Communication Theory

Spring-2024

Assignment 1 Deadline: 23^{rd} January, 11:55pm

Instructions:

- All questions are compulsory.
- Clearly state the assumptions (*if any*) made that are not specified in the questions.
- Submission format: Rollnumber.pdf
- **Cautions:**
  (a) Zero marks for late submissions.
  (b) Zero marks if plagiarism is detected.

Questions

1. Consider the signal $x(t)$ defined as $x(t) = e^{-at}$ for $t \geq 0$ and 0 otherwise. What is the bandwidth required to transmit 95% of the signal?

Hint: It will be a function of ' a '.

2. Consider the tent signal $s(t) = (1 - |t|)I_{[-1,1]}(t)$.

(a) Find and sketch the Fourier transform $S(f)$.
(b) Compute the 99% energy containment bandwidth in KHz, assuming that the unit of time is milliseconds.

3. Let $x(t)$ and $y(t)$ be two periodic signals with period T_0, and let x_n and y_n denote the Fourier series coefficients of these two signals.

(a) Show that

$$\frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)y^*(t)dt = \sum_{n=-\infty}^{\infty} x_n y_n^*$$

This relation is known as Parseval's relation for the Fourier series. Show that the Rayleigh's relation for periodic signals is a special case of this relation. Rayleigh's Relation is shown below.

Rayleigh's Relation:

$$\sum_{n=-\infty}^{\infty} |x_n|^2$$

(b) Show that for all periodic physical signals that have finite power, the coefficients of the Fourier series expansion x_n tend to zero as $n \to \infty$.
(c) Use Parseval's relation in part (a) to prove the following identity.

$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \ldots + \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}$$

Hint: Find Fourier series expansion of $f(x) = x^2, x \in [-\pi, \pi]$ and then use Parseval's identity.

4. Determine the Fourier transform of each of the following signals.

(a) $\text{sinc}^3 t$
(b) t sinct
(c) $te^{-\alpha t}\cos(\beta t)$

5. Using the properties of the Fourier transform, evaluate the following integrals

(a) $\int_0^\infty e^{-\alpha t}\,\text{sinc}^2(t)dt$
(b) $\int_0^\infty e^{-\alpha t}\cos(\beta t)dt$

6. Consider the following two passband signals:

$$u_p(t) = \text{sinc}(2t)\cos(100\pi t)$$

and

$$v_p(t) = \text{sinc}(t)\sin\left(101\pi t + \frac{\pi}{4}\right)$$

(a) Find the complex envelopes $u(t)$ and $v(t)$ for u_p and v_p respectively, with respect to the frequency reference $f_c = 50$ Hz.
(b) What is the bandwidth of $u_p(t)$? What is the bandwidth if $v_p(t)$?
(c) Find the inner product $\langle u_p, v_p\rangle$ using the result in (a).
(d) Find the convolution $y_p(t) = (u_p * v_p)(t)$ using the result in (a).

7. The passband signal $u(t) = I_{[-1;1]}(t)\cos(100\pi t)$ is passed through the passband filter $h(t) = I_{[0;3]}(t)\sin(100\pi t)$. Find an explicit time-domain expression for the filter output.

8. Consider a passband signal of the form

$$u_p(t) = a(t)\cos(200\pi t)$$

where $a(t) = \text{sinc}(2t)$ and the unit of time is in microseconds.
(a) What is the frequency band occupied by $u_p(t)$?
(b) The signal $u_p(t)\cos(199\pi t)$ is passed through a lowpass filter to obtain an output $b(t)$. Give an explicit expression for $b(t)$, and sketch $B(f)$ (if $B(f)$ is complex-valued, sketch its real and imaginary parts separately).
(c) The signal $u_p(t)\sin(199\pi)t$ is passed through a lowpass filter to obtain an output $c(t)$. Give an explicit expression for $c(t)$, and sketch $C(f)$ (if $C(f)$ is complex-valued, sketch its real and imaginary parts separately).
(d) Can you reconstruct $a(t)$ from simple real-valued operations performed on $b(t)$ and $c(t)$? If so, sketch a block diagram for the operations required. If not, say why not.