

2012-2013-1 概率统计标准答案(信二学习部整理)

一、

(1) 设 B = “顾客买下该箱产品”， $A_i (i=0,1,2)$ 为该箱产品中次品数，

$$P(A_0) = 0.8, P(A_1) = 0.1, P(A_2) = 0.1$$

$$P(B|A_0) = 1, P(B|A_1) = \frac{C_{19}^4}{C_{20}^4} = \frac{4}{5}, P(B|A_2) = \frac{C_{18}^4}{C_{20}^4} = \frac{12}{19}$$

$$P(B) = \sum_{i=0}^2 P(A_i)P(B|A_i) = 0.94$$

$$(2) P(A_0|B) = \frac{P(A_0)P(B|A_0)}{P(B)} = 0.85$$

二、

解： X 的密度函数为

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad -\infty < x < \infty$$

先求 $Y = 3X^2$ 的分布函数 $F_Y(y)$

当 $y \leq 0$ 时， $F_Y(y) = 0$ ；

当 $y > 0$ 时， $F_Y(y) = P(Y \leq y) = P(3X^2 \leq y)$

$$\begin{aligned} &= P\left(-\sqrt{\frac{y}{3}} \leq X \leq \sqrt{\frac{y}{3}}\right) \\ &= \Phi\left(\sqrt{\frac{y}{3}}\right) - \Phi\left(-\sqrt{\frac{y}{3}}\right) \end{aligned}$$

因此， $Y = 3X^2$ 的密度函数

$$f_Y(y) = F_Y'(y) = \begin{cases} \frac{1}{2\sqrt{3y}} \left[f_X\left(\sqrt{\frac{y}{3}}\right) + f_X\left(-\sqrt{\frac{y}{3}}\right) \right], & y > 0 \\ 0, & y \leq 0 \end{cases}$$

$$= \begin{cases} \frac{1}{\sqrt{6\pi}\sqrt{y}} e^{-\frac{y}{6}}, & y > 0 \\ 0, & y \leq 0 \end{cases}$$

三、

解：

$$(1) \quad f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy = \begin{cases} \int_{\frac{x}{2}}^1 1 dy = 1 - \frac{x}{2}, & 0 < x < 2, \\ 0, & \text{其他.} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx = \begin{cases} \int_0^{2y} 1 dx = 2y, & 0 < y < 1, \\ 0, & \text{其他.} \end{cases}$$

因为 $f(x, y) \neq f_X(x)f_Y(y)$, $0 < y < 1, 0 < x < 2y$, 所以 X 与 Y 不相互独立.

$$P(X < 1) = \int_0^1 \left(1 - \frac{x}{2}\right) dx = \frac{3}{4}.$$

$$(2) \quad f_Z(z) = \int_{-\infty}^{\infty} f(z-y, y) dy. \quad \text{被积函数的非零域} \begin{cases} 0 < y < 1, \\ 0 < z-y < 2y. \end{cases}$$

$$f_Z(z) = \begin{cases} \int_{\frac{z}{3}}^z 1 dy = \frac{2z}{3}, & 0 < z < 1, \\ \int_{\frac{z}{3}}^1 1 dy = 1 - \frac{z}{3}, & 1 < z < 3, \\ 0, & \text{其他.} \end{cases}$$

四、

解：(1)

$$E(X) = \iint_{R^2} xf(x, y) dxdy = \int_0^\infty dx \int_{0.5x}^\infty x \cdot \frac{1}{2} e^{-y} dy = \int_0^\infty x \cdot \frac{1}{2} e^{-\frac{x}{2}} dx = 2.$$

$$E(X^2) = \iint_{R^2} x^2 f(x, y) dxdy = \int_0^\infty dx \int_{0.5x}^\infty x^2 \cdot \frac{1}{2} e^{-y} dy = \int_0^\infty x^2 \cdot \frac{1}{2} e^{-\frac{x}{2}} dx = 8.$$

所以 $D(X) = E(X^2) - E^2(X) = 8 - 2^2 = 4.$

(2)

$$E(Y) = \iint_{R^2} yf(x, y) dxdy = \int_0^\infty dy \int_0^{2y} y \cdot \frac{1}{2} e^{-y} dx = \int_0^\infty y^2 e^{-y} dy = 2$$

$$E(Y^2) = \iint_{R^2} y^2 f(x, y) dxdy = \int_0^\infty dy \int_0^{2y} y^2 \cdot \frac{1}{2} e^{-y} dx = \int_0^\infty y^3 e^{-y} dy = 6$$

所以 $D(Y) = E(Y^2) - E^2(Y) = 6 - 2^2 = 2.$

(3)

$$E(XY) = \iint_{R^2} xyf(x, y) dxdy = \int_0^\infty dy \int_0^{2y} xy \cdot \frac{1}{2} e^{-y} dx = \int_0^\infty 2y^2 \cdot \frac{y}{2} e^{-y} dy = 6.$$

所以 $\text{cov}(X, Y) = E(XY) - E(X) \cdot E(Y) = 6 - 2 \cdot 2 = 2.$

最终

$$\rho_{xy} = \frac{\text{cov}(X, Y)}{\sqrt{D(X) \cdot D(Y)}} = \frac{2}{\sqrt{4 \cdot 2}} = \frac{\sqrt{2}}{2}.$$

五、解：设 S 为用电高峰时，同时用电的户数，电站至少应具有 xW 发电量，才能以 95% 的概率保证供电，x 需满足

$$P(200S \leq x) \geq 0.95.$$

由题意，

$$S \sim B(10000, 0.9), \quad E(S) = 9000, D(S) = 900,$$

由中心极限定理

$$\frac{S - 9000}{\sqrt{900}} \text{ 近似服从 } N(0, 1),$$

$$P(200S \leq x) = P(S \leq x/200) \approx \Phi\left(\frac{x/200 - 9000}{\sqrt{900}}\right) = \Phi\left(\frac{x - 1800000}{6000}\right),$$

$$\text{由 } \Phi\left(\frac{x - 1800000}{6000}\right) \geq 0.95 \text{ 得到 } \frac{x - 1800000}{6000} \geq 1.65, x \geq 1809900.$$

六、

解：(1). $X_i / \sigma \sim N(0,1), i=1,2,3,4$

由 χ^2 分布的性质知

$$\frac{1}{\sigma^2}(X_1^2 + X_2^2 + X_3^2 + X_4^2)$$

服从 $\chi^2(4)$ 分布。

$$(2). \quad X_1 + X_2 \sim N(0, 2\sigma^2) \quad X_4 - X_3 \sim N(0, 2\sigma^2)$$

$$\frac{X_1 + X_2}{\sqrt{2}\sigma} \sim N(0,1) \quad \frac{X_4 - X_3}{\sqrt{2}\sigma} \sim N(0,1)$$

按 χ^2 分布的性质知

$$\left(\frac{X_1 + X_2}{\sqrt{2}\sigma}\right)^2 \sim \chi^2(1) \quad \left(\frac{X_4 - X_3}{\sqrt{2}\sigma}\right)^2 \sim \chi^2(1)$$

按 F 分布的性质知

$$Y = \frac{(X_1 + X_2)^2}{(X_4 - X_3)^2} = \frac{\left(\frac{X_1 + X_2}{\sqrt{2}\sigma}\right)^2 / 1}{\left(\frac{X_4 - X_3}{\sqrt{2}\sigma}\right)^2 / 1} \sim F(1,1).$$

七、

$$\text{解：(1) 由 } EX = \int_{-\infty}^{\infty} xf(x)dx = \int_2^{\infty} x2^{\alpha} \alpha x^{-(\alpha+1)} dx = \frac{2\alpha}{\alpha-1}$$

得 $\alpha = \frac{EX}{EX - 2}$

用 \bar{x} 代替 EX 得 α 的矩估计为 $\hat{\alpha} = \frac{\bar{x}}{\bar{x} - 2}$

(2) 似然函数为

$$L(\alpha) = \prod_{i=1}^n f(x_i) = (2^\alpha \alpha)^n \prod_{i=1}^n x_i^{-(\alpha+1)}$$

对数似然函数为

$$\ln L(\alpha) = n(\ln \alpha + \alpha \ln 2) - (\alpha + 1) \sum_{i=1}^n \ln x_i$$

对 α 求导并令其为零，得

$$\frac{d \ln L}{d \alpha} = n \left(\frac{1}{\alpha} + \ln 2 \right) - \sum_{i=1}^n \ln x_i = 0$$

解得 α 的最大似然估计为

$$\hat{\alpha} = \frac{1}{\frac{1}{n} \sum_{i=1}^n \ln x_i - \ln 2}$$

(3) 由 $E\left(C \sum_{i=1}^n i X_i\right) = C \sum_{i=1}^n i E(X_i) = C \mu \sum_{i=1}^n i = C \mu \frac{n(n+1)}{2}$

令

$$C \mu \frac{n(n+1)}{2} = \mu$$

得

$$C = \frac{2}{n(n+1)}$$

八、

解 (1) $H_0: \mu=30 \quad H_1: \mu \neq 30$

选取检验统计量 $T = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} \sim t(n-1)$

H_0 的拒绝域 $w_0: |T| = \frac{|\bar{x} - \mu_0|}{s/\sqrt{n}} > t_{\frac{\alpha}{2}}(n-1)$

计算 $|T_0| = \left| \frac{\bar{x} - 30}{0.1/\sqrt{10}} \right| = 3.162$

$$|T_0| = 3.162 > t_{0.05}(9) = 1.833$$

在显著性水平 $\alpha = 0.1$ 下，拒绝原假设，认为 $\mu \neq 30$

(2) 假设 $H_0: \sigma^2 = \sigma_0^2 = 0.04$ $H_1: \sigma^2 \neq \sigma_0^2 = 0.04$

在原假设下，检验统计量

$$\chi^2 = \frac{(n-1)S^2}{\sigma_0^2} = \frac{\sum_{i=1}^n (X_i - \bar{X}_n)^2}{\sigma_0^2} = \frac{(10-1)S^2}{\sigma_0^2}$$

由于在 H_0 下 χ^2 取值过大和过小都是拒绝 H_0 的依据，所以其水平为

α 的拒绝域为

$$\chi^2 \leq \chi_{1-\frac{\alpha}{2}}^2(10-1) \text{ 或 } \chi^2 \geq \chi_{\frac{\alpha}{2}}^2(10-1)$$

$$\chi^2 = \frac{(10-1)S^2}{\sigma_0^2} = \frac{(10-1) \times 0.01}{0.04} = 2.25 < \chi_{0.975}^2(9) = 3.325$$

所以在显著性水平 0.1 下，拒绝原假设. 总体方差与 0.04 有显著性差异。