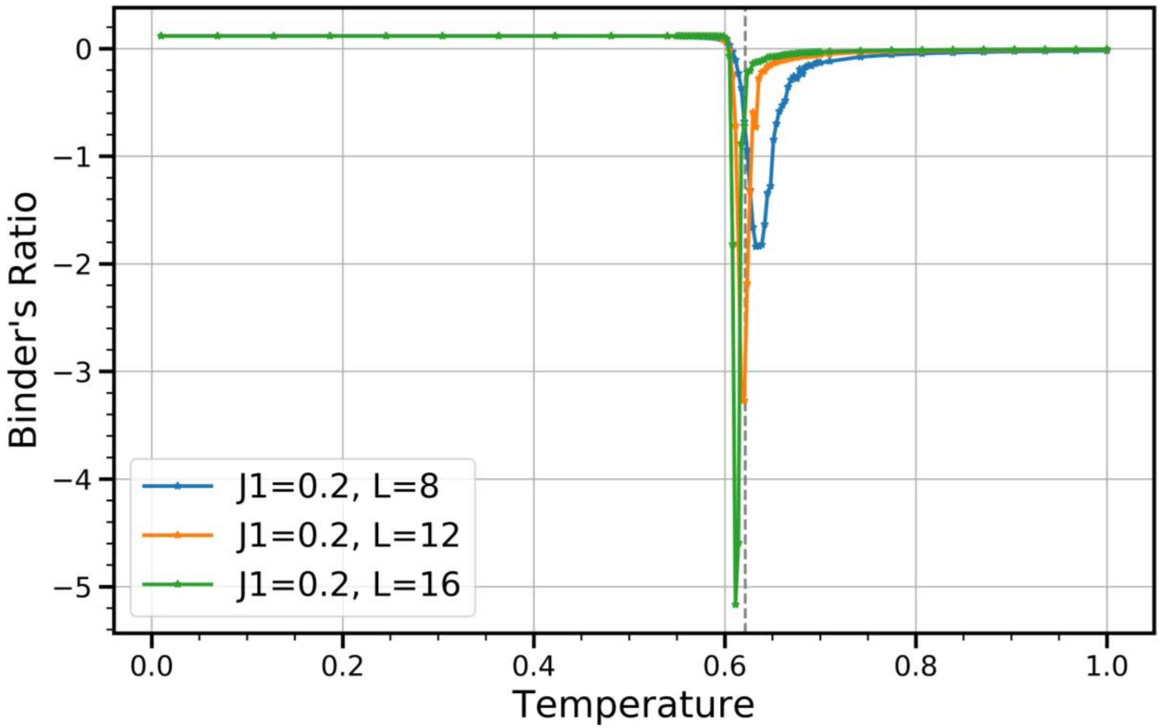
$$B = 1 - \frac{\langle M^4 \rangle}{3\langle M^2 \rangle^2}$$

for the Ising model at $Q = 2$

Binder's Ratio for Low J_1 values

- Show crossing at the theoretical transition temperature T_c for $J1 = 0$(Potts model) limit

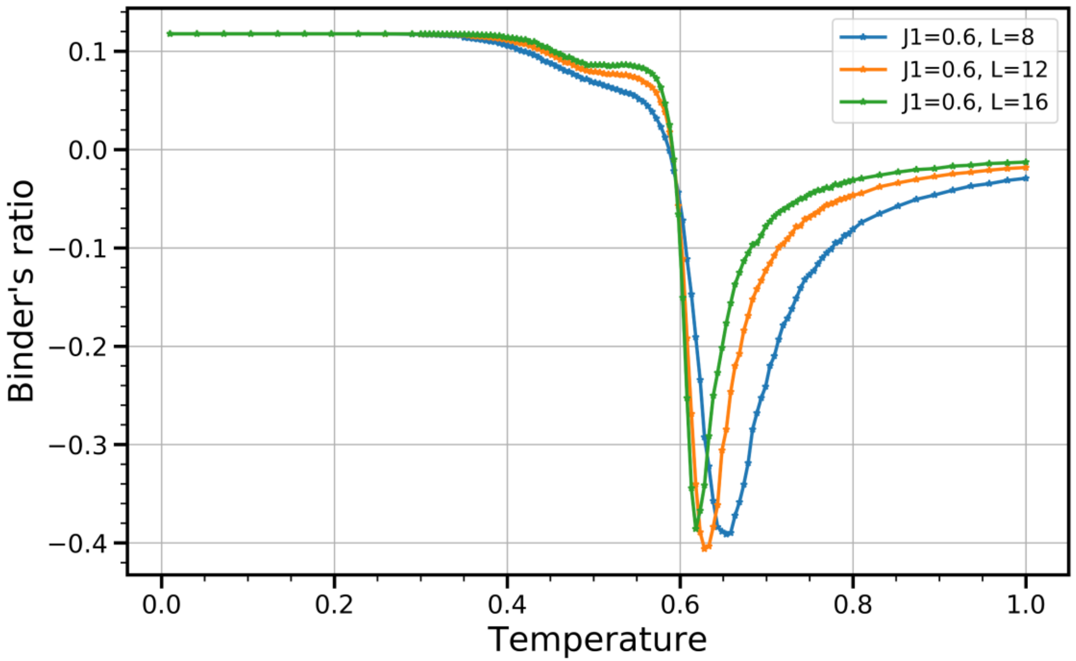


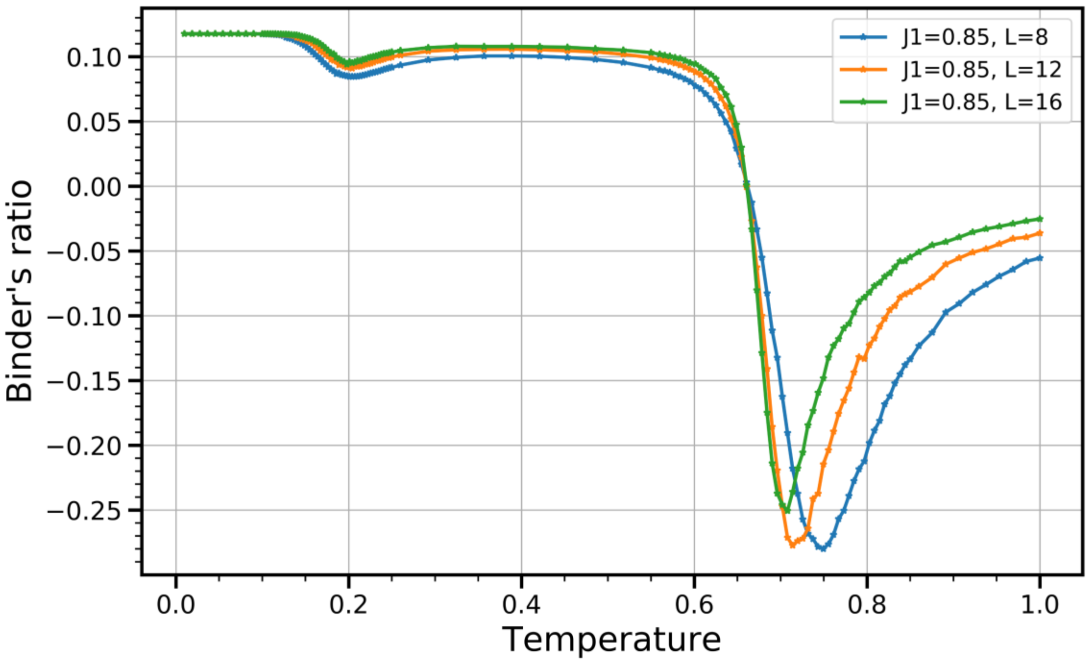


Binder's ratio when J_1 is small

Binder's Ratio for higher J_1 values

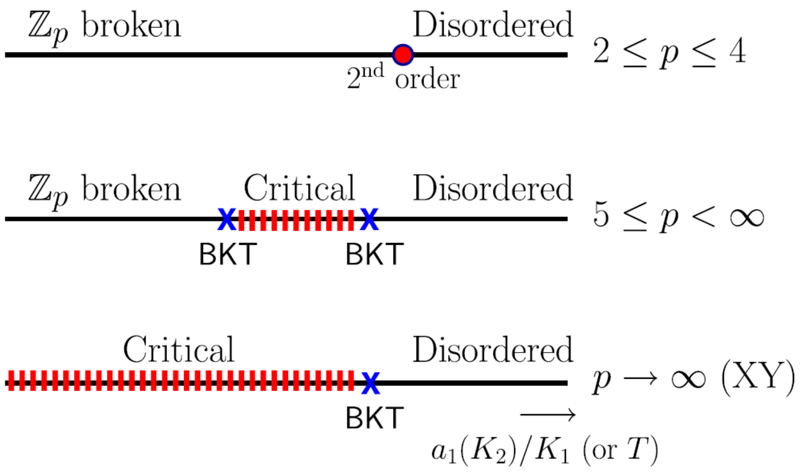
- Showing a dip at the position of the lower T heat capacity peak
- But only one crossing at the high temperature transition point





Binder's ratio when J_1 is higher

KT Phase – XY model & Clock Model



- The XY Model
  - No order at low temperature
  - Power-law decay of the correlation function at low temperature

- The Clock Model

  For $Q \leq 4$
  - values, there's a single 2nd phase transition

  For $Q \geq 5$
  - There are two transition points.
  - Exists an Ordered Phase
  - A critical BKT phase in between

  For $Q \to \infty$
  - Goes to the XY model limit for

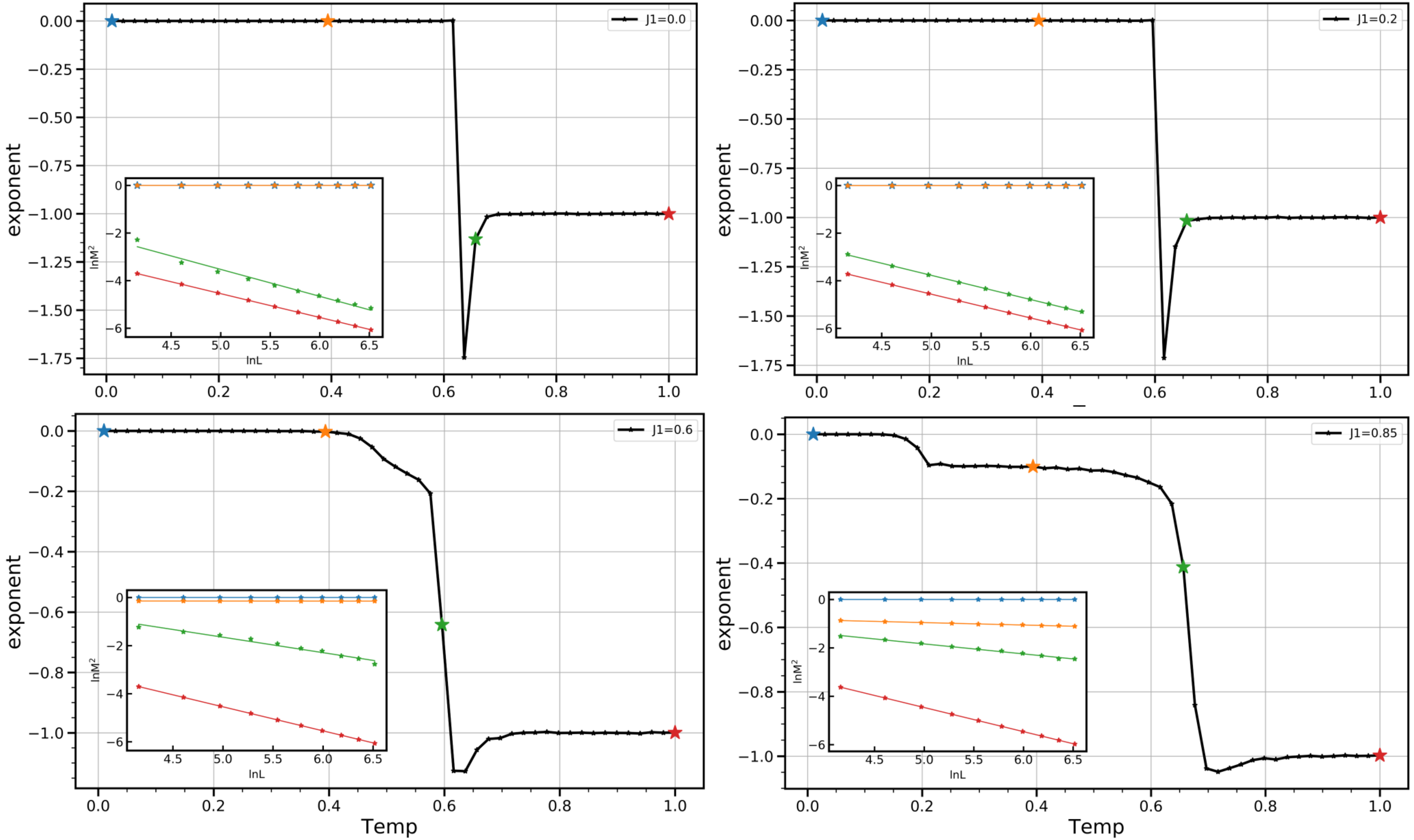
Low Q Results? Hysteresis?

Insert Pic here

M^2 Scaling Analysis

- To obtain the correlation between spins $G(r) = \langle \vec{m}(0) \cdot \vec{m}(r) \rangle$
- $\boldsymbol{M}^2 = \frac{1}{N^2} \sum_{ij} \langle \overrightarrow{m_i} \cdot \overrightarrow{m_j} \rangle = \frac{1}{N} \sum_i \langle \overrightarrow{m_0} \cdot \overrightarrow{m_j} \rangle \cong \frac{1}{N} \int G(r) d^2\vec{r}$
- Long-range order: $\overrightarrow{m_i} = \vec{m}$
  - $\boldsymbol{M}^2 \propto \frac{1}{N} \times N\vec{m}^2 \propto N^2$
- Non-critical phase: $G(r) \propto e^{-\frac{r}{\xi}}$
  - $\boldsymbol{M}^2 \propto \frac{1}{N} \int G(r) d^2r \propto \frac{1}{N} \int e^{-\frac{r}{\xi}} d^2\vec{r} \propto N^{-1}$
- Critical phase: $G(r) \propto r^{-\eta}$
  - $\boldsymbol{M}^2 \propto \frac{1}{N} \int G(r) d^2r \propto \frac{1}{N} \int_0^L r^{-\eta} r dr \propto N^{-\frac{\eta}{2}}$

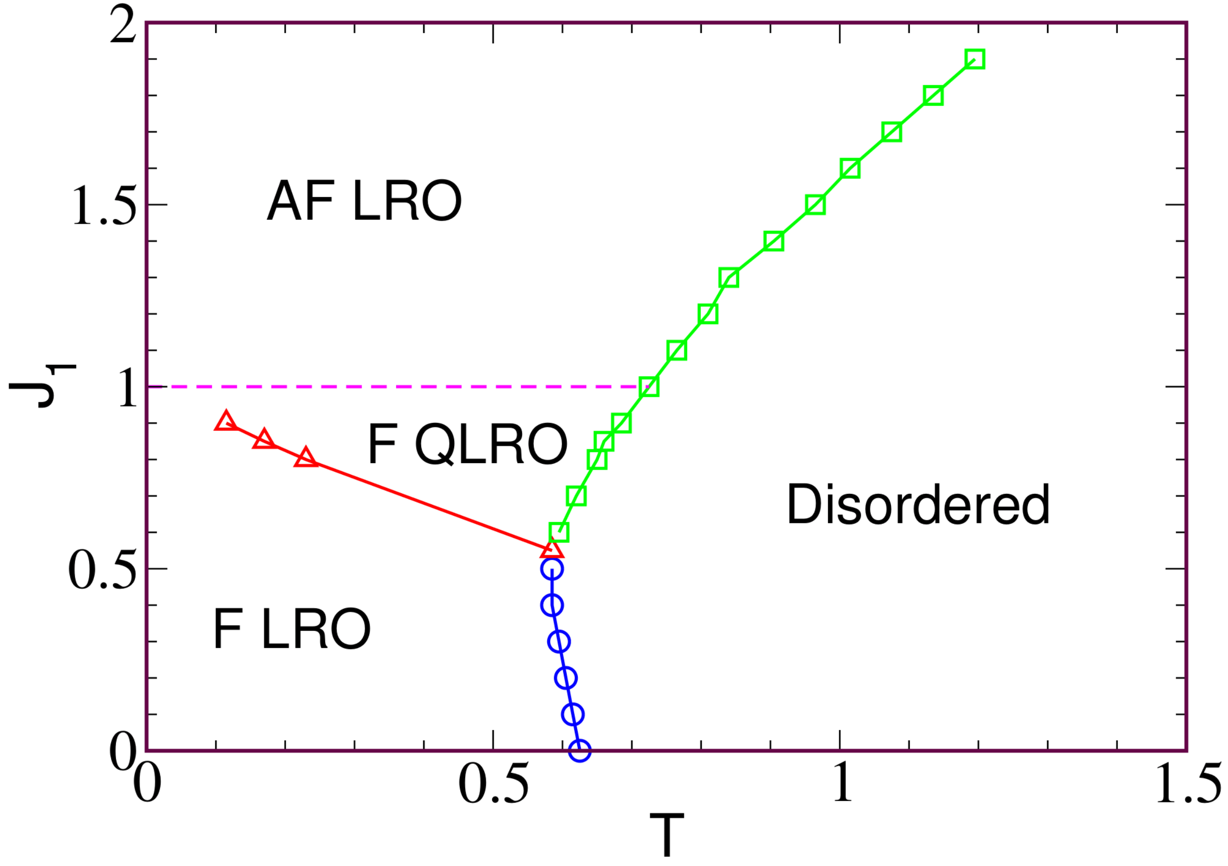
M^2 Scaling Plots for



$$\text{“exponent”} = -\frac{\eta}{2}$$

Rough Phase Diagram

- Reentrance
- 3-phase, critical J_1 to turn on intermediate phase
- Change of order of transition



Future Plan

- Start to do QMC, good initial project
- Chunhan’s started QMC, I can help with
- Quantum tight binding models, SU(N) Hubbard model? Constraint path qmc?

Thank You!

Below Are Backup Slides

Finite Size Scaling

- Finite size scaling hypothesis
- Hyperscaling and relationship between exponents

$$\nu d = 2 - \alpha = 2\beta + \gamma = \beta(\delta + 1) = \gamma \frac{\delta + 1}{\delta - 1}$$

$$2 - \eta = \frac{\gamma}{\nu} = d \frac{\delta - 1}{\delta + 1}$$

- Data Collapsing analyzation

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  - Energy, Heat Capacity
  - Finite Size Scaling
  - Rough Phase Diagram
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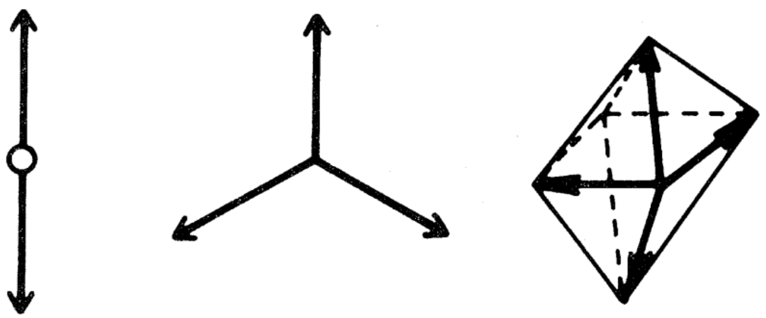
Defining Magnetization $\boldsymbol{M}$

- Each synthetic site $i = 1 \cdots Q$ should correspond to a direction
- Q dimensions in total
- $\boldsymbol{M} = \sum_i \frac{N_i}{N} \hat{\boldsymbol{e}}_{\boldsymbol{i}}$ for a microstate
- Constraint that $\sum_i \frac{N_i}{N} = 1$
- Lives in $Q - 1$ dimensional space in fact
- $\hat{\boldsymbol{e}}_{\boldsymbol{i}}, i = 1 \cdots Q$ forms a $Q - 1$ dimensional hyper-tetrahedron
  - $\delta_{i,j} = \frac{1}{Q}\left(1 + (q-1)\hat{\boldsymbol{e}}_{\boldsymbol{i}} \cdot \hat{\boldsymbol{e}}_{\boldsymbol{j}}\right)$
- Same as the Ising limit when $Q = 2$

Q=2 Q=3



q=2 q=3 q=4

Data collapse analysis-Back up

Simulation Results

- Energy