Dedekind Domains

References for Dedekind domains are Atiyah-Macdonald, and Matsumura (Commutative Ring Theory).

An integral domain is called a **Dedekind domain** if it satisfies the following equivalent properties.

(i) A is either a field or a noetherian ring such that $A_{\mathfrak{m}}$ is a discrete valuation ring for every maximal ideal $\mathfrak{m}$.

(ii) A is normal, noetherian, and $\dim A \leq 1$.

(iii) A is regular and $\dim A \leq 1$.

(iv) Every nonzero ideal of A can be written as a finite product of prime ideals.

(v) Every non-zero ideal of A is a projective A-module of rank 1.

If A is a Dedekind domain, then the factorization in (iv) into prime ideals is unique up to order.

Theorem 1 (Matsumura,Theorem 20.7)**.** A ring A is a PID if and only if A is UFD and a Dedekind ring.

The main examples of Dedekind domains are the following:

Theorem 2. The ring of integers in any algebraic number field is a Dedekind domain.

Theorem 3. Let V be an affine smooth variety over an algebraically closed field k. Then $k[V]$ is a Dedekind domain.

For the proof, see Bump, Algebraic Geometry.

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Here are some historical notes.

Kummer thought that the decomposition of every element in the cyclotomic ring $\mathbb{Z}[\alpha]$ in a product of prime factors of this ring was unique. This hypothesis ought to allow him to prove Fermat's theorem for every integer n.

Several mathematicians discovered that the uniqueness of the decomposition does not hold in general. In order to salvage, in a certain way, the uniqueness of factorization in prime elements, Kummer created his theory of ideal numbers.

With his theory of ideal factors, Kummer succeeded in proving Fermat's theorem for prime numbers less than 100. But his definition of ideal numbers was not sufficiently general.

Dedekind introduced classes of algebraic numbers that he called ideals.