

2014 级概率与数理统计试题 (A 卷)

一、(12 分)

1、解：记 $A_1 = \{\text{任取一件为第一台车床加工的零件}\}$,

$A_2 = \{\text{任取一件为第二台车床加工的零件}\}$,

$B = \{\text{任取一件零件为不合格品}\}$

由全概率公式得到

$$P(B) = P(A_1) \cdot P(B|A_1) + P(A_2) \cdot P(B|A_2)$$

$$= 0.7 \times 0.03 + 0.3 \times 0.05$$

$$= 0.036$$

$$2、\text{解： } P(B|\bar{A}) = \frac{P(\bar{A}B)}{P(\bar{A})} = \frac{P(B) - P(AB)}{1 - P(A)} = 0.85$$

$$P(AB) = P(B) - 0.85(1 - P(A))$$

$$P(A \cup B) = P(A) + P(B) - P(AB) = 0.988$$

二、(12 分)

1、解：(1) 由分布函数的性质 $F(-\infty) = 0, F(+\infty) = 1$

$$\text{可得 } \begin{cases} A + B \arctg(-\infty) = A + B \left(-\frac{\pi}{2}\right) = 0 \\ A + B \arctg(+\infty) = A + B \left(\frac{\pi}{2}\right) = 1 \end{cases}$$

$$\text{解得 } \begin{cases} A = \frac{1}{2} \\ B = \frac{1}{\pi} \end{cases}$$

$$(2) \text{ 由 (1) 得到 } F(x) = \frac{1}{2} + \frac{1}{\pi} \arctg x, \quad -\infty < x < \infty$$

因此

$$\begin{aligned}
 P(-1 < X \leq 1) &= F(1) - F(-1) \\
 &= \left(\frac{1}{2} + \frac{1}{\pi} \arctg(1) \right) - \left(\frac{1}{2} + \frac{1}{\pi} \arctg(-1) \right) \\
 &= \frac{1}{2}
 \end{aligned}$$

2、解： X 的密度函数为

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}, \quad -\infty < x < \infty$$

$Y = e^X$ 的可取值范围是 $(0, \infty)$ ，由 $y = e^x$ 得 $y' = e^x > 0$

故 $y = e^x$ 在 $(-\infty, \infty)$ 上严格单增，其反函数 $h(y) = \ln y$ ，且 $h'(y) = \frac{1}{y}$

因此， $Y = e^X$ 的密度函数

$$\begin{aligned}
 f_Y(y) &= \begin{cases} f_X(\ln y) \left| \frac{1}{y} \right|, & y > 0 \\ 0, & y \leq 0 \end{cases} \\
 &= \begin{cases} \frac{1}{\sqrt{2\pi}\sigma y} e^{-\frac{(\ln y)^2}{2\sigma^2}}, & y > 0 \\ 0, & y \leq 0 \end{cases}
 \end{aligned}$$

三、(16 分)

1、解：(1) 由于 $f_X(x) = \begin{cases} 1, & 0 \leq x < 1 \\ 0, & \text{其它} \end{cases}$ $f_Y(z-x) = \begin{cases} 2(z-x), & 0 \leq z-x < 1 \\ 0, & \text{其它} \end{cases}$

因此 $Z = X + Y$ 的概率密度函数为

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx = \begin{cases} \int_0^z 2(z-x) dx, & 0 \leq z < 1 \\ \int_{z-1}^1 2(z-x) dx, & 1 \leq z < 2 \\ 0, & \text{其它} \end{cases}$$

$$= \begin{cases} z^2, & 0 \leq z < 1 \\ 2z - z^2, & 1 \leq z < 2 \\ 0, & \text{其它} \end{cases}$$

(2) 由于 X 的分布函数为 $F_X(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leq x < 1 \\ 1, & \text{其它} \end{cases}$

Y 的分布函数为 $F_Y(y) = \begin{cases} 0, & y < 0 \\ y^2, & 0 \leq y < 1 \\ 1, & \text{其它} \end{cases}$

因此 $U = \max\{X, Y\}$ 的分布函数为 $F_U(u) = F_X(u)F_Y(u) = \begin{cases} 0, & u < 0 \\ u^3, & 0 \leq u < 1 \\ 1, & \text{其它} \end{cases}$

$U = \max\{X, Y\}$ 的概率密度函数为 $f_U(u) = \begin{cases} 3u^2, & 0 < u < 1 \\ 0, & \text{其它} \end{cases}$

2、解: $P(Z \leq 0.5 | X = 0) = \frac{P(Z \leq 0.5, X = 0)}{P(X = 0)} = \frac{P(X = 0, Y \leq 0.5)}{P(X = 0)}$

$$= \frac{P(X = 0)P(Y \leq 0.5)}{P(X = 0)} = \frac{1}{2}$$

四、(16 分)

1、解: $E(X) = 0, D(X) = 1$.

$$E(Y) = E(X^2) = D(X) + [E(X)]^2 = 1 + 0^2 = 1$$

$$E(Y^2) = E(X^4) = \int_{-\infty}^{\infty} x^4 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 3,$$

$$D(Y) = E(Y^2) - [E(Y)]^2 = 3 - 1^2 = 2.$$

或 $X \sim N(0, 1), Y = X^2 \sim \chi^2(1), E(Y) = 1, D(Y) = 2$.

$$E(XY) = E(X^3) = \int_{-\infty}^{\infty} x^3 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 0,$$

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = 0 - 0 = 0.$$

$$D(X + Y) = D(X) + D(Y) + 2\text{Cov}(X, Y) = 1 + 2 = 3.$$

2、解：由 X 的概率密度函数和分布函数为

$$f_X(x) = \begin{cases} \frac{1}{6}e^{-\frac{1}{6}x}, & x > 0 \\ 0, & \text{其它} \end{cases} \quad F_X(x) = \begin{cases} 1 - e^{-\frac{1}{6}x}, & x > 0 \\ 0, & \text{其它} \end{cases}$$

得到 $U = \min\{X, Y, Z\}$ 的分布函数和密度函数分别为

$$F_U(u) = 1 - (1 - F_X(u))(1 - F_Y(u))(1 - F_Z(u)) = \begin{cases} 1 - e^{-\frac{1}{2}u}, & u > 0 \\ 1, & \text{其它} \end{cases}$$

$$f_U(u) = \begin{cases} \frac{1}{2}e^{-\frac{1}{2}u}, & u > 0 \\ 1, & \text{其它} \end{cases}$$

因此 $EU = 2, DU = 4$.

五、(8 分)

解：由题知 X_i 服从 $U[-0.5, 0.5]$ 分布,

$$E(X_i) = 0, D(X_i) = \frac{1}{12}, i = 1, 2, \dots, 1200.$$

根据中心极限定理知, 所求的概率为

$$\begin{aligned} P\left\{\left|\sum_{i=1}^{1200} X_i\right| < 10\right\} &= P\left\{\left|\frac{\sum_{i=1}^{1200} X_i - 0}{\sqrt{1200} \sqrt{\frac{1}{12}}}\right| < \frac{10}{\sqrt{1200} \sqrt{\frac{1}{12}}}\right\} \\ &\approx \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 = 0.6826. \end{aligned}$$

六、(8 分)

解:

$$(1) \quad \frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$$
$$\frac{9S^2}{\sigma^2} = \frac{1}{\sigma^2} \sum_{i=1}^{10} (X_i - \bar{X})^2 \sim \chi^2(9)$$

$$(2) P\left(0.2088\sigma^2 \leq \frac{1}{10} \sum_{i=1}^{10} (X_i - \bar{X})^2 \leq 2.1665\sigma^2\right)$$
$$= P\left(2.088 \leq \frac{1}{\sigma^2} \sum_{i=1}^{10} (X_i - \bar{X})^2 \leq 21.665\right)$$
$$= P\left(\frac{1}{\sigma^2} \sum_{i=1}^{10} (X_i - \bar{X})^2 \geq 2.088\right) - P\left(\frac{1}{\sigma^2} \sum_{i=1}^{10} (X_i - \bar{X})^2 \geq 21.665\right)$$
$$= 0.99 - 0.01 = 0.98$$

七、(16分)

1、解: 由于 $EX = \int_0^\lambda x \frac{2}{\lambda^2} (\lambda - x) dx = \frac{\lambda}{3}$

令 $EX = \bar{X}$

得 λ 的矩估计为 $\hat{\lambda} = 3\bar{X}$

由于 $E\hat{\lambda} = E(3\bar{X}) = 3EX = \lambda$

因此 $\hat{\lambda}$ 是 λ 的无偏估计.

2、解: 似然函数为

$$L(\theta) = \prod_{i=1}^3 P(X = x_i) = P(X=1)P(X=2)P(X=3)$$
$$= \theta^2 \cdot 2\theta(1-\theta) \cdot (1-\theta)^2 = 2\theta^3(1-\theta)^3$$

对数似然函数为 $\ln L(\theta) = \ln 2 + 3\ln \theta + 3\ln(1-\theta)$

对 θ 求导并令其为零, 得

$$\frac{d \ln L(\theta)}{d\theta} = \frac{3}{\theta} - \frac{3}{1-\theta} = 0$$

解得 θ 的最大似然估计值为

$$\hat{\theta} = \frac{1}{2}$$

八、(12 分)

1、解：假设 $H_0: \mu=5$, $H_1: \mu \neq 5$

$$\text{检验统计量为 } t = \frac{\bar{X} - \mu}{S / \sqrt{n}}$$

$$\text{拒绝域为 } W = \{|t| \geq t_{0.025}(8) = 2.306\}$$

$$\text{由 } n=9, \bar{x}=5.9, s=0.9 \text{ 计算得: } |t|=3 > 2.306$$

因此, 拒绝 H_0 认为该零件的长度与 5mm 有显著差异.

2、解：该检验犯第一类错误的概率为

$$\begin{aligned} P(\text{拒绝 } H_0 | H_0 \text{ 成立}) &= P(S^2 \leq 0.349 | \sigma^2 = 0.8) = P\left(\frac{(n-1)S^2}{\sigma^2} \leq \frac{8 \cdot 0.349}{0.8}\right) \\ &= P\left(\frac{(n-1)S^2}{\sigma^2} \leq 3.49\right) = P\left(\frac{(n-1)S^2}{\sigma^2} \leq \chi_{0.90}^2(8)\right) = 0.10 \end{aligned}$$