Probability HW4

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Problem/1.
a). For any 670;
$P(X_n-c ^2\xi) = P(X_n-c ^2]$ $P(X_n-c ^2,\xi^2) \leq \frac{E[X_n-c ^2]}{\xi^2} \qquad (Markov's inequality)$
if $n \to \infty$ we will get
0< lim p(Xn-c 7) < lim =[Xr-c ²] = 0.
i lim P(Xn-c 3E)=0., which means that.
Xn Pac.

De Let
$$X_n = \{0^n, \text{ with probability}(h^n)\}$$

So, with probability (h^n) .

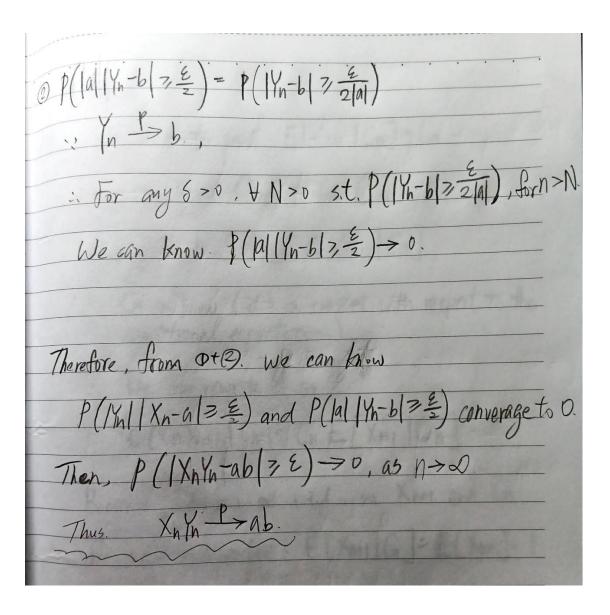
We have $P(|X_n - o| \ge 2) \le \frac{1}{h}$, for any $h \ge 2$, $E > 0$.

Which implies that $\lim_{n \to \infty} P(|X_n - o| \ge 2) = 0$, $(X_n \to 0)$

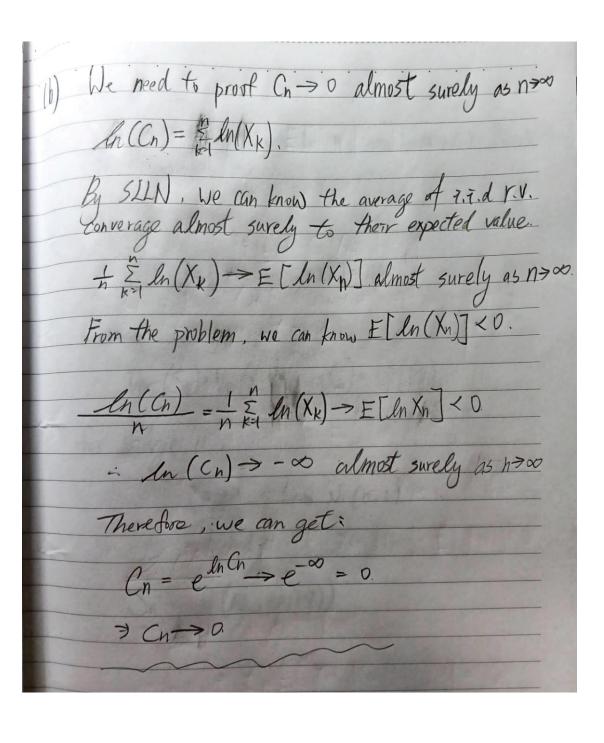
But $E[(X_n - o)^2] = \frac{1}{h}(\sqrt{h}) + (1 + \frac{1}{h}) \cdot o^2 = 1 + 0$. For all $n = 1$.

So, X_n doesn't converage to 0 in the mean square.

Problem 2
ble know Xn > a and Yn > b, we need to proof that
- Xn Yn - ab (lim P(Xn Yn - ab 24) = 0)
by triangle inequality we can know:
1 Xn Yn-ab/= /Xn/n-a/n+a/n-ab/= Yn//xn-a + a /n-b
> P(Xn7n-ab 36) < P(Yn Xn-a 3€)+P(a Yn-b 3€)
Φ (Y _n X _n -a = ½) ≤ P(Y _n = M) + P(X _n -a = ½) For M>0.
We know In b. md Xn Da.
: for any 6.20, \mo s.t P(Yn > M) < 6.
also UN=06.t P(Xn-1===)<6, for n=N
We can know $P(Y_n X_n-a ^3\frac{\epsilon}{2}) \rightarrow 0$, as $n \rightarrow \infty$



Problem3
We need to proof E[Cn+1 Cn] = Cn
We can know Cn+1 = Xn+1 Cn
=) E[Cn+1 Cn] = E[Cn Xn+1 Cn]
conditional expectation)
:. We can rewrite it as E
E[Cn Xn+1 Cn] = Cn E[X+1 Cn]
Because [Xi] are i.i.d., so Xn+1 and Ch
are also Independent: E[Xn+1]Gn]=E[Xn+1]=1.
Therefor, We can get!
E[Cht, Ch] = Ch E[Xm, Ch] = Ch = Ch
E Chillians Chillians Chillians



Problem 4.
1). Sn=Xi+X2++Xn, which Xi are independent Bernoultir.v with parameter p. So:
Mean :
E[Sn]=E[X1+X5++Xn]=np.
Variance:
Var (Sh)= Var (X1+Xe++Xn)= np(17)
for large n,
SN-NP d > N (0,1).
: Sn ~ N (np, np(1-p))
On Welf of the

 $Q_n(x) = p'(S_n = x) = {n \choose x} p^x (1-p)^{n \times x} = {n \choose x} {n \choose n}^x (1-n)^{n \times x}$ $2 Q_n(x) = \binom{n}{x} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{nx}$ For large n:

**Monday (n+1) nx

**X2 (n+x)? ** X! $\Rightarrow Q_n(x) \approx \frac{x^2}{x!} \cdot \frac{x^2}{n!} \cdot \left(1 - \frac{x}{n}\right)^n \cdot \left(1 - \frac{x}{n}\right)^{-x}$ = 1x · e -1, as n -> 0 $\lim_{N\to\infty} Q_n(x) = \frac{\lambda^x}{x!} e^{\lambda}$

O Why (b) doesn't contradict to CLT: CLT applies to scaled sums of r.v. In contrast, part (b) is talk about exact distribution of Sn. (not standardized) when p= 1. Also the p in part (b) will change will n, and making the variance of Sn strink as n grows: $Var(S_n) = np(1-p) \approx n(1-\frac{1}{h}) \rightarrow \lambda \quad \text{as } n \rightarrow \infty$. This leads to a possion limit, not normal limit. Thus. (b) and CLT describe different scenarios: 5 CLT assumpt p 13 fixed as n→∞ (b) assumpt $p=\frac{n}{n}$, leading to different limiting i. part (b) doesn't contridict to CLT

Q When P= \$\frac{\partial}{n}\$, the variance of Sn become small,

making the normal approximation from CLT

less accurate (time limiting distribution is

possion, not normal). In such case, using

CLT approximate \$\mathbb{N}(np, np(1p))\$ can lead

to the error because the variance is too small

For the normall distribution to be a good fit.

Problem 5.
a). From the question, we have m students who
take the RL course and distributed randomly among
n roomate pairs. Also this pair can only contribute
$ \begin{array}{c} \text{to } X. \\ \text{if } P(X \mid M=m) = \frac{\binom{m}{2}}{\binom{2n}{2}} \end{array} $
$=\frac{m(m-1)}{2h(2h-1)}$
$\exists \left[\times \left[M=m \right] = M \cdot \frac{m \left(m+1 \right)}{2 \pi \left(2 m+1 \right)} = \frac{m \left(m+1 \right)}{2 \left(2 n+1 \right)}$

b) By | TE:

$$E[X] = E[E[X|M^{2}m]] = \sum_{n=0}^{2n} \frac{m(n-1)}{2(2n-1)} \cdot p(M=m)$$

$$\therefore p(M^{2}m) = \binom{2m}{m} p^{m} (p^{2n-m})$$

$$\Rightarrow E[X] = \sum_{m>12}^{2n} \frac{m(m-1)}{2(2n-1)} \cdot \binom{2n}{m} p^{m} (p^{2n-m})$$

$$= \frac{1}{2(2n-1)} (\sum_{m=0}^{2n} m^{2n} p^{m} (p^{2n-m}) p^{m} (p^{2n-m}) p^{m} (p^{2n-m})$$

$$= \frac{1}{2(2n-1)} (\sum_{m=0}^{2n} m^{2n} p^{2n-m} (p^{2n-m}) p^{m} (p^{2n-m}) p^{m} (p^{2n-m})$$

$$= \frac{1}{2(2n-1)} (\sum_{m=0}^{2n} m^{2m} p^{2m} (p^{2n-m}) p^{m} (p^{2n-$$