Thm: Let A be an $m \times n$ matrix.
If A can be transitioned to echelon form without interchanging rows, then there exist an $m \times m$ lower triangular matrix L and an $m \times n$ echelon matrix U such that $A = LU$.

pf)
$$(E_k \cdots E_1) A = U$$

where $E_1, \ldots, E_k$ are type III elementary matrices.

$$A = \overbrace{\underbrace{E_1^{-1} \cdots E_k^{-1}}_{\text{lower triangular}}}^{L} \underbrace{U}_{\text{in echelon form}}$$

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

$R_2 \to R_2 + (-3) R_1$

$$E_1 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

$$\underset{E_1}{\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}} = \underset{U}{\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}}$$

$$E_1^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

$$A = \underbrace{E_1^{-1}}_{L} U$$

$$A = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

Ex:

$$A = \begin{bmatrix} 2 & 8 & 0 \\ 4 & 18 & -4 \\ -2 & -2 & -13 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightsquigarrow E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_1 A = \begin{bmatrix} 2 & 8 & 0 \\ 0 & 2 & -4 \\ -2 & -2 & -13 \end{bmatrix}$$

$R_3 \to R_3 + R_1$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \rightsquigarrow E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$E_2 E_1 A = \begin{bmatrix} 2 & 8 & 0 \\ 0 & 2 & -4 \\ 0 & 6 & -13 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \rightarrow E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

$$E_3 E_2 E_1 A = \begin{bmatrix} 2 & 8 & 0 \\ 0 & 2 & -4 \\ 0 & 0 & -1 \end{bmatrix} = U$$

$$A = \underbrace{E_1^{-1} E_2^{-1} E_3^{-1}}_{L} U$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix} = L$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix} \boxed{\begin{bmatrix} 2 & 8 & 0 \\ 0 & 2 & -4 \\ 0 & 0 & -1 \end{bmatrix}}$$

LU factorization is not unique.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & & \\ & 2 & \\ & & -1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\quad L \qquad\qquad D \qquad\qquad U$$

unique (where L and U have 1's on main diagonal)

If A is symmetric

$$A^T = A$$

$$U^T D L^T = L D U$$

By uniqueness, $L = U^T$ $\quad D = \begin{bmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{bmatrix}$

$$A = U^T D U = L D L^T$$

If D has positive entries on diagonal,

$$D^{1/2} = \begin{bmatrix} \sqrt{d_1} & & \\ & \sqrt{d_2} & \\ & & \sqrt{d_3} \end{bmatrix}$$

$$A = L D^{1/2} D^{1/2} L^T$$

$$= L\,(D^{1/2})^T\, D^{1/2}\, L^T$$

$$= \underbrace{\left[L(D^{1/2})^T\right]}\ \underbrace{\left[L\,(D^{1/2})^T\right]^T}$$

$$A = \quad R\, R^T$$

Cholesky factorization.