EECS 336 Problem 1.3

(Proof by induction on k)

**Claim:** Given an arbitrary outcome graph $G_k$ with $k > 0$ vertices, there always exists a path that traverse all the vertices along the direction of the edges. (Which is equivalent to the definition of total ordering in the problem statement.)

**Base case:** $k = 1$, there is only one vertex in the outcome graph $G$. The vertex itself forms a total ordering.

**Inductive hypothesis:** if there is a total ordering $(i_1, i_2, ..., i_k)$ in $G_k$ such that $i_1$ beats $i_2$, $i_2$ beats $i_3$, etc. (Note: $(i_1, i_2, ..., i_k)$ is a sorted ordering)

**Inductive steps:** There always exists a total ordering when $G_k$ is expanded by adding $i_{k+1}$ to form $G_{k+1}$.

Let a binary variable $d_j$ denote if $i_{k+1}$ could beat $i_j$, where $d_j = 1$ if $i_{k+1}$ could beat $i_j$, and $d_j = 0$ if $i_{k+1}$ could not beat $i_j$. Then $\mathbf{d} = (d_1, d_2, ..., d_k)$.

1) if $d_1 = 1$, then $i_{k+1}$ could beat $i_1$. So $i_{k+1}$could be inserted in front of the first element to form a total ordering regardless of the other components in $\mathbf{d}$. The total ordering would be $(i_{k+1}, i_1, i_2, ..., i_k)$.

2) if $d_1 = 0$,

(a) if there exists at least one component of $\mathbf{d}$ equals to one, let $d_m (m \geq 2)$ be the first component of $\mathbf{d}$ that equals to one from the very left. Then $d_{m-1}$ is always 0. In this case, $i_{k+1}$ could be inserted between $i_{m-1}$ and $i_m$ to form a total ordering as $(i_1, i_2, ..., i_{m-1}, i_{k+1}, i_m, ..., i_k)$

(b) if there is no component of $\mathbf{d}$ equals to one. $\Rightarrow$ $i_{k+1}$ cannot beat anyone in the total ordering of $G_k$. Then $i_{k+1}$ could be placed at the end of the previous total ordering in $G_k$ to form a new total ordering in $G_{k+1}$ as $(i_1, i_2, ..., i_k, i_{k+1})$

Proved.