EECS 336 Fall 2015
Homework Problem 8.1

Algorithm

Construct multiple cycles of directed edges:

1. For graph $G = (V, E)$, construct a new graph G', which contain two complete identical copies of V, where the entire set of $2|V|$ vertices is denoted as V'. Denote each vertex in V as v, and each corresponding pair of vertices in V' as v_1, v_2.

2. For each directed edge $e = (u, v) \in G$, connect vertices in V' such that $e' = (u_1, v_2) \in G$.

3. If there exists a perfect matching in G', returns TRUE. Otherwise, returns FALSE.

Correctness

We observe that in order for each vertex $v \in G$ to be included in exactly one cycle of $C_1, ..., C_k$, there must be exactly one edge leading into v, and another edge going out of v.

Connecting edges in $e \in E$ among vertices in V' transforms the original problem into a bipartite matching problem. If there exists a perfect matching in G', then each vertex $v \in V$ is connected to exactly two different vertices u, w (or just one vertex u), since v is represented by two vertices v_1, v_2 in bipartite G', where exists edges $(v_1, u_2), (w_1, v_2)$ (or $(v_1, u_2), (v_2, u_1)$, in which case C_j has size of two). Thus, v must be a member of some C_j and is in C_j only.

If each $v \in V$ is a member of exactly one C_j, then v must be connected with two different vertices u, w by $(v, u), (w, v)$ (or just one vertex u when C_j is of size two). From the construction above, v_1 must be connected with u_2, and v_2 must be connected with w_1 (or u_1 for C_j of size two). Thus, all vertices $v_1, v_2 \in V'$ constitute a perfect matching in G' if and only if each corresponding $v \in V$ belongs to exactly one cycle C_j.

Note that v_1, v_2 cannot be connected to each other in any perfect matching in G', which is equivalent to the fact that v cannot become a cycle by itself in G.

Runtime Analysis

Construction of G' takes $O(|V| + |E|)$, since each vertex is copied twice, and each directed edge is connected once in G. Since bipartite matching takes $O(|V||E|)$, total algorithm takes $O(|V||E|)$, which is **polynomial**.