Reducing Zoo Tycoon to max flow

Step 1:
a) Connect s to each $j \in m$ with capacity T_j.
b) Connect each $i \in n$ to t with capacity F_i.
c) For each $i \in n$, connect i with j for each $j \in S_i$ with capacity T_j.
d) For each $i \in n$, create a node x_i and 1) connect x_i with i with capacity D_i, 2) connect x_i with j for each $j \notin S_i, j \in m$ with capacity T_j.

Step 2:
Compute max flow f from s to t.

Step 3:
If $|f| = \sum_{i=1}^{n} F_i$, then food is enough to feed all animals. Otherwise, food is not enough.

Correctness

Conservation of total food means that total amount of food supplied equals to total consumption. Food supply only flows in one direction from s to t.

From s, total actual supply is $|f| = \sum_{j=1}^{m} t_j$, where t_j is the actual supply amount for each type of food. For each i, t_j, part of food is consumed in preference, where $t_{j,i} = (j, i), j \in S_i$; another part of food may be consumed non-preference, where $t_{j,i} = (j, x_i), j \notin S_i$. It holds that $|f| = \sum_{j=1}^{m} t_{j,i}, \forall i$.

Each animal consumes $t_{j,i}$ if $j \in S_i$ from preference plus $f(x_i, i)$ from non-preference. It holds that $|f| = \sum_{i=1}^{n} f(x_i, i) + t_{j,i} \forall j \in S_i$.

The only condition where food is enough for all animals is that all (i, t) edges are in full capacity. Thus, only when $|f| = \sum_{i=1}^{n} F_i$, food is enough.

Thus, each flow in the graph corresponds to a food supply chain from s to t, and max flow can be checked with total demand to see whether supply is enough.

Runtime Analysis

Max flow runs in $O(|E|C)$. Here, $C = \sum_{i=1}^{n} F_i$, $|E| = O(mn)$. Thus, algorithm runs in $O(mnC)$, where $C = \sum_{i=1}^{n} F_i$