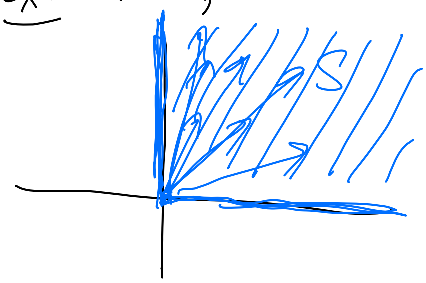
Closure under Vector Addition

Def'n: A nonempty subset S of $\mathbb{R}^n$ is called closed under vector addition if, whenever $\vec{v}$ and $\vec{w}$ are vectors in S, $\vec{v}+\vec{w}$ is also in S.

Ex: $\mathbb{R}^2$, $S = \{\langle x,y\rangle : x \geq 0 \text{ and } y \geq 0\}$.



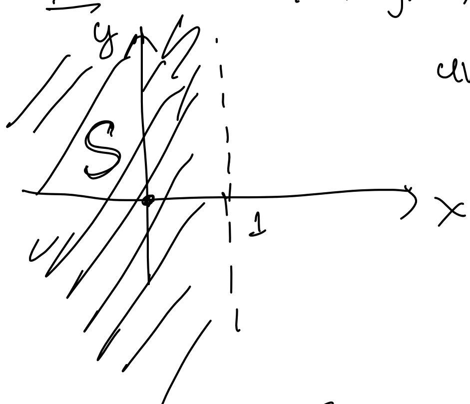
S is closed under vector addition because if $\vec{v} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$,

then $a_1 \geq 0$, $a_2 \geq 0$, $b_1 \geq 0$, $b_2 \geq 0$, so

$$\vec{v}+\vec{w} = \begin{bmatrix} a_1+b_1 \\ a_2+b_2 \end{bmatrix} \in S,$$ because

$a_1+b_1 \geq 0$ and $a_2+b_2 \geq 0$.

Ex: $S = \{\langle x,y\rangle : x<1\}$ is not closed under vector addition



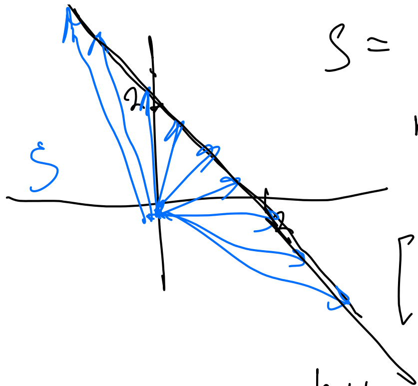
because $\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \in S$ and

$\begin{bmatrix} 2/3 \\ 0 \end{bmatrix} \in S$ but

$$\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} + \begin{bmatrix} 2/3 \\ 0 \end{bmatrix} = \begin{bmatrix} 7/6 \\ 1/2 \end{bmatrix} \notin S$$

because $7/6 \geq 1$.

<u>Ex:</u>



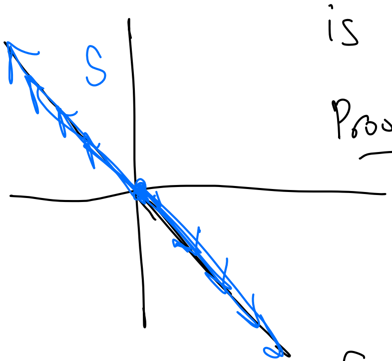
$S = \{\langle x,y\rangle : x+y=2\}$ is not closed under addition, because $\begin{bmatrix}0\\2\end{bmatrix} \in S$ and $\begin{bmatrix}2\\0\end{bmatrix} \in S$

but $\begin{bmatrix}0\\2\end{bmatrix} + \begin{bmatrix}2\\0\end{bmatrix} = \begin{bmatrix}2\\2\end{bmatrix} \notin S$

because $2+2 \neq 2$.

<u>Ex:</u> $S = \{\langle x,y\rangle : x+y=0\} \subseteq \mathbb{R}^2$

is closed under vector addition.



<u>Proof:</u> Suppose $\vec{v} \in S$ and $\vec{w} \in S$.

Then we can write

$\vec{v} = \begin{bmatrix}a_1\\a_2\end{bmatrix}$ where $a_1+a_2=0$

and $\vec{w} = \begin{bmatrix}b_1\\b_2\end{bmatrix}$, where $b_1+b_2=0$.

So $\vec{v}+\vec{w} = \begin{bmatrix}a_1+b_1\\a_2+b_2\end{bmatrix}$. Then $\vec{v}+\vec{w} \in S$ because

$(a_1+b_1)+(a_2+b_2) = (a_1+a_2)+(b_1+b_2)$

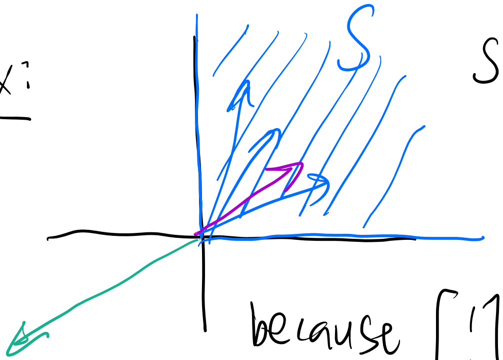
$= 0+0 = 0.$

So S is closed under vector addition.

Closure under Scalar Multiplication

Def'n: A subset S of $\mathbb{R}^n$ is **closed under scalar multiplication** if for every $\vec{v} \in S$ and every scalar c, $c\vec{v}$ is also in S.

Ex:

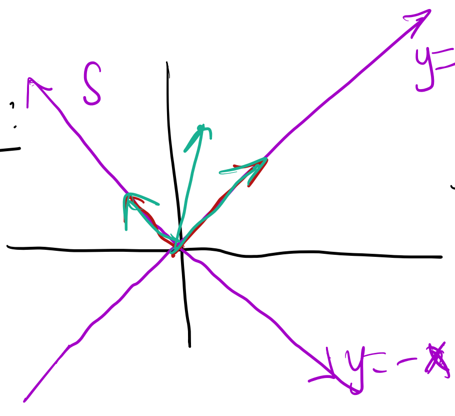


$S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0 \text{ and } y \geq 0 \right\}$ is not closed under scalar multiplication, because $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is in S but

$-5\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -5 \end{bmatrix}$ is not in S.

A subset S of $\mathbb{R}^n$ is NOT closed under scalar multiplication if there is some vector $\vec{v} \in S$ and some scalar c such that $c\vec{v}$ is not in S.

Ex:



$S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : y = x \text{ or } y = -x \right\}$ is closed under scalar multiplication.

If $\vec{v} \in S$, where $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$, then $y = x$ or

$y=-x$. For any scalar c, $c\vec{v}=\begin{bmatrix} cx \\ cy \end{bmatrix}$. If $y=x$,
then $cy=cx$, so $c\vec{v}$ is in S. If $y=-x$,
then $cy=-cx$, so $c\vec{v}$ is in S.

S is not closed under vector addition because
$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is in S (because $y=x$) and $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$ is in S because $2=-(-2)$;
but the sum, $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ is not in S because
$3\neq -1$ and $3\neq -(-1)$.