On the analytical representability of so-called arbitrary functions of a real variable

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Abstract

A crude translation of Weierstrass' original two papers titled *Über die analytische Darstellbarkeit sogenannter willkürlicher Functionen einer reellen Veränderlichen.*

First paper (*Erste Mitteilung*)

(7/9/1885)

Let $f(x)$ be a real continuous function uniquely defined for every $x \in \mathbb{R}$ and suppose $|f(x)|$ has a finite upper bound for all x. Then the following equation holds, where u is another real-valued variable and k is a positive quantity independent of x and u:

$$\lim_{k\to 0} \frac{1}{k\sqrt{\pi}} \int_{-\infty}^{+\infty} f(u) e^{-\left(\frac{u-x}{k}\right)^2} du = f(x). \tag{1}$$

This statement is easy to generalize. Suppose $\psi(x)$ is any function with the same properties as $f(x)$ such that $\psi(x)$ does not change signs, satisfies $\psi(x) = \psi(-x)$, and meets the condition $\omega := \int_0^{+\infty} \psi(x)dx$ is finite. Define $F(x, k)$ to be

$$F(x,k) = \frac{1}{2k\omega} \int_{-\infty}^{+\infty} f(u)\psi\left(\frac{u-x}{k}\right) du. \tag{2}$$

Then

$$\lim_{k\to 0} F(x,k) = f(x). \tag{3}$$

To prove Equations (1) and (3), the following observation needs to be made.

Let a_1, a_2, b_1, b_2 be positive quantities such that $b_1 > a_1$ and $b_2 > a_2$, then

$$\begin{aligned}
&\frac{1}{k}\int_{-b_1}^{b_2} f(u)\psi\left(\frac{u-x}{k}\right)du - \frac{1}{k}\int_{-a_1}^{a_2} f(u)\psi\left(\frac{u-x}{k}\right)du \\
&= \frac{1}{k}\int_{-b_1}^{-a_1} f(u)\psi\left(\frac{u-x}{k}\right)du + \frac{1}{k}\int_{a_2}^{b_2} f(u)\psi\left(\frac{u-x}{k}\right)du \\
&= f(-b_1\cdots-a_1)\int_{\frac{a_1+x}{k}}^{\frac{b_1+x}{k}} \psi(u)du + f(a_2\cdots b_2)\int_{\frac{a_2-x}{k}}^{\frac{b_2-x}{k}} \psi(u)du.[1]
\end{aligned}$$

Related to the assumptions made for the functions $f(x)$ and $\psi(x)$, this equation implies that, if one assigns specific values to x, k and let a_1, a_2 become arbitrarily large independent of each other, the integral

$$\frac{1}{k}\int_{-a_1}^{a_2} f(u)\psi\left(\frac{u-x}{k}\right)du$$

approaches a certain finite limit value. Therefore, the integral

$$\frac{1}{k}\int_{-\infty}^{+\infty} f(u)\psi\left(\frac{u-x}{k}\right)du$$

has a well-defined value.

Now that we have established the above, let δ be an arbitrarily small positive quantity, then

$$\begin{aligned}
F(x,k) &= \frac{1}{2k\omega}\int_{-\infty}^{x-\delta} f(u)\psi\left(\frac{u-x}{k}\right)du + \frac{1}{2k\omega}\int_{x+\delta}^{+\infty} f(u)\psi\left(\frac{u-x}{k}\right)du \\
&\quad + \frac{1}{2k\omega}\int_{x-\delta}^{x} f(u)\psi\left(\frac{u-x}{k}\right)du + \frac{1}{2k\omega}\int_{x}^{x+\delta} f(u)\psi\left(\frac{u-x}{k}\right)du \\
&= \frac{1}{2\omega}f(-\infty\cdots x-\delta)\int_{\delta/k}^{+\infty}\psi(u)du + \frac{1}{2\omega}f(x+\delta\cdots+\infty)\int_{\delta/k}^{+\infty}\psi(u)du \\
&\quad + \frac{1}{2\omega}\int_0^{\delta/k}\left(f(x-ku)+f(x+ku)\right)\psi(u)du.
\end{aligned}$$

Then it follows that

$$\begin{aligned}
F(x,k) - f(x) &= \frac{f(-\infty\cdots+\infty) - f(x)}{\omega}\int_{\delta/k}^{\infty}\psi(u)du \\
&\quad + \frac{1}{2\omega}\int_0^{\delta/k}\left(f(x-ku)+f(x+ku)-2f(x)\right)\psi(u)du \\
&= \frac{f(-\infty\cdots+\infty) - f(x)}{\omega}\int_{\delta/k}^{\infty}\psi(u)du \\
&\quad + \frac{1}{2}\epsilon_1\left(f(x-\epsilon\delta)+f(x+\epsilon\delta)-2f(x)\right),
\end{aligned} \tag{4}$$

[1] Here $f(x_1\cdots x_2)$ denotes an intermediate value between the smallest and largest value of $f(x)$ on $x\in[x_1,x_2]$.

where ϵ, ϵ_1 are positive quantities between 0 and 1.

Now let x_1, x_2 be two certain values of x, let G be the upper bound for $|f(x)|$, and let g_1, g_2 be two arbitrarily small positive quantities. Then, first of all, one can take δ sufficiently small such that, for $x \in (x_1, x_2)$ and $u \in (0, \delta)$, the absolute value of

$$\left| \frac{1}{2} \left(f(x-u) + f(x+u) - 2f(x) \right) \right| < g_1.$$

Moreoever, once we fix such a value of δ, we can determine $k' > 0$ such that $\forall k < k'$,

$$\frac{2G}{\omega} \int_{\delta/k}^{+\infty} \psi(u) du < g_2.$$

Thus by Equation (4), the absolute difference between $F(x, k)$ and $f(x)$ can be made smaller than $g_1 + g_2$ for any value of x.

Here, we have shown that not only does $F(x, k)$ converge to $f(x)$ for any specific value of x when k is infinitely small, but also that the convergence is uniform for all values of x that belong to finite intervals.

Now I draw a noteworthy conclusion from Equation (3).

Among the functions $\psi(x)$ that satisfy the conditions posed above, there are infinitely many transcendental entire functions whose associated functions $F(x, k)$ have continuously convergent[2] power series in x for every value of k. Let us take $\psi(x)$ to be such a function, e.g. $\psi(x) = e^{-x^2}$, then the following statement arises, which appears remarkable and fruitful to me:

A. "If $f(x)$ is a uniformly continuous function uniquely defined only for real values of x, then one can construct a transcendental entire function $F(x, k)$ in many different ways such that $F(x, k)$, apart from x, also includes a variable/alterable parameter $k > 0$ and has the property that for every real value of x, the following equation holds:

$$\lim_{k \to 0} F(x, k) = f(x)." \tag{5}$$

Let g' be arbitrarily small. Under the condition that the variable x is confined to some finite interval, one can (as previously shown) assign a sufficiently small value k' to the parameter k such that $|F(x, k') - f(x)| < g'$ for every value of x. Thereupon let us represent $F(x, k')$ in the form of a power series

$$A_0 + A_1 x + A_2 x^2 + \cdots$$

and use $G(x)$ to denote the n^{th} partial sum. Let g'' be another positive quantity. Then one can find n sufficiently large such that $|F(x, k') - G(x)| < g''$ for every x that belongs to the previously assumed finite interval. Therefore, with this construction, $|f(x) - G(x)| < g' + g''$.

[2][YZ] This book translates "*beständig convergierend*" as "continuously convergent", but points out that Weierstrass likely means "convergent for any finite value of x" by this term. This applies to all the following instances of "continuously convergent".

Hence we have shown that:

B. "If $f(x)$ has the properties specified above and the variable x is confined to some finite interval, then, assuming g is an arbitrarily small positive quantity, there are many ways to determine an entire rational function $G(x)$ which approaches $f(x)$ so closely in this finite interval that $|f(x)-G(x)|$ is always less than g."

Now let us take two infinite positive sequences

$$a_1, a_2, a_3, \cdots$$
$$g_1, g_2, g_3, \cdots$$

such that $\lim_{n\to\infty} a_n = \infty$ and $\sum_{\nu=1}^{\infty} g_\nu$ has a finite value. Then according to the previous discussions, one can find a sequence of entire rational functions $G_1(x), G_2(x), G_3(x), \cdots$ such that, for $x \in (-a_\nu, a_\nu)$ and $\nu = 1, 2, \cdots, \infty$,

$$|f(x) - G_\nu(x)| < g_\nu.$$

Then let us set

$$\begin{aligned} f_0(x) &= G_1(x), \\ f_\nu(x) &= G_{\nu+1}(x) - G_\nu(x), \end{aligned}$$

then

$$\sum_{\nu=0}^{n} f_\nu(x) = G_{n+1}(x),$$

and for every specific value of x,

$$\lim_{n\to\infty} G_{n+1}(x) = f(x),$$

which implies

$$f(x) = \sum_{\nu=0}^{\infty} f_\nu(x).$$

Now let x_1, x_2 be two specific finite values of x, then the inequalities

$$\begin{aligned} &|f(x) - G_\nu(x)| < g_\nu, \quad (-a_\nu \le x \le a_\nu) \\ &|f(x) - G_{\nu+1}(x)| < g_{\nu+1}, \quad (-a_{\nu+1} \le x \le a_{\nu+1}) \end{aligned}$$

imply that $\forall x \in (x_1, x_2)$,

$$|f_\nu(x)| < g_\nu + g_{\nu+1},$$

as long as ν is larger than some ν', which is defined such that every interval $(-a_\nu, a_\nu)$ contains both x_1, x_2 for any $\nu > \nu'$. Then we have

$$\sum_{\nu=\nu'+1}^{\infty} |f_\nu(x)| < \sum_{\nu=\nu'+1}^{\infty} (g_\nu + g_{\nu+1}), \text{ if } x_1 \le x \le x_2;$$

hence the series

$$\sum_{\nu=\nu'+1}^{\infty} f_\nu(x)$$

and therefore the series

$$\sum_{\nu=0}^{\infty} f_\nu(x)$$

converge absolutely and uniformly for this $x \in (x_1, x_2)$. The choice of x_1, x_2 is subject to no other constraints than that they have finite real values, and the functions $f_\nu(x)$ are independent of them. So the previous series converge absolutely for every value of x and uniformly in every interval $x \in (x_1, x_2)$ with finite endpoints. Thus the following theorem holds:

C. "Every function $f(x)$ with the properties specified above can be represented (in many ways) in the form of an infinite series whose terms are entire rational functions of x; this series converges absolutely for every finite value of x and uniformly on every interval (x_1, x_2) with finite endpoints."

It should be remarked that, to justify statement (B.), we need only assume $\psi(x)$ to be a transcendental entire function that has the previously specified properties for real values of x, but not that $F(x,k)$ is also an entire function of x, which is not a necessary consequence of the first assumption.

Let a, b be two arbitrary real numbers and let

$$F_1(x,k) = \frac{1}{2k\omega} \int_a^b f(u)\psi\left(\frac{u-x}{k}\right) du.$$

Then for real x, we have

$$F(x,k) = F_1(x,k) + \frac{1}{2\omega}\int_{\frac{x-a}{k}}^{+\infty} f(x-ku)\psi(u)du + \frac{1}{2\omega}\int_{\frac{b-x}{k}}^{+\infty} f(x+ku)\psi(u)du.^3$$

Assume $a < x_1 < x_2 < b$ and let g_1 be an arbitrarily small positive quantity, then we can fix the value k such that $|f(x) - F_1(x,k)| < g_1 \; \forall x \in (x_1, x_2)$. Given this, since $F_1(x,k)$ is always[4] a (transcendental) entire function of x, then for a second arbitrarily small positive g_2, we can find an entire rational function $G(x)$ such that for $x \in [x_1, x_2]$,

$$|G(x) - F_1(x,k)| < g_2,$$

and so

$$|f(x) - G(x)| < g_1 + g_2,$$

which gives statement (B.).

I consider this proof of the statement in question to be perfectly rigorous and sufficient to show that such entire rational functions $G(x)$ that can approximate

[3][YZ] The lower limit for the second integral originally had a on the denominator, which is most likely a typo.

[4][YZ] The original word is *unbedingt*, which also means "unconditionally"

a given function $f(x)$ at every point in an arbitrarily given interval (x_1, x_2) as accurately as desired exist and can actually be determined. However, the above method of forming such functions has a crucial drawback. If we set

$$F_1(x, k) = \sum_{\nu=0}^{\infty} (k)_\nu x^\nu,$$

where $(k)_\nu$ is a function of k that admits the following representation

$$(k)_\nu = \frac{(-1)^\nu}{\nu! \omega k^\nu} \int_{a/k}^{b/k} f(ku) \frac{d^\nu \psi(u)}{du^\nu} du,$$

and

$$G^{(n)}(x, k) = \sum_{\nu=0}^{n-1} (k)_\nu x^\nu.$$

Then for any given positive δ, there admittedly exist values of k and n for which

$$|f(x) - G^{(n)}(x, k)| < \delta$$

holds for $x \in [x_1, x_2]$. However, if δ is infinitely small, then k is also infinitely small. Then the trouble occurs: from the expression of $(k)_\nu$ above, we are not able to tell whether it approaches a finite limit value if k is infinitely small, or whether it at least remains finite, which is absolutely necessary if the method in question is to yield a viable approximating expression to $f(x)$ for an arbitrarily small δ.

I will show how to fix this trouble in a following paper.

Second paper (*Zweite Mitteilung*)

(7/30/1885)

As specified in the previous note presented on 7/9, we let $f(x)$ be a real-valued continuous function uniquely defined for every $x \in \mathbb{R}$ with absolute value bounded by G ($|f(x)| < G$). On the other hand, let $\psi(x)$ be a transcendental entire function, for which it is assumed that it has real values for real x and satisfies the condition $\psi(-x) = \psi(x)$. Furthermore, let $u, v \in \mathbb{R}$ be real variables independent of each other. Then we have

$$\sqrt{\psi(u+vi)\psi(u-vi)} = \psi(u,v),$$

where the square root takes the positive value. Then the absolute value of $\frac{\psi(u+vi)}{\psi(u,v)}$ is exactly 1, and therefore we have, for $a, b \in \mathbb{R}$,

$$\int_a^b f(u)\psi(u+vi)du = \int_a^b f(u)\frac{\psi(u+vi)}{\psi(u,v)}\psi(u,v)du = \epsilon G\int_a^b \psi(u,v)du,$$

where ϵ denotes a complex number with modulus less than 1. Assuming that $\psi(x)$ has the property that the integral $\int_0^{+\infty} \psi(u,v)du$ is finite for every v, then for positive real numbers a_1, a_2, b_1, b_2 with $b_1 > a_1$ and $b_2 > a_2$, the two integrals

$$\begin{aligned} &\int_{a_2}^{b_2} \psi(u,v)du, \\ &\int_{-b_1}^{-a_1} \psi(u,v)du \;= \int_{a_1}^{b_1} \psi(u,v)du \quad (\text{since } \psi(-u,v) = \psi(u,v)) \end{aligned}$$

both take infinitely small values if a_1, b_1 become infinitely large. Following the previous equation, the same also applies for the integrals

$$\int_{-b_1}^{-a_1} f(u)\psi(u+vi)du, \quad \int_{a_2}^{b_2} f(u)\psi(u+vi)du.$$

So the integral

$$\int_{-\infty}^{+\infty} f(u)\psi(u+vi)du$$

has a certain finite value for every value of x.

I will further assume that, if a_2 becomes infinitely large, the integral

$$\int_{a_2}^{+\infty} \psi(u,v)dv$$

converges uniformly to the limit value 0 for all x whose absolute value doesn't exceed an arbitrarily set threshold. Then, by Equation (1), the same statement applies to the integral

$$\int_{a_2}^{+\infty} f(u)\psi(u+vi)du,$$

and similarly, if a_1 becomes infinitely large, to

$$\int_{-\infty}^{-a_1} f(u)\psi(u+vi)du.$$

So if V, g are assigned positive values, where V is arbitrarily large and g is arbitrarily small, then we can always find two positive quantities a_1, a_2 such that, for every $v \in [-V, V]$,

$$\left|\int_{-\infty}^{+\infty} f(u)\psi(u+vi)dv - \int_{-a_1}^{a_2} f(u)\psi(u+vi)du\right| < g.$$

Now let $x = \xi + \xi' i$ be a complex variable, k be a positive constant, and $\omega = \int_0^{+\infty} \psi(u)du$, as in the first paper. Then by the reasoning above, the integral

$$\frac{1}{2\omega}\int_{-\infty}^{+\infty} f(\xi+ku)\psi\left(u - \frac{\xi' i}{k}\right)du = \frac{1}{2k\omega}\int_{-\infty}^{+\infty} f(u)\psi\left(\frac{u-x}{k}\right)du \qquad (1)$$

has a finite value that is uniquely defined for every finite x, which we denoted $F(x, k)$ in the previous paper.

Now we need to prove that $F(x, k)$ is a transcendental entire function of x.

If we set an upper bound r for the modulus of x, then given two arbitrarily small positive quantities g', g'', we can find (a_1, a_2) such that

$$\frac{1}{2\omega}\int_{-\infty}^{-a_1} f(\xi+ku)\psi\left(u - \frac{\xi' i}{k}\right)du + \frac{1}{2\omega}\int_{a_2}^{+\infty} f(\xi+ku)\psi\left(u - \frac{\xi' i}{k}\right)du < g'$$

for all ξ, ξ' satisfying

$$\xi^2 + \xi'^2 \leq r^2.$$

Then we have

$$F(x,k) = \frac{1}{2k\omega}\int_{-a_1}^{a_2} f(u)\psi\left(\frac{u-x}{k}\right)du + \epsilon' g',$$

where ϵ' has absolute value less than 1. The integral on the right hand side of this equation can be written as a continuously convergent (*beständig convergierend*) power series $\mathcal{P}(x)$. Let $G^{(n)}(x)$ denote the sum of the first n terms in $\frac{1}{2k\omega}\mathcal{P}(x)$. Then n can be taken sufficiently large such that for every value of x with $|x| \leq r$,

$$|F(x,k) - G^{(n)}(x)| < g' + g''.$$

With this established, we can apply the same procedure that grounded Theorem (C.) in the first paper to show that $F(x, k)$ can be represented by an infinite series whose terms are entire rational functions of x, and that this series converges uniformly for all x values contained in some finite interval. In the end we assume there are two sequences of positive quantities

$$r_1, r_2, r_3, \cdots$$
$$g_1, g_2, g_3, \cdots$$

such that $\lim_{n\to\infty} r_n = \infty$ and $\sum_{n=1}^{\infty} g_n$ is finite. Thereupon we can find a series of entire rational functions $G_1(x), G_2(x), \cdots$ such that for every x satisfying $|x| \leq r_\nu$,

$$|F(x,k) - G_\nu(x)| < g_\nu, \qquad (\nu = 1, 2, \cdots, \infty) \tag{2}$$

and

$$f_0(x) = G_1(x), \tag{3}$$

$$f_\nu(x) = G_{\nu+1}(x) - G_\nu(x). \tag{4}$$

Then

$$F(x,k) = \sum_{\nu=0}^{\infty} f_\nu(x). \tag{5}$$

According to a statement that I proved in an elementary way earlier (Monthly Reviews/Bulletins of the Academy of 1880, Page 723), if the series on the right hand side converges uniformly in every finite interval, it can be transformed into a power series $\mathcal{P}(x)$ that converges for every finite x.

Let us take

$$\psi(x) = e^{-x^2},$$

then

$$\psi(u,v) = e^{-u^2+v^2},$$

and the function $\psi(u,v)$ has the properties postulated earlier. The same holds if we set

$$\psi(x) = e^{-(c_1x^2+c_2x^4+\cdots+c_\rho x^{2\rho})},$$

where we assign the constant c_ρ a positive value and $c_1, \cdots, c_{\rho-1}$ can take arbitrary real values.

Indeed, there exist countless functions $\psi(x)$ whose associated $F(x,k)$ are transcendental entire functions, as given in the justification of Theorem (A.) in the first paper.

Now let $F(x,k)$ be any such function, then the power series $\mathcal{P}(x)$, through which $F(x,k)$ can be represented, can be transformed into a series expanded in terms of spherical functions and converges for every finite x. By Carl Neumann's famous theorem[5] regarding the expansion of unique analytical functions of a complex variable x in spherical functions of the first kind[6], it is staightforward that every entire function $G(x)$, whether transcendental or rational, can be represented as a series of the following form that converges for every finite value of x:

$$G(x) = \sum_{\nu=0}^{\infty} C_\nu P^{(\nu)}(x).^7$$

[5] Paper is probably this one.

[6] [YZ] Do these refer to Legendre polynomials?

The coefficients of this series are such that $\sum_{\nu=0}^{\infty} |C_\nu| r^\nu$ is finite for every positive r. Furthermore, the series converges uniformly for all x values that belong to finite intervals. Based on this last property, we have

$$\int_{-1}^{+1} G(x')P^{(\mu)}(x')dx' = \sum_{\nu=0}^{\infty} C_\nu \int_{-1}^{+1} P^{(\mu)}(x')P^{(\nu)}(x')dx',$$

where x' denotes a real variable; from this, it follows that

$$C_\mu = \frac{2\mu+1}{2} \int_{-1}^{+1} G(x')P^{(\mu)}(x')dx' \qquad (\mu = 0, 1, \cdots, \infty).$$

For the function $F(x,k)$, we then have

$$\begin{aligned} C_\nu &= \frac{2\nu+1}{2} + \frac{1}{2k\omega} \int_{-1}^{+1} P^{(\nu)}(x')dx' \int_{-\infty}^{+\infty} f(u)\psi\left(\frac{u-x'}{k}\right) du \\ &= \frac{2\nu+1}{4\omega} \int_{-1}^{+1} P^{(\nu)}(x') \int_{-\infty}^{+\infty} f(x'+ku)\psi(u)du, \end{aligned}$$

where, if we set

$$\frac{2\nu+1}{2} \int_{-1}^{+1} f(x'+u)P^{(\nu)}(x')dx' = f_\nu(u),$$

[7] This can also be derived in the following way: By the definition of spherical functions, if we determine the value $\sqrt{x^2-1}$ such that $|x+\sqrt{x^2-1}| \geq 1$, then

$$|P^{(n)}(x)| \leq |x+\sqrt{x^2-1}|^n.$$

Furthermore,

$$x^n = c_{n,0}P^{(n)}(x) + c_{n,1}P^{(n-2)}(x) + \cdots,$$

and so if $\sum_{n=0}^{\infty} A_n x^n$ is a continuously convergent power series of x, then

$$\sum_{n=0}^{\infty} A_n x^n = \sum_n \sum_\nu A_n c_{n,\nu} P^{(n-2\nu)}(x).$$

$c_{n,\nu}$ are all positive and $\sum_\nu c_{n,\nu} = 1$, so

$$\sum_\nu |c_{n,\nu}P^{(n-2\nu)}(x)| \leq |x+\sqrt{x^2-1}|^n.$$

Then it follows that $\sum_{n=1}^{\infty}\sum_\nu |A_n c_{n,\nu} P^{(n)}(x)|$ is finite, and therefore if we set

$$C_\mu = \sum_{\substack{n,\nu \\ n-2\nu=\mu}} c_{n,\nu}A_n = \sum_{\nu=0}^{\infty} c_{\mu+2\nu,\nu}A_{\mu+2\nu}, \qquad (\text{for } \mu = 0, 1, 2, \cdots, \infty)$$

the following equality holds

$$\sum_{n=0}^{\infty} A_n x^n = C_0 + \sum_{\mu=1}^{\infty} C_\mu P^{(\mu)}(x).$$

we have

$$C_\nu = \frac{1}{2\omega}\int_{-\infty}^{+\infty} f_\nu(ku)\psi(u)du.$$

Like $f(u)$, the function $f_\nu(u)$ is also a uniformly continuous function; and as long as $|P^{(\nu)}(x')| \leq 1$ for all $x' \in (-1,+1)$, the absolute value of $f_\nu(u)$ can be at most $(2\nu+1)G$. On the other hand, if a is an arbitrary positive quantity,

$$\begin{aligned}
C_\nu &= \frac{1}{2\omega}\int_{-a}^{a} f_\nu(ku)\psi(u)du + \frac{1}{2\omega}\int_{a}^{+\infty} f_\nu(ku)\psi(u)du + \frac{1}{2\omega}\int_{-\infty}^{-a} f_\nu(ku)\psi(u)du \\
&= \frac{1}{2\omega}\int_{-a}^{a} f_\nu(ku)\psi(u)du \\
&\quad + \frac{1}{2\omega} f_\nu(ka\cdots+\infty)\int_{a}^{+\infty}\psi(u)du + \frac{1}{2\omega} f_\nu(-\infty\cdots-ka)\int_{-\infty}^{-a}\psi(u)du.
\end{aligned}$$

If we then take k to be infinitely small, we have

$$\lim_{k\to 0} C_\nu = f_\nu(0)\frac{1}{2\omega}\int_{-a}^{+a}\psi(u)du + \cdots,$$

where the omitted terms on the right hand side have infinitely small values if a is infinitely large. Since we can take a to be arbitrarily large, this implies

$$\lim_{k\to 0} C_\nu = f_\nu(0) = \frac{2\nu+1}{2}\int_{-1}^{+1} f(x)P^{(\nu)}(x)dx.$$

Let us set

$$\frac{1}{2\omega}\int_{-\infty}^{+\infty} f_\nu(ku)\psi(u)du = \phi_\nu(k),$$

and let δ be a small real number, then

$$\phi_\nu(k+\delta) - \phi_\nu(k) = \frac{1}{2\omega}\int_{-a}^{+a}\left(f_\nu(ku+\delta u) - f_\nu(ku)\right)\psi(u)du + \cdots,$$

where, again, the omitted terms on the right hand side have arbitrarily small values for sufficiently large a. Let δ_1 be a given arbitrarily small quantity, then we can find a specific value of a for which the absolute value of

$$\phi_\nu(k+\delta) - \phi_\nu(k) = \frac{1}{2\omega}\int_{-a}^{+a}\left(f_\nu(ku+\delta u) - f_\nu(ku)\right)\psi(u)du$$

is smaller than δ_1 for any values of k, δ. Moreover, let δ_2 be a second arbitrarily small quantity. Then we can find an upper bound δ' for $|\delta|$ such that

$$\frac{1}{2\omega}\int_{-a}^{+a}\left(f_\nu(ku+\delta u) - f_\nu(ku)\right)\psi(u)du$$

has an absolute value smaller than δ_2, so that

$$|\phi_\nu(k+\delta) - \phi_\nu(k)| < \delta_1 + \delta_2$$

holds for $|\delta| < \delta'$. Therefore, $\phi_\nu(k)$ is a continuous function of k.

Hereby we have shown the following fact:

"Let $\psi(x)$ be a function with the properties postulated above, and

$$F(x,k) = \frac{1}{2k\omega} \int_{-\infty}^{+\infty} f(u) \psi\left(\frac{u-x}{k}\right) du,$$

then for every finite complex value of x, we have

$$F(x,k) = \sum_{\nu=0}^{\infty} \phi_\nu(k) P^{(\nu)}(x), \tag{6}$$

if we set

$$\begin{cases} f_\nu(u) = \dfrac{2\nu+1}{2} \displaystyle\int_{-1}^{+1} f(x'+u) P^{(\nu)}(x') dx', \\ \phi_\nu(k) = \dfrac{1}{2\omega} \displaystyle\int_{-\infty}^{+\infty} f_\nu(ku) \psi(u) du; \end{cases} \tag{7}$$

and then $\phi_\nu(k)$ are continuous functions of k."

Now let x be a real variable again, such that

$$f(x) = \lim_{k \to 0} F(x,k).$$

If x is confined to the interval $x \in [-a,a]$, where a denotes an arbitrary positive quantity, then given an arbitrarily small positive quantity g', we can find a value k' for the parameter k such that

$$|f(x) - F(x,k')| < g'.$$

Furthermore, let R denote the largest value that $|\sqrt{x^2-1}+x|$ can take for the current $x \in [-a,a]$, then we have

$$R = \begin{cases} 1, & \text{if } a \leq 1, \\ a + \sqrt{a^2-1}, & \text{if } a > 1. \end{cases}$$

Therefore, as we noticed earlier,

$$|P^{(\nu)}(x)| \leq R^\nu.$$

Now that the series $\sum_{\nu=0}^{\infty} |\phi_\nu(k')| R^\nu$ has a finite value, then, for a second positive quantity g'', it is always possible to find a positive integer n such that

$$\left| \sum_{\nu=n+1}^{\infty} \phi_\nu(k') P^{(\nu)}(x) \right| < g''.$$

Let

$$G^{(n)}(x,k) = \sum_{\nu=0}^{\infty} \phi_\nu(k) P^{(\nu)}(x), \tag{8}$$

then

$$\left| f(x) - G^{(n)}(x,k') \right| < g + g''.$$

Hereafter we can express Theorem (B.) in my first paper in the following way:

"Let a, g be positive quantities such that a can be arbitrarily large and g can be arbitrarily small. Then, through the expression $G^{(n)}(x,k)$ defined in Equation 8, which is an n^{th} order entire rational function of x, it is always possible to find values for the parameter k and the number n such that for $x \in (-a, a)$, the absolute difference $|f(x) - G^{(n)}(x,k)|$ does not exceed a prescribed threshold that can be arbitrarily small."

The expression for the function $F(x,k)$ developed above has an essential advantage over a power series representation, in that the coefficients of the former – the $\phi_\nu(k)$ – are represented in such a form that it is easy to recognize them as continuous functions of k, and that the magnitude of each coefficient is bounded for all values of k, while at the same time $\lim_{\nu\to\infty} \phi_\nu(k) = 0$ for every given value of k.[8]

In this way, we get rid of the trouble stated at the end of the first paper, which occurs if the function $G(x)$ in Theorem (B.) is defined the way it is there.

So far, we assume that $|f(x)|$ has a finite upper bound. This assumption can be dropped if we are merely trying to determine an entire rational function $G(x)$ that approximates the function $f(x)$ so well in (x_1, x_2) that for an arbitrarily set threshold g, $|f(x) - G(x)| < g$ for every value of x.

Indeed, we can define the function $f_1(x)$ such that

$$\begin{aligned} f_1(x) &= f(x_1), \text{ if } x < x_1 \\ f_1(x) &= f(x), \text{ if } x_1 \le x \le x_2 \\ f_1(x) &= f(x_2), \text{ if } x > x_2. \end{aligned}$$

Then $f_1(x)$ has the same properties as $f(x)$, and according to this we can find a function $G(x)$ such that for every $x \in (x_1, x_2)$,

$$|f_1(x) - G(x)| < g,$$

and so

$$|f(x) - G(x)| < g.$$

In the proof of Theorem (C.) given in the first paper, it is assumed for the function $f(x)$ that for two arbitrary positive quantities a_ν, g_ν, it is possible to construct an entire rational function $G_\nu(x)$ such that

$$|f(x) - G_\nu(x)| < g, \text{ if } -a_\nu \le x \le a_\nu.$$

[8] [YZ] The original text has $\nu \to a$ here instead of $\nu \to \infty$, which I think is a typo.

So Theorem (C.) also applies in the same setting if $f(x)$ is now only assumed to take a finite value varying continuously with x for every finite real value x.

It remains to be investigated how the theorems developed so far should be modified if we also drop the assumption that $f(x)$ is a uniformly continuous function. I intend to address this in a following treatise. The investigation can be extended to unique-valued functions of several real variables, which is not difficult to do for uniformly continuous functions.

I now take $f(x)$ to be a periodic function, i.e. it does not change its value if its argument is increased by a certain positive quantity $2c$. Then the corresponding function $F(x,k)$ can also be represented as a Fourier series that converges for every complex value of x, whose coefficients are continuous functions of k.

From Equation (1) above, it follows that

$$F(x+2c,k) = F(x,k);$$

let z be another complex variable and define

$$\bar{F}(z) = F\left(\frac{c}{\pi i}\log z, k\right),$$

then $\bar{F}(z)$ is a unique analytic function of z with only two singularities in the entire domain, namely 0 and ∞. So it can be expanded as a continuously convergent series of the form

$$\sum_{\nu=-\infty}^{\nu=+\infty} C_\nu z^\nu.$$

Let $z = e^{\frac{\pi x}{c}i}$, then $\bar{F}(z) = F(x,k)$, and therefore for every finite value of x,

$$F(x,k) = \sum_{\nu=-\infty}^{\nu=+\infty} C_\nu e^{\frac{\nu\pi x}{c}i}.$$

Given that this expansion of $F(x,k)$ converges uniformly in every finite domain of x, if we let x' be another real variable and n an integer, we have

$$\frac{1}{2c}\int_{-c}^{c} F(x',k)e^{-\frac{n\pi x'}{c}i}dx' = \frac{1}{2c}\sum_{\nu=-\infty}^{\nu=+\infty} C_\nu \int_{-c}^{c} e^{\frac{(\nu-n)\pi x'}{c}i}dx' = C_n.$$

Then we have

$$\begin{aligned}
2cC_n &= \frac{1}{2k\omega}\int_{-c}^{c} e^{-\frac{n\pi x'}{c}i}dx' \int_{-\infty}^{+\infty} f(u)\psi\left(\frac{u-x'}{k}\right)du \\
&= \frac{1}{2\omega}\int_{-c}^{c} e^{-\frac{n\pi x'}{c}i}dx' \int_{-\infty}^{+\infty} f(x'+ku)\psi(u)du \\
&= \frac{1}{2\omega}\int_{-\infty}^{+\infty} \psi(u)e^{-\frac{nku\pi}{c}i}du \int_{-c}^{c} f(x'+ku)e^{-\frac{n\pi}{c}(x'+ku)i}dx'.
\end{aligned}$$

Now if we set

$$f_1(x') = f(x')e^{-\frac{n\pi x'}{c}i}$$

and let x_0 be a quantity independent of x', we have

$$\begin{aligned}
\int_{-c}^{c} f_1(x')dx &= \int_{-c}^{x_0-c} f_1(x')dx' + \int_{x_0-c}^{c} f_1(x')dx \\
&= \int_{x_0-c}^{c} f_1(x')dx + \int_{-c}^{x_0-c} f_1(x'+2c)dx' \\
&= \int_{x_0-c}^{c} f_1(x')dx + \int_{c}^{x_0+c} f_1(x')dx \\
&= \int_{x_0-c}^{x_0+c} f_1(x')dx' \\
&= \int_{-c}^{c} f_1(x'+x_0)dx';
\end{aligned}$$

and

$$\int_{-c}^{c} f(x'+ku)e^{-\frac{n\pi}{c}(x'+ku)i}dx' = \int_{-c}^{c} f(x')e^{-\frac{n\pi x'}{c}i}dx'.$$

Let v denote an arbitrary real value, and define

$$\phi(v) = \frac{1}{2\omega}\int_{-\infty}^{+\infty} \psi(u)e^{-uvi}du = \frac{1}{\omega}\int_{0}^{\infty} \psi(u)\cos(vu)du, \tag{9}$$

then

$$C_n = \phi\left(\frac{nk\pi}{c}\right)\int_{-c}^{c} \frac{1}{2c}f(x')e^{-\frac{n\pi x'}{c}i}dx'. \tag{10}$$

Let us define

$$\begin{aligned}
A_n &= \frac{1}{2c}\int_{-c}^{c} f(x')\cos\left(\frac{n\pi}{c}x'\right)dx' \\
A'_n &= \frac{1}{2c}\int_{-c}^{c} f(x')\sin\left(\frac{n\pi}{c}x'\right)dx',
\end{aligned} \tag{11}$$

then

$$C_n = (A_n - iA'_n)\phi\left(\frac{nk\pi}{c}\right), \tag{12}$$

and so we have

$$F(x,k) = A_0 + 2\sum_{n=1}^{\infty}\phi\left(\frac{nk\pi}{c}\right)\cdot\left(A_n\cos\left(\frac{n\pi}{c}x\right) + A'_n\sin\left(\frac{n\pi}{c}x\right)\right). \tag{13}$$

By the formula above, $\phi(v)$ is a continuous function of v and takes the value 1 for $v = 0$.

Setting $k = 0$ in the right hand side of the equation above reduces it to

$$A_0 + 2\sum_{n=1}^{\infty}\left(A_n \cos\left(\frac{n\pi}{c}x\right) + A'_n \sin\left(\frac{n\pi}{c}x\right)\right),$$

that is, one now turns to the series where the function $f(x)$ can be expanded by Fourier's Theorem in general (i.e. if one refrains from special functions that have not been sufficiently characterized so far). However, just as Mr. P. Du Bois-Reymond first proved with an example, such functions $f(x)$ do exist which cannot be represented as the series above for certain values of x (and there exist infinitely many of them in however small an interval (x_1, x_2)). This demonstrates that, to find the limit of the right hand side of Equation (13) as $k \to 0$, it is not necessarily allowed to set $k = 0$ for every single term in the series.

As has been shown, the series $\sum_{n=-\infty}^{n=+\infty} C_n z^n$ converges for every value of z, with the exception at 0 and ∞. So the series $\sum_{n=0}^{\infty} C_n z^n$ converges for every finite value of z.

Let us take, for example,

$$f(x) = \sum_{n=-\infty}^{n=+\infty} \frac{1}{n^2}\cos\left(\frac{n\pi}{c}x\right),$$

then

$$A_n = \frac{1}{n^2}, \quad A'_n = 0,$$

and so

$$C_n = \frac{1}{n^2}\phi\left(\frac{nk\pi}{c}\right);$$

from this it follows that the series

$$\sum_{n=1}^{\infty}\phi\left(\frac{nk\pi}{c}\right)z^n$$

also converges for every finite value of z.

Let us set

$$\chi(x; v) = \sum_{n=-\infty}^{n=+\infty}\phi(nv)e^{nxi}, \tag{14}$$

then $\chi(x; v)$ is an analytic function uniquely defined for every complex value of x and every real value of v; in addition, because $\phi(-v) = \phi(v)$, the following expression for $\chi(x;v)$ holds:

$$\chi(x;v) = \sum_{n=-\infty}^{n=+\infty} \phi(nv)\cos(nx) = 1 + 2\sum_{n=1}^{\infty} \phi(nv)\cos(nx); \tag{15}$$

and then the function $F(x,k)$ can be expressed in the following way:

$$F(x,k) = \frac{1}{2c}\int_{-c}^{c} f(x')\chi\left(\frac{x-x'}{c}\pi; \frac{k\pi}{c}\right) dx'. \tag{16}$$

Now let g' and g'' be arbitrarily small positive quantities, and let k' be taken to be a certain value of k such that

$$|f(x) - F(x,k')| < g'$$

for every real value of x. Let us then determine a positive integer n such that

$$\left|\sum_{\nu=n+1}^{\infty} \phi\left(\frac{\nu k'\pi}{c}\right)\left(A_\nu \cos\left(\frac{\nu\pi}{c}x\right) + A'_\nu \sin\left(\frac{\nu\pi}{c}x\right)\right)\right| < g''$$

for every real value of x, and set

$$f(x) = A_0 + 2\sum_{\nu=1}^{n} \phi\left(\frac{\nu k'\pi}{c}\right)\left(A_\nu \cos\left(\frac{\nu\pi}{c}x\right) + A'_\nu \sin\left(\frac{\nu\pi}{c}x\right)\right) + R_n, \tag{17}$$

then $|R_n| < g' + g''$ always holds.

This yields the following theorem:

D. "Let $f(x)$ be a uniformly continuous, real-periodic function uniquely defined for every real value of x. Let g be an arbitrarily small positive quantity. Then there are many ways in which one can construct a finite Fourier series that approximates the function $f(x)$ so closely that the difference between them is less than g for every value of x."

By the reasoning for proving Theorem (C.), where we now regard the functions $G_1(x), G_2(x), \cdots$ there as finite Fourier series with the same primitive period as $f(x)$, the following can be derived from Theorem (D.) above:

E. "Every function $f(x)$ with the same properties as in (D.) with the same primitive period $2c$ can be written as a sum whose terms are all finite Fourier series with period $2c$. This series converges absolutely for every value of x and uniformly in every finite interval."

To elucidate the previous statements through a simple example, let us take

$$\psi(x) = e^{-x^2}.$$

Then $2\omega = \sqrt{\pi}$, and

$$\phi(v) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-u^2-uvi}du = \frac{e^{-\frac{v^2}{4}}}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-(u+\frac{vi}{2})^2}du = e^{-\frac{v^2}{4}}.$$

Then it follows that

$$\chi(x;v) = \sum_{\nu=-\infty}^{\nu=+\infty} e^{-\frac{\nu^2 v^2}{4}} \cos(\nu x), \tag{18}$$

so if we set

$$q = e^{-\frac{v^2}{4}}, \tag{19}$$

then

$$\chi(x;v) = \mathcal{J}_3\left(\frac{x}{2}, q\right), \tag{20}$$

where $\mathcal{J}_3(x,q)$ is the Jacobi function

$$1 + 2q\cos(2x) + 2q^4\cos(4x) + 2q^9\cos(6x) + \cdots$$

Hereafter we have

$$F(x,k) = \frac{1}{2c}\int_{-c}^{c} f(x')\mathcal{J}_3\left(\frac{x-x'}{2c}\pi, q\right) dx', \quad \text{where } q = e^{-\frac{k^2\pi^2}{4c^2}}. \tag{21}$$

The formula on the right hand side of this equation already occurs in Fourier's work (*The Analytical Theory of Heat*, Chapter X). To determine the temperature state of an infinitely thin, nondiffusive[9], homogeneous ring with length $2c$ at an arbitrary time point, where the temperature state is known at some point[10], one has to determine a function ϕ of two real variables x, t such that the differential equation

$$\frac{\partial\phi}{\partial t} = \mu\frac{\partial^2\phi}{\partial x^2}$$

holds, where μ is a positive constant regarded as a function of x with period $2c$ that is equal to a given arbitrary function $F(x)$ in the interval $x \in [-c, c]$ at $t = 0$; the function $F(x)$ is assumed to be continuous with $F(-c) = F(c)$.

Fourier found the following expression for this ϕ:

$$\begin{aligned}\phi &= \frac{1}{2c}\sum_{\nu=-\infty}^{\nu=+\infty}\int_{-c}^{c} F(x')e^{-\frac{\nu^2\mu\pi^2}{c^2}t}\cos\left(\nu\frac{x-x'}{c}\pi\right)dx' \\ &= \frac{1}{2c}\int_{-c}^{c} F(x')\sum_{\nu=-\infty}^{\nu=+\infty} e^{-\frac{\nu^2\mu\pi^2}{c^2}t}\cos\left(\nu\frac{x-x'}{c}\pi\right)dx',\end{aligned} \tag{22}$$

so we would have

$$\phi = F(x,k), \tag{23}$$

[9] [YZ] I hope this is the right word for "not radiating heat"?
[10] [YZ] i.e. an initial condition is given.

if we define the function $f(x)$ such that it coincides with $F(x)$ on the interval $x \in [-c, c]$, and take

$$k = 2\sqrt{\mu t}. \tag{24}$$

To show that ϕ is the same as $F(x)$ in the interval $x \in [-c, c]$ at $t = 0$, Fourier set $t = 0$ in the individual terms in his expression, which turns the series into

$$\frac{1}{2c} \sum_{\nu=-\infty}^{\nu=+\infty} \int_{-c}^{c} F(x') \cos\left(\nu \frac{x - x'}{c} \pi\right) dx',$$

for which he assumed that it represents $F(x)$ throughout the interval $[-c, c]$. However, it is worth mentioning that, regardless of the objections that could be made to Fourier's approach, the expression for ϕ constructed above is correct without exception. Because it can be rearranged as $F(x, 2\sqrt{\mu t})$, as shown above, it follows, without the help of Fourier's Theorem, that

$$\lim_{t \to 0} \phi = F(x).$$

Moreover, the individual terms satisfy the differential equation

$$\frac{\partial \phi}{\partial t} = \mu \frac{\partial^2 \phi}{\partial x^2}.$$

From this it follows that ϕ itself also satisfies this equation, since its series is a unique analytic function of x and t if we impose the condition that the real part of t is positive, and moreover because this series converges uniformly in every finite region of x and t. Finally, the series does not change value if x is replaced by $x + 2c$. Thus, the function that it represents is consistent with the imposed conditions.

It is highly noteworthy that for a problem in mathematical physics where we seek a function of two variables (which, by their physical meanings, can only take real values) which, for a fixed value of one of its variables, should coincide with a certain arbitrary function of the other variable, there is an expression for this function we are seeking which is analytic in the variable, and so is meaningful if the variable takes complex values.[11]

Let n be a positive integer, and let

$$\chi(x; v)_n = \sum_{\nu=-n}^{\nu=+n} e^{-\frac{\nu^2 v^2}{4}} \cos(\nu x).$$

Then, by Equation (18),

$$\chi(x; v) = \chi(x; v)_n + 2 \sum_{\nu=n+1}^{\infty} e^{-\frac{\nu^2 v^2}{4}} \cos(\nu x).$$

[11] [YZ] This is an extremely long sentence. Basically it is saying: we are looking for $\phi(x, t)$ where $\phi(x, 0) = F(x)$ for some arbitrary $F(x)$; then there is an expression for ϕ that is analytic in x, i.e. it is also meaningful if we extend x to the complex domain.

For real values of x, the absolute value of the second term on the right hand side of this equation, which we denote as R_n, is never larger than

$$2\sum_{\nu=n+1}^{\infty} e^{-\frac{\nu^2 v^2}{4}} = 2\sum_{\nu=1}^{\infty} e^{-\frac{n^2+2n\nu+\nu^2}{4}v^2},$$

so

$$R_n < e^{-\frac{n^2+2n}{4}v^2} \cdot 2\sum_{\nu=1}^{\infty} e^{-\frac{\nu^2 v^2}{4}}.$$

If, in this well-known identity (where $\tau > 0$)

$$1 + 2\sum_{\nu=1}^{\infty} e^{-\nu^2\tau\pi} = \frac{1}{\sqrt{\tau}}\left(1 + 2\sum_{\nu=1}^{\infty} e^{-\frac{v^2\pi}{\tau}}\right),$$

we set $\tau = \frac{v^2}{4\pi}$, then for positive v, this yields

$$2\sum_{\nu=1}^{\infty} e^{-\frac{\nu^2 v^2}{4}} = \frac{2\sqrt{\pi}}{v} - 1 + \frac{4\sqrt{\pi}}{v}\sum_{\nu=1}^{\infty} e^{-\frac{4\nu^2\pi^2}{v^2}}.$$

Then

$$R_n < \frac{2\sqrt{\pi}}{v} e^{-\frac{(n+1)^2}{4}v^2} \cdot \left\{1 - \frac{v}{2\sqrt{\pi}} + 2\sum_{\nu=1}^{\infty} e^{-\frac{4\nu^2\pi^2}{v^2}}\right\} e^{\frac{v^2}{4}}.$$

Now let us set

$$v = \frac{2\sqrt{m\log(n+1)}}{n+1}, \tag{25}$$

where m is a positive value, then

$$R_n < \frac{(n+1)^{-m+1}}{2\sqrt{m\log(n+1)}} \cdot (1 + [n]),$$

where $[n]$ denotes a value that is infinitely small for infinitely large n. Let us take

$$m \geq 1,$$

then

$$\lim_{n\to\infty} R_n = 0.$$

But then,

$$e^{-\frac{v^2}{4}} = e^{-\frac{m\log(n+1)}{(n+1)^2}} = (n+1)^{-\frac{m}{(n+1)^2}},$$

so if we denote

$$\{n\} = (n+1)^{-\frac{m}{(n+1)^2}}, \tag{26}$$

and set

$$\chi(x,n) = \sum_{\nu=-n}^{\nu=+n} \{n\}^{\nu^2} \cos(\nu x) = 1 + 2\sum_{\nu=1}^{n} \{n\}^{\nu^2} \cos(\nu x), \tag{27}$$

then we have

$$\chi(x;v)_n = \chi(x,n).$$

From the equation

$$F(x,k) = \frac{1}{2c}\int_{-c}^{c} f(x')\chi\left(\frac{x-x'}{c}\pi; \frac{k\pi}{c}\right) dx',$$

it follows that

$$\begin{aligned} F\left(x, \frac{cv}{\pi}\right) &= \frac{1}{2c}\int_{-c}^{c} f(x')\chi\left(\frac{x-x'}{c}\pi; v\right) dx' \\ &= \frac{1}{2c}\int_{-c}^{c} f(x')\chi\left(\frac{x-x'}{c}\pi, n\right) dx' + R_n', \end{aligned}$$

where R_n' denotes a quantity that is infinitely small for infinitely large n. Now because $\lim_{n\to\infty} v = 0$ and $\lim_{v\to 0} F\left(x, \frac{cv}{\pi}\right) = f(x)$, it follows that

$$f(x) = \lim_{n\to\infty} \frac{1}{2c}\int_{-c}^{c} f(x')\chi\left(\frac{x-x'}{c}\pi, n\right) dx'. \tag{28}$$

Precisely written, the Fourier Theorem says that

$$f(x) = \lim_{n\to\infty} \frac{1}{2c}\int_{-c}^{c} f(x')\bar{\chi}\left(\frac{x-x'}{c}\pi, n\right) dx', \tag{29}$$

where

$$\bar{\chi}\,(x,n) = \sum_{\nu=-n}^{\nu=+n} \cos(\nu x). \tag{30}$$

This equality, which does not always hold under all circumstances, can be substituted by the earlier universally valid expression, where we replace $\bar{\chi}(x,n)$ by another $\chi(x,n)$, which, like $\bar{\chi}(x,n)$, has the form

$$1 + 2(n,1)\cos(x) + 2(n,2)\cos(2x) + \cdots + 2(n,n)\cos(nx),$$

where (n,ν) denotes a positive quantity depending o n and ν, which reduces to one in $\bar{\chi}(x)$. For every particular value of ν,

$$\lim_{n\to\infty} (n,\nu) = 1.$$

Thus we can take n sufficiently large such that the first $(\nu+1)$ terms of $\chi(x,n)$ coincide with those of $\bar{\chi}(x,n)$ as closely as desired.

Functions $\chi(x,n)$ with the same form and properties, like those studied here, to which Equation (28) applies unconditionally, can also be derived from the above function $\chi(x;v)$, which originates from an arbitrary function $\psi(u)$. We can always determine v_n, a positive quantity dependent on n, such that the difference

$$\chi(x;v_n) - \sum_{\nu=-n}^{\nu=+n} \phi(\nu v_n)\cos(\nu x)$$

converges to 0 if n grows infinitely. And then

$$\chi(x,n) = \sum_{\nu=-n}^{\nu=+n} \phi(\nu v_n)\cos(\nu x) = 1 + 2\sum_{\nu=1}^{n} \phi(\nu v_n)\cos(\nu x) \tag{31}$$

has the properties stated above.

Obviously, this is not to say that all functions can be obtained in this way.

Finally it should be remarked that Equation (28) also holds for a function $f(x)$ that is uniquely defined and continuous on $[-c,c]$, without requiring that $f(c)=f(-c)$. For $x=\pm c$, we set the left hand side of the equation to be

$$\frac{1}{2}\left(f(-c)+f(c)\right),$$

instead of $f(\pm c)$.