Probability HW4

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Problem1

Problem 1.

a).

For any $\varepsilon > 0$:

$$P(|X_n - c| \geq \varepsilon) \geq 0.$$

$$\Rightarrow P(|X_n - c|^2 \geq \varepsilon^2) \leq \frac{E[|X_n - c|^2]}{\varepsilon^2} \quad \text{(Markov's inequality)}$$

if $n \to \infty$, we will get

$$0 \leq \lim_{n\to\infty} P(|X_n - c| \geq \varepsilon) \leq \lim_{n\to\infty} \frac{E[|X_n - c|^2]}{\varepsilon^2} = 0.$$

$\therefore \lim_{n\to\infty} P(|X_n - c| \geq \varepsilon) = 0.$, which means that

$X_n \xrightarrow{P} c$

b). Let $X_n = \begin{cases} 0, & \text{with probability } (1 - \frac{1}{n}) \\ \sqrt{n}, & \text{with probability } (\frac{1}{n}). \end{cases}$

We have $P(|X_n - 0| \geq \varepsilon) \leq \frac{1}{n}$, for any $n \geq \varepsilon^2$, $\varepsilon > 0$.

which implies that $\lim_{n\to\infty} P(|X_n - 0| \geq \varepsilon) = 0$, $(X_n \xrightarrow{P} 0)$

But. $E[(X_n - 0)^2] = \frac{1}{n}(\sqrt{n})^2 + (1 - \frac{1}{n}) \cdot 0^2 = 1 \neq 0$. for all n.

So, X_n doesn't converage to 0 in the mean square.

Problem2

Problem 2.

We know $X_n \xrightarrow{P} a$ and $Y_n \xrightarrow{P} b$, we need to proof that

$X_n Y_n \xrightarrow{P} ab$. $\left(\lim_{n\to\infty} P(|X_nY_n - ab| \geq \varepsilon) = 0\right)$

by triangle inequality, we can know:

$|X_n Y_n - ab| = |X_nY_n - aY_n + aY_n - ab| \leq |Y_n||X_n - a| + |a||Y_n - b|$

$\Rightarrow P(|X_nY_n - ab| \geq \varepsilon) \leq P(|Y_n||X_n - a| \geq \frac{\varepsilon}{2}) + P(|a||Y_n - b| \geq \frac{\varepsilon}{2})$

① $P(|Y_n||X_n - a| \geq \frac{\varepsilon}{2}) \leq P(|Y_n| \geq M) + P(|X_n - a| \geq \frac{\varepsilon}{2M})$, for $M > 0$.

$\because$ We know $Y_n \xrightarrow{P} b$ and $X_n \xrightarrow{P} a$.

$\therefore$ for any $\delta > 0$, $\forall M > 0$ s.t $P(|Y_n| \geq M) < \delta$.

also $\forall N > 0$ s.t $P(|X_n - a| \geq \frac{\varepsilon}{2M}) < \delta$, for $n > N$

We can know $P(|Y_n||X_n - a| \geq \frac{\varepsilon}{2}) \to 0$, as $n \to \infty$

② $P\left(|a||Y_n-b| \geq \frac{\varepsilon}{2}\right) = P\left(|Y_n-b| \geq \frac{\varepsilon}{2|a|}\right)$

$\because Y_n \xrightarrow{P} b$,

$\therefore$ For any $\delta > 0$, $\forall N > 0$ s.t. $P\left(|Y_n-b| \geq \frac{\varepsilon}{2|a|}\right)$, for $n > N$.

We can know $P\left(|a||Y_n-b| \geq \frac{\varepsilon}{2}\right) \to 0$.

Therefore, from ①+②. we can know

$P\left(|Y_n||X_n-a| \geq \frac{\varepsilon}{2}\right)$ and $P\left(|a||Y_n-b| \geq \frac{\varepsilon}{2}\right)$ converage to 0.

Then, $P\left(|X_nY_n-ab| \geq \varepsilon\right) \to 0$, as $n \to \infty$.

Thus. $X_nY_n \xrightarrow{P} ab$.

Problem3

Problem3

(a). We need to proof $E[C_{n+1} | C_n] = C_n$

We can know $C_{n+1} = X_{n+1} C_n$

$\Rightarrow E[C_{n+1} | C_n] = E[C_n X_{n+1} | C_n]$

$\because$ C_n is know (it's a constant with respect to the conditional expectation)

$\therefore$ We can rewrite it as

$E[C_n X_{n+1} | C_n] = C_n E[X_{n+1} | C_n]$

Because $\{X_i\}_{i=1}^{n}$ are i.i.d, so X_{n+1} and C_n are also independent: $E[X_{n+1} | C_n] = E[X_{n+1}] = 1$

Therefor, We can get

$E[C_{n+1} | C_n] = C_n E[X_{n+1} | C_n] = C_n \cdot 1 = C_n$ #

(b) We need to proof $C_n \to 0$ almost surely as $n \to \infty$

$$\ln(C_n) = \sum_{k=1}^{n} \ln(X_k).$$

By SLLN, we can know the average of i.i.d r.v. converage almost surely to their expected value.

$\frac{1}{n}\sum_{k=1}^{n} \ln(X_k) \to E[\ln(X_n)]$ almost surely as $n \to \infty$.

From the problem, we can know $E[\ln(X_n)] < 0$.

$$\frac{\ln(C_n)}{n} = \frac{1}{n}\sum_{k=1}^{n} \ln(X_k) \to E[\ln X_n] < 0$$

$\therefore \ln(C_n) \to -\infty$ almost surely as $n \to \infty$

Therefore, we can get:

$$C_n = e^{\ln C_n} \to e^{-\infty} = 0.$$

$\Rightarrow C_n \to 0$

Problem4

Problem 4.

1). $S_n = X_1 + X_2 + \dots + X_n$, which X_i are independent Bernoulli r.v with parameter p. So:

Mean:

$$E[S_n] = E[X_1 + X_2 + \dots + X_n] = np.$$

Variance:

$$Var(S_n) = Var(X_1 + X_2 + \dots + X_n) = np(1-p)$$

for large n,

$$\frac{S_n - np}{\sqrt{np(1-p)}} \xrightarrow{d} N(0,1).$$

$$\therefore \; S_n \sim N(np, np(1-p))$$

b)

$$Q_n(x) = P(S_n = x) = \binom{n}{x} p^x (1-p)^{n-x} = \binom{n}{x} \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$\therefore Q_n(x) = \binom{n}{x} \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

For large n:

$$\frac{\cancel{n!}\; n(n-1)(n-2)\cdots(n-k+1)}{x!\,\cancel{(n-x)!}} = \approx \frac{n^x}{x!}$$

$$\Rightarrow Q_n(x) \approx \frac{\cancel{n^x}}{x!} \cdot \frac{\lambda^x}{\cancel{n^x}} \cdot \left(1-\frac{\lambda}{n}\right)^n \cdot \left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^x}{x!} \cdot e^{-\lambda} \cdot 1, \text{ as } n \to \infty$$

Therefore,

$$\lim_{n\to\infty} Q_n(x) = \frac{\lambda^x}{x!} e^{-\lambda}$$

c).

① Why (b) doesn't contradict to CLT:

CLT applies to "scaled sums" of r.v. In contrast, part (b) is talk about "exact distribution" of S_n (not standardized) when $p=\frac{\lambda}{n}$.

Also the p in part (b) will change will n, and making the variance of S_n shrink as n grows:

$$Var(S_n) = np(1-p) \approx \lambda\left(1-\frac{\lambda}{n}\right) \to \lambda \quad \text{as } n \to \infty.$$

This leads to a posson limit, not normal limit.

Thus. (b) and CLT describe different scenarios:

$\begin{cases} \text{CLT assumpt } p \text{ is fixed as } n\to\infty \\ \text{(b) assumpt } p=\frac{\lambda}{n}\text{, leading to different limiting distribution} \end{cases}$

∴ part (b) doesn't contridict to CLT.

② When $p=\frac{\lambda}{n}$, the variane of S_n become small, making the normal approximation from CLT less accurate (the limiting distribution is posson, not normal). In such case, using CLT approximate $N(np, np(1-p))$ can lead to the error because the variance is too small for the normall distribution to be a good fit.

Problem5

Problem 5.

a). From the question, we have m students who take the RL course and distributed randomly among n roomate pairs. ~~Also~~ this pair can only contribute to X.

$$\therefore P(X \mid M=m) = \frac{\binom{m}{2}}{\binom{2n}{2}}$$

$$= \frac{m(m-1)}{2n(2n-1)}$$

$$\Rightarrow E[X \mid M=m] = \cancel{n} \cdot \frac{m(m-1)}{2\cancel{n}(2n-1)} = \frac{m(m-1)}{2(2n-1)}$$

b). By LIE:

$$E[X] = E\left[E[X \mid M=m]\right] = \sum_{m=0}^{2n} \frac{m(m-1)}{2(2n-1)} \cdot P(M=m)$$

$\because M \sim \text{Binomial}(2n, p)$

$$\therefore P(M=m) = \binom{2n}{m} p^m (1-p)^{2n-m}.$$

$$\Rightarrow E[X] = \sum_{m=0}^{2n} \frac{m(m-1)}{2(2n-1)} \cdot \binom{2n}{m} p^m (1-p)^{2n-m}.$$

$$= \frac{1}{2(2n-1)} \left(\sum_{m=0}^{2n} (m^2 - m) \binom{2n}{m} p^m (1-p)^{2n-m} \right)$$

$$= \frac{1}{2(2n-1)} \left(\sum_{m=0}^{2n} m^2 \binom{2n}{m} p^m (1-p)^{2n-m} - \sum_{m=0}^{2n} m \binom{2n}{m} p^m (1-p)^{2n-m} \right)$$

$$= \frac{1}{2(2n-1)} \left(E[M^2] - E[M] \right)$$

$$= \frac{1}{2(2n-1)} \left(Var(M) + (E[M])^2 - E[M] \right)$$

$$= \frac{1}{2(2n-1)} \left(2np(1-p) + (2np)^2 - 2np \right)$$

$$= \frac{(2np(1-p) + 4n^2p^2 - 2np)}{2(2n-1)} = \frac{2n^2p^2 - np^2}{(2n-1)}$$