EECS 336: Introduction to Algorithms	**Lecture 3**
Greedy Algorithms	Interval Scheduling

Reading: 4.1–4.2

Last Time:

- computational tractibility
- Big-oh

Today:

- Big-Oh (cont.)
- Graph Review
- Greedy Algorithms
- Interval Scheduling

Some common runtimes

- constant,
- logarithmic,
- linear,
- $n \log n$,
- quadratic,
- cubic,
- exponential

Recall: $T(n)$ is worst-case runtime for instance of size n.

Recall: $T(n)$ is $O(f(n))$ if $\exists n_0, c > 0$ such that $\forall n > n_0,\ T(n) < cf(n)$.

Recall: $\Omega(\cdot)$ and $\Theta(\cdot)$.

Example: $T(n) = 5n^2 - n$ is:

	$O(\cdot)$	$\Omega(\cdot)$	$\Theta(\cdot)$
n^3			
n^2			
n			

Greedy Algorithms

- build solution in steps.
- each step myopically optimal
- hard part: prove final solution is optimal

Question: For what problems are greedy algorithms optimal?

Scheduling

- many tasks competing for limited resources.
- temporal constraints.
  - start & end times,
  - deadlines, and
  - one job at a time.
- find most efficient schedule.
  - most tasks schedules, or
  - best tasks scheduled

Example: CPU scheduling.

Interval Scheduling

"sharing a single resource"

Input:

- n jobs
- one machine
- requests: job i needs machine between times $s(i)$ and $f(i)$

Goal: schedule to maximize # of jobs scheduled.

Examples: Greedy by ...

- "start time"

```
  ---  ---  ---  ---  ---
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```

- "smallest size"

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```

- "fewest incompatibilities"

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```

Greedy Algorithm for Interval Scheduling

Idea: scheduling the earliest finish time first, leaves the least constraints on remaining schedule.

Def: jobs i and j are

- **incompatible** if $[s(i), f(i)] \cap [s(j), f(j)] \neq \emptyset$
- otherwise **compatible**.
- set S is **compatible** if all $i, j \in S$ are compatible.

Examples:

```
-----    or   ----   or  ---
 -----       ------          ---
```

Algorithm: Greedy by Min. Finish Time

1. $S = \emptyset$
2. Sort jobs by increasing finish time.
3. For each job j (in sorted order):
    - if j if compatible with S

      schedule j: $S \leftarrow S \cup \{j\}$
    - else discard j

Analysis

Runtime

$$\begin{aligned} T(n) &\le \underbrace{n \log n}_{\text{sort}} + \overbrace{\sum\nolimits_j j}^{\text{check compatibility}} \\ &\approx n \log n + n^2 \\ &= O(n^2). \end{aligned}$$

Idea: Job j in alg. is compatible if it is compatible with last scheduled job.

$$\begin{aligned} T(n) &= n \log n + n \\ &= \Theta(n \log n) \end{aligned}$$

Correctness

"schedule is compatible and optimal"

Lemma 1: schedule of algorithm is compatible

Proof: (by induction, straightforward)

Def:

- let $i_1, \ldots, i_k$ be jobs scheduled by greedy
- let $j_1, \ldots, j_m$ be jobs scheduled by OPT

Goal: show $k = m$.

Approach: "Greedy Stays Ahead"

Lemma 2: for $r \leq k$, $f(i_r) \leq f(j_r)$

Proof: (induction on r)

base case: $r = 0$

- add dummy job 0 with $s(0) = f(0) = -\infty$
- only change: OPT and GREEDY schedule dummy
- so $f(i_0) = f(j_0)$

inductive hypothesis: $f(i_r) \leq f(j_r)$

inductive step:

- Let $I = \{\text{jobs starting after } f(i_r)\}$

  $J = \{\text{jobs starting after } f(j_r)\}$
- IH $\Rightarrow J \subseteq I$
- Alg $\Rightarrow f(i_{r+1}) = \min_{j \in I} f(j)$

  $\leq \min_{j \in J} f(j)$

  $\leq f(j_{r+1})$.

Theorem: Greedy alg. is optimal

Proof: (by contradiction)

- OPT has job j_{k+1} but greedy terminates at k.
- lemma 2 (with $r = k$)

  $$\Rightarrow f(i_k) \leq f(j_k) \tag{1}$$
- j_{k+1} is compatible with j_k

  $$\Rightarrow f(j_k) \leq s(k_{k+1}) \tag{2}$$
- (1)&(2)

  $\Rightarrow f(i_k) \leq s(j_{k+1})$

  $\Rightarrow j_{k+1}$ is compatible with i_k

  $\Rightarrow$ alg doesn't terminate at k

$\rightarrow\leftarrow$

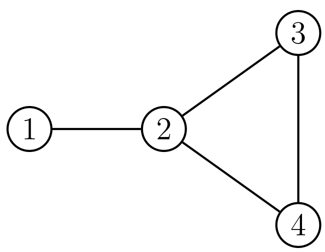
Graphs

“encode pair-wise relationships”

Examples: computer networks, social networks, travel networks, dependencies.

$$G = (\overset{\text{vertices}}{V}, \underset{\text{edges}}{E})$$

Example:



- $V = \{1, 2, 3, 4\}$
- $E = \{(1, 2), (2, 3), (2, 4), (3, 4)\}$

Concepts

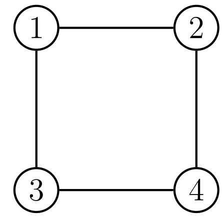
- degree
- neighbors
- paths, path length
- distance
- connectivity, connected components
- directed graphs.

Graph Traversals

“visit all the vertices in a connected component of graph”

- Breadth First Search (BFS).

  Example:

  BFS from 1: 1, 2, 3, 4 or 1, 3, 2, 4.

- Depth First Search (DFS).

  Example: DFS from 1: 1, 2, 4, 3 or 1, 3, 4, 2.