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Notes on: Ultrafilters

1 Introduction

These notes are a general introduction to filters and ultrafilters on sets. Some material here is from [BRV01], particularly the exercises in section 2.5. We begin with some definitions and elementary properties of filters and ultrafilters. Then, we finish with the ultraproduct constrution on models of first order logic, and Łos's theorem.

2 Filters

Let S be a set.

Definition 1. A *filter* on S is a nonempty subset $E \subseteq \mathcal{P}S$ such that:

- (Non-empty) The set S is in E
- ($\cap$-closed) For every $x, y \in E$ we have $x \cap y \in E$
- (Upwards closed) When $x \in E$ and $y \in P$ with $x \subseteq y$ then $y \in E$.

Call a filter *proper* if it doesn't contain $\emptyset$ (for if it does then by upwards-closedness it will contain every subset of S).

Remark 2. We can think of a filter on S as choosing a collection of sets that are "large". The conditions then say that S itself is large, the intersection of two large sets is large, and any superset of a large set is also large.

An interesting property of filters is that they are closed under intersection.

Proposition 3. *Let $\{E_i\}$ be a collection of filters. Then $E = \bigcap E_i$ is a filter.*

Proof. We check each condition.

- We have $S \in E_i$ for all i.
- Let $x, y \in E$. So for all i, $x \cap y \in E_i$
- Let $x \in E$ and $x \subseteq y$. Then $y \in E_i$ for all i.

□

Let $T \subseteq \mathcal{P}S$.

Definition 4. The *filter generated by* T, denoted $\langle T \rangle$ is the smallest filter on S that contains T as a subset. Note that this always exists, for $\mathcal{P}S$ is a filter.

Remark 5. By Proposition 3 we can characterize the filter generated by T as

$$\langle T \rangle = \bigcap_{\substack{F \text{ a filter on } S \\ T \subseteq F}} F.$$

In particular, if T is a filter, then $\langle T \rangle = T$.

In fact, we have another characterization of the filter generated by T.

Proposition 6. *The elements of $\langle T \rangle$ the subsets x of S such that contain a finite intersection of sets in T— that is, for some $y_1, \dots y_n \in T$ we have $\bigcap y_i \subseteq x$.*

Proof. Let F be the set of all $x \subseteq S$ with $x = S$ or $x \supseteq \bigcap y_i$ for some $y_1, \dots y_n \in T$. We will first show that F is a filter containing T.

- Clearly, $S \in F$.
- Suppose $x, y \in F$. So there are $y_1, \dots y_n, z_1, \dots z_m \in T$ with $\bigcap y_i \subseteq x$ and $\bigcap z_i \subseteq y$. But now we see that $(\bigcap y_i) \cap (\bigcap z_i) \subseteq x \cap y$, so $x \cap y \in F$.
- Upwards-closedness is obvious.

So we deduce that $\langle T \rangle \subseteq F$. The final step is to show the converse, that $F \subseteq \langle T \rangle$.

So let x contain a finite intersection of sets $y_1, \dots y_n \in T$. Let F' be a filter that contains T. As F' is closed under finite intersections we see that $\bigcap y_i \in F'$. As F' was an arbitrary filter containing T, it follows that $\bigcap y_i \in F$ too. And, by upwards-closedness we see that $x \in F$. □

Definition 7. A collection of sets T has *the finite intersection property (FIP)* if no finite intersection of sets in T is empty.

From Proposition 6 it follows that $\langle T \rangle$ is proper if and only if T has the FIP. In particular, every proper filter itself has the FIP.

3 Ultrafilters

Now we turn our attention to a special sort of filter.

Definition 8. An *ultrafilter* on S is a filter that is maximal among the proper filters. That is, a filter E is an ultrafilter if it is proper and there is no other proper filter F with $E \subset F$.

Proposition 9. *A filter E is an ultrafilter if and only if for every subset x of S we have*

$$x \in E \iff S \setminus x \notin E.$$

Proof. Let E be an ultrafilter, and suppose $x \subseteq S$ is not in E. We will show that $S \setminus x \in E$. To that end, let $G = \langle E \cup \{x\} \rangle$. Since E is a proper subset of G, we know that $G = \mathcal{P}S$. In particular, $\emptyset \in G$, so we may express it as a finite intersection of sets $y_1, \ldots y_n \in E \cup \{x\}$. Because E is proper, we know that no finite intersection of sets in E can be empty. So, WLOG, $y_n = x$. So we see that $y_1 \cap \ldots \cap y_{n-1} \subseteq S \setminus x$, and hence $S \setminus x \in E$, as desired.

For the other direction, suppose that E is a filter and for every $x \in S$, $x \in E$ if and only if $S \setminus x \notin E$.

Let $x \notin E$. We show that any filter extension of E containing x is in fact $\mathcal{P}S$. The smallest filter that contains E and x is the filter F generated $E \cup \{x\}$. Note that $S \setminus x \in E$, so both x and $S \setminus x$ are in F. Hence $\emptyset \in F$ and in fact $F = \mathcal{P}S$.

□

We said in Remark 2 that a filter is choosing a collection of "large" sets. An ultrafilter then, says that every set is either large or co-large. Yet another characterization of ultrafilters is the following.

Proposition 10. *A filter E is an ultrafilter if and only if it is proper, and for all $x, y \subseteq S$ we have*

$$x \cup y \in E \iff x \in E \text{ or } y \in E..$$

Proof. Let E be an ultrafilter. Suppose that $x \cup y \in E$ and neither x nor y are themselves in E. So by Proposition 9 we have that $S \setminus x, S \setminus y \in E$ and hence the intersection $(x \cup y) \cap (S \setminus x) \cap (S \setminus y)$ is also in E. But this is the empty set!. On the other hand, suppose that $x \in E$. Must we have $x \cup y \in E$? If not, we have $S \setminus (x \cup y) = (S \setminus x) \cap (S \setminus y) \in E$, and hence $x \cap (S \setminus x) \cap (S \setminus y) = \emptyset \in E$

For the converse, let E be a filter with the above property, and x a subset of S. Since $x \cup S \setminus x = S \in E$ we know that at either one of x and $S \setminus x$ are in E. And, they can't both be in E for E is proper. So E contains exactly one of x and $S \setminus x$ for any set x, as desired. □

We may summarise the situation like so:

Theorem 11 (Ultrafilter Theorem)**.** *Let E be a proper filter on S. The following are equivalent:*

- *The filter E is maximal among the proper filters—that is, whenever F is a proper filter containing E we have $E = F$*
- *For every subset x of S precisely one of x and $S \setminus x$ are in E*
- *For every $x, y \subseteq S$ we have $x \cup y \in E$ if and only if $x \in E$ or $y \in E$*

A well known result is the following.

Theorem 12. *Every proper filter is contained in an ultrafilter.*

As a corollary, every set with the FIP can be extended to an ultrafilter.

I have not yet seen a satisfactory proof of Theorem 12, so we will now prove it in great detail. The main tool is Zorn's Lemma (Lemma 15), for which we will need to introduce some terminology.

Definition 13. Let $(P, \leq)$ be a poset.

A *chain in* P is a subset $\mathfrak{C}$ of P with the property that for every $x, y \in \mathfrak{C}$ we have either $x \leq y$ or $y \leq x$.

Suppose $T \subseteq P$. An element x of P is an *upper bound for* T if for every $y \in Y$ we have $y \leq x$. We call an element $x \in P$ *maximal* if whenever we have $y \in P$ with $x \leq y$ it turns out that $x = y$.

Remark 14. We are all familiar with finite or even countable chains of (e.g.) sets:

$$x_1 \subseteq x_2 \subseteq \ldots \subseteq x_n \subseteq \ldots.$$

But in general, chains can be bigger. Also, by our definition a chain may be empty, which is why in Lemma 15 we stipulate that the chains in question are non-empty.

Now we will recall Zorn's lemma.

Lemma 15 (Zorn)**.** *Let P be a non-empty poset with the property that every non-empty chain in P has an upper bound in P. Then P has a maximal element.*

Remark 16. Somewhat worryingly, Zorn's lemma is equivalent to the axiom of choice. We will attempt to ignore this, and any other set-theoretic intricacies.

We are now ready to prove Theorem 12.

Proof of Theorem 12. Let E be a proper filter on S. Take P to be the poset of proper filters on S that contain E, ordered by inclusion. Note that $E \in P$, so P is non-empty.

We first show that every non-empty chain in P has an upper bound. So, let $\mathfrak{C}$ be a chain in P. Let $C = \bigcup \mathfrak{C}$. We must first demonstrate that C is a proper filter containing E.

- The chain $\mathfrak{C}$ is non-empty, so it contains some filter F. As $S \in F$, we have $S \in C$.
- Suppose $x, y \in C$, so $x \in F$ and $y \in G$ for some $F, G \in \mathfrak{C}$. WLOG we assume that $F \subseteq G$, so $x, y \in G$. Hence the intersection $x \cap y$ is in G, and in C too.
- Suppose $x \in C$ and $x \subseteq y$. So $x \in F$ for some $F \in \mathfrak{C}$. Because F is a filter, $y \in F \subseteq C$.
- Can we have $\emptyset \in C$? No, because C is a union of proper filters, none of which can contain $\emptyset$.

So C is in P. It is also clear that C is an upper bound for $\mathfrak{C}$, for every filter F in $\mathfrak{C}$ is a subset of $C = \bigcup \mathfrak{C}$.

Hence we may apply Zorn's lemma to P and take U to be a maximal element. This means that whenever F is a proper filter containing E and U, we have $F = U$. Note that $E \subseteq U$, so F contains E and U if and only if it contains U. So, by Theorem 11 we see that U is indeed an ultrafilter.

As U contains E, we are finished. $\square$

Two concrete examples of filter are the trivial $\mathcal{P}S$ and $\{S\}$. What do other filters and ultrafilters look like? One interesting and easy to describe class is the *principal ultrafilters.*

Definition 17. Let $a \in S$. The *principal ultrafilter (on S) generated by a* is the ultrafilter $\pi_a = \{x \subseteq S : a \in x\}$.

It is easy to see that π_a is indeed an ultrafilter.

Proposition 18. *If S is finite, all ultrafilters on S are principal.*

Proof. Let S be a finite set and U an ultrafilter on S. We know that $\bigcap U \neq \emptyset$, for U itself is finite. Hence let $a \in \bigcap U$. This means that every $x \in U$ contains a, so $U \subseteq \pi_a$. On the other hand, let x be a set containing a. If x is not itself in U then we must have $S \setminus x \in U$, but $S \setminus x$ doesn't contain a!.

It follows that $U = \pi_a$, as desired. $\square$

Now, suppose that S is infinite. In this case, we see that

Proposition 19. *The collection of cofinite subsets of S has the FIP.*

Proof. Let x, y be cofinite. Note that $S \setminus (x \cap y) = (S \setminus x) \cup (S \setminus y)$, which, being a union of two finite sets, is finite. So any finite intersection of cofinite sets is itself cofinite. Finally, note that $\emptyset$ is not cofinite, as S is infinite. $\square$

Remark 20. In fact, the set of all cofinite subsets of S is itself a filter on S, sometimes called the *Fréchet filter.*

We now have the following result

Proposition 21. *Let U be an ultrafilter on S. The following are equivalent:*

- *U is non-principal*
- *U contains only infinite sets*
- *U contains all cofinite sets*

Proof. First, suppose that U is non-principal, so it doesn't contain any singleton sets. We also know (Theorem 11) that if $x, y \notin U$ then $x \cup y \notin U$. Since a finite set is a finite union of singletons, no finite set can be in U.

Now, suppose that U contains only infinite sets. So, for any finite set x we have $x \notin U$. Hence, $S \setminus x \in U$. So U contains every cofinite set.

Finally, suppose that U contains all cofinite sets. For the sake of contradiction, suppose further that $U = \pi_a$. Hence $\{a\} \in U$, but $S \setminus \{a\} \in U$ too, for it is cofinite. Then we see that $\emptyset \in U$, which is impossible. So U must not be principal. $\square$

4 Ultraproducts

Let A_i be a family of sets over I, U an ultrafilter over I, and write $C = \Pi_I A_i$. The elements of C are functions $f : I \to \bigcup_I A_i$ such that for all i, $f(i) \in A_i$.

We define an equivalence relation $\sim_U$ on C by saying that $f \sim_U g$ if they agree on a large set, that is, if

$$\{i \in I : f(i) = g(i)\} \in U.$$

It is easy to see that $\sim_U$ is indeed an equivalence relation. We denote its equivalence classes as $[f]$.

Definition 22. The *ultraproduct of the sets A_i modulo U* is the quotient of C by $\sim_U$, and is denoted $\Pi_U A_i$.

We can also define an ultraproduct on families of other structures. Let $\mathcal{L}$ be a first order language, $\mathfrak{M}_i$ an I-indexed family of models of $\mathcal{L}$, and U an ultrafilter on I.

Definition 23. The *ultraproduct of the models $\mathfrak{M}_i$ modulo U* is the model $\Pi_U \mathfrak{M}_i$ where:

- The universe of $\Pi_U \mathfrak{M}_i$ is the ultraproduct $\Pi_U A_i$, where A_i is the universe of $\mathfrak{M}_i$.
- Suppose c is a constant symbol, interpreted as $c_i \in \mathfrak{M}_i$. Then c is interpreted in $\Pi_U \mathfrak{M}_i$ as the equivalence class of $i \mapsto c_i$.
- Suppose F is an n-ary function symbol, interpreted as $F_i : A_i^n \to A_i$ in $\mathfrak{M}_i$. Then F is interpreted in $\Pi_U \mathfrak{M}_i$ as the function

$$\begin{aligned} &F_U : (\Pi_U A_i)^n \to \Pi_U A_i \\ &F_U([f_1], \ldots, [f_n])(i) = F_i(f_1(i), \ldots, f_n(i)) \end{aligned}$$

- Suppose that R is an n-ary relation, interpreted as $R_i \subseteq A_i^n$ in each $\mathfrak{M}_i$. Then R is interpreted in $\Pi_U \mathfrak{M}_i$ as the relation R_U on $(\Pi_U A_i)^n$, where

$$(f_1, \ldots f_n) \in R_U \iff \{i \in I : (f_1(i), \ldots, f_n(i)) \in R_i\} \in U.$$

When all the components M_i are equal, we call this the *ultrapower*.

Remark 24. Of course we ought to check that this definition doesn't depend on the representatives of the equivalence classes. This is routine and uninteresting.

The main result is a characterization of the semantics of an ultraproduct.

Theorem 25 (Łos's Theorem)**.** *Let $\mathfrak{M}_i$ be a family of models over I and U an ultrafilter on I. For any formula $\phi(x_1, \ldots, x_n)$ and $[f_1], \ldots, [f_n] \in \Pi_U \mathfrak{M}_i$ we have*

$$\Pi_U \mathfrak{M}_i \vDash \phi([f_1], \ldots, [f_n]) \iff \{i \in I : \mathfrak{M}_i \vDash \phi(f_1(i), \ldots, f_n(i))\} \in U.$$

Remark 26. That is, a formula holds in the ultraproduct if it holds in a "large" set of component models.

Proof. The proof is by induction on ϕ. For clarity, we will assume that ϕ is a formula of a single variable. This changes nothing.

- The atomic case is clear
- Suppose $\phi(x) = \neg\psi(x)$. Then we have

$$\begin{aligned} \Pi_U \mathfrak{M}_i \vDash \phi([f],) &\iff \Pi_U \mathfrak{M}_i \nvDash \psi([f]) \\ &\iff \{i \in I : \mathfrak{M}_i \vDash \psi(f(i))\} \notin U \\ &\iff \{i \in I : \mathfrak{M}_i \nvDash \psi(f(i))\} \in U \\ &\iff \{i \in I : \mathfrak{M}_i \vDash \phi(f(i))\} \in U \end{aligned}$$

- Suppose $\phi(x) = \psi_1(x) \vee \psi_2(x)$. Then we have

$$\begin{aligned} \Pi_U \mathfrak{M}_i \vDash \phi([f]) &\iff \Pi_U \mathfrak{M}_i \vDash \psi_1([f]) \text{ or } \Pi_U \mathfrak{M}_i \vDash \psi_2([f]) \\ &\iff \{i \in I : \mathfrak{M}_i \vDash \psi_1(f(i))\} \in U \text{ or } \{i \in I : \mathfrak{M}_i \vDash \psi_2(f(i))\} \in U \\ &\iff \{i \in I : \mathfrak{M}_i \vDash \psi_1(f(i))\} \cup \{i \in I : \mathfrak{M}_i \vDash \psi_2(f(i))\} \in U \\ &\iff \{i \in I : \mathfrak{M}_i \vDash \psi_1(f(i)) \vee \psi_2(f(i))\} \in U \end{aligned}$$

  The third line follows from Theorem 11.

- Finally, suppose that $\phi(x) = \exists y.\psi(y, x)$. Then, we see that $\Pi_U \mathfrak{M}_i \vDash \phi([f])$ if and only if for some g we have $\Pi_U \mathfrak{M}_i \vDash \psi([g], [f])$, which we know happens precisely when $\{i \in I : \mathfrak{M}_i \vDash \psi(g(i), f(i))\} \in U$.

  So, suppose that $\{i \in I : \mathfrak{M}_i \vDash \psi(g(i), f(i))\}$ is indeed in U. By definition, we have $\{i \in I : \mathfrak{M}_i \vDash \phi(f(i))\}$ in U too.

  On the other hand, suppose that $J = \{i \in I : \mathfrak{M}_i \vDash \phi(f(i))\} \in U$. So, for every $i \in J$ we have an element y_i in $\mathfrak{M}_i$ such that

$$\mathfrak{M}_i \vDash \psi(y_i, f(i)).$$

  Hence we can define an element $[g]$ of $\Pi_U \mathfrak{M}_i$ that has $g(i) = y_i$ for $i \in J$. For i not in J we can pick any element of $\mathfrak{M}_i$. Now we know that $\{i \in I : \mathfrak{M}_i \vDash \psi(g(i), f(i))\}$ contains J. It may not be equal to J—for we might have $\mathfrak{M}_i \vDash \psi(g(i), f(i))$ for some other i, depending on the value of $g(i)$—but it is certainly in U. So $\Pi_U \mathfrak{M}_i \vDash \psi([g], [f])$, and the result follows.

$\square$

References

[BRV01] Patrick Blackburn, Maarten de Rijke and Yde Venema. *Modal Logic*. New York, NY, USA: Cambridge University Press, 2001. ISBN: 0-521-80200-8.