EECS 336 Fall 2015
Homework Problem 8.3

Idea

Denote More Cat People problem as MCP with n schools each of at most k delegates and list L of m pairs of fighting delegates. We need to show that MCP can reduce from k-SAT, which can reduce from 3-SAT.

Def: k-SAT: boolean formulas with clauses containing at most k literals.

Plan: Two-step procedure: convert 3-SAT formula to k-SAT formula, then convert k-SAT formula to MCP.

Step One: Convert 3-SAT formula to k-SAT formula.

Case One: $k = 1$. If $m > 0$, return FALSE. Else, return TRUE.

Each school has exactly one delegate. If there exists any pair of delegates that fight each other, then there is no possible selection of delegates that works. Problem is solved in P.

Case Two: $k = 2$. Then the problem is in P.

Algorithm:

a) Create vertex for each literal and a negation of the literal.

b) Create edge (u, v) if and only if there exists a clause equivalent to $(\neg u \vee v)$.

c) For each literal u, find if there is a path from u to $\neg u$, and vice versa.

d) Return FALSE if any of above tests succeeded. Else, return TRUE.

Correctness: Prove by contradiction.

Assume there exists a path from u to $\neg u$ and the formula is SAT. Along the path u to $\neg u$, if u is TRUE, then $\neg u$ is FALSE, and there has to be a directed edge (i, j) along the path where i is TRUE and j is FALSE. Then $\neg i \vee j$ is FALSE, which corresponds to a clause in the formula. If u is FALSE, then $\neg u$ is TRUE. Using similar analysis, we can find a contradiction on path $\neg u$ to u such that $\neg i \vee j$ is FALSE, which contradicts the assumption. Thus, the algorithm is correct.

Runtime:

Starting from each vertex, the search takes $O(|E|)$. Therefore, total runtime is $O(|V||E|)$.

Case Three: $k \geq 3$.

Construct k-SAT from 3-SAT.

In each clause in 3-SAT, c, add $k - 3$ FALSE literals, which converts c to $\{c \vee F \vee F...\}$ until c has k literals.

Runtime: for each clause, $O(k)$ literals are added. So a total of $O(kn)$ literals are added, and total runtime is $O(kn)$.

Correctness

a) 3-SAT to k-SAT. Apparently it follows the construction.

b) k-SAT to 3-SAT. For each clause $c = \{z_1, ..., z_k\}$, create $k - 3$ new variables and $k - 2$ new clauses in formula. For $3 \leq j \leq k - 3$, $c_j = \{r_{j-1}, z_j, \neg v_j\}$, $c_1 = \{z_1, z_2, \neg r_1\}$, $c_{k-2} = \{r_{k-2}, z_{k-1}, z_k\}$.

Step Two: Convert k-SAT formula to MCP.

1. **Construction** from k-SAT formula $f(z)$ into MCP.

Constraints in k-SAT are a) literals are consistent across clauses and b) at least one TRUE literal in each clause. Construct a graph G with vertices and edges that satisfy:

i) delegates $d_{i,j}$ corresponds to literals $l_{i,j}$.

ii) edges for clause (in polygon of size up to k): "at most one vertex selected per clause" corresponds to "each school selects one delegate"

iii) edges for conflicting literals, which corresponds to the m pairs of fighting delegates in list L.

2. Construction is **polynomial** time with one delegate per literal. Thus, the total construction time is $O(kn)$.

3. **Correctness**

i) Show that f is k-SAT, then MCP works.

If f is SAT, then there exists a set of literals z such that $f(z)$ is TRUE.

Let S' denotes the vertices in G that corresponds to true literals. If more than one vertex in S' belongs to the same polygon, drop all but one for each polygon, which converts S' to S with $|S| = n$. Then for all $u, v \in S$, u, v are not in the same polygon (selected delegates do not come from the same school), and l_u, l_v are both true (u, v cannot fight each other). Thus, S is a working selection of delegates.

ii) Show that if MCP works (can select set of n delegates, S, one from each school, such that no pair of delegates fight each other), then $f(z)$ is SAT.

Construct z from S. For each literal z_r,

a) if $v \in S$ labeled by z_r, then set z_r as TRUE.

b) if $v \in S$ labeled by $\neg z_r$, then set z_r as FALSE

c) if no $v \in S$ labeled by z_r or $\neg z_r$, then set z_r as TRUE.

Then for each $v \in S$, l_v is TRUE. Exactly one vertex in each polygon is true, so $|S| = n$. Thus, $f(z)$ is TRUE.

Conclusion

If $k = 1, 2$, then MCP is in P.

For $k \geq 3$, MCP is NP-hard. For each of the two steps, certificates are the list of delegates, and can be transformed into literals of k-SAT and then 3-SAT in polynomial time. Additionally, testing the formulas in SAT is in polynomial time. Thus, MCP is $\in NP$. Therefore, MCP is NP-complete.