1. **What is Olber's paradox and how was it resolved?** [3]

The olbers paradox was that even though the universe was assumed to be infinitely large, eternally old and static; the night sky appeared dark. Ideally if the universe assumptions were true then the sky would always appear as hot as the surface of the sun at all times as eternally old implies that all the light has reached us.

Few way of resolving involved correcting the observations. Firstly, our line of sight does not tend to infinity, it may be blocked by a non luminous object. Also the assumption of eternally old was challenged as we can assume that some light has not yet reached us.

3

2. **A galaxy emits light of wavelength of 1.21567×10^{-7}m. When observed at earth, the wavelength is 1.21445×10^{-7}m. If the comoving distance of the galaxy is 2.5 million light years then find the magnitude and direction of the velocity of the galaxy with respect to the earth.** *1 light year = 0.3067 parsec.* [3]

Given, emitted $\lambda_e = 1.21567 \times 10^{-7} m$

observed $\lambda_o = 1.21445 \times 10^{-7} m$

comoving distance $r = 2.5 \times 10^6 \, ly = 0.767 \times 10^6 \, pc$

The redshift is given by $z = \dfrac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}} = \dfrac{-0.00122 \times 10^{-7} m}{1.21567 \times 10^{7} m} \sim -10^{-3}$

using Hubbles eq. $v = H_0 r$

$= 72 km s^{-1} pc^{-1} \times r$

$= 72 \times 0.767 \times 10^6 \, km s^{-1}$

$= 55.2 \times 10^6 \, km s^{-1}$ towards the earth.

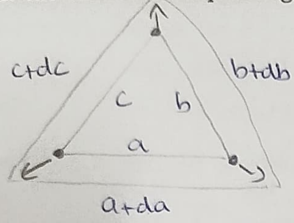
——— Ans.

since z is -ve, the galaxy is moving closer to the earth.

3

Turn the page.

3. Three galaxies which are not in the same line form an acute triangle. Using the cosmological principle, and the fact that the universe is expanding, show that the expansion is according to the Hubble's law. [6]



cosmological principle → universe is isotropic and homogeneous

universe is expanding.

triangle similarity is preserved due to cosmological principle.

↳ ratio of sides is constant.

$\Rightarrow \frac{a+da}{a} = \frac{b+db}{b} = \frac{c+dc}{c} = k.$

$\frac{da}{dt}$

~~[illegible] expansion is same~~

Let $v_1 = \frac{da}{dt}$ $v_2 = \frac{db}{dt}$ $v_3 = \frac{dc}{dt}$

we have $1 + \frac{da}{a} \Rightarrow \frac{da}{dt\, a} = \frac{(k-1)}{dt}$

similarly $1 + db/b \Rightarrow \frac{db}{dt\, b} = \frac{(k-1)}{dt}$

$1 + dc/c \Rightarrow dc/dtc = (k-1)/dt$

all are same.

$\therefore \frac{\dot{a}}{a} = \frac{\dot{b}}{b} = \frac{\dot{c}}{c} = H_0$ (Hubbles constant).

and $\frac{da}{dt} = a \cdot \frac{(k-1)}{dt} = H_0 a. \rightarrow v_1 = H_0 a$

similarly $v_2 = H_0 b$ $v_3 = H_0 c$

} Hubbles Law.

Ans

6

4. **Suppose the universe was expanding at much faster rate in the past than it is now. If the present Hubble constant is H_0, explain if the universe older or younger than H_0^{-1}.** [2]

Say H_0' is the ~~[illegible]~~ hubbles const in the faster expanding universe, with $\dot{a}' > \dot{a}$

↳ present universe

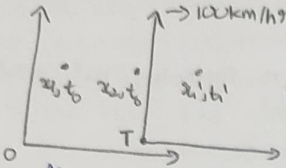
$\therefore H_0' > H_0$ as $H_0 = \frac{\dot{a}}{a}$

$\Rightarrow \frac{1}{H_0'} < \frac{1}{H_0}$

$\Rightarrow t' < t$ $\therefore$ The new universe would be younger than our universe.

2

5. A train 1/2 Km long (as measured by an observer on the train) is traveling at a speed of 100 km/hr. Two ligtning bolts strike the ends of the train simultaneously as determined by an observer on the ground. What is the time separation of the strikes as measured by an observer on the train? [5]



Ground frame - end of trains ⇒ (x_1) and (x_2) at time t_0 (simultaneous)

Train frame - ends: (t_1', x_1') (t_2', x_2')

Also, $x_2' - x_1' = 500m$ and $V = 100\frac{km}{hr} = \frac{100\,000}{3600} = \frac{1000}{36} m/s$

speed of train

Assume relative vel along x axis only.

using Lorentz Transformation : $x' = \begin{bmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{bmatrix} x$ $\beta = \frac{v}{c}$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$

$\therefore t_1' = \gamma t_0 - \gamma\beta x_1$

$x_1' = -\gamma\beta t_0 + \gamma x_1$

$t_2' = \gamma t_0 - \gamma\beta x_2$

$x_2' = -\gamma\beta t_0 + \gamma x_2$

now, $x_2' - x_1' = \gamma(x_2 - x_1) = 500$

and $t_2' - t_1' = -\gamma\beta(x_2 - x_1) = -\frac{\gamma\beta \times 500}{\gamma} = -500\beta$

$\beta = \frac{v}{c} = \frac{1000}{36 \times 3 \times 10^8 s} = \frac{1}{108 \times 10^5} s$

$\therefore t_2' - t_1' = \frac{-500}{108 \times 10^5} = -4.65 \times 10^{-5}$ sec

∴ The front end (x_2') of the train strike is observed 4.65×10^{-5} s before the back end (x_1')

— Ans.

6. Starting from the fluid equation

$$\frac{\partial\rho}{\partial t} + \frac{\dot{a}}{a}(3\rho + 3P) = 0\,,$$

show that in a universe dominated by radiation only, the energy density scales as $\rho \propto a^{-4}$. [5]

Dominated by radiation ⇒ $\rho_M = \rho_\Lambda = 0$

Let the pressure energy density $P/\rho = w$. →

then $\frac{\partial\rho}{\partial t} + \frac{da}{a\,dt}(3\rho + 3P) = 0$ ⇒ $\frac{d\rho}{\rho}$

⇒ $\frac{d\rho}{\rho}$ + $\frac{da}{a} 3(w+1) = 0$ ⇒ $\int \frac{d\rho}{\rho} = \int \frac{da}{a} -3(w+1)$

⇒ $\ln(\rho) = -3(w+1)\ln(a) + c = \ln(a^{(-3(w+1))})$

∴

⇒ $\rho \propto a^{-3(w+1)}$

for radiation dominated ∋ $P = \rho/3$ ⇒ $w = \frac{P}{\rho} = \frac{1}{3}$

⇒ $\rho \propto a^{-3(1+\frac{1}{3})} = a^{-3-1} = a^{-4}$

$\therefore \rho \propto a^{-4}$

— Ans.

5

$\frac{\partial\rho}{\partial t} + \frac{\dot{a}}{a} 3\rho = 0$

$\frac{d\rho}{\rho} + \frac{da}{a}(3w+3) = 0$

$\frac{d\rho}{\rho} + \frac{da}{a}(3)(w+1) = 0$

$\int w \frac{d\rho}{\rho} = \int -3(w+1) \frac{da}{a}$

$w \ln(\rho) = \ln(a)(\frac{-3(w+1)}{w}$

$\rho = a^{-3(w+1)}$

$w \ln((\frac{\rho}{w}))^w$

$\frac{\rho - 0}{w}$

Turn the page.

7. Consider a light source at a redshift $z = 3$ in an Einstein-de Sitter universe, where $\Omega_M = 1.0$ and $\Omega_\Lambda = 0$, $\Omega_R = 0$, $\Omega_K = 0$, *i.e.*, the universe is dominated by non-relativistic matter only. Calculate the distance the light has travelled to reach us. *You can use* $H_0 = 70km/s/Mpc$. [7]

~~Einstein de sitter universe, $\Omega_M=1$, $\Omega_\Lambda=0$, $\Omega_R=0$, $\Omega_K=0$~~ ~~matter dominated~~ ~~$\frac{1}{8}$ & $\frac{1}{8}$~~

~~$z=3 = 1+\frac{a(t_0)}{a(t_e)}$~~

~~$\Rightarrow \frac{a(t_0)}{a(t_e)} = 2$~~ ~~$\frac{\rho_0}{\rho_e} = \left(\frac{a(t_e)}{a(t_0)}\right)^3 = \frac{1}{8}$~~

~~0 → present e → emitted.~~ ~~$\Rightarrow \rho_0 = \frac{1}{8}\rho_e$~~

~~at the time of emission $\Omega_M^0 = 1 = \frac{\rho_M^0}{\rho_{critical}}$~~ ~~$\Rightarrow \rho_c = \rho_M^0 = \rho_e$~~

~~currently.~~

~~$\Omega_M = \frac{\rho_0}{\rho_c} = \frac{1}{8}\frac{\rho_e}{\rho_e} = \frac{1}{8}$~~ ~~$H^2 = H_0^2 \Omega_M$~~ ~~$\frac{a_0}{a} = \frac{a(t_e)}{a(t_0)} = \frac{1}{2}$~~

~~$\therefore H^2 = H_0^2 \left(\Omega_M^0 \left(\frac{a_0}{a}\right)^3\right) = \frac{H_0^2}{8}$~~

~~$H^2 = H_0^2 \left(\Omega_M^0 \left(\frac{a_0}{a}\right)^3\right) = H_0^2 \left(1.\left(\frac{1}{2}\right)^3\right)$~~

~~$= \frac{H_0^2}{8} \Rightarrow H = \frac{H_0}{2\sqrt{2}}$~~

$$t = \int dt = \int_0^1 \frac{dx}{H_0 x \sqrt{\Omega_M^0 x^{-3}}} \qquad \left(x = \frac{1}{1+z}\right)$$

↳ rest $\Omega_R = \Omega_K = \Omega_\Lambda = 0$. $\Omega_M = 1$

$$= \int_0^1 \frac{dx}{H_0 x^1 . x^{-3/2}} = \frac{1}{H_0}\int_0^1 x^{1/2} dx = \frac{1}{H_0} \left.\frac{x^{3/2}}{3/2}\right]_0^1$$

$$= \frac{1}{H_0}\left[\frac{1}{3/2} - 0\right] = \frac{2}{3H_0}$$

5

$t = \frac{2}{3H_0} = \frac{dist}{speed \to c=1}$ $\Rightarrow dist = \frac{2}{3H_0} = \frac{2}{3\times 70 \ km/s/Mpc}$

~~1000~~ $\frac{1000}{3.26 \times 9.46 \times 10^{15}}$

30.8

$\frac{100}{30.8 \times 10^{14}}$

~~$= \frac{2}{3} \times 3.25 \times 10^{-14} m$~~

~~$= 2.16 \times 10^{-14} m$~~

$d = \frac{2}{3 \times 3.25 \times 10^{-14} m}$

$= \frac{20}{3 \times 3.25} \times 10^{13} m$

$= 2.05 \times 10^{13} m$ Ans.

8. The definition of the horizon distance is

$$d_h = a(t_0)\int_0^{r_h} \frac{dr}{\sqrt{1-Kr^2}},$$

where $(r_h, 0, 0)$ is any point in the particle horizon, and t_0 is the present age of the universe. Show that in a radiation dominated universe $d_h = 2t_0$ (in the unit $c = 1$). How do you explain the fact that d_h is greater than the length of the path that light travels in time t_0.

[6]

as particle horizon is based on the lights $ds^2=0$ property,

$$\int_0^{r_h}\frac{dr}{\sqrt{1-Kr^2}} = \int_0^{t_0}\frac{dt}{a(t)}$$

$$\hookrightarrow ds^2 = -dt^2 + a(t)^2\left(\frac{1}{1-Kr^2}\right)dr^2 + a(t^2)(0)$$

only radial distance

$$\Rightarrow \frac{a(t)\,dr}{\sqrt{1-Kr^2}} = dt$$

∴ horizon distance

$$d_h = a(t_0)\int_0^{t_0}\frac{dt}{a(t)}$$

For radiation dominated universe,

$$a(t) = \left(2\sqrt{\frac{8\pi G \rho_0 a_0^4}{3}}\; t - t^2\right)^{1/2} \quad \text{for } K=1$$

$$t \quad \text{for } K=0$$

$$+t^2 \quad \text{for } K=-1$$

$$\therefore a(t) \propto t^{1/2}$$

Let $a(t) = t^{1/2}$

then $\int_0^{t_0} t^{-1/2}\,dt = \left.\frac{t^{1/2}}{1/2}\right]_0^{t_0}$

$= 2t_0^{1/2}$

also $a(t_0) = t_0^{1/2}$

$$\therefore d_h = a(t_0)\int_0^{t_0}\frac{dt}{a(t)} = a(t_0)\int_0^{t_0} t^{-1/2}\,dt = t_0^{1/2}\cdot 2\,t_0^{1/2} = 2t_0$$

Ans.

5

For light we would assume that $d_h = t_0$ and in this case $d_h = 2t_0$, the reason being that for horizon distance the measurements are simultaneous.

Ans

9. The number density of photons in thermal equilibrium at temperature T and between frequency ν and $\nu + d\nu$ is

$$n_T(\nu)d\nu = \frac{8\pi\nu^2 d\nu}{\exp(h\nu/k_BT) - 1}.$$

Show that the equilibrium temperature T scales as inversely with the scale factor. [5]

Given, $n_T(\nu)\,d\nu = \dfrac{8\pi\nu^2 d\nu}{\exp(h\nu/k_BT) - 1}$

There is no change in number density ~~per~~ in a comoving volume.

$\therefore n_T(\cancel{a}\,\nu,t)\,a(t^{\cancel{3}})^3\,d\nu = n_{T_L}(\nu_L,t_L)\,a(t_L)^3\,d\nu_L$

↳ Equilibrium temperature.

we know, $\dfrac{\nu_L}{\nu} = \dfrac{a(t)}{a(t_L)} = \dfrac{d\nu_L}{d\nu}$

$$\frac{\cancel{8\pi}\,\nu^2 d\nu \cdot a(t)^3}{\exp\left(\frac{h\nu}{k_BT}\right) - 1} = \frac{\cancel{8\pi}\,\nu_L^2 d\nu_L\, a(t_L)^3}{\exp\left(\frac{h\nu_L}{k_BT_L}\right) - 1}$$

now, $\dfrac{\nu^2 d\nu \cdot a(t)^3}{\nu_L^2\, d\nu_L\, a(t_L)^3} = \dfrac{\nu^2}{\nu_L^2} \cdot \dfrac{d\nu}{d\nu_L} \cdot \dfrac{a(t)^3}{a(t_L)^3} = \dfrac{a(t_L)^2}{a(t)^2} \cdot \dfrac{a(t_L)}{a(t)} \cdot \dfrac{a(t)^3}{a(t_L)} = 1$

$\therefore \exp\left(\dfrac{h\nu}{k_BT}\right) - 1 = \exp\left(\dfrac{h\nu_L}{k_BT_L}\right) - 1$

$\Rightarrow \dfrac{\nu}{T} = \dfrac{\nu_L}{T_L} \Rightarrow \dfrac{T}{T_L} = \dfrac{\nu}{\nu_L} = \dfrac{a(t_L)}{a(t)}$

$\therefore T \propto \dfrac{1}{a}$

— Ans.

5

10. Assume that the present universe is dominated by the cosmological constant $\Lambda = 8\pi G\rho_\Lambda > 0$ only, *i.e.*, $\rho_M = 0, \rho_R = 0, K = 0$. At present the temperature of the CMB is $T_0 = 2.7$K. Calculate the time (from the present epoch) when the CMB temperature will be 2K. You are given $G = 6.67 \times 10^{-11} m^3 kg^{-1} s^{-2}$, $\rho_\Lambda = \Omega_\Lambda \rho_{crit}$, $\Omega_\Lambda = 0.685$, and $\rho_{crit} = 8.5 \times 10^{-27} kg/m^3$. [5]

Given, $\Lambda = 8\pi G \rho_\Lambda > 0$ $\rho_M = \rho_R = K = 0$ $T_0 = 2.7K$ time when CMB Temp will be 2K

using friedman Equation, $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G \rho}{3} - \frac{K}{a^2}$

$$\rho = \rho_\Lambda, \quad K = 0$$

$$\Rightarrow \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G \rho_\Lambda}{3} = \frac{\Lambda}{3}$$

$$\Rightarrow \frac{\dot{a}}{a} = \sqrt{\frac{\Lambda}{3}} = \frac{da}{a\,dt}$$

2nd friedman eq $\Rightarrow$ $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho_\Lambda - 3\rho_\Lambda)$ as $P = -\rho_\Lambda$

$$= \frac{8}{3}\pi G \rho_\Lambda = \frac{\Lambda}{3}$$

conformal time $\rightarrow$ $d\eta = \frac{dt}{a}$ $\Rightarrow \eta = \int \frac{dt}{a(t)}$

$\dot{a} = \frac{da}{dt} = \frac{da}{d\eta\, a}$ $\Rightarrow \frac{\dot{a}}{a} = \frac{da}{d\eta\, a^2} = \sqrt{\frac{\Lambda}{3}}$

$\Rightarrow \frac{da}{a^2} = \sqrt{\frac{\Lambda}{3}}\, d\eta$ integrate both sides.

-2

$\Rightarrow -\frac{1}{a_f} + \frac{1}{a_i} = \sqrt{\frac{\Lambda}{3}}\, d\eta$ $a \propto T^{-1}$

$\therefore -T_f + T_i = \sqrt{\frac{\Lambda}{3}}\, d\eta = 2.7 - 2.0\,k = 0.7K$

$$\Rightarrow d\eta = 0.7 \times \sqrt{\frac{3}{\Lambda}}$$

$\Lambda = 8\pi G \rho_\Lambda$ $\Omega_\Lambda \times \rho_{crit}$

$= 1.68 \times 10^2 \times 10^{-11} \times 0.685 \times 8.5 \times 10^{-27}$

$= 796 \times 10^{-38}$

$3/\Lambda = 3.77 \times 10^{35}$ $\sqrt{\frac{3}{\Lambda}} = 6.14 \times 10^{17}$

$\Rightarrow$ time $= 0.7 \times \sqrt{\frac{3}{\Lambda}} = 4.3 \times 10^{17}$ sec

$= \frac{4.13 \times 10^{17}}{3.15 \times 10^7}$ years $= 1.3 \times 10^{10}$ years ——— Ans.

$\rightarrow$ 10 billion years

Sa^3

$S'a^3 + 3a^2 a S = 0$

$a^3 \frac{dS}{dT}$

$\frac{\dot{a}}{a} = \sqrt{\frac{\Lambda}{3}} \frac{\dot{a}}{a}$

a^{-2}

$\frac{a^{-1}}{-1}$

$-\frac{1}{a}$.

11. The number density of particles of type i at equilibrium temperature T is given by

$$n_i = g_i \left(\frac{m_i k_B T}{2\pi \hbar^2} \right)^{3/2} \exp\left(-\frac{m_i c^2}{k_B T} \right). \tag{1}$$

(a) Assuming the neutrons ($i = n$) and the protons ($i = p$) to be in thermal equilibrium, calculate the temperature and the age of the universe (since Big Bang) when n_n/n_p was 0.50. *You may use* $g_p = g_n = 2$. *Neutron and proton mass difference* $(m_n - m_p)c^2 = 1.29$ MeV $= 1.29 \times 10^6$eV, *and the ratio* $(m_n/m_p)^{3/2} = 1.002$. *The Boltzmann constant* $k_B = 8.6 \times 10^{-5}$ *eV/K. Reduced Planck constant* $\hbar = 6.582 \times 10^{-16}$ *eV.s. You may need the numerical value* $\ln(0.50/1.002) = 0.695145$. [4]

(b) After the neutrinos decouple and freeze out, the number densities of protons and neutrons evolve independently of each other. Since, the neutrons decay to protons ($n \to p + e^- \bar{\nu}$), the number density of neutrons deplete according to $n_n(t) = n_n^{\rm eq} e^{-t/\tau_n}$, where $n_n^{\rm eq}$ is the equilibrium number densities at the moment of freezeout, and $\tau_n = 880$s is the neutron lifetime. Assuming $n_n^{\rm eq}/n_p = 1/5$ at the time of freezeout, calculate the value of $X_n = n_n/(n_n + n_p)$ at time $t = 250$s. *Hint: You may need the numerical value* $e^{-250/880} = 0.752698$. [6]

(c) Assume that all the neutrons end in helium during the BBN. Calculate the ratio of number density of helium $n_{\rm He}$ to hydrogen $n_H = n_p$ at $t = 250$s and show that it is close to 0.07. *Hint:* $e^{-250/880} = 0.752698$. [3]

a) when neutrons and protons are in thermal equilibrium

$$\frac{n_n}{n_p} = \frac{g_n (m_n k_B T/2\pi\hbar^2)^{3/2} \exp(-m_n c^2/k_B T)}{g_p (m_p k_B T/2\pi\hbar^2)^{3/2} \exp(-m_p c^2/k_B T)} = \left(\frac{m_n}{m_p}\right)^{3/2} \exp\left(-\frac{c^2(m_n - m_p)}{k_B T}\right)$$

$$= 1.002 \times \exp\left(\frac{-1.29\times10^6 eV}{k_B T}\right) = \frac{1}{2} = 0.5$$

$$\Rightarrow \frac{-1.29\times10^6 eV}{k_B T} = \ln\left(\frac{0.5}{1.002}\right) = -0.695145$$

$$\Rightarrow k_B T = \cancel{\frac{0.695145}{1.29\times10^6 eV}} \quad \frac{1.29\times10^6 eV}{0.695145} = 1.86\times10^6 eV$$

4

$$\Rightarrow T = \frac{1.86\times10^6 eV}{k_B = 8.6\times10^{-5} eV/K} = 0.216\times10^{11} K$$

$$T = 2.1\times10^{10} K$$ Ans.

time $t = 0.994 \text{ sec} \left(\frac{10^{10} K}{T}\right)^2$

$$= 0.994\left(\frac{1}{2.1}\right)^2 = 0.225 \text{ sec}$$ Ans.

b) $n \to p + \bar{e} + \nu$ using saha equation, ~~$\frac{n_n}{n_n^{eq}} = \frac{n_p}{n_p^{eq}} \Rightarrow \frac{n_n^{eq} e^{-t/\tau_n}}{n_n^{eq}} = 5$ at freezeout.~~

at freezeout $\frac{n_n^{eq}}{n_p} = \frac{1}{5}$ $\Rightarrow$ ~~n_p~~

∴ num density of protons evolve as ⇒

$n_p(t) = n_p + (n_n^{eq} - n_n(t))$

↑ at freezeout

↳ no of neutrons converted to p.

$\therefore n_p(t) = 5n_n^{eq} + n_n^{eq} - n_n(t) = 6n_n^{eq} - n_n(t)$

$= n_n^{eq}(6 - e^{-t/\tau_n})$

at 250s, $X_n = \frac{n_n^{eq} e^{-250/880}}{n_n^{eq} e^{-250/880} + n_n^{eq}(6 - e^{-250/880})} = \frac{0.752698}{6} = 0.125$ Ans.

6