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29th September 2019
Last modified: 28th December 2019

Notes on: The Yoneda Lemma

Let $\mathcal{C}$ be a category. We write $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ for the category of contravariant functors $\mathcal{C} \to \mathbf{Set}$. This is also denoted as $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$, or *the category of presheaves over* $\mathcal{C}$.

We can define the Yoneda embedding, a functor $Y : \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ that takes:

- objects A of $\mathcal{C}$ to the functor $\mathrm{Hom}(-, A) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$
- arrows $f : A \to B$ to the natural transformation $\mathrm{Hom}(-, f) : \mathrm{Hom}(-, B) \implies \mathrm{Hom}(-, A)$, which has components:

$$\begin{aligned} \mathrm{Hom}(-, f)_C : \mathrm{Hom}(C, B) &\to \mathrm{Hom}(C, A) \\ (g : C \to B) &\mapsto f \circ g : C \to A \end{aligned}$$

  For convenience, we may be use $(f \circ -)$ to denote both the components and the natural transformation itself.

Note that for any C in $\mathcal{C}$, the hom-functor $\mathrm{Hom}(-, C)$ sends arrows $f : A \to B$ to the function

$$\begin{aligned} \mathrm{Hom}(f, C) : \mathrm{Hom}(B, C) &\to \mathrm{Hom}(A, C) \\ (g : B \to C) &\mapsto g \circ f \end{aligned}$$

which we may write as $(- \circ f)$, and is not to be confused with the *natural transformation* $(f \circ -)$ above.

Now, let $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ be a presheaf on $\mathcal{C}$.

Lemma 1 (Yoneda)**.** *For every object C of $\mathcal{C}$ there is a canonical isomorphism*

$$\mathrm{Hom}(YC, F) \cong FC$$

Proof. Fix an object C of $\mathcal{C}$. Note that YC, F are themselves presheaves on $\mathcal{C}$, so the elements of the homset on the left are actually natural transformations. So we can equivalently write:

$$\mathrm{Nat}(\mathrm{Hom}(-, C), F) \cong FC.$$

We begin by defining a function

$$\begin{aligned} &\varphi : \mathrm{Nat}(\mathrm{Hom}(-, C), F) \to FC \\ &\varphi(\eta) = \eta_C(\mathrm{id}_C) \end{aligned}$$

Next we verify that φ is injective. Suppose that $\varphi(\eta) = \varphi(\theta)$, so that

$$\eta_C(\mathrm{id}) = \theta_C(\mathrm{id}).$$

We want to show that $\eta_B = \theta_B : \mathrm{Hom}(B, C) \to FB$ for any object B of $\mathcal{C}$. That is, that for any $f : B \to C$ we have $\eta_B(f) = \theta_B(f)$. So pick an arrow $f : B \to C$.

Now we form the naturality square of η:

$$\begin{array}{ccc} \mathrm{Hom}(C, C) & \xrightarrow{(-\circ f)} & \mathrm{Hom}(B, C) \\ \downarrow{\scriptstyle \eta_C} & & \downarrow{\scriptstyle \eta_B} \\ FC & \xrightarrow{\quad Ff \quad} & FB \end{array}$$

If we follow $\mathrm{id}_C \in \mathrm{Hom}(C, C)$ around both ways we see that:

$$\begin{aligned} \eta_B \circ (f \circ -)(\mathrm{id}) &= Ff \circ \eta_C(\mathrm{id}) \\ \eta_B(f) &= Ff(\eta_C(\mathrm{id})) \end{aligned}$$

By forming the naturality square for θ we similarly see that

$$\begin{aligned} \theta_B(f) &= Ff(\theta_C(\mathrm{id})) \\ &= Ff(\eta_C(\mathrm{id})) \\ &= \eta_B(f) \end{aligned}$$

Therefore $\eta = \theta$, as desired, and we see that φ is indeed injective.

The final step is to show that φ is surjective. Pick an element x of FC. We want to find an η such that

$$\begin{aligned} \varphi(\eta) &= x \\ \eta_C(\mathrm{id}) &= x \end{aligned}$$

We define η by

$$\eta_B(f) = Ff(x).$$

For if $f : B \to C$ then $Ff : FC \to FB$, and so $Ff(x)$ is an element of FB. So we have defined a family of arrows $\eta_B : \mathrm{Hom}(B, C) \to FB$. It just remains to verify that η is natural.

Let $f : A \to B$, and form the naturality square:

$$\begin{array}{ccc} \mathrm{Hom}(B, C) & \xrightarrow{(-\circ f)} & \mathrm{Hom}(A, C) \\ \downarrow{\scriptstyle \eta_B} & & \downarrow{\scriptstyle \eta_A} \\ FB & \xrightarrow{\quad Ff \quad} & FA \end{array}$$

Pick a g in $\mathrm{Hom}(B,C)$, so $g : B \to C$, and follow it around the square:

$$\begin{aligned} Ff \circ \eta_B(g) &= Ff(Fg(x)) \\ &= F(g \circ f)(x) && \text{for } F \text{ is contravariant} \\ &= \eta_A(g \circ f) \\ &= \eta_A \circ (- \circ f)(g) \end{aligned}$$

Hence we see that the square commutes and η is indeed natural. Therefore φ is surjective, and in fact bijective, which is what we wanted to show.

□

As a corollary, we see that

Proposition 2. *The Yoneda functor is in fact an embedding (fully faithful functor).*

Proof. We want to show that for every A, B of $\mathcal{C}$ the map

$$\begin{aligned} Y : \mathrm{Hom}(A,B) &\to \mathrm{Nat}(YA, YB) \\ (f : A \to B) &\mapsto (f \circ -) : YA \implies YB \end{aligned}$$

is a bijection.

By above we have an iso:

$$\begin{aligned} \varphi : \mathrm{Nat}(YA, YB) &\to (YB)A \\ &= (\mathrm{Hom}(-,B))A \\ &= \mathrm{Hom}(A,B) \end{aligned}$$

Suppose $f : A \to B$. Observe that:

$$\begin{aligned} \varphi \circ Y(f) &= Yf(\mathrm{id}) \\ &= (f \circ -)(\mathrm{id}) \\ &= f \end{aligned}$$

So we see that in fact $\varphi \circ Y = \mathrm{id} : \mathrm{Hom}(A,B) \to \mathrm{Hom}(A,B)$. Therefore (as we know φ to be a bijection) it follows that Y is a bijection, as desired. □