4. A tensor $T^{\mu\nu}$ is defined as

$$T^{\mu\nu} = \frac{1}{4\pi}\left(F^{\mu\lambda}F^{\nu}{}_{\lambda} - \frac{1}{4}\eta^{\mu\nu}F^{\lambda\sigma}F_{\lambda\sigma}\right).$$

where $F^{\mu\lambda}$ is also a tensor, and $\eta^{\mu\nu}$ is the Minkowski metric and the Greek alphabets μ, ν, λ run from 0 to 3. Show that $T^{\mu}_{\ \mu} = 0$. *Hint: Be careful about the order of the indices in $F^{\nu}{}_{\lambda}$. Here ν is the first and λ is the second index.* [4]

$$T^{\mu}_{\mu} = \eta_{\nu\mu} T^{\mu\nu}$$

$$= \eta_{\nu\mu}\left(\frac{1}{4\pi}\left(F^{\mu\lambda}F^{\nu}_{\lambda} - \frac{1}{4}\eta^{\mu\nu}F^{\lambda\sigma}F_{\lambda\sigma}\right)\right)$$

$$= \frac{1}{4\pi}\left(\eta_{\nu\mu}F^{\mu\lambda}F^{\nu}_{\lambda} - \left(\frac{1}{4}\eta_{\nu\mu}\eta^{\mu\nu}F^{\lambda\sigma}F_{\lambda\sigma}\right)\right)$$

-4

$$= \frac{1}{4\pi}F_{\nu}{}^{\lambda}F^{\nu}{}_{\lambda} - \frac{1}{4\pi}\frac{1}{\cancel{4}}\cancel{\delta^{\mu}_{\mu}}F^{\lambda\sigma}F_{\lambda\sigma} \qquad [\because \textstyle\sum \delta^{\nu}_{\mu} = 4]$$

$$= \frac{1}{4\pi}F_{\nu}{}^{\lambda}F_{\lambda}{}^{\nu} - \frac{1}{4\pi}F^{\lambda\sigma}F_{\lambda\sigma}$$

since indices run from 0 to 3

$$= 0$$

5. The covariant derivatives of 1 and 2 rank tensors are defined as

$$\nabla_\lambda V^\mu \equiv V^\mu_{;\lambda} = \frac{\partial V^\mu}{\partial x^\lambda} + \Gamma^\mu_{\lambda\kappa} V^\kappa, \quad \nabla_\lambda T^{\mu\nu} \equiv T^{\mu\nu}_{;\lambda} = \frac{\partial T^{\mu\nu}}{\partial x^\lambda} + T^{\alpha\nu}\Gamma^\mu_{\alpha\lambda} + T^{\mu\alpha}\Gamma^\nu_{\alpha\lambda}.$$

Prove that $(A^\mu B^\nu)_{;\lambda} = A^\mu_{;\lambda} B^\nu + A^\mu B^\nu_{;\lambda}$. [6]

$$A^\mu_{;\lambda} B^\nu + A^\mu B^\nu_{;\lambda} = (\nabla_\lambda A^\mu) B^\nu + (\nabla_\lambda B^\nu) A^\mu$$

$$= \left(\frac{\partial A^\mu}{\partial x^\lambda} + \Gamma^\mu_{\lambda k} A^k\right) B^\nu + \left(\frac{\partial B^\nu}{\partial x^\lambda} + \Gamma^\mu_{\lambda k} B^k\right) A^\mu$$

$$= B^\nu \frac{\partial A^\mu}{\partial x^\lambda} + \Gamma^\mu_{\lambda k} A^k B^\nu + A^\mu \frac{\partial B^\nu}{\partial x^\lambda} + \Gamma^\mu_{\lambda k} B^k A^\mu$$

$$= \frac{B^\nu \partial A^\mu}{\partial x^\lambda} + \frac{A^\mu \partial B^\nu}{\partial x^\lambda} + \Gamma^\mu_{\lambda k} A^k B^\nu + \Gamma^\mu_{\lambda k} B^k A^\mu$$

$$= \frac{\partial A^\mu B^\nu}{\partial x^\lambda} + \Gamma^\mu_{\lambda k} A^k B^\nu + \Gamma^\mu_{\lambda k} B^k A^\mu$$

since A^μ & B^ν are tensors we can write

$$A^\mu B^\nu = T^{\mu\nu}$$

$$= \frac{\partial T^{\mu\nu}}{\partial x^\lambda} + T^{k\nu}\Gamma^\mu_{\lambda k} + T^{\mu k}\Gamma^\nu_{\lambda k}$$

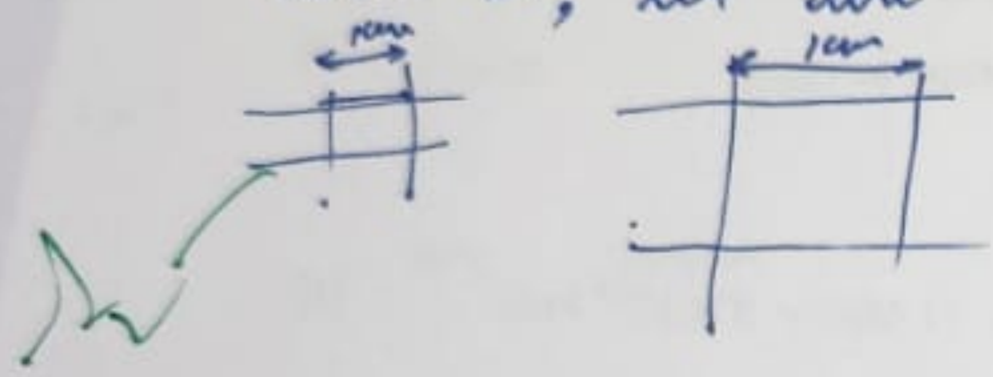
$$= T^{\mu\nu}_{;\lambda} = \nabla_\lambda T^{\mu\nu} = \nabla_\lambda (A^\mu B^\nu)$$

$$= (A^\mu B^\nu)_{;\lambda}$$

$\frac{1}{2}$

6. Since the universe is expanding, are we and everything (including the atoms and molecules) around us expanding, too? Justify your answer. [1]

Since the universe is expanding and we are moving along with it, we are called as co moving.

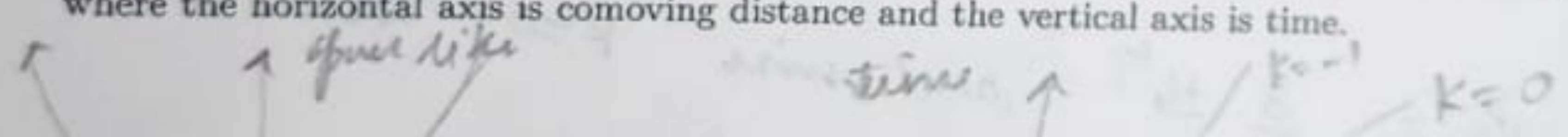


since it is expanding, the ~~fo~~ spacetime fabric is expanding with it. hence we could scale we scale along with it.

7. In the $c = 1$ unit, the flat ($K = 0$) Friedmann-Robsertson-Walker metric is

$$ds^2 = -dt^2 + a(t)^2 \left(dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right),$$

where $a(t)$ is the scale factor of our expanding universe. Draw the past light-cone in a spacetime diagram where the horizontal axis is comoving distance and the vertical axis is time. [3]



8. Consider a universe that is flat ($K = 0$) and dominated by non-relativistic (cold) matter only.

(a) Show that in this universe

$$\frac{a}{a_0} = \left(\frac{3}{2} H_0 t\right)^{2/3},$$

where a is the scale factor at any time t, and a_0 is the scale factor at the present epoch $t = t_0$. [5]

(b) Show that the density of the matter in this universe evolves with time as [3]

$$\rho(t) = \frac{1}{6\pi G t^2}.$$

(c) Calculate the age of this universe. At present $H_0 = 70\text{Km/s/Mpc}$. ($1\ \text{Mpc} = 3 \times 10^{19}\ \text{km}$) [1]