Modules

Informally a module is a vector space over a ring. More precisely, if R is a ring, then a **left** R **module** M is an abelian group with the additional structure of a scalar multiplication $R \times M \to M$ such that

(i) $r(r'm) = (rr')m$ for r and r' in R and m in M,

(ii) $(r + r')m = rm + r'm$ and $r(m + m') = rm + rm'$ if r and r' are in R and m and m' are in M.

In addition, if R has unity, we say that M is **unital** if

(iii) $1m = m$ for all $m \in M$.

One may also speak of **right** R**-modules**. For these the scalar multiplication is usually written as mr with m in M and r in R, and the expected analogs of (i) and (ii) are to hold.

When R is commutative, it is immaterial which side is used for the scalar multiplication, and one speaks simply of an R**-module**.

Let R be a ring, and let M and N be two left R-modules. A **homomorphism of left** R**-modules** is an additive group homomorphism $\varphi : M \to N$ such that for all $r \in R$, $\varphi(rm) = r\varphi(m)$.

If R is a ring and M is a left R-module, then an R**-submodule** N of M is an additive subgroup of M that is closed under scalar multiplication, i.e., has rm in N whenever $r \in R, m \in N$.

If the ring R has unity and M is a unital left R-module, then the R-submodule of M **generated** by $m \in M$, i.e., the smallest R-submodule containing m, is Rm, the set of products rm with $r \in R$. However, if the left R-module M is not unital, then the R-submodule generated by m may not equal Rm. More generally, the R-submodule of M **generated** by a finite set $\{m_1, ..., m_n\}$ in M is $Rm_1 + ... + Rm_n$ if the left R-module M is unital.

As an example, we can treat R itself as a left R-module. In this setting, the left R-submodules are called **left ideals** in R. If the ring R has a unity, then the left R-module R is automatically unital, and the left ideal of R generated by an element a is Ra, the set of all products ra with $r \in R$.