Problem 2.2

To show that the solution given by the minimum spanning tree algorithm is optimal, we show that:

Correctness: The solution generated by MST satisfy the connectivity constraint.

Proof: By the definition of the minimum spanning tree, the vertices would be connected by the MST without leaving any of the vertices unconnected.

Optimal:

Claim: The MST solution is at least as good as the optimal solution. (Note: there might be more than one optimal solutions).

Proof:

Let us invent some notations first before we start the proof: Let $s_i(i = 1, 2, ..., n)$ be the edges in the optimal solution. And let $t_j(j = 1, 2, ..., m)$ be the edges in the MST solution. Without loss of generality, we assume that s_i and t_j are sorted in the ascending order by their lengths. Then the claim turns out to be proving $s_n \geq t_m$.

In the MST solution, $t_m = (u, v)$ is the longest edge in the graph, and it partitions the whole set of vertices V into two sets A and B, where vertex u is contained in A and vertex v in contained in B. According to the Cut property of MST, t_m has to be the shortest edge connecting A and B.

In the optimal solution, there should be at least one edge connecting A and B (Note: in the optimal solution, there might be two or more edges with the same length connecting A and B, which could be the longest edge, so the optimal solution is not necessary to be a tree.) Let $s_k(1 \leq k \leq n)$ be one of the edges connecting A and B in the optimal solution, then according to the Cut property, $s_k \geq t_m$.

Also, $s_k \leq max\{s_i\} = s_n$

$\Rightarrow s_n \geq t_m$

Proved.