Problem Set Five

Zhixian(Jason) Yu

04/25/2018

1 Theory

Problem 3

a) Because x is a single vector, we have:

$$\begin{aligned}
\|x - \sum_{j\in N} w_j x_j\|^2 &= (x - \sum_{j\in N} w_j x_j)^T (x - \sum_{j\in N} w_j x_j) \\
&= (x^T - \sum_{j\in N} w_j x_j^T)(x - \sum_{j\in N} w_j x_j) \\
&= x^T x - \sum_{j\in N} w_j x^T x_j - \sum_{j\in N} w_j x_j^T x + \sum_{j\in N} w_j x_j^T \sum_{j\in N} w_j x_j \\
&= x^T x - \sum_{j\in N} w_j x^T x_j - \sum_{j\in N} w_j x_j^T x + \sum_{j\in N} w_j x_j^T \sum_{k\in N} w_k x_k \\
&= x^T x - \sum_{j\in N} w_j x^T x_j - \sum_{j\in N} w_j x_j^T x + \sum_{k\in N}\sum_{j\in N} w_j x_j^T (w_k x_k) \\
&= x^T x - \sum_{k\in N} w_k x^T x_k - \sum_{j\in N} w_j x_j^T x + \sum_{jk} w_j w_k x_j^T x_k
\end{aligned}$$

Because $\sum_j w_j = 1$, we have $\sum_k w_k = 1$ and $\sum_{jk} w_j w_k = 1$. Therefore, the above equation becomes:

$$\begin{aligned}
\|x - \sum_{j\in N} w_j x_j\|^2 &= x^T x - \sum_{k\in N} w_k x^T x_k - \sum_{j\in N} w_j x_j^T x + \sum_{jk} w_j w_k x_j^T x_k \\
&= \sum_{jk} w_j w_k x^T x - \sum_{j} w_j \sum_{k\in N} w_k x^T x_k - \sum_{k} w_k \sum_{j\in N} w_j x_j^T x + \sum_{jk} w_j w_k x_j^T x_k \\
&= \sum_{jk} w_j w_k x^T x - \sum_{jk} w_j w_k x^T x_k - \sum_{jk} w_j w_k x_j^T x + \sum_{jk} w_j w_k x_j^T x_k \\
&= \sum_{jk} w_j w_k (x^T x - x^T x_k - x_j^T x + x_j^T x_k) \\
&= \sum_{jk} w_j w_k (x - x_j)^T (x - x_k)
\end{aligned}$$

Thus,

$$E(w) = \sum_{jk} w_j w_k C_{jk}$$

where $C_{jk} = (x - x_j)^T (x - x_k)$.

b) Our goal is to minimize $\sum_{jk} w_j w_k C_{jk}$ subject to $\sum_j w_j = 1$.

2 Programming

The following code sets up the environment and import packages.

```
import numpy as np
import matplotlib.pyplot as plt
import matplotlib.image as mpimg
import cv2
import random
```

Problem 1