Primary Decomposition

Reference: Lawrence and Zarzitto

If A is an abelian group and p is a prime, let

$$A(p) = \{x \in A : p^k x = 0 \text{ for some exponent } k\}.$$

Such $A(p)$ will be called the p-**primary component** of A.

The set $A(p)$ is the set of elements whose order is a power of p.

If $A(p)$ is finite, then $|A(p)| = p^d$ for some $d \geq 0$. In other words, the p-primary component $A(p)$ is a p-group.

Proposition 1. For each finite abelian group A and each prime p, the order $|A(p)|$ is the highest power of p that divides $|A|$.

Theorem 2 (Primary Decomposition)**.** Let A be a finite abelian group (written additively) of order $n = \prod p_i^{e_i}$. The mapping

$$\psi \colon \prod A(p_i) \to A$$

given by $(x_1, ..., x_k) \mapsto x_1 + \cdots + x_k$ is an isomorphism. The decomposition is unique up to arrangement of factors.

This result is used in the study of cohomology of finite groups. Basically, some results in cohomology can be reduced to studying cohomology of p-groups by using the Primary Decomposition theorem.