Quiz 2

(MA6.102) Probability and Random Processes, Monsoon 2023

Question cum Answer Booklet

19 October, 2023

Max. Duration: 45 Minutes

Max. Marks: 20

Roll Number:_________________

Programme:______________

Special Instruction: - Please don't do rough work anywhere in this booklet except on the last 3 pages.

Marks Table

Q1	Q2	Q3

Question 1. Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of independent and identically distributed continuous random variables with the common PDF $f_X(x)$. Note that $P(X = \alpha) = 0$, for all $\alpha \in \mathbb{R}$, and that $P(X_i = X_j) = 0$, for all $i \neq j$. For $n \geq 2$, define X_n as a *record-to-date* of the sequence if $X_n > X_i$, for all $i < n$.

(a) [5 Marks] Find an expression for the expected number of records-to-date that occur over the first m trails for any given integer m.
Hint: Use indicator random variables.

(b) [5 Marks] Let A_n denote the event that X_n is a record-to-date, for $n \geq 2$. Is A_n independent of A_{n+1}, i.e., is $P(A_n \cap A_{n+1}) = P(A_n)P(A_{n+1})$?

(a)

Let I_i denote the indicator R.V. for the event that X_n is a record-to-date, for $i \geq 1$

Let N denote the no. of records-to-date that occur over first m trails

We then have $N = \sum_{i=1}^{m} I_i$

$$\Rightarrow \mathbb{E}[N] = \mathbb{E}\left[\sum_{i=1}^{m} I_i\right] = \sum_{i=1}^{m} \mathbb{E}[I_i] \quad \left[\text{By Linearity of Expectations}\right]$$

Now, $\mathbb{E}[I_i] = 1.\, P(I_i = 1) + \cancel{0.\, P(I_i = 0)}^{\,0}$

$$= P(X_i \text{ is record-to-date})$$

$$= P(X_i > \max\{X_1, \ldots, X_{i-1}\})$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{x_i} \cdots \int_{-\infty}^{x_i} f_{X_1,\ldots,X_i}(x_1, x_2, \ldots, x_i)\, dx_1\, dx_2 \ldots dx_i$$

X_i's are independent

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{x_i} \cdots \int_{-\infty}^{x_i} f_{X_1}(x_1) f_{X_2}(x_2) \ldots f_{X_i}(x_i)\, dx_1\, dx_2 \ldots dx_i$$

X_i's are identically distr.

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{x_i} \cdots \int_{-\infty}^{x_i} f_X(x_1) f_X(x_2) \ldots f_X(x_i)\, dx_1\, dx_2 \ldots dx_i$$

$$= \int_{-\infty}^{\infty} f_X(x_i)\left(\int_{-\infty}^{x_i} f_X(x_{i-1})\,dx_{i-1}\right)\cdots\cdots\left(\int_{-\infty}^{x_i} f_X(x_1)\,dx_1\right) dx_i$$

By defn. of CDF

$$= \int_{-\infty}^{\infty} f_X(x_i)\, F_X(x_i) \cdots\cdots F_X(x_i)\, dx_i$$

$$= \int_{-\infty}^{\infty} f_{X_i}(x_i)\left(F_X(x_i)\right)^{i-1} dx_i$$

$$\boxed{\begin{array}{l} \text{let } F_X(x_i) = t \\ \Rightarrow f_X(x_i)\,dx = dt \\ x_i = -\infty \Rightarrow F_X(x_i) = 0 \Rightarrow t = 0 \\ x_i = \infty \Rightarrow F_X(x_i) = 1 \Rightarrow t = 1 \end{array}}$$

$$\mathbb{E}[I_i] = \int_0^1 t^{i-1}\,dt = \left[\frac{t^i}{i}\right]_0^1 = \frac{1}{i}$$

$$\Rightarrow \mathbb{E}[N] = \mathbb{E}[I_1] + \sum_{i=2}^{m} \mathbb{E}[I_i] = \sum_{i=2}^{m} \frac{1}{i} = \underline{\underline{\frac{1}{2} + \frac{1}{3} + \cdots\cdots + \frac{1}{m}}} \quad \underline{\text{Ans}}.$$

[Note $\mathbb{E}[I_1] = 0$, $\because$ $P(X_1$ is record-to-date$) = 0$ as X_1 being r-t-d is an empty event]

(b) $P(A_n) = P(X_n$ is record-to-date$) = \frac{1}{n}$ [from part (a), as our choice of i was arbitrary, the result in part (a) holds $\forall i \geq 2$]

Similarly, $P(A_{n+1}) = \frac{1}{n+1}$

Now, $P(A_n \cap A_{n+1}) = P(X_n$ is record-to date $\cap$ X_{n+1} is record-to-date$)$

$$= P(X_n > \max\{X_1, \ldots, X_{n-1}\} \cap X_{n+1} > \max\{X_1, \ldots, X_n\})$$

$$= P(X_{n+1} > X_n > \max\{X_1, \ldots, X_{n-1}\})$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{x_{n+1}}\int_{-\infty}^{x_n}\cdots\cdots\int_{-\infty}^{x_n} \overbrace{f_X(x_1) f_X(x_2) \cdots\cdots f_X(x_n) f_X(x_{n+1})}^{\text{again from i.i.d.}}\, dx_1 \cdots dx_{n+1}$$

$$= \int_{-\infty}^{\infty} f_X(x_{n+1})\left[\int_{-\infty}^{x_{n+1}} f_X(x_n)\left[\int_{-\infty}^{x_n} f_X(x_{n-1})\,dx_{n-1} \cdots\cdots \int_{-\infty}^{x_n} f_X(x_1)\,dx_1\right] dx_n\right] dx_{n+1}$$

$$= \int_{-\infty}^{\infty} f_X(x_{n+1}) \int_{-\infty}^{x_{n+1}} f_X(x_n)\left(F_X(x_n)\right)^{n-1} dx_n\, dx_{n+1} = \int_{-\infty}^{\infty} f_X(x_{n+1})\left[\frac{F_X(x_n)^n}{n}\right]_{-\infty}^{x_{n+1}}$$

$$= \int_{-\infty}^{\infty} f_X(x_{n+1}) \frac{F_X(x_{n+1})^n}{n} dx_{n+1} = \frac{1}{n} \int_{-\infty}^{\infty} f_X(x_{n+1}) F_X(x_{n+1})^n dx_{n+1} = \frac{1}{n} \left[\frac{F_X(x_{n+1})^{n+1}}{n+1} \right]_{-\infty}^{\infty} = \frac{1}{n} \frac{1}{n+1}$$

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$$= P(A_n) P(A_{n+1})$$

Question 2 (5 Marks)**.** Consider a discrete integer-valued random variable Y with CDF

$$F_Y(y) = 1 - \frac{2}{(y+1)(y+2)}, \text{ for integer values } y \geq 0.$$

Let Z be another integer valued random variable with the conditional PMF

$$P_{Z|Y}(z|y) = \frac{1}{y^2}, \text{ for } 1 \leq z \leq y^2.$$

Find $\mathbb{E}[Z]$.

$$P_Y(y) = F_Y(y) - F_Y(y-1), \text{ for } y \geq 1 \quad [P_Y(0) = 0, \because F_Y(0) = 0]$$

$$= \cancel{1} - \frac{2}{(y+1)(y+2)} - \cancel{1} + \frac{2}{y(y+1)}$$

$$= \frac{2}{y+1} \left[\frac{1}{y} - \frac{1}{y+2} \right] = \frac{2}{y+1} \frac{2}{y(y+2)}$$

$$\mathbb{E}[Z | Y=y] = \sum_{z=1}^{y^2} z P_{Z|Y}(z|y) = \sum_{z=1}^{y^2} z \frac{1}{y^2} = \frac{1}{\cancel{y^2}} \frac{\cancel{y^2}}{2} (y^2+1)$$

$$\Rightarrow \mathbb{E}[Z|Y] = \frac{Y^2+1}{2}$$

$$\underbrace{\mathbb{E}[Z] = \mathbb{E}\left[\mathbb{E}[Z|Y]\right]}_{\text{By Law of Iterated Expectations}} = \mathbb{E}\left[\frac{Y^2+1}{2}\right] = \frac{1}{2}\mathbb{E}[Y^2] + \frac{1}{2}$$

$$\text{Now, } \mathbb{E}[Y^2] = \sum_{y=0}^{\infty} y^2 P_Y(y) = \sum_{y=1}^{\infty} y^2 P_Y(y)$$

$$= \sum_{y=1}^{\infty} y^{\cancel{2}} \frac{4}{\cancel{y}(y+1)(y+2)}$$

$$= 4 \sum_{y=1}^{\infty} \frac{y}{(y+1)(y+2)}$$

$$= 4\sum_{y=1}^{\infty} \frac{2(y+1)-(y+2)}{(y+1)(y+2)} = 4\sum_{y=1}^{\infty}\left(\frac{2}{y+2} - \frac{1}{y+1}\right)$$

$$= 4 \lim_{N\to\infty}\left[\sum_{y=1}^{N} \frac{2}{y+2} - \frac{1}{y+1}\right]$$

This splitting of sum is allowed since the summation on right is a finite quantity

$$= 4 \lim_{N\to\infty}\left[\sum_{y=1}^{N} \frac{1}{y+2} + \sum_{y=1}^{N}\left(\frac{1}{y+2} - \frac{1}{y+1}\right)\right]$$

In general,

$$\lim_{N\to\infty}\sum_{n=1}^{N} A(n)+B(n) = \lim_{N\to\infty}\sum_{n=1}^{N}A(n) + \lim_{N\to\infty}\sum_{n=1}^{N}B(n)$$

if either one of the limits is finite

$$= 4 \lim_{N\to\infty}\left(\sum_{y=1}^{N}\frac{1}{y+2}\right) + 4\lim_{N\to\infty}\left(\frac{1}{N+2} - \frac{1}{2}\right)$$

$$= \underbrace{4\lim_{N\to\infty}\left(\sum_{y=1}^{N}\frac{1}{y+2}\right)}_{=\infty} - 2$$

$= \infty$ (since, this is a divergent series)

Hence, $\underline{\underline{\mathbb{E}[Z] = \infty}}$

Question 3 (5 Marks)**.** Let X be a continuous random variable with PDF f_X such that $f_X(x) > 0$, for all $x \in \mathbb{R}$. Suppose $g : \mathbb{R} \to \mathbb{R}$ is a strictly monotonic differentiable function. Find the PDF of $Y \triangleq g(X)$ and use it to show that

$$\mathbb{E}[Y] = \int_{-\infty}^{\infty} g(x) f_X(x)\ dx. \tag{1}$$

Let g be monotonically increasing (strictly)

Ineq. sign remains unchanged $\because$ g is incr.

$$F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = P(X \leq g^{-1}(y))$$

$$= F_X(g^{-1}(y))$$

($\because$ g is increasing, g^{-1} is well-defined)

$$\Rightarrow f_Y(y) = \frac{dF_X(g^{-1}(y))}{d(g^{-1}(y))} \frac{dg^{-1}(y)}{dy} = f_X(g^{-1}(y))\,(g^{-1})'(y)$$

Now, $$\mathbb{E}[Y] = \int_{-\infty}^{\infty} y f_Y(y)\, dy = \int_{-\infty}^{\infty} y\, f_X(g^{-1}(y))\,(g^{-1})'(y)\, dy$$

let $g^{-1}(y) = t \Rightarrow (g^{-1})'(y)\, dy = dt$

$$\Rightarrow \mathbb{E}[Y] = \int_{-\infty}^{\infty} g(t)\, f_X(t)\, dt = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

change of variables

Let g be monotonically decreasing (strictly)

$$F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = P(X \geq g^{-1}(y))$$

[Inequality gets reversed, $\because$ g is decr.]

$$= P(X > g^{-1}(y))$$

$$= 1 - P(X \leq g^{-1}(y)) = 1 - F_X(g^{-1}(y))$$

$$\Rightarrow f_Y(y) = -f_X(g^{-1}(y))\,(g^{-1})'(y)$$

$$\Rightarrow \mathbb{E}[Y] = \int_{-\infty}^{\infty} y\, f_Y(y)\,dy = -\int_{-\infty}^{\infty} y\, f_X(g^{-1}(y))\,(g^{-1})'(y)\,dy$$

let $g^{-1}(y) = t \Rightarrow (g^{-1})'(y)\,dy = dt$

$$\Rightarrow \mathbb{E}[Y] = -\int_{\infty}^{-\infty} g(t)\, f_X(t)\,dt \qquad \left[\begin{array}{l} y = -\infty \Rightarrow g^{-1}(y) = \infty \Rightarrow t = \infty \\ y = \infty \Rightarrow g^{-1}(y) = -\infty \Rightarrow t = -\infty \end{array}\right]$$

$$= -\Big[\underbrace{-\int_{-\infty}^{\infty} g(t)\, f_X(t)\,dt}_{\text{using property of definite integrals}}\Big] = \int_{-\infty}^{\infty} g(t)\, f_X(t)\,dt$$

$$= \int_{-\infty}^{\infty} g(x)\, f_X(x)\,dx$$

ROUGH SHEET-1

ROUGH SHEET-2

ROUGH SHEET-3