| EECS 336: Introduction to Algorithms | Lecture 11 |
|---|---|
| Dynamic Programming (cont) | Bellman-Ford |

**Reading:** 6.4, 6.8

**Last time:**

- Integer Knapsack
- Interval Pricing
- "finding a first decision"

**Today:**

- Shortest Paths.

## Suggested Approach

I. identify subproblem in english

   OPT($i$) = "optimal schedule of $\{i, \ldots, n\}$ (sorted by start time)"

II. specify subproblem recurrence

   $\text{OPT}(i) = \max(\text{OPT}(i+1), v_i + \text{OPT}(\text{next}(i)))$

III. identify base case

   $\text{OPT}(n+1) = 0$

IV. write iterative DP.

   (see last thurs)

## Finding Optimal Schedule

"traverse memoization table to find schedule"

**Algorithm:** schedule

$i = 1$

while $i < n$

  if memo[$i+1$] $< v_i +$ memo[next($i$)]

    schedule $i$; $i \leftarrow \text{next}(i)$.

  else

    $i \leftarrow i + 1$.

  endif

endwhile

# Shortest Paths with Negative Weights

"e.g., currency exchange: nodes are currencies, path weights are exchange rates, minimize product of path weights."

**Note:** to minimize product of path weights, can minimize sum of logs of path weights.

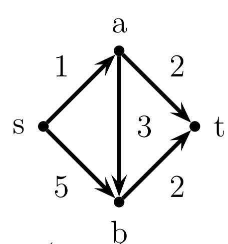
**Example:** $r_1 r_2 = 2^{\log_2 r_1} 2^{\log_2 r_2} = 2^{\log_2 r_1 + \log_2 r_2}$.

**Note:** if $r \leq 1$ then $\log r$ is negative.

**Recall:** Dijkstra's Algorithm

1. initialize known distance from $s$ as $\infty$, except $d(s) = 0$
2. take closest unknown vertex $v$
   (a) declare $v$ known.
   (b) update known distances to neighbors of $v$ if closer via $v$.
3. repeat (2) until $t$ known.

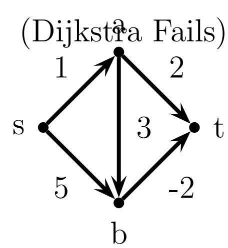
**Example:**




Shortest Path: d($s$-$a$-$t$) = 3.

## Negative Edge Weights

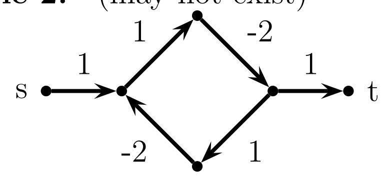
**Example 1:** (Dijkstra Fails)




Dijkstra's Path: d($s$-$a$-$t$) = 3

Shortest Path: d($s$-$a$-$b$-$t$) = 2.

**Example 2:** (may not exist)

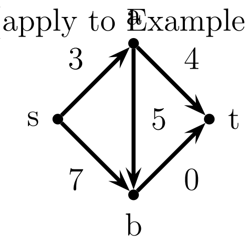



Negative cycle $\Rightarrow$ no shortest path.

First try:

- find most negative edge "$-c$"
- add $c$ to all edges.
- run Dijkstra

**Example:** (apply to Example 1)




Shortest Paths: $s$-$a$-$t$ or $s$-$b$-$t$, not shortest in original problem.

Second Try: Dynamic programming

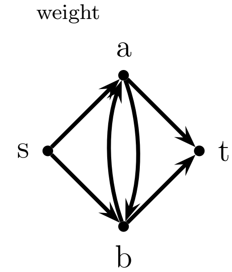
subproblem:

OPT($v$)

= shortest path from $v$ to $t$.

$= \min_{u \in N(v)} [\underbrace{c(v,u)}_{\text{weight}} + \text{OPT}(u)]$.

**Example:**




Subproblems have cyclic dependencies!

## Imposing measure of progress

"parameterize subproblems to keep track of progress"

**Lemma:** if $G$ has no negative cycles, then minimum cost path is **simple** (i.e., does not repeat nodes); therefore, it has at most $n-1$ edges.

**Proof:** (contradiction)

- let $P$ be the min-length path with fewest number of edges.
- suppose (for contradiction) that $P$ is not simple.

  $\Rightarrow$ $P$ repeats a vertex $v$.

- no negative cycle $\Rightarrow$path from $v$ to $v$ non-negative.

  $\Rightarrow$ can "splice out" cycle and not increase length.

  $\Rightarrow$ new path has fewer edges than $p$.

$\rightarrow\leftarrow$

**Idea:** if simple path goes $s \rightsquigarrow v \rightarrow u \rightsquigarrow t$ then $u$-$t$ path has one fewer edge than $v$-$t$ path.

## Part I: identify subproblem in english

$\mathrm{OPT}(v,k)$

$=$ "length of shortest path from $v$ to $t$ with at most $k$ edges."

## Part II: write recurrence

$\mathrm{OPT}(v,k)$

$= \min_{u \in N(v)} [c(v,u) + \mathrm{OPT}(u, k-1)]$

## Part III: base case

- for all $k$: $\mathrm{OPT}(t,k) = 0$.
- for all $v \neq t$: $\mathrm{OPT}(v,0) = \infty$.

## Part IV: iterative DP

**Algorithm:** Bellman-Ford

1. initialize

   for all $k$: $\mathrm{memo}[t,k] = 0$.

   for all $v \neq t$: $\mathrm{memo}[v,0] = \infty$.

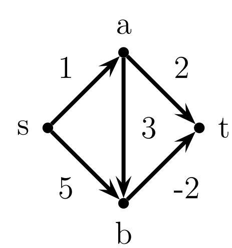
2. for $k = 1$ up to $n-1$,

   for all $v$

   $\mathrm{memo}[v,k] = \min_{u \in N(v)} \mathrm{OPT}(u, k-1)$.

3. return $\mathrm{memo}[s, n-1]$.

**Example:**




| | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $s$ | $\infty$ | $\infty$ | 3 | 2 |
| $a$ | $\infty$ | 2 | 1 | 1 |
| $b$ | $\infty$ | $-2$ | $-2$ | $-2$ |
| $t$ | 0 | 0 | 0 | 0 |

## Correctness

lemma + induction.

## Runtime

$$
\begin{aligned}
T(n,m) &= \overbrace{\text{“size of table”}}^{n^2} \times \overbrace{\text{“cost per entry”}}^{n} \\
&= O(n^3)
\end{aligned}
$$

(better accounting: $T(n,m) = O(n^2+nm) = O(nm)$)