Probability HW1

109350008 張詠哲

Problem1

1-a

if A & B are mutually exclusive, $P(A\cap B) = 0$, then by axiom (3), $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$, we have $P(A\cup B) = P(A)+P(B)$

Therefore $P(A\cup B)+P(A\cap B) = P(A)+P(B)$

else by inclusion-exclusion principle

we have $P(A\cup B) = P(A)+P(B)-P(A\cap B)$, then we get

$P(A\cup B)+P(A\cap B) = P(A)+P(B)$ #

1-b

1. $P(A\cup B) \geq P(A)+P(B)$ × $\rightarrow$ $P(A\cup B) \leq P(A)+P(B)$

2. additivity axiom is correct for 2 set, which is $P(A\cup B) = P(A)+P(B)-P(A\cap B)$. But when apply to n events, the principle should focus on bounding the union, not applying exact additivity

$$\Rightarrow P\left(\bigcup_{i=1}^{k+1} A_i\right) = P\left(\bigcup_{i=1}^{k} A_i\right)+P(A_{k+1}) - P\left(\left(\bigcup_{i=1}^{k} A_i\right)\cap A_{k+1}\right) \quad \times$$

$$\Rightarrow P\left(\bigcup_{i=1}^{k+1} A_i\right) \leq P\left(\bigcup_{i=1}^{k} A_i\right)+P(A_{k+1})$$

Problem2

2-a. Let $S := \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} S_n$

$T := \{x : x \in S_k \text{, for infinitely many } k\}$

for $S \subseteq T$: so we can know $x \in \bigcup_{n=k}^{\infty} S_n$

Let $x \in S$. Suppose $x \notin T$, then x only appears in $S_1, S_2 \cdots S_k$ for finity many K, then we will get $x \notin \bigcup_{n=k+1}^{\infty} S_n$, which contradict to our assumption.

Therefore $S \subseteq T$.

for $T \subseteq S$:

Let $x \in T$. Then there exist a countable infinity sequence $a_1, a_2 \ldots a_m$ s.t. $x \in S_{a_m}$ for all $m \in \mathbb{N}$.

That implies $x \in \bigcup_{n=k}^{\infty} S_n$. for all $K \in \mathbb{N}$

Therefore, we can imply that $x \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} S_n$

2-b.

1. For each finite m. we have $\bigcap_{k=1}^{m} \bigcup_{n=k}^{m} S_n$, as the m grows, the interval $[k, m]$ become wider, meaning we are considering larger and lager sets in the intersection. Taking the limit as $m \to \infty$, we cover an infinite sequence of sets.

2. by the proof from (2-a), we know $\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} S_n := \{x \mid x \in S_n, \text{for infinite many } n\}$

Therefore from 1 & 2. we can get H is equivalent to $\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} S_n$

2-C

1. $A_1 = B - C$
$A_2 = B - C$
$A_3 = B - C$
$A_4 = C - B$
$\Rightarrow$ $A_1 \cap A_2 \cap A_3 \cap A_4 = \phi$. Then, we can know $A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 \cap \dots A_n = \phi$, for $n \in N$. So, we can imply that $\bigcap_{n=1}^{\infty} A_n = \phi$

2. Because A_n only be $B-C$ or $C-B$, which is defined by whether n is in Fibonacci sequence. Also, $(B-C) \cup (C-B) = (B \cup C) - (B \cap C)$. Then, we can imply $\bigcup_{n=1}^{\infty} A_n = (B \cup C) - (B \cap C)$

3. Because A_n must be $B-C$ or $C-B$, even for large n.

Thus, for sufficiently large k, $\bigcap_{n=k}^{\infty} A_n = \phi$.

Therefore, $\bigcap_{n=k}^{\infty} A_n$, for all $k \in \mathbb{N}$. Then we can get.

$$\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n = \phi$$

4. $\bigcup_{n=k}^{\infty} A_n$ covers all elements that appear in A_n for $n \geq k$.

Thus, $\bigcup_{n=k}^{\infty} A_n$ still equal to $(B \cup C) - (B \cap C)$.

Because $B-C$ and $C-B$ will eventually appear in some A_n

Even k is very large.

$$\therefore \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n = (B \cup C) - (B \cap C)$$

d.

(i) Suppose the set of $\mathbb{R}$ in $(0,1)$ is countably infinite.
That is we can list all such real numbers as:
$x_1, x_2, x_3 \ldots$. Each x_i is a real number in $(0,1)$

(ii). and for x_i, $x_i = 0.a_{i1}a_{i2}a_{i3}\ldots$, where a_{ij} is the j-th decimal digit of x_i

(iii). Let $y \in (0,1)$ also $y \in \mathbb{R}$ that is different from each x_i. And $y = 0.b_1b_2b_3\ldots$ where b_j is either 1 or 2. b_j is chosen as follows:

- If a_{jj} of x_j is 1, set $b_j = 2$.
- If a_{jj} of x_j is 2, set $b_j = 1$.
- If a_{jj} is other digit $(0, 3, 4 \ldots)$, arbitrarily choose $b_j = 1$ or 2.

This ensure that y is different from x_j at j-th digit.

$\Rightarrow$ Since y differs from each x_i at i-th decimal place, y can't not equal to any x_i. This contradicts the assumption that we list all real numbers in $(0,1)$. Thus, the real numbers in $(0,1)$ aren't countable, which are uncountable infinite.

Problem3

3-a

Approach 1.

Suppose $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n) = 0$

Let $B_k = \bigcup_{n=k}^{\infty} A_n$, for $n = 1, 2, 3, \dots$, also $B_1 \supseteq B_2 \supseteq B_3 \cdots B_k$.

Then, $P(\bigcap_{k=1}^{\infty} B_k) = \lim_{k \to \infty} P(B_k)$. We can get

$$P\left\{\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n\right\} = P\left(\bigcap_{k=1}^{\infty} B_k\right) = \lim_{k \to \infty} P(B_k)$$

$\because P(B_k) = P(\bigcup_{n=k}^{\infty} A_n) \leq \sum_{n=k}^{\infty} P(A_n)$, also $\sum_{n=k}^{\infty} A_n$ is finite

that means as the k growing, $\sum_{n=k}^{\infty} A_n$ will become smaller

Thus, when $k \to \infty$, $\sum_{n=k}^{\infty} P(A_n) \to 0$, then we can know

$$\lim_{k \to \infty} P(B_k) \leq \sum_{n=k}^{\infty} P(A_n) \to 0 \quad \#$$

Approach 2.

Let $B_k = \bigcup_{n=k}^{\infty} A_n$, $B_1 \supseteq B_2 \supseteq B_3 \dots \supseteq B_k \dots$,

Suppose $P(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n) > 0$, we are given $P(\bigcap_{k=1}^{\infty} B_k) = \lim_{k\to\infty} P(B_k) > 0$

By the assumption, this implied for sufficiently large k, $P(B_k)$ must remain bounded away from 0, which implied $P(\bigcup_{n=k}^{\infty} A_n) > 0$. This shows that for infinitely many events A_n, $\sum_{n=1}^{\infty} P(A_n)$ must diverge. Therefore, $\sum_{n=1}^{\infty} P(A_n) = \infty$.

3-b

for odd k, $P(A_k) = 0$.

for even k, $P(A_k) = \frac{1}{10} k^{-N}$.

$$\Rightarrow \sum_{k=1}^{\infty} P(A_k) = \sum_{\substack{k=2 \\ k \text{ even}}}^{\infty} \frac{1}{10} k^{-N} = \frac{1}{10} \sum_{m=1}^{\infty} (2m)^{-N}, \text{ for } k = 2m$$

Then we can get $\sum_{k=1}^{\infty} P(A_k) = \frac{1}{10} \sum_{m=1}^{\infty} (2m)^{-N} = \frac{1}{10} 2^{-N} \sum_{m=1}^{\infty} m^{-N}$

Since $\sum_{m=1}^{\infty} m^{-N}$ is a p-series, we can discuss it in 2 senario.

① $N>1$.

$\frac{1}{w}2^{-N}\sum_{m=1}^{\infty} m^{-N}$ will converge, which means $\sum_{k=1}^{\infty} P(A_k)$ is finite

$\sum_{k=1}^{\infty} P(A_k) < \infty$, then by Borel-Cantelli Lemma; this implies

$P(I)=0$

② $0<N\leq 1$.

$\frac{1}{w}2^{-N}\sum_{m=1}^{\infty} m^{-N}$ will diverge, which means $\sum_{k=1}^{\infty} P(A_k)$ is infinite

$\sum_{k=1}^{\infty} P(A_k)=\infty$, then by Borel-Cantelli Lemma, this implies

$P(I)=1$ (Guarante to observe an infinite number of Gryffindor)

Problem4

4-1.

Because we have N doors, M of them are empty. $N-M$ doors come with a prize. Therefore the sample space is. $\Omega = \{x \subseteq \{1,2,3\dots N\} \mid |x| = N-M\}$, which x is a subset from $\{1,2,3,\dots N\}$. The elements in x are C^{N}_{N-M}.

4-2.

(i). Bill's initial choice has a prize.

If Bill first choose a door with prize, and also win a prize after randomly switch a door the $P(E_{M,N})$ will be $\frac{N-M}{N} \cdot \frac{(N-M)-1}{N-2}$

(ii) Bill's initial choice doesn't have a prize.

If Bill first choose a door without prize, then win the prize after randomly switch a door.

$P(E_{M,N})$ will be $\frac{M}{N} \cdot \frac{N-M}{N-2}$

∴ from (i)+(ii), we can get $P(E_{M,N}) = \frac{N-M}{N} \cdot \frac{N-M-1}{N-2} + \frac{M}{N} \cdot \frac{N-M}{N-2}$

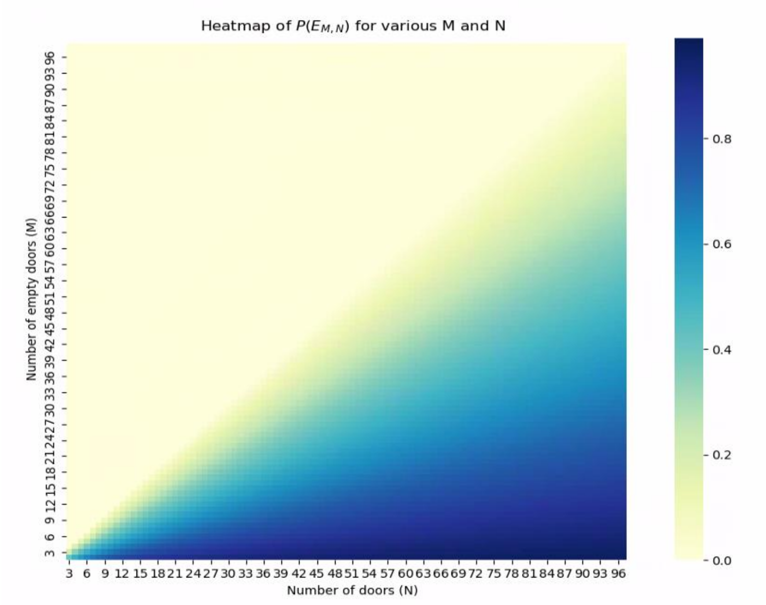
4-3

Code:

```
import numpy as np
import matplotlib.pyplot as plt
import seaborn as sns

def calculate_probability(M, N):
```

```python
if N < 3 or M < 2 or M >= N:
return 0

Init_prize = ((N-M)/N) * ((N-M-1)/(N-2))
Init_without_prize = (M/N) * ((N-M)/(N-2))

return Init_prize + Init_without_prize


if __name__ == "__main__":

N_values = range(3, 101)
M_values = range(2, 101)
probabilities = np.zeros((len(M_values), len(N_values)))

for i, M in enumerate(M_values):
for j, N in enumerate(N_values):
if M < N:
probabilities[i, j] = calculate_probability(M, N)

plt.figure(figsize=(10, 8))
sns.heatmap(probabilities, cmap="YlGnBu")
plt.ylim(2, 100)
plt.xlim(3, 100)
plt.xlabel('Number of doors (N)')
plt.ylabel('Number of empty doors (M)')
plt.title('Heatmap of $P(E_{M,N})$ for various M and N')
plt.show()
```

Result



Result explanation

- As N increases, the probability tends to stabilize.
- For smaller values of M, the probability increases as N increases.
- For larger values of M (closer to N), the probability of winning by switching decreases.

Problem5

5-1. SGRGGH GRSH
2 5 2 1

$$\begin{cases} P(A_1|C) = \dfrac{P(C|A_1)P(A_1)}{P(C)} = \dfrac{(0.25)^6 \times \frac{1}{2}}{P(C)} \\[2ex] P(A_2|C) = \dfrac{P(C|A_2)P(A_2)}{P(C)} = \dfrac{0.2 \times 0.5 \times 0.2 \times 0.5 \times 0.5 \times 0.1 \times \frac{1}{4}}{P(C)} \\[2ex] P(A_3|C) = \dfrac{P(C|A_3)P(A_3)}{P(C)} = \dfrac{0.2 \times 0.2 \times 0.5 \times 0.2 \times 0.2 \times 0.1 \times \frac{1}{4}}{P(C)} \end{cases}$$

, where $P(C) = P(C|A_1)P(A_1) + P(C|A_2)P(A_2) + P(C|A_3)P(A_3)$

$$\Rightarrow \begin{cases} P(A_1|C) \approx \dfrac{0.00012207}{P(C)} \approx 0.4576 \\[2ex] P(A_2|C) \approx \dfrac{0.00012500}{P(C)} \approx 0.4685 \\[2ex] P(A_3|C) \approx \dfrac{0.00002000}{P(C)} \approx 0.0749 \end{cases}$$

5-2.

A_2