End Sem exam 2023 Info & Communication Solutions.

1.

(a) Video : 30 fps. Each frame is 720×1080p

Each pixel needs 8 bits (as it can take 255 values)

$\Rightarrow$ Per second, we need bit rate of $30 \times 720 \times 1080 \times 8$

$$\approx 0.1867 \times 10^8 \text{ bps}$$

(b) Shannon entropy of each pixel is $\frac{3}{4}\log 4 + \frac{253}{1012}\log 1012$

$$\approx 3.95 \text{ bits}$$

Optimal source code can have at most avg length $= 4.95 \approx 5$ bits

$\hookrightarrow H(X)+1$

Thus now bit rate requirement will be

$$\approx 30 \times 720 \times 1080 \times 5$$

$$= 0.116 \times 10^9 \text{ bps.}$$

(c) Now in this, the entropy of the frame itself is only 1.371 bits.

Thus, we need bit rate $1.371 \times 30 \approx 42$ bps.

The channel cannot support this as the channel has ~~sink~~ rate only 1.2 bps.

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2. (a) $\underline{m} = (0\,0\,1\,1)$, $\underline{m}G = (0\,0\,1\,1)\left(I_4 \middle| \begin{matrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{matrix}\right)$

Codeword is $= (0\,0\,1\,1\,1\,0\,0)$

(b) Suppose 6^{th} bit is flipped, that is, we receive $(0\,0\,1\,1\,1\,1\,0)$.

The H matrix is $[P^T \; I_3] = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$

Computing the syndrome Hy^T = sum of columns 3,4,5,6, which gives us syndrome $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

(Some students may not know this term 'syndrome')

Thus, comparing with H matrix, we see that it is the sixth column of the matrix. Hence $\underline{e} = (0\,0\,0\,0\,0\,1\,0)$

$\Rightarrow$ Estimated codeword is $(0\,0\,1\,1\,1\,0\,0)$, which was the transmitted codeword. Hence decoding succeeds.

(c) Assume that first two coordinates are flipped. Then we get the syndrome $\underline{s} = Hy^T$ as sum of first 2 columns $= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ which is the 3^{rd} column. Hence, decoder 'thinks' the error was in the third position. & flips the third position to get a codeword estimate, which will be wrong.

This happens for any tx codeword & any two bit flips as the sum of any two columns of a H matrix results in some other third column.

Some students may not write this, which may incur only 0.5 mark loss

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$$C_{BSC} = \max_{P_X} I(X;Y).$$

3. BSC channel capacity, Assume $X \sim Bern(\alpha)$

(a)

$$I(X;Y) = H(Y) - H(Y|X)$$

$$H(Y|X=0) = H(Y|X=1) = H_2(p) \Rightarrow H(Y|X) = H_2(p)$$

$$P_Y(1) = \left(P_X(1)(1-p) + P_X(0)p\right) = \alpha(1-p) + (1-\alpha)p$$

$$P_Y(0) = (1-\alpha)(1-p) + \alpha p.$$

$$\Rightarrow H(Y) = -\left(\alpha(1-p) + (1-\alpha)p\right)\log\left(\alpha(1-p) + (1-\alpha)p\right)$$
$$-\left((1-\alpha)(1-p) + \alpha p\right)\log\left((1-\alpha)(1-p) + \alpha p\right)$$

$$= -(\alpha + p - 2p\alpha)\log(\alpha + p - 2p\alpha)$$
$$- (1-\alpha-p+2p\alpha)\log(1-\alpha-p+2p\alpha)$$

$$\frac{dH(Y)}{d\alpha} = -(1-2p)\log(\alpha+p-2p\alpha)$$
$$-(-1+2p)\log(1-\alpha-p+2p\alpha)$$

cancels
$$\left\{ \begin{array}{l} -\dfrac{(\alpha+p-2p\alpha)}{(\alpha+p-2p\alpha)}(1-2p)\log_e 2 \\[2ex] -\dfrac{(1-\alpha-p+2p\alpha)}{(1-\alpha-p+2p\alpha)}(-1+2p)\log_e 2 \end{array} \right.$$

The above is 0 when $\log(\alpha+p-2p\alpha) = \log(1-\alpha-p+2p\alpha)$

(or) $\alpha + p - 2p\alpha = 1 - \alpha - p + 2p\alpha$

$\Rightarrow 2\alpha + 2p = 1 + 4p\alpha$ 5

Since this holds for every p, we must get $\alpha = 0.5$.

Plugging $\alpha = 0.5$ in $H(Y)$ gives us that $H(Y) = 1$.

We know that $H(Y) \leq 1$ (as o/p is binary valued & $\log_2 |\mathcal{Y}| = 1$)

Thus $\alpha = 0.5$ is the highest value.

Hence $\max_{p_X} I(X;Y) = \max_{p_X} H(Y) - H(Y|X)$

$$= 1 - H_2(\alpha).$$

(Procedure could be bit simpler when student uses $H(X) - H(X|Y)$)

(For convenience we assume n is odd. Roughly idea works for n even))

(b) Prob of error when using repetition code:

$$\mathcal{C}_{rep} : \{\underbrace{(0 \ldots 0)}_{\text{all-zero}}, \underbrace{(1, \ldots, 1)}_{\text{all-one}}\} \subseteq \{0,1\}^n.$$

Decoding rule: $\hat{\underline{x}}_{MAP} = \underset{\underline{x} \in \mathcal{C}_{rep}}{\operatorname{argmax}} \; p(\underline{x}|\underline{y})$ (Prob that $\underline{x}$ is tx, given $\underline{y}$ is received)

$$= \underset{\underline{x} \in \mathcal{C}_{rep}}{\operatorname{argmax}} \frac{p(\underline{y}|\underline{x})\, p(\underline{x})}{p(\underline{y})}$$

Gn $p(\underline{x}) = \frac{1}{2}$ for $\underline{x} \in \mathcal{C}_{rep}$. Further $p(\underline{y})$ doesn't change.

Hence $\hat{\underline{x}}_{MAP} = \underset{\underline{x} \in \mathcal{C}_{rep}}{\operatorname{argmax}} \; p(\underline{y}|\underline{x})$

Now suppose $\underline{x} = (0, \ldots, 0)$. Then we make an error when $p(\underline{y}|0 \ldots 0) < p(\underline{y}|1 \ldots 1)$

Let $w(\underline{y})$ denote no. of 1s in $\underline{y}$.

Thus $p(y|0\dots0) = p^{w(y)}(1-p)^{n-w(y)}$

& $p(y|1\dots1) = (1-p)^{w(y)}p^{n-w(y)}$

Thus when $\underline{x} = (0\dots0)$ is trx, we make an error when

$$p^{w(y)}(1-p)^{n-w(y)} < (1-p)^{w(y)}p^{n-w(y)}$$

$\Rightarrow$ when $$\left(\frac{p}{1-p}\right)^{w(y)}\left(\frac{1-p}{p}\right)^{n-w(y)} < 1$$

$$\Rightarrow \left(\frac{1-p}{p}\right)^{n-2w(y)} < 1$$

Now, as $p<0.5$, $\frac{1-p}{p} > 1$.

Thus $\left(\frac{1-p}{p}\right)^{n-2w(y)} < 1$ iff $2w(y) > n$.

(ie) when more than $n/2$ flips happen.

Thus $p(\text{error} \mid 00\dots0 \text{ is trx}) = P(\text{more than } \frac{n}{2} \text{ bit flips})$

$$= \sum_{i=\frac{n}{2}+1}^{n} \binom{n}{i} p^i (1-p)^{n-i}$$

III^{ly} we can calculate $P(\text{error}|1\dots1 \text{ is trx})$

$$= \sum_{i=\frac{n}{2}+1}^{n} \binom{n}{i} p^i (1-p)^{n-i}.$$

The average of the two is $P(\text{error})$

$$P(\text{error}) = P_{\underline{X}}(\underline{0})\, P(\text{error} \mid \underline{0}) + P_{\underline{X}}(\underline{1})\, P(\text{error} \mid \underline{1})$$

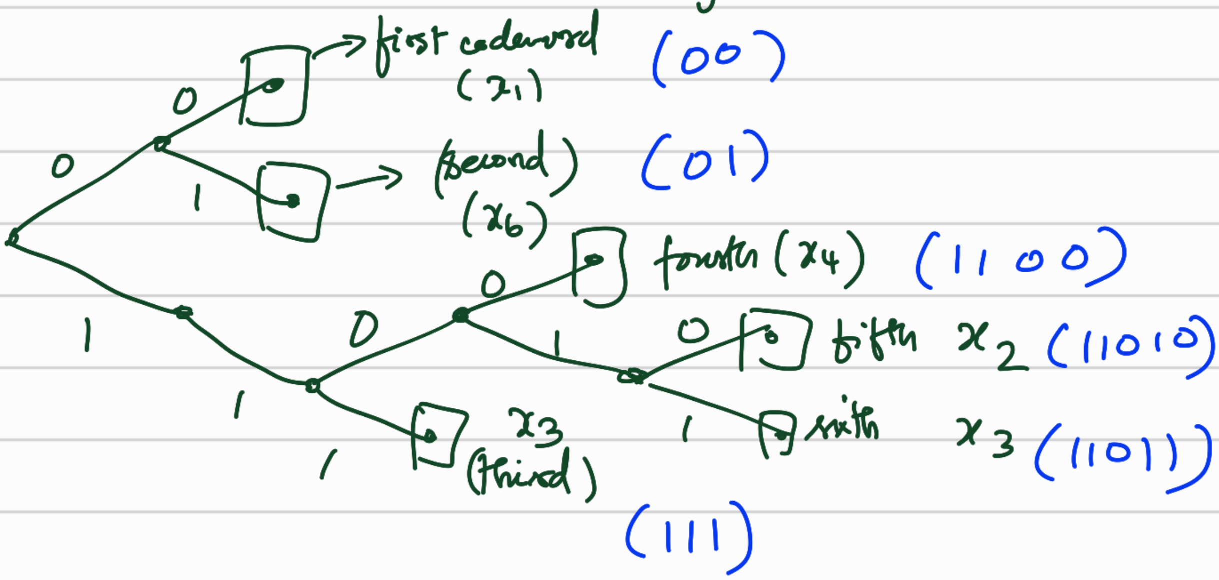
$$= \sum_{i=n/2+1}^{n} \binom{n}{i} p^i (1-p)^{n-i}.$$

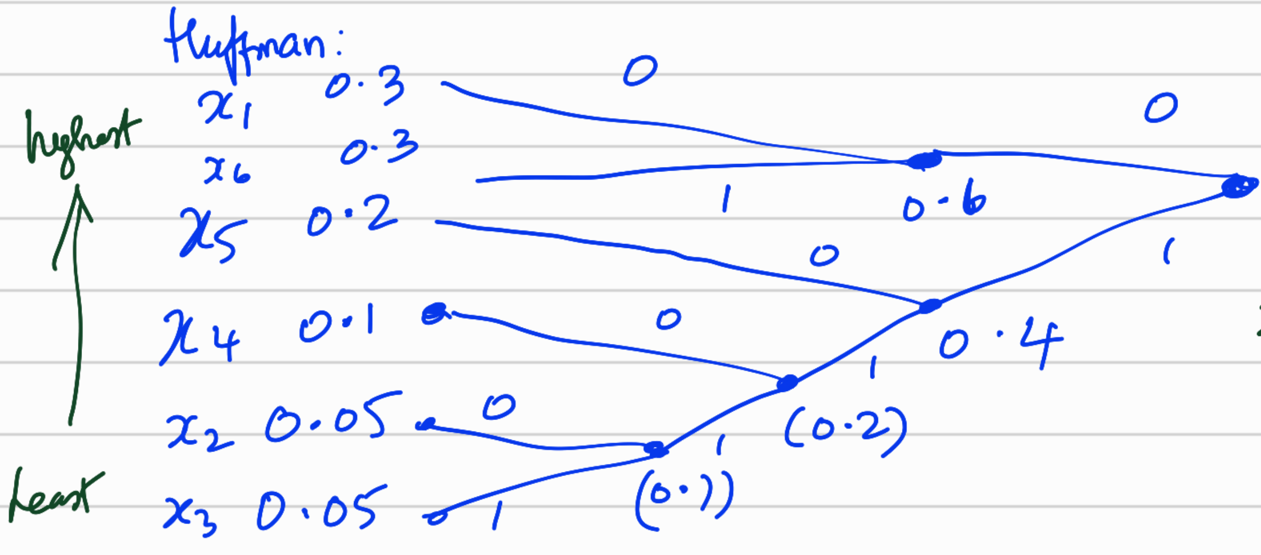
4\. (a) Shannon-Fano technique: Choose lengths as follows

$\lceil -\log 0.3 \rceil$ (2) x_1; $\lceil -\log 0.05 \rceil$ (5) x_2; $\lceil -\log 0.05 \rceil$ (5) x_3; $\lceil -\log 0.1 \rceil$ (4) x_4; $\lceil -\log 0.2 \rceil$ (3) x_5 & $\lceil -\log 0.3 \rceil$ (2) x_6

(b)

Thus a SF code is obtained as follows





Thus codewords are

x_1: 00
x_6: 01
x_5: 10
x_4: 110
x_2: 1110
x_3: 1111

(b)

Expected lengths are

$$\mathcal{L}_{SF} = 0.3\times4 + 0.05\times10 + 0.1\times4 + 0.2\times3$$

$$= 1.2 + 0.5 + 0.4 + 0.6 = 2.7 \text{ bits}$$

$$\mathcal{L}_{Huff} = 0.8\times2 + 0.1\times3 + 0.1\times4$$

$$= 1.6 + 0.7 = 2.3 \text{ bits.}$$

Thus $\mathcal{L}_{Huff} < \mathcal{L}_{SF}$.

(c) Entropy is 2.271. $\Rightarrow$ $\mathcal{L}_{Huff}$ is close to $H(X)$ but $\mathcal{L}_{SF}$ is larger.

5. [Hard question as students have not seen].

Recall for Gaussian channel: $\boxed{p(y|x) = \frac{1}{\sqrt{2\pi N_0/2}} e^{-\frac{(y-x)^2}{N_0/2}}}$

MAP rule is as before $\hat{x}_{MAP} = \underset{x\in\{A,-A\}}{\operatorname{argmax}} \; p(x|y)$

$$\hat{x}_{MAP} = \underset{x\in\{A,-A\}}{\operatorname{argmax}} \; \frac{p(y|x)\,p(x)}{p(y)}$$

$$= \underset{x\in\{A,-A\}}{\operatorname{argmax}} \; p(y|x)$$

when x is uniform.

But $p(y|x) = \frac{1}{\sqrt{2\pi \frac{N_0}{2}}} e^{-(y-x)^2 / 2\frac{N_0}{2}}$

$= \frac{1}{\sqrt{\pi N_0}} e^{-(y-x)^2/N_0}$

$\Rightarrow$ MAP rule essentially becomes

$$\hat{x} = \underset{x \in \{A, -A\}}{\operatorname{argmax}} \; e^{-(y-x)^2/N_0}$$

$$= \underset{x \in \{A, -A\}}{\operatorname{argmin}} \; |y-x|$$

($|y-x|$ is absolute diff between x & y)

Thus when trx symbol is A, we have an error when y is closer to $-A$ than A.

$\Rightarrow y \in [-\infty \text{ to } 0]$

When trx symbol is $-A$, we have an error when y is closer to A than $-A$. $\Rightarrow y \in [0 \text{ to } \infty]$

$\Rightarrow P(\text{error}) = P(\text{error} | A \text{ is trx}) P(A \text{ is trx})$
$+ P(\text{error} | -A \text{ is trx}) P(-A \text{ is trx})$

$$P(\text{error} | A \text{ is trx}) = \int_{y=-\infty}^{0} p(y|x=A)\, dy$$

$$= \frac{1}{\sqrt{\pi N_0}} \int_{-\infty}^{0} e^{-(y-A)^2/N_0}\, dy$$

$$P(\text{error} \mid -A \text{ is tx}) = \frac{1}{\sqrt{\pi N_0}} \int_{0}^{\infty} e^{-(y+A)^2/N_0}\, dy.$$

$$= \frac{1}{\sqrt{\pi N_0}} \int_{-\infty}^{0} e^{-(y-A)^2/N_0}\, dy$$

$\Rightarrow$ Total error prob is

$$P(\text{error}) = \frac{1}{\sqrt{\pi N_0}} \int_{-\infty}^{0} e^{-(y-A)^2/N_0}\, dy.$$

6. Clearly we see that $q(x,y) \geq 0 \;\; \forall x,y.$

Now $$\sum_{x,y} q(x,y) = \sum_{x,y} P_X(x) P_Y(y)$$

$$= \left(\sum_x P_X(x)\right)\left(\sum_y P_Y(y)\right)$$

$$= 1 \times 1 = 1.$$

Hence $q(x,y)$ is a valid distribution.

(Same holds in continuous case)

Now $I(X;Y) = \sum_{x,y \in supp(P_{X,Y})} p_{X,Y}(x,y) \log \frac{p_{X,Y}(x,y)}{p_X(x)\, p_Y(y)}$

$$-I(X;Y) = \sum_{(x,y) \in supp(P_{X,Y})} p(x,y) \log \frac{P_X(x) P_Y(y)}{P_{XY}(x,y)}$$

as log is concave, we get that

avg of log $\leq$ log (avg)

$$\Rightarrow -I(X;Y) \leq \log \left[\sum_{(x,y) \in supp P_{X,Y}} p(x,y) \left(\frac{P_X(x) P_Y(y)}{P_{XY}(x,y)} \right) \right]$$

$$\leq \log \sum_{(x,y) \in supp(P_{X,Y})} P_X(x) P_Y(y)$$

Now $\sum_{(x,y) \in supp P_{X,Y}(x,y)} P_X(x) P_Y(y) \leq 1$

$$\Rightarrow -I(X;Y) \leq 0 \Rightarrow I(X;Y) \geq 0.$$

(If student has missed $supp(P_{X,Y})$ it is OK).

Further as $H(Y|X)$ & $H(X|Y)$ are nonnegative,

we have that

(Simple proof is good if written, or even otherwise it is ok)

$$I(X;Y) = H(Y) - H(Y|X)$$
$$\leq H(Y)$$
$$\& \ I(X;Y) = H(X) - H(X|Y) \leq H(X)$$
$$\Rightarrow I(X;Y) \leq \min(H(X), H(Y)).$$

7. (i) $$X_1(f) = \int_{t=-\infty}^{\infty} x(3t-6)\, e^{-j2\pi f t}\, dt$$

Let $t' = 3t - 6 \Rightarrow t = \left(\frac{t'+6}{3}\right)$

$$\Rightarrow X_1(f) = \int_{t=-\infty}^{\infty} x(t')\, e^{-j2\pi f\left(\frac{t'+6}{3}\right)}\, dt$$

$$= e^{-j4\pi f} \int_{t=-\infty}^{\infty} x(t')\, e^{-j2\pi t' f/3}\, dt$$

$$X_1(f) = e^{-j4\pi f}\, X(f/3).$$

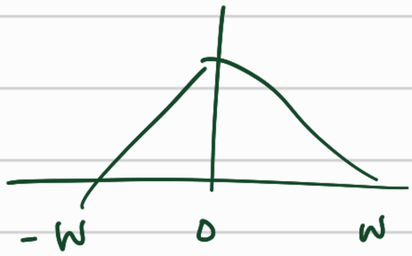
(ii) $$X_2(f) = \int_{t=-\infty}^{\infty} x(t)\, e^{j2\pi f_0 t}\, e^{-j2\pi f t}\, dt$$

$$= \int_{t=-\infty}^{\infty} x(t)\, e^{-j2\pi (f-f_0)t}\, dt$$

$$= X(f-f_0)$$

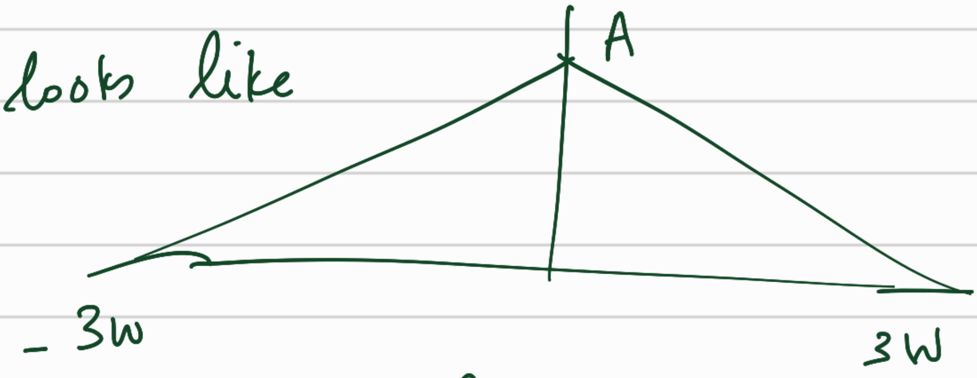
Answers to other parts:

(a) Assume $X(f)$



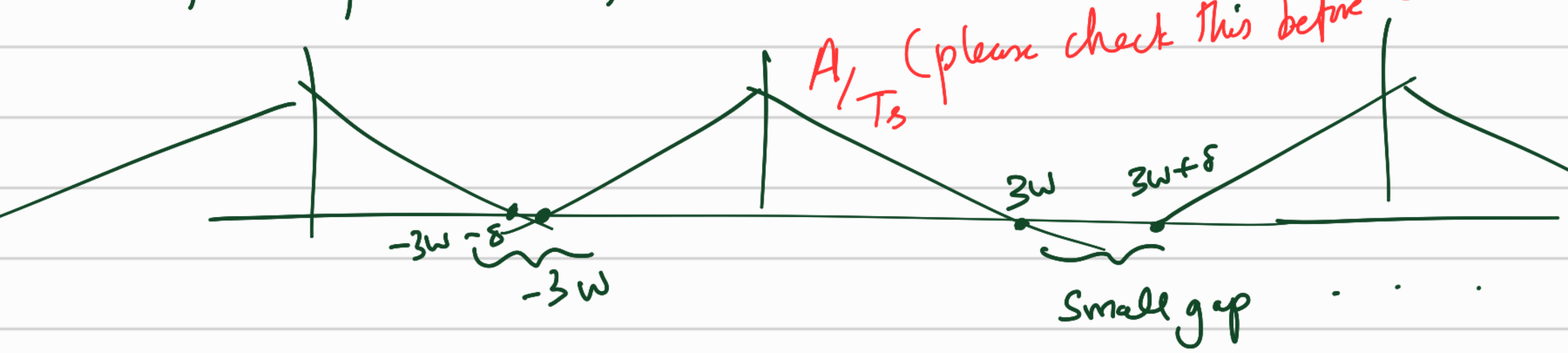
(if such general $X(f)$ is not assumed then cut 1 mark)

Then $|X_1(f)|$ looks like



(bandwidth blow up)

Sample spectrum repeats, so we get



(b) $x_2(t)$ is closer to the idea of modulation as it looks like the spectrum has undergone freq shift, just like $x_2(t)\cos 2\pi f_0 t$.

(c) If $x(t)$ is like a baseband signal, we take only +ve part of the spectrum for bandwidth. This is given as W

(i) $x_1(t)$ has BW of 3W as there is bw overlap

(ii) $x_2(t)$ has BW 2W as f_0 is given as large enough & $x_2(t)$ is more like passband signal

8. (a) In given question, $g(X,Y) = X+Y$.

Thus $\mathbb{E}[X+Y] = \sum_{x,y} P_{X,Y}(x,y)(x+y)$

$$= \sum_{x,y} P_{X,Y}(x,y)x + \sum_{x,y} P_{X,Y}(x,y)y$$

$$= \sum_x \Big(\sum_y P_{X,Y}(x,y)\Big)x + \sum_y \Big(\sum_x P_{X,Y}(x,y)\Big)y$$

$$= \sum_x p_X(x)\, x + \sum_y p_Y(y)\, y$$

$$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y].$$

(b) Given n RVs $X_1, \ldots, X_n$ with joint prob mass fn $p_{X_1, \ldots, X_n}$. Let $g(X_1, \ldots, X_n)$ be any fn of the n RVs. Then the expectation

$$\mathbb{E}[g(X_1, \ldots, X_n)] \triangleq \sum_{(x_1, \ldots, x_n)} p_{X_1 \ldots X_n}(x_1, \ldots, x_n)\, g(x_1, \ldots, x_n)$$

(defined as).

Then show that

$$\mathbb{E}[X_1 + \ldots + X_n] = \mathbb{E}[X_1] + \ldots + \mathbb{E}[X_n]$$