Fields of Fractions

Reference: Fraleigh, A First Course in Abstract Algebra

We work with commutative rings only. We show how an integral domain can be embedded canonically in a field.

Let R be an integral domain. Form the set

$$\tilde{F} = \{(a, b) : a \in R, b \in R, b \neq 0\}.$$

Impose the equivalence relation $(a, b) \sim (c, d)$ if $ad = bc$. The relation $\sim$ is certainly reflexive and symmetric. To see that it is transitive, suppose that $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. Then $ad = bc$ and $cf = de$, and these together force $adf = bcf = bde$. In turn, this implies $af = be$ since R is an integral domain and d is assumed $\neq 0$. Thus $\sim$ is transitive and is an equivalence relation.

Let F be the set of equivalence classes.

The definition of addition in $\tilde{F}$ is $(a, b) + (c, d) = (ad + bc, bd)$, the expression we get by naively clearing fractions, and we want to see that addition is consistent with the equivalence relation. In checking this...

The element $(0, 1)$ is a two-sided identity for addition in $\tilde{F}$, and hence the class of $(0, 1)$ is a two-sided identity for addition in F. We denote this class by 0. Let us identify this class. A pair (a, b) is in the class $(0, 1)$ if and only if $0b = 1a$, hence if and only if $a = 0$. In other words, the class of $(0, 1)$ consists of all $(0, b)$ with $b \neq 0$.

...Therefore F is a field.

The field F is called the **field of fractions** of the integral domain R. The function $\eta : R \to F$ defined by saying that $\eta(r)$ is the class of $(r, 1)$ is easily checked to be a homomorphism of rings sending 1 to 1. It is injective. Let us call it the canonical embedding of R into F. The pair (F, η) has the following universal property.

Proposition 1. Let R be an integral domain, let F be its field of fractions, and let η be the canonical embedding of R into F. Whenever φ is an injective ring homomorphism of R into a field F' carrying 1 to1, then there exists a unique ring homomorphism $\widetilde{\varphi} : F \to F'$ such that $\varphi = \widetilde{\varphi}\eta$, and $\widetilde{\varphi}$ is injective as a homomorphism of fields.

We say that $\widetilde{\varphi}$ is the extension from R to F. Once this proposition has been proved, it is customary to drop η from the notation and regard R as a subring of its field of fractions.

If K is a field, then $R = K[X]$ is an integral domain. The field of fractions consists in effect of formal rational expressions $P(X)Q(X)^{-1}$ in the indeterminate X, with the expected identifications made. We write $K(X)$ for this field of fractions. More generally, the field of fractions of the integral domain $K[X_1, ..., X_n]$ consists of formal rational expressions in the indeterminates $X_1, ..., X_n$, with the expected identifications made, and is denoted by $K(X_1, ..., X_n)$.