EECS 336 Fall 2015
Homework Problem 2.3

MST Trivia

a) **False**

By a counter example. Suppose the shortest path between any two vertices $e = (u, v)$ must be part of some MST. From Theorem 4.20, e cannot be the heaviest edge in any cycle in an arbitrary graph G. Say length of edge $e = (a, b) = 3$ with length of edges $(b, c) = (a, c) = 2$, then e is the shortest path between ab and the heaviest edge in cycle abc. This contradicts with the inferred fact that e cannot be heaviest edge in a cycle. Thus, the statement is false.

b) **False**

This proof assumes that graph $G = (V, E)$ contains at least a pair of vertices $u, v \in V$ that are not connected by some edge $e \in E$.

By a counter example. Suppose edge $e_m = (u, v)$ is the unique heaviest edge in G, and $e_1 = (v, w) \in E$ while $f_1 = (w, u) \notin E$. Graph $G = (\{u, v, w\}, \{e_m, e_1\}$. Then MST $T \in G$ connects u, v, w with edges e_m, e_1, and thus $e_m \in T$ which is an MST.

c) **True**

Let $e = (v, w)$ be the unique lightest edge in some cycle C' of graph $G = (V, E)$. Let T be an MST of G that does not contain e. We prove the statement by contradiction showing that T does not have the minimum possible total cost.

Since T is an MST, there exists a path P in T from v to w. Since $e = (v, w)$, cycle C consisting of P and e is the only cycle in $(V, T \cup \{e\})$. Let u be the first node after v along P and edge $e' = (v, u)$. Since e is the lightest edge in C, $|e'| > |e|$. We claim that $T' = (T - \{e'\}) \cup \{e\}$ is also a MST: T' is connected, since T is connected, and any path that uses e' can be rerouted via e; T is also acyclic, since C is the only cycle in $(V, T \cup \{e\})$, and no long exists after breaking e'. Since $|e'| > |e|$, the total cost of T' is less than that of T, as desired. Thus, e must be part of any MST.

d) **True**

For the set of non-unique lightest edges E' consisting of $e_1 = (u_1, v_1), e_2 = (u_2, v_2), ..., e_k = (u_k, v_k)$ $(k \geq 1)$ where u's and v's may refer to the same vertices, each e_i is a lightest edge in graph $G = (V, E)$. Taking u_i as either vertex of e_i, e_i is then the shortest path between u_i and $(V - \{u_i\})$. According to Theorem 4.17, with u_i as the subset of nodes S in theorem statement, e_i is contained in some MST of G. Therefore, the statement is true.

e) **True**

According to Theorem 4.20, if edge $e = (u, v)$ is the heaviest edge in a cycle C in graph G, then e does not belong to any MST. Thus, the statement is true.