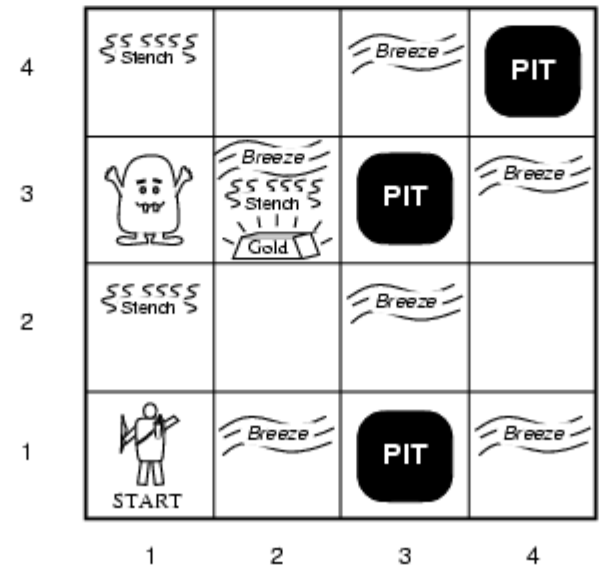
Propositional Logic and AI

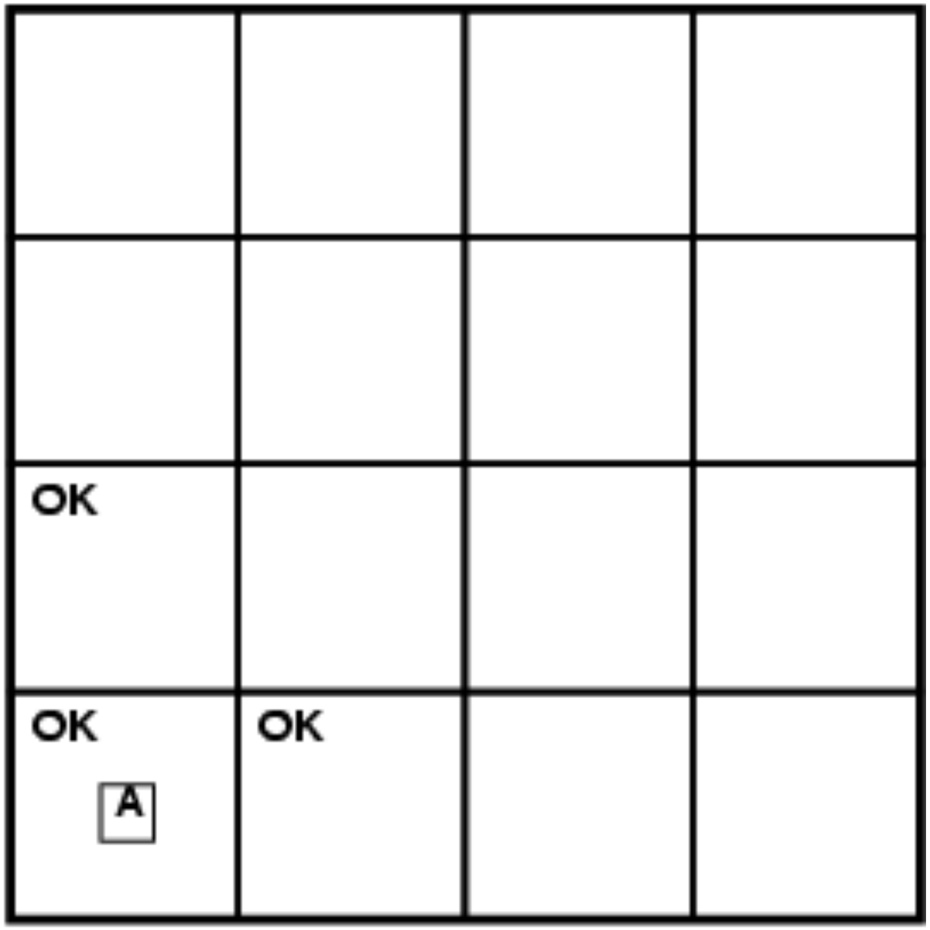
The Wumpus World

- Performance measure
  - gold +1000, death -1000
  - -1 per step, -10 for using the arrow
- Environment
  - Squares adjacent to wumpus are smelly
  - Squares adjacent to pit are breezy
  - Glitter iff gold is in the same square
  - Shooting kills wumpus if you are facing it
  - Shooting uses up the only arrow
  - Grabbing picks up gold if in same square
  - Releasing drops the gold in same square
  - Walking into a wall makes the agent perceive a bump
  - When the wumpus is killed, it emits a scream that is heard throughout the cave



- Sensors: Stench, Breeze, Glitter, Bump, Scream
- Actuators: turn left, turn right, move forward, Grab, Release, Shoot

Exploring a wumpus world



Logic for reasoning

- Goal: Deduce new facts (α) using:
  - The rules about the world
  - Information we gather through perception

  $\left.\begin{array}{l}\text{The rules about the world}\\ \text{Information we gather through perception}\end{array}\right\}$ KB

- Entailment means that one thing follows from another:

$$KB \models \alpha$$

Last night I ate either spaghetti or pizza
I did not eat pizza last night
⊨ Last night I ate spagett

Model

- Assignment of a truth value – true or false – to every atomic sentence

S v P
¬P

KB

M_1: S = False, P = False

M_2: S = False, P = True

M_3: S = True, P = False

M_4: S = True, P = True

A model m is a model of KB iff it is a model of all sentences in KB, that is, all sentences in KB are true in m

Satisfiability of a KB

A KB is satisfiable iff it admits at least one model; otherwise it is unsatisfiable

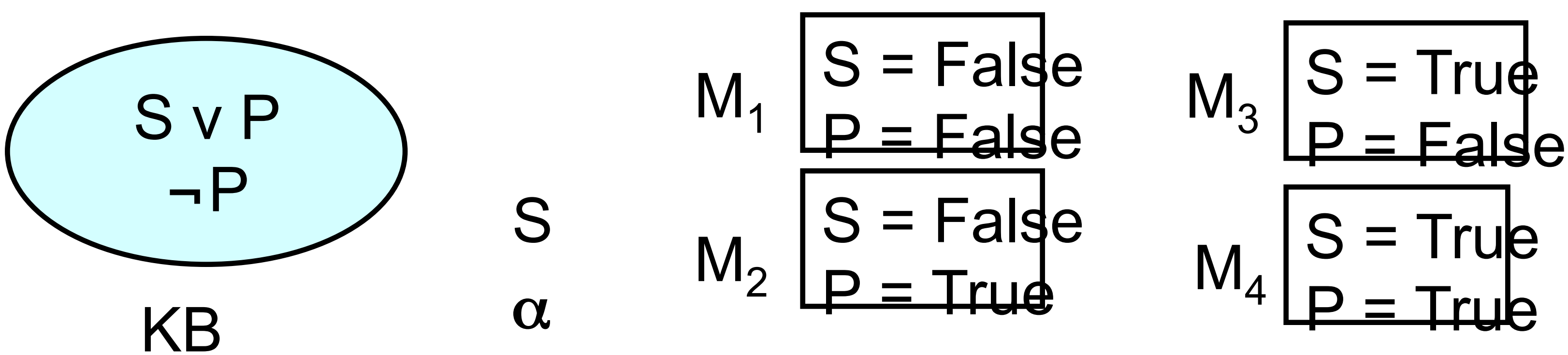
KB1 = $\{P, \neg Q \wedge R\}$ is ____________________

KB2 = $\{\neg P \vee P\}$ is ____________________

KB3 = $\{P, \neg P\}$ is ____________________

Logical Entailment

- KB : set of sentences
- α : arbitrary sentence
- KB entails α – written KB $\models \alpha$ – iff every model of KB is also a model of α
- Alternatively, KB $\models \alpha$ iff
  - $\{KB, \neg\alpha\}$ is unsatisfiable
  - KB $\Rightarrow \alpha$ is valid

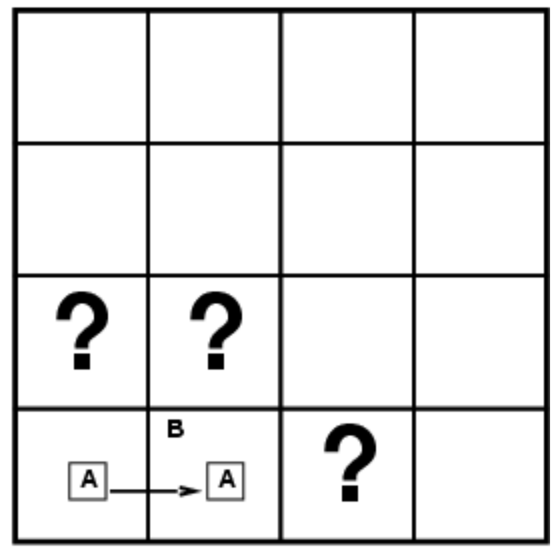


Entailment in the wumpus world

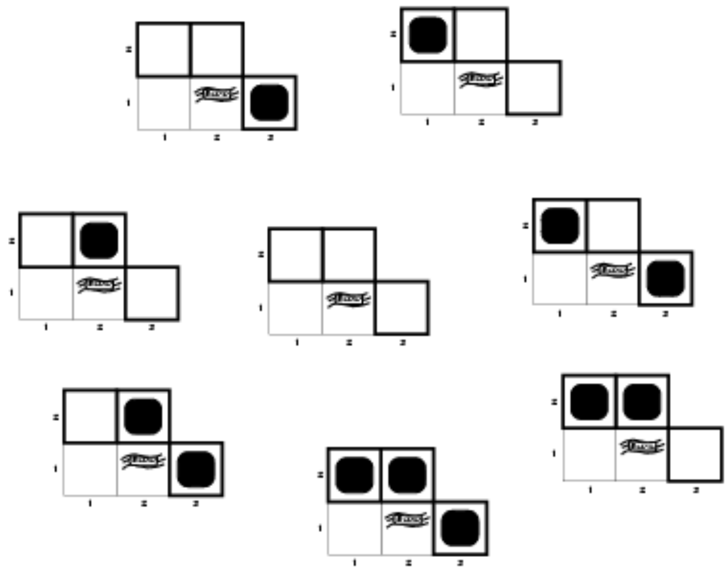
Situation after detecting nothing in [1,1], moving right, breeze in [2,1]

Consider possible models for *KB* assuming only pits

3 Boolean choices ⇒ 8 possible models

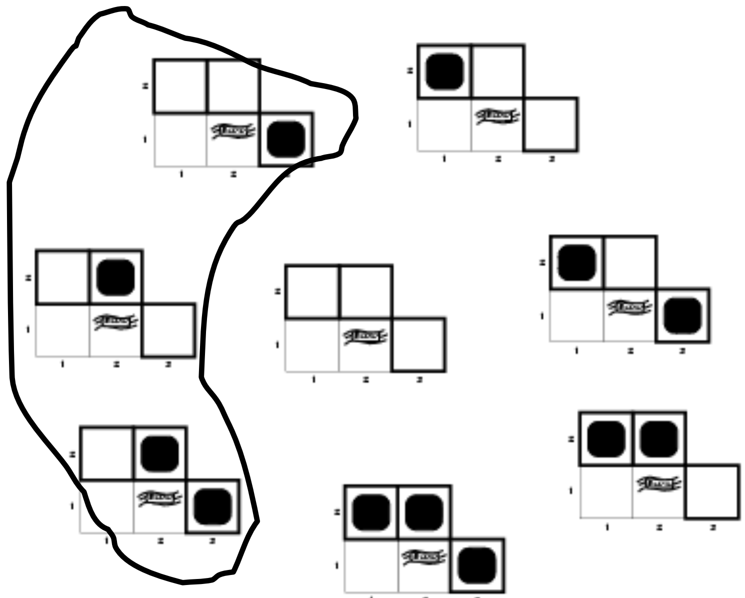


Wumpus models



- *KB* = wumpus-world rules +2observations
- α_1 = "[1,2] is safe"

Wumpus models



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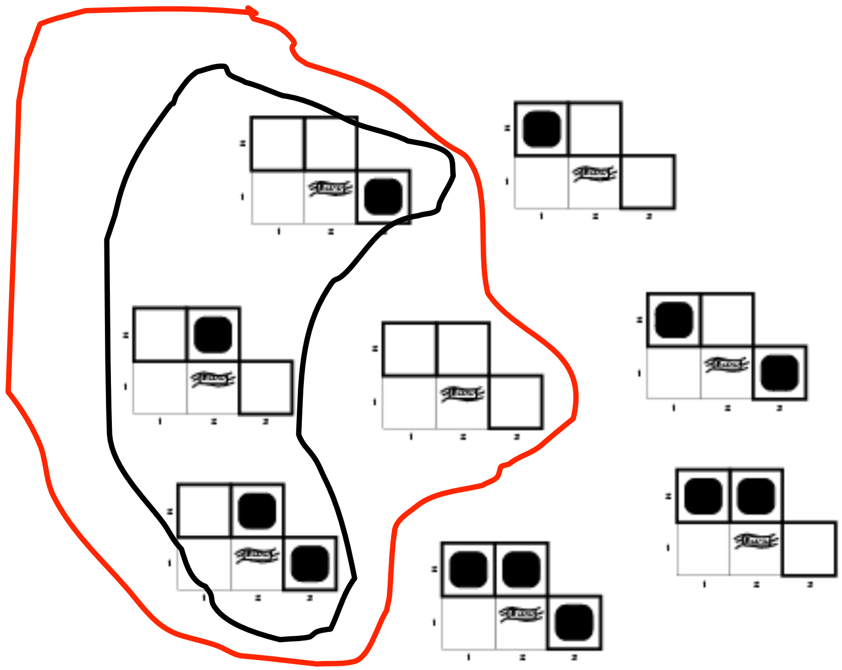

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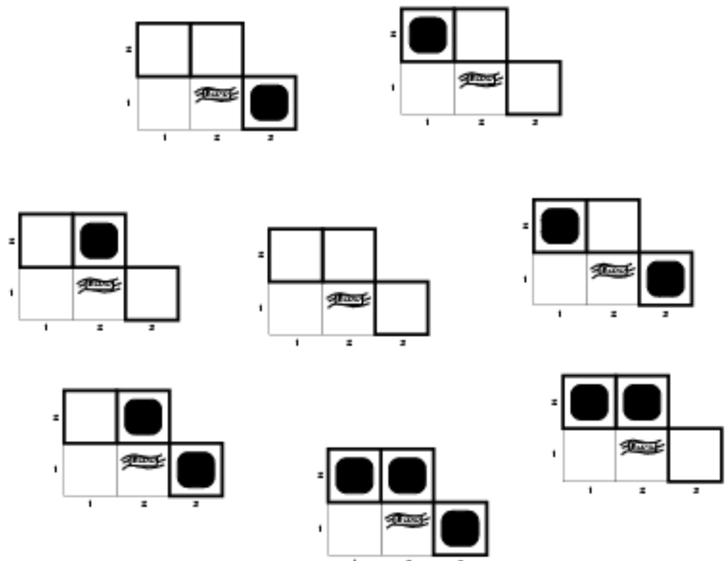
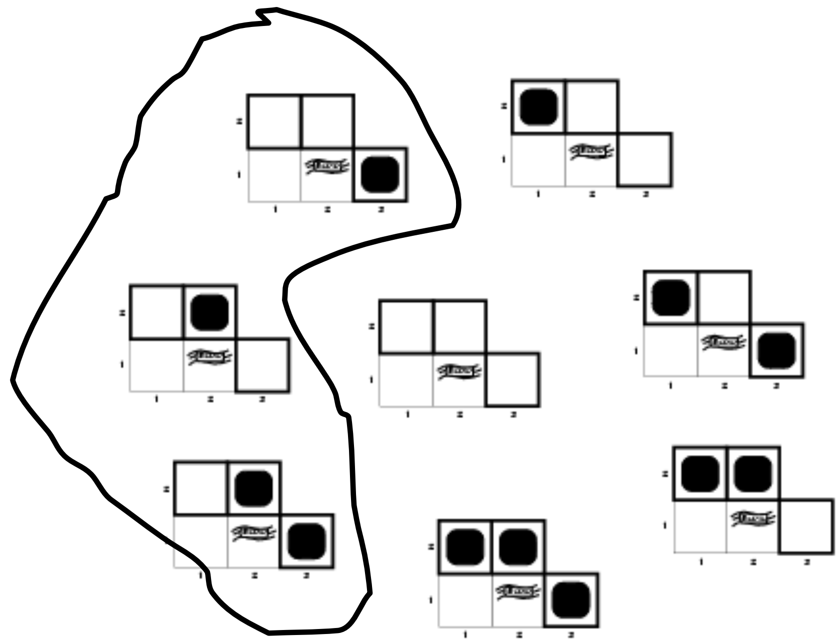
Wumpus models



- *KB* = wumpus-world rules + 2observations
- α_2 = "[2,2] is safe"

Wumpus models



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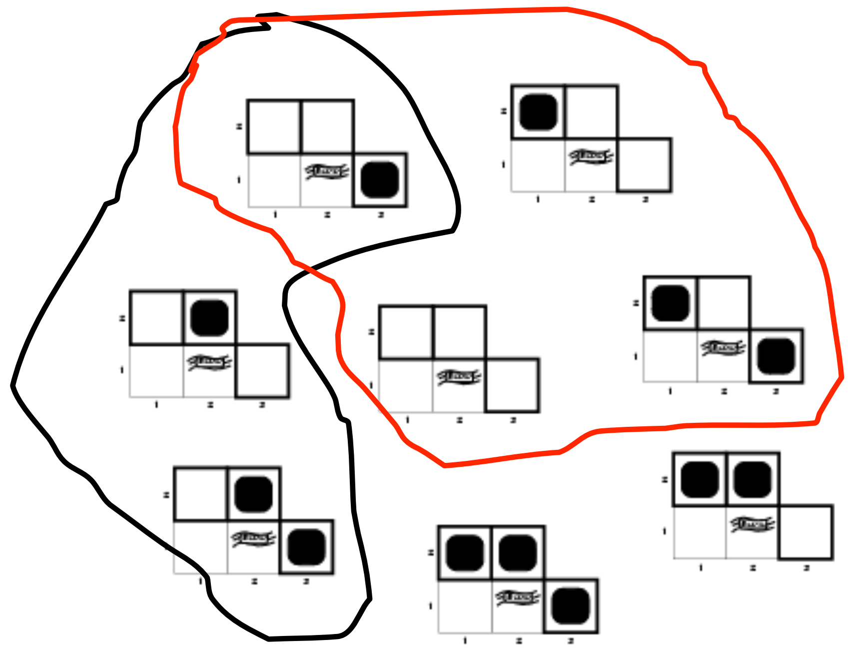
Wumpus models



- *KB* = wumpus-world rules + observations
- α_2 = "[2,2] is safe"

Propositional logic: Syntax

- Propositional logic is the simplest logic – illustrates basic ideas
- The proposition symbols P_1, P_2 etc are sentences

  – If S is a sentence, $\neg S$ is a sentence (negation)
  – If S_1 and S_2 are sentences, $S_1 \wedge S_2$ is a sentence (conjunction)
  – If S_1 and S_2 are sentences, $S_1 \vee S_2$ is a sentence (disjunction)
  – If S_1 and S_2 are sentences, $S_1 \Rightarrow S_2$ is a sentence (implication)
  – If S_1 and S_2 are sentences, $S_1 \Leftrightarrow S_2$ is a sentence (biconditional)

Propositional logic: Semantics

Each model specifies true/false for each proposition symbol

E.g.	$P_{1,2}$	$P_{2,2}$	$P_{3,1}$
	false	true	false

With these symbols, 8 possible models, can be enumerated automatically.

Rules for evaluating truth with respect to a model *m*:

$\neg S$	is true iff	S is false	
$S_1 \wedge S_2$	is true iff	S_1 is true and	S_2 is true
$S_1 \vee S_2$	is true iff	S_1 is true or	S_2 is true
$S_1 \Rightarrow S_2$	is true iff	S_1 is false or	S_2 is true
$S_1 \Leftrightarrow S_2$	is true iff	$S_1 \Rightarrow S_2$ is true and	$S_2 \Rightarrow S_1$ is true

Simple recursive process evaluates an arbitrary sentence, e.g.,

$$\neg P_{1,2} \wedge (P_{2,2} \vee P_{3,1}) = \mathit{true} \wedge (\mathit{true} \vee \mathit{false}) = \mathit{true} \wedge \mathit{true} = \mathit{true}$$

Inference

- Just because a KB entails a sentence doesn't mean we can find (infer) it
- Inference is the process of generating sentences entailed by the KB

Inference Rule

- An inference rule $\{\xi, \psi\} \vdash \varphi$ consists of 2 sentence patterns ξ and ψ called the **conditions** and one sentence pattern φ called the **conclusion**
- If ξ and ψ match two sentences of KB then the corresponding φ can be inferred according to the rule

Example: Modus Ponens

$$\{\ \alpha \Rightarrow \beta\ ,\ \alpha\ \}\ \vdash\ \beta$$

$$\{\xi,\ \psi\} \vdash \varphi$$

Battery-OK ∧ Bulbs-OK ⇒ Headlights-Work ($\alpha \Rightarrow \beta$)
Battery-OK ∧ Starter-OK ∧ ¬Empty-Gas-Tank ⇒ Engine-Starts
Engine-Starts ∧ ¬Flat-Tire ⇒ Car-OK
Battery-OK ∧ Bulbs-OK (α)

Can be used as inference rules…

$$
\begin{aligned}
(\alpha \wedge \beta) &\equiv (\beta \wedge \alpha) \quad \text{commutativity of } \wedge \\
(\alpha \vee \beta) &\equiv (\beta \vee \alpha) \quad \text{commutativity of } \vee \\
((\alpha \wedge \beta) \wedge \gamma) &\equiv (\alpha \wedge (\beta \wedge \gamma)) \quad \text{associativity of } \wedge \\
((\alpha \vee \beta) \vee \gamma) &\equiv (\alpha \vee (\beta \vee \gamma)) \quad \text{associativity of } \vee \\
\neg(\neg\alpha) &\equiv \alpha \quad \text{double-negation elimination} \\
(\alpha \Rightarrow \beta) &\equiv (\neg\beta \Rightarrow \neg\alpha) \quad \text{contraposition} \\
(\alpha \Rightarrow \beta) &\equiv (\neg\alpha \vee \beta) \quad \text{implication elimination} \\
(\alpha \Leftrightarrow \beta) &\equiv ((\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)) \quad \text{biconditional elimination} \\
\neg(\alpha \wedge \beta) &\equiv (\neg\alpha \vee \neg\beta) \quad \text{de Morgan} \\
\neg(\alpha \vee \beta) &\equiv (\neg\alpha \wedge \neg\beta) \quad \text{de Morgan} \\
(\alpha \wedge (\beta \vee \gamma)) &\equiv ((\alpha \wedge \beta) \vee (\alpha \wedge \gamma)) \quad \text{distributivity of } \wedge \text{ over } \vee \\
(\alpha \vee (\beta \wedge \gamma)) &\equiv ((\alpha \vee \beta) \wedge (\alpha \vee \gamma)) \quad \text{distributivity of } \vee \text{ over } \wedge
\end{aligned}
$$

igure 7.11 Standard logical equivalences. The symbols α, β, and γ stand for arbitrary entences of propositional logic.

$\Rightarrow$ Connective symbol (implication)

$\models$ Logical entailment

KB $\models \alpha$ iff KB $\Rightarrow \alpha$ is valid

$\vdash$ Inference

Soundness

- An inference rule is sound if it generates only entailed sentences
- All inference rules previously given are sound, e.g.:
  modus ponens: $\{\alpha \Rightarrow \beta , \alpha\} \vdash \beta$
- Is the following rule sound?
  $$\{\alpha \Rightarrow \beta , \beta\} \vdash \alpha$$

Completeness

- A set of inference rules is complete if every entailed sentences can be obtained by applying some finite succession of these rules
- Modus ponens *alone* is not complete, e.g.:
  from $A \Rightarrow B$ and $\neg B$, we cannot get $\neg A$

Proof

The proof of a sentence α from a set of sentences KB is the derivation of α by applying a series of sound (legal) inference rules

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1. Battery-OK $\wedge$ Bulbs-OK $\Rightarrow$ Headlights-Work
2. Battery-OK $\wedge$ Starter-OK $\wedge$ $\neg$Empty-Gas-Tank $\Rightarrow$ Engine-Starts
3. Engine-Starts $\wedge$ $\neg$Flat-Tire $\Rightarrow$ Car-OK
4. Headlights-Work
5. Battery-OK
6. Starter-OK
7. $\neg$Empty-Gas-Tank
8. $\neg$Car-OK

Proof

The proof of a sentence α from a set of sentences KB is the derivation of α by applying a series of sound inference rules

1	1.	Battery-OK $\wedge$ Bulbs-OK $\Rightarrow$ Headlights-Work
2	2.	Battery-OK $\wedge$ Starter-OK $\wedge$ $\neg$Empty-Gas-Tank $\Rightarrow$ Engine-Starts
3	3.	Engine-Starts $\wedge$ $\neg$Flat-Tire $\Rightarrow$ Car-OK
4	4.	Headlights-Work
5	5.	Battery-OK
6	6.	Starter-OK
7	7.	$\neg$Empty-Gas-Tank
8	8.	$\neg$Car-OK
9	9.	Battery-OK $\wedge$ Starter-OK $\leftarrow$ (5+6)
10	10.	Battery-OK $\wedge$ Starter-OK $\wedge$ $\neg$Empty-Gas-Tank $\leftarrow$ (9+7)
11	11.	Engine-Starts $\leftarrow$ (2+10)
12	12.	Engine-Starts $\Rightarrow$ Flat-Tire $\leftarrow$ (3+8)
13	13.	Flat-Tire $\leftarrow$ (11+12)

Inference Problem

- Given:
  - KB: a set of sentence
  - α: a sentence
- Answer:
  - KB $\models \alpha$?

We require an automatic, sound and complete inference method

Inference by enumeration

- We can enumerate all possible models and test whether every model of KB is also a model of α

Wumpus world sentences

Let $P_{i,j}$ be true if there is a pit in [i, j]
Let $B_{i,j}$ be true if there is a breeze in [i, j]

KB:

$\neg P_{1,1}$
$\neg B_{1,1}$
$B_{2,1}$

Rules of the environment:

"Pits cause breezes in adjacent squares"

Truth tables for inference

$B_{1,1}$	$B_{2,1}$	$P_{1,1}$	$P_{1,2}$	$P_{2,1}$	$P_{2,2}$	$P_{3,1}$	KB	α_1
$false$	$false$	$false$	$false$	$false$	$false$	$false$	$false$	$true$
$false$	$false$	$false$	$false$	$false$	$false$	$true$	$false$	$true$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$false$	$true$	$false$	$false$	$false$	$false$	$false$	$false$	$true$
$false$	$true$	$false$	$false$	$false$	$false$	$true$	$\underline{true}$	$\underline{true}$
$false$	$true$	$false$	$false$	$false$	$true$	$false$	$\underline{true}$	$\underline{true}$
$false$	$true$	$false$	$false$	$false$	$true$	$true$	$\underline{true}$	$\underline{true}$
$false$	$true$	$false$	$false$	$true$	$false$	$false$	$false$	$true$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$true$	$true$	$true$	$true$	$true$	$true$	$true$	$false$	$false$

¬ P1,1

¬B1,1

B2,1

B1,1 ⇔ (P1,2 ∨ P2,1)

B2,1 ⇔ (P1,1 ∨ P2,2 ∨ P3,1)

α_1 ¬$P_{1,2}$

Inference by enumeration

- How efficient is this? How much time/space does it take?

Proof methods

- Proof methods divide into (roughly) two kinds:
  - Application of inference rules
    - Legitimate (sound) generation of new sentences from old
    - Proof = a sequence of inference rule applications
      Can use inference rules as operators in a standard search algorithm
    - Typically require transformation of sentences into a normal form
  - Model checking
    - truth table enumeration (always exponential in *n*)
    - improved backtracking, e.g., Davis--Putnam-Logemann-Loveland (DPLL)
    - heuristic search in model space (sound but incomplete)
      e.g., min-conflicts-like hill-climbing algorithms

Resolution Inference Rule

Conjunctive Normal Form (CNF)

conjunction of disjunctions of literals clauses

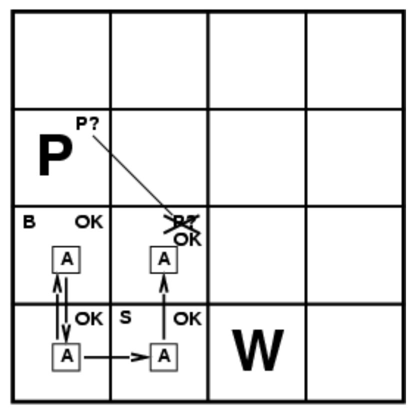
E.g., $(A \lor \neg B) \land (B \lor \neg C \lor \neg D)$

- Resolution inference rule (for CNF):

$$\frac{\ell_i \lor \ldots \lor \ell_k, \qquad m_1 \lor \ldots \lor m_n}{\ell_i \lor \ldots \lor \ell_{i-1} \lor \ell_{i+1} \lor \ldots \lor \ell_k \lor m_1 \lor \ldots \lor m_{j-1} \lor m_{j+1} \lor \ldots \lor m_n}$$

where ℓ_i and m_j are complementary literals.

E.g., $\dfrac{P_{1,3} \lor P_{2,2}, \quad \neg P_{2,2}}{P_{1,3}}$



- Resolution is sound and complete for propositional logic

Conjunctive Normal Form

- Resolution rule only applies to disjunctions of literals
- How could it possibly be compete?
- EVERY sentence in prop logic is equivalent to a conjunction of clauses (disjunction of literals)
- This equivalent form is called CNF
- We have a standard procedure for converting to CNF

Prove that KB ⊨ α by proving that KB $\wedge \neg\alpha$ **is unsatisfiable**

function PL-RESOLUTION(KB, α) **returns** *true* or *false*
  inputs: KB, the knowledge base, a sentence in propositional logic
    α, the query, a sentence in propositional logic

  clauses ← the set of clauses in the CNF representation of $KB \wedge \neg\alpha$
  new ← { }
  loop do
    for each C_i, C_j **in** *clauses* **do**
      resolvents ← PL-RESOLVE(C_i, C_j)
      if *resolvents* contains the empty clause **then return** *true*
      new ← *new* ∪ *resolvents*
    if *new* ⊆ *clauses* **then return** *false*
    clauses ← *clauses* ∪ *new*

Figure 7.12 A simple resolution algorithm for propositional logic. The function PL-RESOLVE returns the set of all possible clauses obtained by resolving its two inputs.

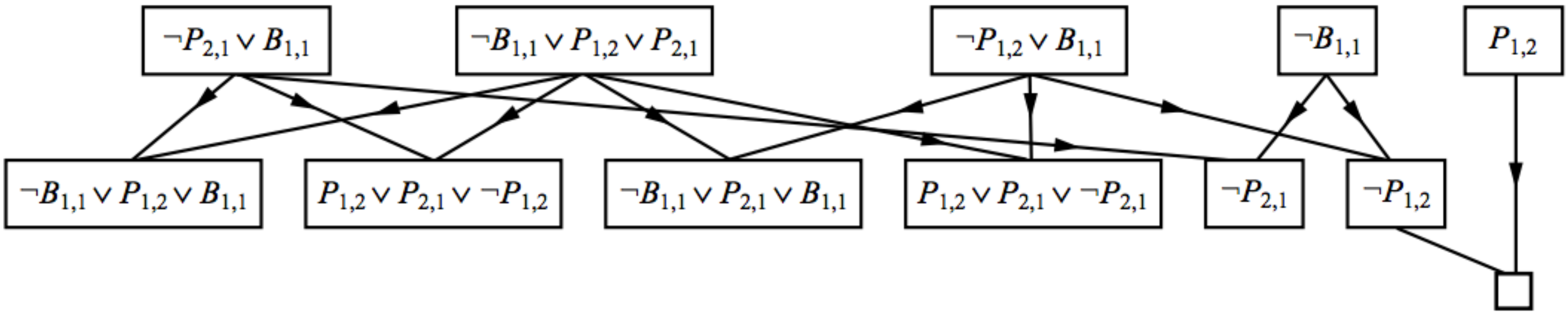


Figure 7.13 Partial application of PL-RESOLUTION to a simple inference in the wumpus world. $\neg P_{1,2}$ is shown to follow from the first four clauses in the top row.

Forward and backward chaining

- Horn Form (restricted)
  KB = conjunction of Horn clauses

  – Horn clause =
    - proposition symbol; or
    - (conjunction of symbols) $\Rightarrow$ symbol
  – E.g., $C \wedge (B \Rightarrow A) \wedge (C \wedge D \Rightarrow B)$

- Modus Ponens (for Horn Form): complete for Horn KBs

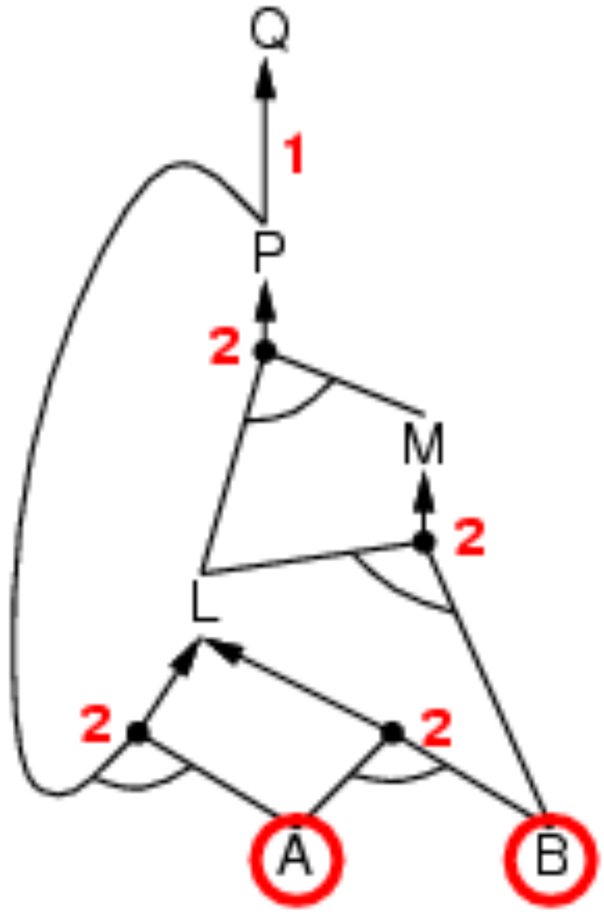
$$\frac{\alpha_1, \ldots, \alpha_n, \qquad \alpha_1 \wedge \ldots \wedge \alpha_n \Rightarrow \beta}{\beta}$$

- Can be used with forward chaining or backward chaining.
- These algorithms are very natural and run in linear time

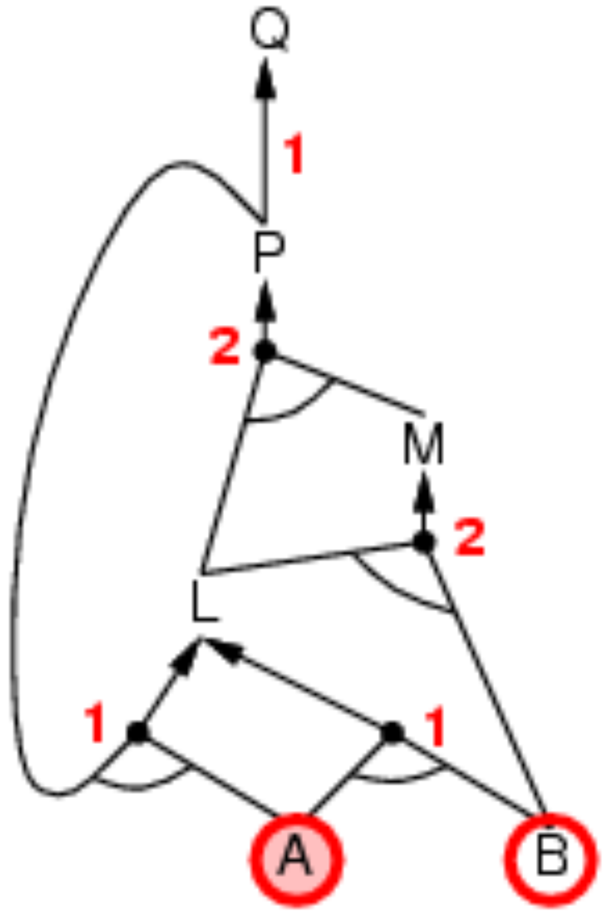
Forward chaining

- Idea: fire any rule whose premises are satisfied in the *KB*,
  - add its conclusion to the *KB*, until query is found

Forward chaining example



Forward chaining example



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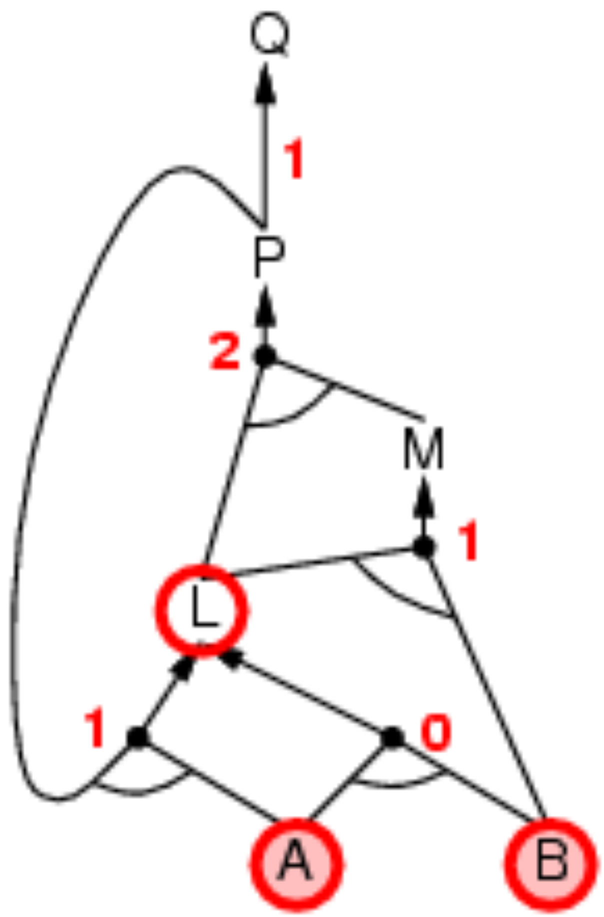

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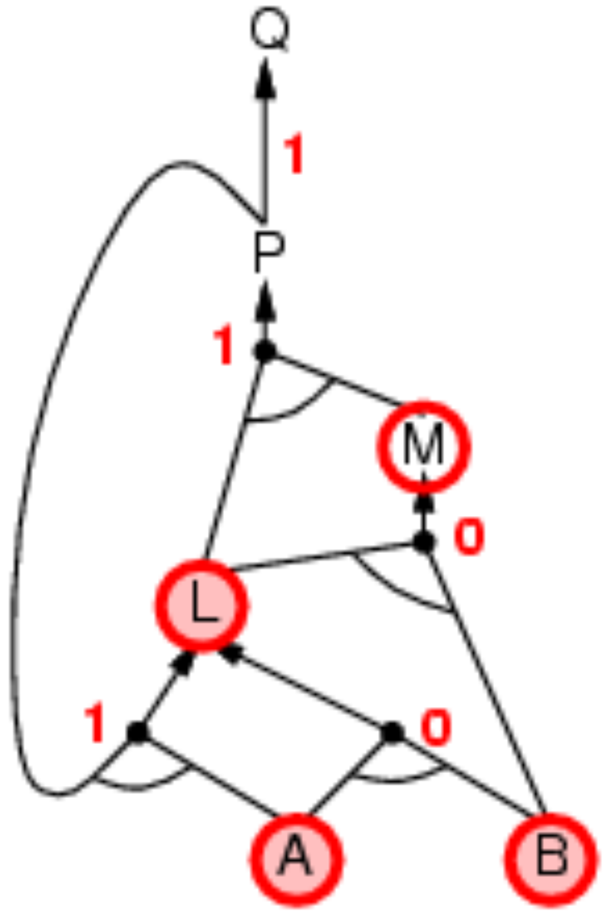

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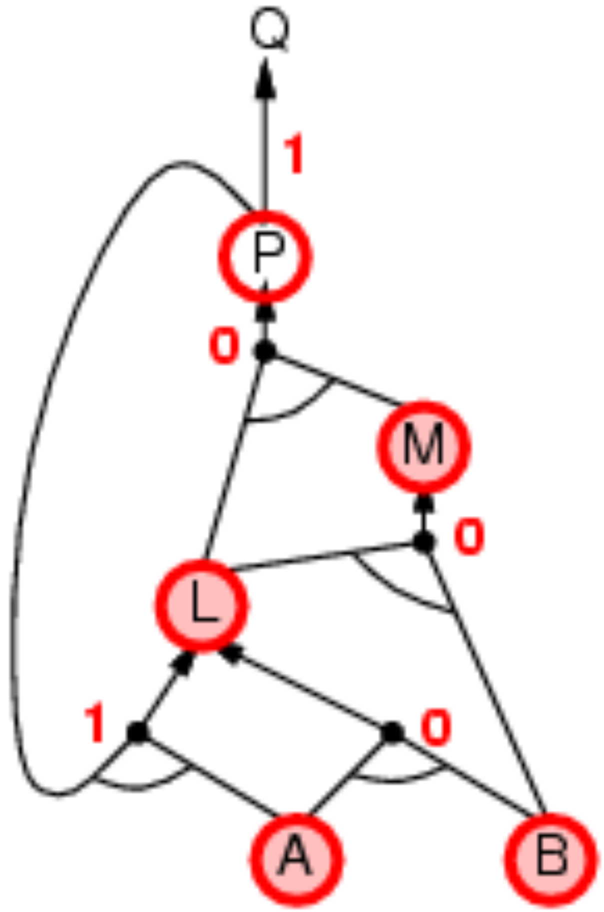

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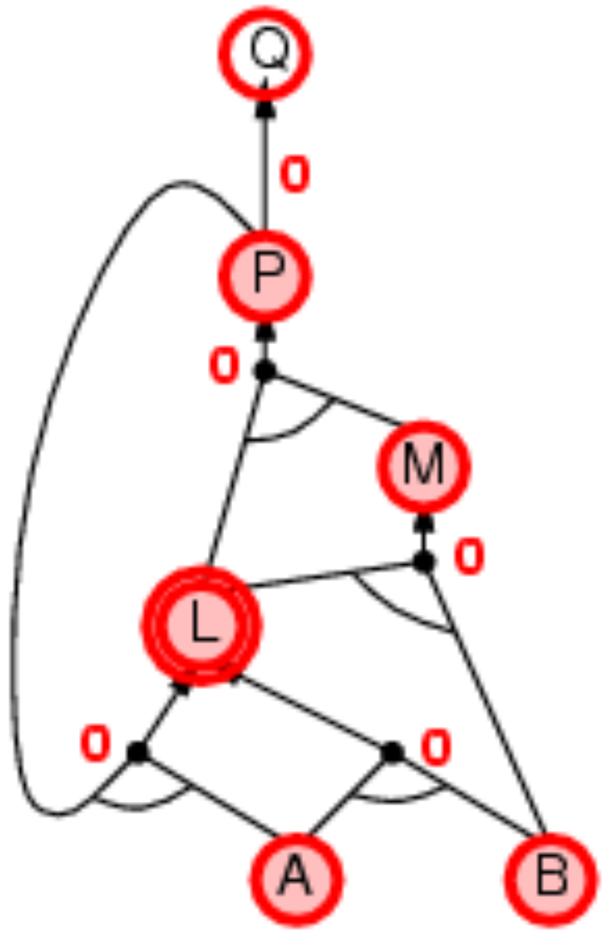

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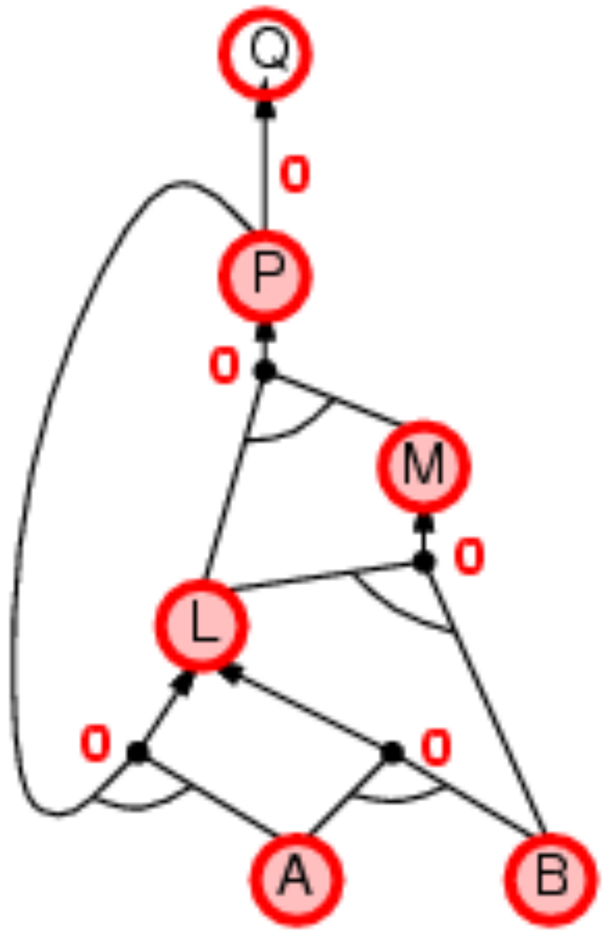

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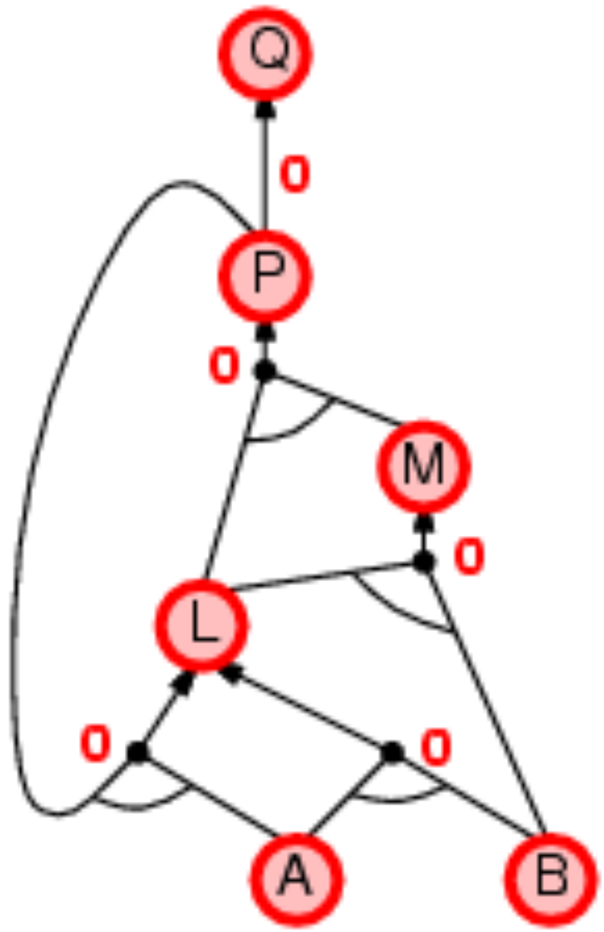


Backward chaining

Idea: work backwards from the query *q*:

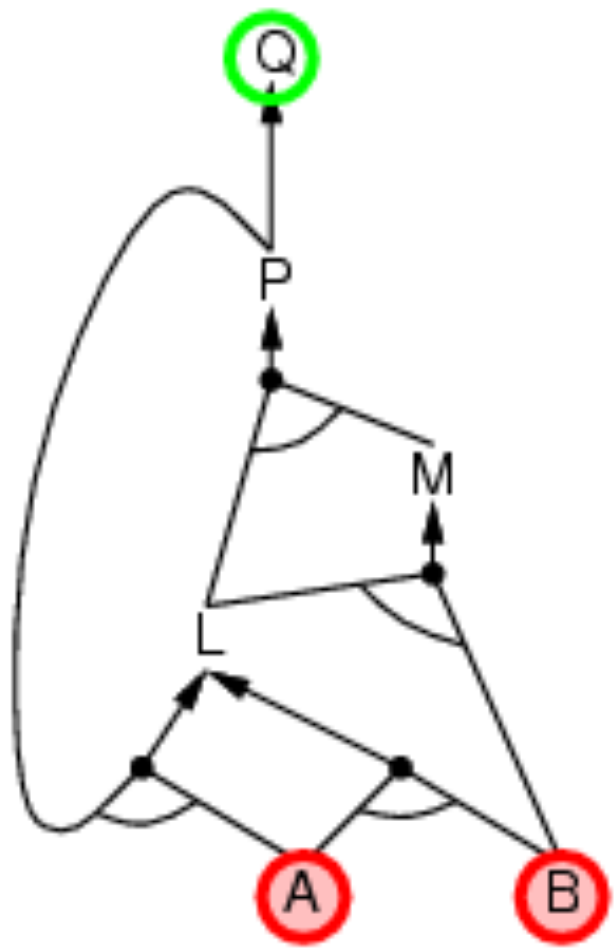
- to prove *q* by BC,
  - check if *q* is known already, or
  - prove by BC all premises of some rule concluding *q*

Avoid loops: check if new subgoal is already on the goal stack

Avoid repeated work: check if new subgoal

1. has already been proved true, or
2. has already failed

Backward chaining example



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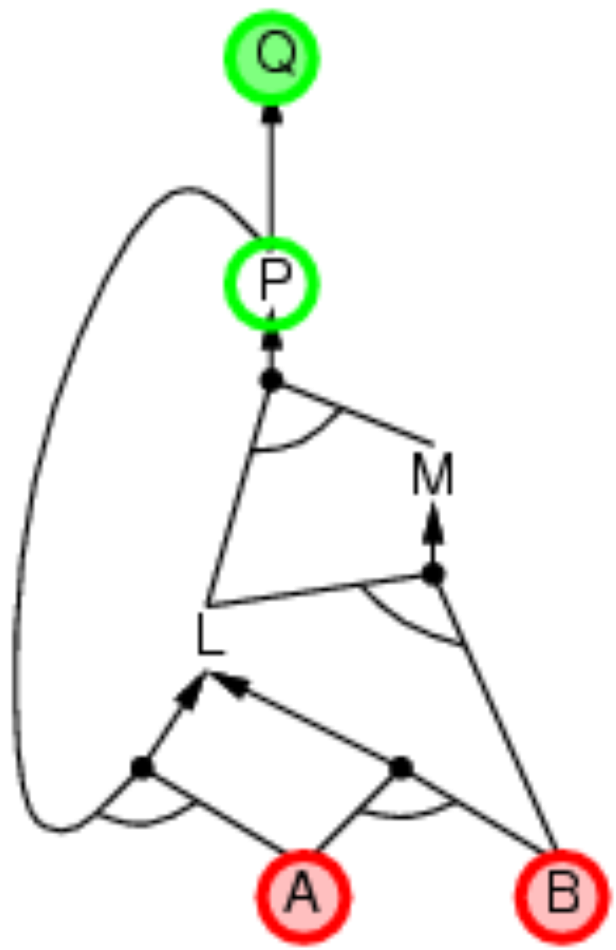

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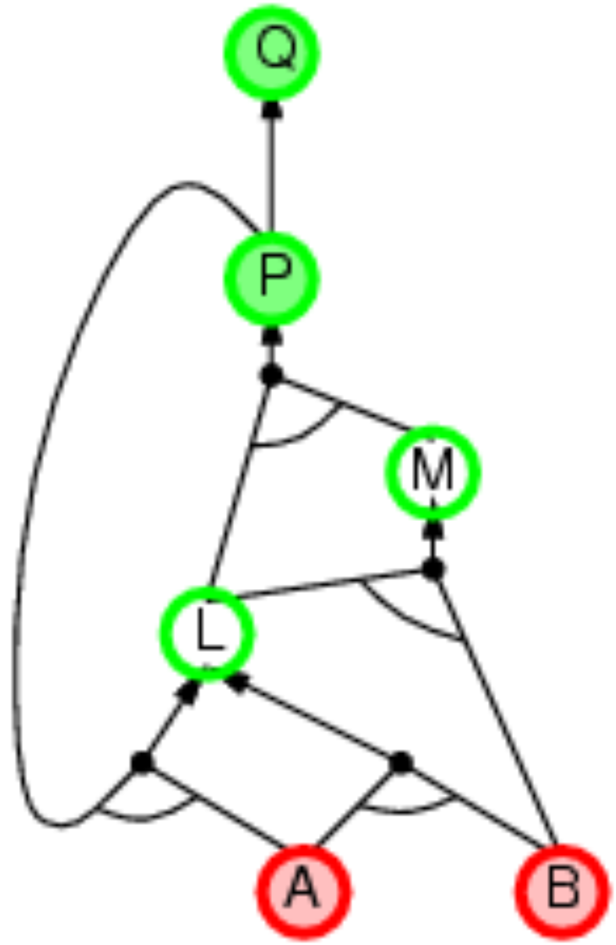

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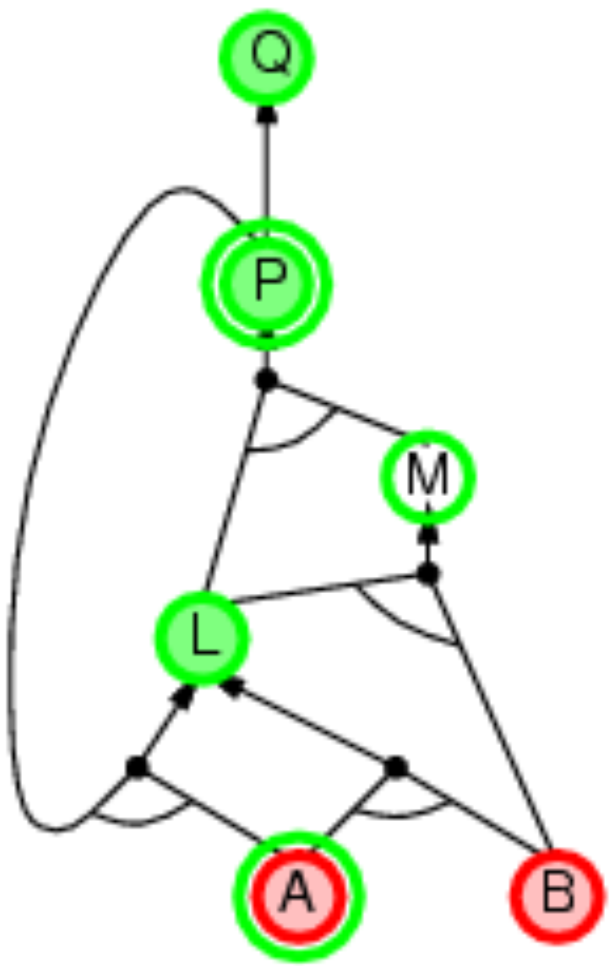

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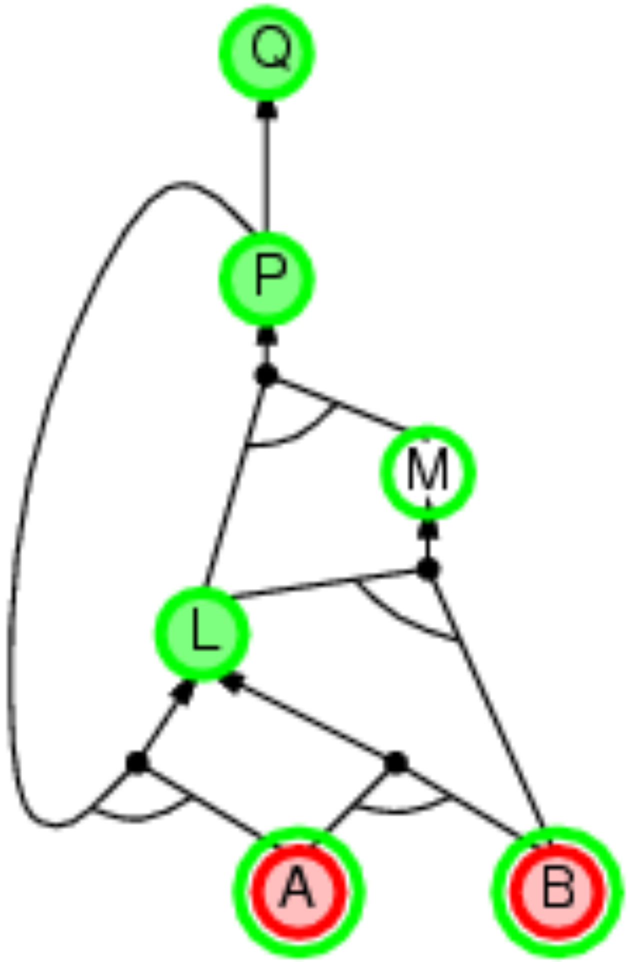

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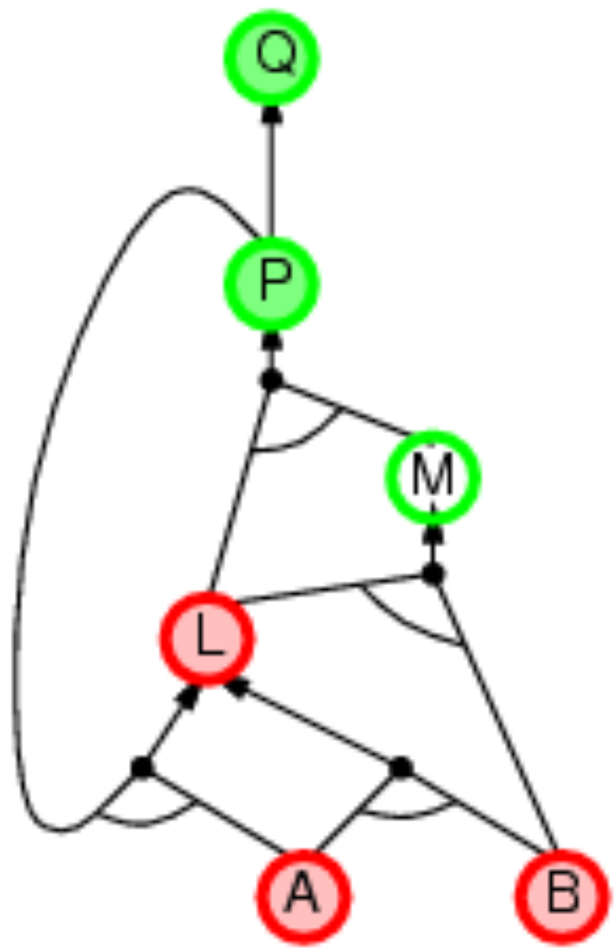

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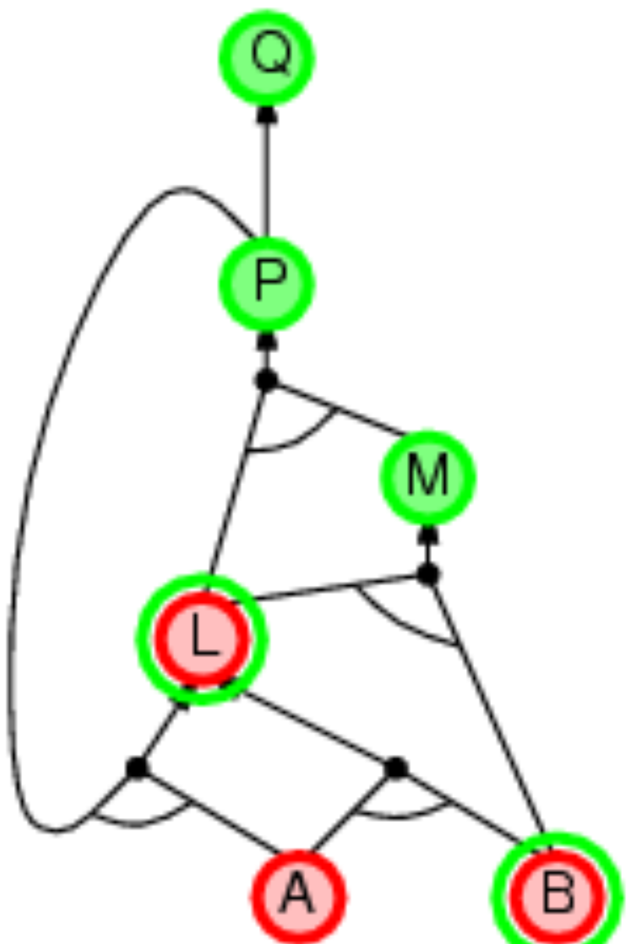

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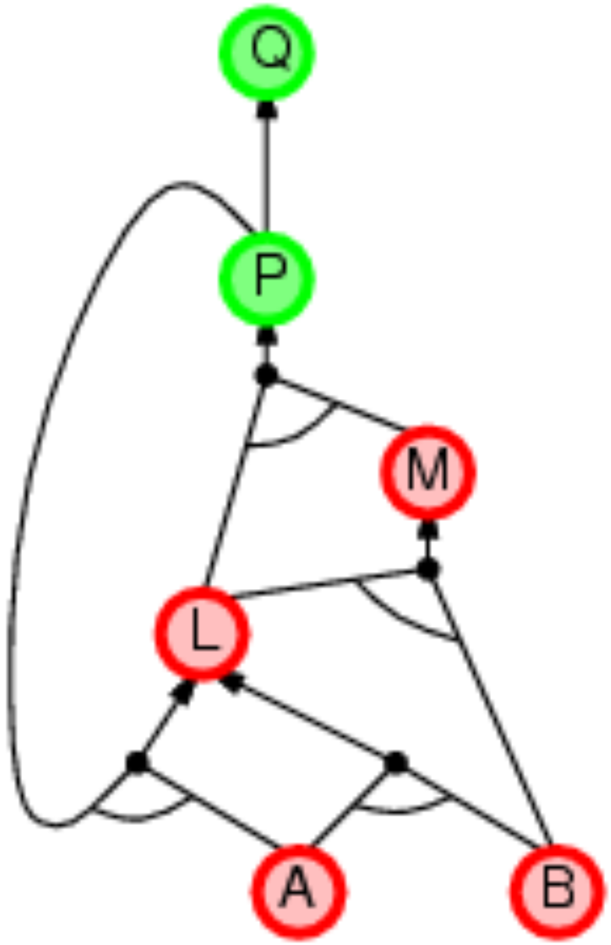

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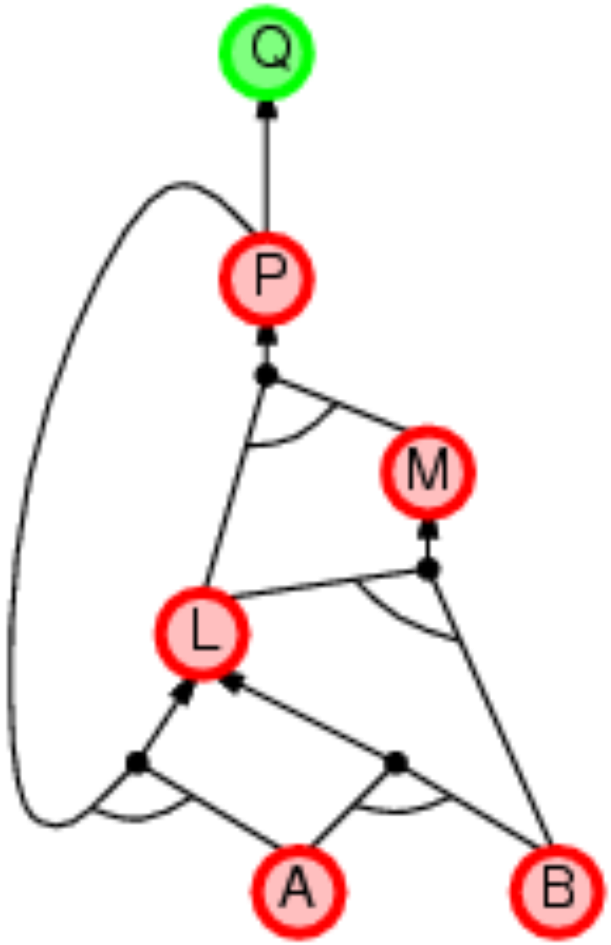

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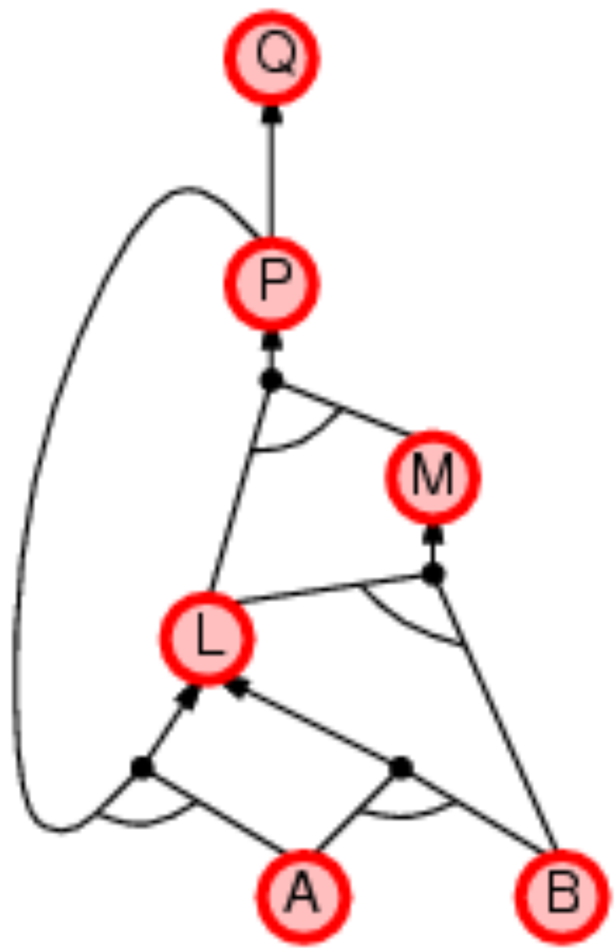

Forward vs. backward chaining

- FC is data-driven, automatic, unconscious processing,
  - e.g., object recognition, routine decisions
- May do lots of work that is irrelevant to the goal
- BC is goal-driven, appropriate for problem-solving,
  - e.g., Where are my keys? How do I get into a PhD program?
- Complexity of BC can be much less than linear in size of KB