EECS 336 Fall 2015
Homework Problem 8.2

Step One: Undirected Hamiltonian Cycle is NP-complete

Algorithm

Reduce directed Hamiltonian Cycle (HC) $G = (V, E)$ to undirected Hamiltonian Cycle (UHC) $G' = (V', E')$ and show that G' is TRUE HC if and only if G is TRUE UHC.

1. Construct G'.

For each $v \in V$, create three vertices v_1, v_2, v_3 in V'. For each directed edge $e = (u, v) \in E$, connect $(u_3, v_1) \in E'$.

2. The construction can be done in polynomial time, since G' has $3|V$ vertices and $2|V| + |E|$ edges.

3. **Correctness**

i) If G is TRUE HC, then G' is a TRUE UHC.

If G is TRUE, then there exists a directed cycle $C \in G$, where $C = \{v_1, ..., v_n, v_1\}$. Thus, for $1 \leq i < n$, there is a directed edge (v_i, v_{i+1}). Then it is possible to construct an undirected cycle C', where $C' = \{v'_{1,1}, v'_{1,2}, v'_{1,3}, ..., v'_{n,1}, v'_{n,2}, v'_{n,3}, v'_{1,1}\}$ with length of $3|V|$, which goes through each vertex exactly once except for the starting point. All edges in C' are undirected and can be created in G'. Therefore, given G is TRUE, we can obtain a UHC C' for G'.

ii) If G' is TRUE UHC, then G is a TRUE HC.

If G' is TRUE, then there exists a undirected cycle $C' \in G'$. For each set of vertices v_1, v_2, v_3, v_2 is only accessible via v_1 or v_3. Given the rule of construction, we can assume $C' = \{v'_{i,1}, v'_{i,2}, v'_{i,3}, ..., v'_{j,1}, v'_{j,2}, v'_{j,3}, v'_{i,1}\}(j > i)$. Then it is possible to construct a directed cycle $C = \{v_i, ..., v_j\} \in G$ with length of $|V|$, which contains all vertices in V and goes through each vertex exactly once except for the starting point. Therefore, given G' is TRUE, we can obtain a HC C for G.

Note

UHC can be transformed from HC in polynomial time, so UHC is $\in NP$. HC is known to be NP-hard. By way of construction and correctness proof, UHC is also NP-hard. Thus, UHC is NP-complete.

Step Two: From Lemma 8.19 on textbook, undirected Hamiltonian Path (UHP) is also NP-complete.

Step Three: Movie Seating (MS) problem can be reduced from UHP, G', and is also NP-complete.

i) Construction

Construct a complete graph G" containing list of friends V' from G' with weight $w_{u,v}$ of edge $(u_3, v_1) \in G'$ as the value point of directed relationship from u to v such that:

i-1) If u loves v, $w_{u,v} = 6$

i-2) If u befriends v, $w_{u,v} = 5$

i-3) If u is acquaintance with v, $w_{u,v} = 4$

i-4) If u is stranger with v, $w_{u,v} = 3$

i-5) If u hates v, $w_{u,v} = 0$

All internal edges (v_1, v_2) and (v_2, v_3) are given weights of 0. G' is a UHC if and only if G" has a total cost of $k + 6(n - 1)$.

ii) The construction can be done in polynomial time, since the number of edges in G' is in $O(|V|)$.

iii) **Correctness**

a) If G' has a TRUE UHC, then MS has total point of at least k.

If G' is TRUE, then there exists a undirected cycle $C' \in G'$, where $C' = \{v'_{i,1}, v'_{i,2}, v'_{i,3}, ..., v'_{j,1}, v'_{j,2}, v'_{j,3}, v'_{i,1}\}(j > i)$. The worst case of lowest points is with weights of all $(v'_{i,3}, v'_{i+1,1})$ edges being 0 (everyone hates each other). With $|V|$ people, there are $2(|V| - 1)$ edges that connect the 3-node with 1-node. Since the 'hate' edge is boosted by three points, total point is increased by $3 * 2(n - 1)$. Thus, if there exists C' in G', all vertices are traveled by exactly once, and the total weight in G" is at least $k + 6(n - 1)$, which corresponds to k total point in the original MS problem.

b) If MS has a total point of at least k, then G' has a TRUE UHC.

If MS has a lowest possible total point of k, then the boosted version G" has a total weights of $k+6(n-1)$ as a result of the construction. Then it is possible to create $C' = \{v'_{i,1}, v'_{i,2}, v'_{i,3}, ..., v'_{j,1}, v'_{j,2}, v'_{j,3}, v'_{i,1}\}(j > i)$, such that all vertices are connected in G' with all 3-node to 1-node vertices representing 'hatred'. Thus, if MS has a total point of at least k, then G' has a TRUE UHC.

Note

Movie Seating problem can be transformed from UHP in polynomial time, so MS is $\in NP$. UHP is known to be NP-hard, so MS is also NP-hard. Thus, MS is NP-complete.