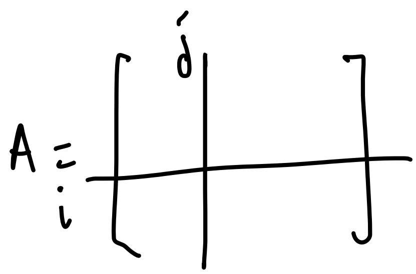
Let A be an n by n matrix.



The $n-1$ by $n-1$ matrix obtained by deleting row i and column j is called the $\underline{(i,j)\text{-minor of } A}$, written A_{ij}.

$\underline{\text{Ex}}$: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ The $(2,3)$-minor is $A_{23} = \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}$.

The number $(-1)^{i+j} \det A_{ij}$ is called the $\underline{(i,j)\text{-cofactor of } A}$.

$\underline{\text{Ex}}$: The $(2,3)$-cofactor of A is

$$(-1)^{2+3} \det A_{23} = -(8-14) = 6.$$

$\underline{\text{Thm}}$ (Cofactor expansion). Pick i from $1, \dots, n$. Then

$$\det A = \sum_{k=1}^{n} (-1)^{i+k} a_{ik} \det A_{ik}.$$

Ex: Pick $i=1$.

$$\det A = (1)\,a_{11}\det A_{11} + (-1)\,a_{12}\det A_{12} + (1)\,a_{13}\det A_{13}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$A_{11} = \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}, \quad A_{13} = \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}$$

so $\det A = 1(-3) - 2(-6) + 3(-3)$

$$= -3 + 12 - 9$$
$$= 0.$$

Note: · can also use $i=2$ or 3 in example above.

· can also go along a column since $\det A^T = \det A$.

columns

$$\boxed{\det A = \sum_{k=1}^{n} (-1)^{i+k} a_{ki} \det A_{ki}}$$

Ex: cofactor expansion down column 2.

(i=2)

$$A_{12} = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}, \quad A_{32} = \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}$$

$$|A_{12}| = -6, \quad |A_{22}| = -12, \quad |A_{32}| = -6.$$

so
$$\det A = (-1)\, a_{12} |A_{12}| + (1)\, a_{22} |A_{22}| + (-1)\, a_{32} |A_{32}|$$

$$= (-1)\, 2(-6) + 5(-12) - 8(-6)$$

$$= 12 - 60 + 48$$

$$= 0.$$

For a proof of this thm, see

- Meckes, Meckes, "Linear Algebra."