

Sylow Theorems

This section continues the use of group actions to obtain results concerning structure theory for abstract groups. We state the three theorems as the parts of one theorem.

Theorem 1. (Sylow Theorems). Let G be a finite group of order $p^m r$, where p is a prime and p does not divide r . Then

(a) G contains a subgroup of order p^m , and any subgroup of G of order p^l with $0 \leq l < m$ is contained in a subgroup of order p^m ,

(b) any two subgroups of order p^m in G are conjugate in G , i.e., any two such subgroups P_1 and P_2 have $P_2 = aP_1a^{-1}$ for some $a \in G$,

(c) the number of subgroups of order p^m is of the form $pk + 1$ and divides r .

A subgroup of order p^m as in the theorem is called a **Sylow p -subgroup** of G . A consequence of (a) with $m \geq 1$ is that G has a subgroup of order p ; this special case is sometimes called Cauchy's theorem in group theory.

Before coming to the proof, let us carefully give two simple applications to structure theory.

Proposition 2. If p and q are primes with $p < q$, then there exists a nonabelian group of order pq if and only if p divides $q - 1$, and in this case the nonabelian group is unique up to isomorphism. It may be taken to be a semidirect product of the cyclic groups C_p and C_q with C_q normal.

Proposition 3. If G is a group of order 12, then G contains a subgroup H of order 3 and a subgroup K of order 4, and at least one of them is normal. Consequently there are exactly five groups of order 12, up to isomorphism - two abelian and three nonabelian.

Historical notes: Sylow (1832-1918) worked as a math teacher at a high school in Norway from 1858-1898. He became extraordinary professor at the university in 1898. His mathematical career consisted largely in further developing Abel's ideas.