EECS 336 Problem 1.2

a. False

Counterexample: $f(n) = 1, g(n) = 2$

First we prove $f(n)$ is $\Omega(g(n))$

$\exists n_0 = 1, C = \frac{1}{3}$ (both are constants), such that $\forall n > n_0$,

$f(n) = 1 > \frac{1}{3} \cdot 2 = Cg(n)$

Therefore, $f(n)$ is $\Omega(g(n))$

.

Then we prove that $log(f(n))$ is not $\Omega(log(g(n)))$ by contradiction:

Suppose $log(f(n))$ is $\Omega(log(g(n)))$, which implied by definition

$\exists n_1, C_1 > 0$ (both are constants), such that $\forall n > n_1$,

$log(f(n)) \geq C_1 log(g(n))$.

$0 \geq C_1 log(2)$

Since $C_1 > 0 \Rightarrow C_1 log(2) > 0$,

This contradicts the supposition that C_1 is a positive constant. Therefore, the supposition is false.

b. False

Counterexample: $f(n) = 2e^n, g(n) = e^n$

Firstly we prove $f(n)$ is $\theta(g(n))$:

$\exists n_0 = 1, C_0 = 3$, such that, $\forall n > n_0, f(n) = 2e^n < 3e^n = C_0 g(n)$
$\Rightarrow f(n)$ is $\Omega(g(n))$

Similarly, $\exists n_1 = 1, C_1 = 1$, such that, $\forall n > n_0, f(n) = 2e^n > e^n = C_1 g(n)$
$\Rightarrow f(n)$ is $O(g(n))$

Therefore, $f(n)$ is $\theta(g(n))$

Then we prove that $2^{f(n)}$ is not $\theta(2^{g(n)})$ by contradiction:

Suppose that $2^{f(n)}$ is $\theta(2^{g(n)})$, then $2^{f(n)}$ is $O(2^{g(n)})$, which implies

$\exists n_2, C_2 (constants)$, such that $\forall n > n_2, 2^{f(n)} \leq C_2 2^{g(n)} \Rightarrow C_2 \geq 2^{f(n)-g(n)} = 2^{e^n}$

$Since \lim_{n\to+\infty} 2^{e^n} = +\infty$, C_2 cannot be a constant. This contradicts the supposition that C_2 is a constant. Hence the supposition is false.

c. True

Since $f(n)$ is $\theta(g(n))$, $f(n)$ is both $\Omega(g(n))$ and $O(g(n))$.

1). Since $f(n)$ is $\Omega(g(n))$, $\exists n_0, C_0$ (constants), $\forall n > n_0, f(n) \geq C_0 g(n)$

Because $x^{0.5}$ is a monotonically increasing function, $f(n)^{0.5} \geq (C_0 g(n))^{0.5}$ $\Rightarrow \sqrt{f(n)} \geq \sqrt{C_0}\sqrt{g(n)}$

Then $\exists n_1 = n_0, C_1 = \sqrt{C_0}$, such that $\forall n > n_1, \sqrt{f(n)} \geq C_1\sqrt{g(n)}$

So $\sqrt{f(n)}$ is $\Omega(\sqrt{g(n)})$

2). Since $f(n)$ is $O(g(n))$, $\exists n_2, C_2$ (constants), $\forall n > n_2, f(n) \leq C_2 g(n)$

Because $x^{0.5}$ is a monotonically increasing function, $f(n)$, C_2 and $g(n)$ are all positive, $f(n)^{0.5} \leq (C_2 g(n))^{0.5} \Rightarrow \sqrt{f(n)} \leq \sqrt{C_2}\sqrt{g(n)}$

Then $\exists n_3 = n_2, C_3 = \sqrt{C_2}$ (so that $C_3 > 0$), such that $\forall n > n_1, \sqrt{f(n)} \leq C_3\sqrt{g(n)}$

So $\sqrt{f(n)}$ is $O(\sqrt{g(n)})$

By combining 1) and 2), $\sqrt{f(n)}$ is $\theta(\sqrt{g(n)})$

d. False

counterexample: $f(n) = 2log(n), g(n) = log(n) + 1$

Firstly we prove that $f(n)$ is $\theta(g(n))$:

$\exists n_0 = 5, C_0 = 1$, such that $\forall n > n_0$, $f(n) = 2log(n) \geq log(n) + 1 = C_0 g(n)$ $\Rightarrow f(n)$ is $\Omega(g(n))$

$\exists n_1 = 1, C_1 = 3$, such that $\forall n > n_1$, $f(n) = 2log(n) \leq 3log(n) + 3 = 3g(n) = C_1 g(n) \Rightarrow f(n)$ is $O(g(n))$

Therefore $f(n)$ is $\theta(g(n))$.

Secondly we prove that $f(n)-log(n)$ is not $\theta(g(n)-log(n))$ by contradiction:

Suppose that $f(n)-log(n)$ is $\theta(g(n)-log(n))$, then $f(n)-log(n)$ is $O(g(n)-log(n))$.
So $\exists n_2, C_2$, such that $\forall n > n_2,\ f(n) - log(n) \leq C_2(g(n) - log(n))$

$\Rightarrow 2log(n) - log(n) = log(n) \leq C_2(log(n) - log(n) + 1) = C_2$

Since $\lim_{n\to+\infty} log(n) = +\infty$, C_2 cannot be a constant. This contradicts the supposition that C_2 is a constant. So the supposition is false.