1. (2) $f(x) = x^4 - 2x + 5$, $g(x) = x^2 - x + 2$

$$x^2 + x - 1$$

$$x^2 - x + 2 \enclose{longdiv}{x^4 + 0x^3 + 0x^2 - 2x + 5}$$

$$x^4 - x^3 + 2x^2$$

$$x^3 - 2x^2 - 2x$$

$$x^3 - x^2 + 2x$$

$$-x^2 - 4x + 5$$

$$-x^2 + x - 2$$

$$-5x + 7$$

(3) $f(x) = 2x^5 - 5x^3 - 8x$, $g(x) = x + 3$

$$2x^4 - 6x^3 + 13x^2 - 39x + 109$$

$$x + 3 \enclose{longdiv}{2x^5 + 0x^4 - 5x^3 + 0x^2 - 8x}$$

$$2x^5 + 6x^4$$

$$-6x^4 - 5x^3$$

$$-6x^4 - 18x^3$$

$$13x^3 + 0x^2$$

$$13x^2 + 39x^2$$

$$-39x^2 - 8x$$

$$-39x^2 - 117x$$

$$109x$$

$$109x + 327$$

$$-327$$

2. (1) $f(x) = x^4 + x^3 - 3x^2 - 4x - 1$, $g(x) = x^3 + x^2 - x - 1$

$$
\begin{array}{r|l}
 & x \\
\hline
x^3+x^2-x-1 & x^4+x^3-3x^2-4x-1 \\
 & x^4+x^3-x^2-x \\
\cline{2-2}
 & -2x^2-3x-1
\end{array}
$$

$f(x) = x \cdot g(x) + r_1(x)$

$g(x) = (-\frac{1}{2}x + \frac{1}{4}) r_1(x) - \frac{3}{4} r_2(x)$

$r_1(x) = (-2x-1) r_2(x)$

$$
\begin{array}{r|l}
 & -\frac{1}{2}x+\frac{1}{4} \\
\hline
-2x^2-3x-1 & x^3+x^2-x-1 \\
 & x^3+\frac{3}{2}x^2+\frac{1}{2}x \\
\cline{2-2}
 & -\frac{1}{2}x^2-\frac{3}{2}x-1 \\
 & -\frac{1}{2}x^2-\frac{3}{4}x-\frac{1}{4} \\
\cline{2-2}
 & -\frac{3}{4}x-\frac{3}{4}
\end{array}
$$

$(f(x), g(x)) = r_2(x) = x+1$

$$
\begin{array}{r|l}
 & -2x-1 \\
\hline
x+1 & -2x^2-3x-1 \\
 & -2x^2-2x \\
\cline{2-2}
 & -x-1 \\
 & -x-1 \\
\cline{2-2}
 & 0
\end{array}
$$

(2) $f(x) = x^4 - 4x^3 + 1$, $g(x) = x^3 - 3x^2 + 1$

$$
\begin{array}{r|l}
 & x \\
\hline
x^3-3x^2+1 & x^4-4x^3+0x^2+0x+1 \\
 & x^4-3x^3+\quad x \\
\cline{2-2}
 & -x+1
\end{array}
$$

$(f(x), g(x)) = (g(x), r(x)) = 1$

$$
\begin{array}{r|l}
 & -x^2+2x+2 \\
\hline
-x+1 & x^3-3x^2+0x+1 \\
 & x^3-x^2 \\
\cline{2-2}
 & -2x^2 \\
 & -2x^2+2x \\
\cline{2-2}
 & -2x+1 \\
 & -2x+2 \\
\cline{2-2}
 & -1
\end{array}
$$

3. $\quad (f(x), g(x)) = u(x) f(x) + v(x) g(x)$

(1) $f(x) = x^4 + 2x^3 - x^2 - 4x - 2$, $g(x) = x^4 + x^3 - x^2 - 2x - 2$

$$f(x) = g(x) + x^3 - 2x = g(x) + r_1(x)$$

$$\begin{array}{r|l} & \qquad\qquad x+1 \\ x^3-2x & x^4+x^3-x^2-2x-2 \\ & x^4 \quad -2x^2 \\ \hline & \quad x^3+x^2-2x-2 \\ & \quad x^3 \quad -2x \\ \hline & \qquad x^2-2 \end{array}$$

$g(x) = (x+1)\, r_1(x) + x^2 - 2 = (x+1)\, r_1(x) + r_2(x)$

$r_1(x) = x\, r_2(x)$

$\Rightarrow (f(x), g(x)) = r_2(x)$

$= g(x) - (x+1)\, r_1(x)$

$= g(x) - (x+1)\,(f(x) - g(x))$

$= (x+2)\, g(x) - (x+1) f(x)$

$$\begin{array}{r|l} & \quad x \\ x^2-2 & x^3-2x \\ & x^3-2x \\ \hline & \quad 0 \end{array}$$

$$\Rightarrow \begin{cases} u(x) = -(x+1) \\ v(x) = x+2 \end{cases}$$

(3) $f(x) = x^4 - x^3 - 4x^2 + 4x + 1$, $g(x) = x^2 - x - 1$

$$\begin{array}{r|l} & \qquad x^2 \quad -3 \\ x^2-x-1 & x^4-x^3-4x^2+4x+1 \\ & x^4-x^3-x^2 \\ \hline & \qquad -3x^2+4x+1 \\ & \qquad -3x^2+3x+3 \\ \hline & \qquad\qquad x-2 \end{array}$$

$f(x) = (x^2-3)\, g(x) + (x-2) = (x^2-3)\, g(x) + r_1(x)$

$g(x) = (x+1)\, r_1(x) + 1 = (x+1)\, r_1(x) + r_2(x)$

$\Rightarrow (f(x), g(x)) = r_2(x)$

$= g(x) - (x+1)\, r_1(x)$

$= g(x) - (x+1)\left[f(x) - (x^2-3)\, g(x) \right]$

$= -(x+1) f(x) + \left[(x+1)(x^2-3) + 1 \right] g(x)$

$$\begin{array}{r|l} & \quad x+1 \\ x-2 & x^2-x-1 \\ & x^2-2x \\ \hline & \quad x-1 \\ & \quad x-2 \\ \hline & \qquad 1 \end{array}$$

$$\Rightarrow \begin{cases} u(x) = -(x+1) \\ v(x) = x^3 + x^2 - 3x - 2 \end{cases}$$

9. (1) $f(x) = x^5 - 5x^4 + 7x^3 - 2x^2 + 4x - 8$

$f'(x) = 5x^4 - 20x^3 + 21x^2 - 4x + 4$

$$\frac{1}{5}x - \frac{1}{5}$$

$$5x^4 - 20x^3 + 21x^2 - 4x + 4 \;\big)\; x^5 - 5x^4 + 7x^3 - 2x^2 + 4x - 8$$

$$x^5 - 4x^4 + \frac{21}{5}x^3 - \frac{4}{5}x^2 + \frac{4}{5}x$$

$$-x^4 + \frac{14}{5}x^3 - \frac{6}{5}x^2 + \frac{16}{5}x - 8$$

$$-x^4 + 4x^3 - \frac{21}{5}x^2 + \frac{4}{5}x - \frac{4}{5}$$

$$-\frac{6}{5}x^3 + 3x^2 + \frac{12}{5}x - \frac{36}{5}$$

$$-\frac{5}{2}x + \frac{15}{4}$$

$$-2x^3 + 5x^2 + 4x - 12 \;\big)\; 5x^4 - 20x^3 + 21x^2 - 4x + 4$$

$$5x^4 - \frac{25}{2}x^3 - 10x^2 + 30x$$

$$-\frac{15}{2}x^3 + 31x^2 - 34x + 4$$

$$-\frac{15}{2}x^3 + \frac{75}{4}x^2 + 15x - 45$$

$$\frac{49}{4}x^2 - 49x + 49$$

$$-2x - 3$$

$$x^2 - 4x + 4 \;\big)\; -2x^3 + 5x^2 + 4x - 12$$

$$-2x^3 + 8x^2 - 8x$$

$$-3x^2 + 12x - 12$$

$$-3x^2 + 12x - 12$$

$$0$$

$\Rightarrow (f(x), f'(x)) = (x^2 - 4x + 4) \neq 1 \Rightarrow$ 有重因式

10. x^3+px+q 有重根的条件

$f'(x)=3x^2+p$

$$3x^2+p \;\big)\; x^3+px+q \quad \text{商 } \frac{1}{3}x$$

$$x^3+\frac{p}{3}x$$

$$\frac{2}{3}px+q$$

$$\frac{2}{3}px+q \;\big)\; 3x^2+p \quad \text{商 } \frac{9}{2p}x$$

$$3x^2+\frac{9}{2}\cdot\frac{q}{p}x$$

$$-\frac{9}{2}\cdot\frac{q}{p}x+p$$

① $\frac{2}{3}px+q=0 \Rightarrow p=q=0$

② $-\frac{9}{2}\frac{q}{p}x+p=0$ ，无解

③ $\frac{2}{3}px+q=k\left(-\frac{9}{2}\frac{q}{p}x+p\right)$

$$\frac{2}{3}p^2=-\frac{9}{2}\frac{q}{p}\cdot q$$

$$-4p^3=27q^2$$

综上，有重根的条件为 $4p^3+27q^2=0$

12. $\mathbb{Z}_5[x]$ 中，$f(x)=x^{214}+3x^{152}+2x^{47}+1$ ，求 $f(3)$

$3^0=1$ ，$3^1=3$，$3^2=4$ ，$3^3=2$

$3^4=1$ ，…

$3^{214}=3^{53\times4+2}=3^2=4$

$3^{152}=3^{38\times4}=3^0=1$

$3^{47}=3^{11\times4+3}=3^3=2$

$$\Rightarrow f(3)=3^{214}+3\times3^{152}+2\times3^{47}+1$$
$$=4+3\times1+2\times2+1$$
$$=2$$