Directional Derivatives and the Gradient

1 Directional Derivatives

Let f be differentiable at (a, b) and let $\mathbf{u} = \langle u_1, u_2 \rangle$ be a unit vector in the xy-plane. The *directional derivative* of f at (a, b) in the direction of $\mathbf{u}$ is

$$D_{\mathbf{u}} f(a, b) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h},$$

provided the limit exists.

Theorem. The directional derivative of f at (a, b) in the direction of $\mathbf{u}$ is

$$D_{\mathbf{u}} f(a, b) = \langle f_x(a, b), f_y(a, b) \rangle \cdot \mathbf{u}$$

Example 1. Consider the paraboloid $z = f(x, y) = \frac{1}{4}(x^2 + 2y^2) + 2$. Let P_0 be the point $(3, 2)$ and consider the unit vector $\mathbf{u} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$. Find the directional derivative of f at P_0 in the direction of $\mathbf{u}$. Interpret this number.

2 The Gradient Vector

The **gradient** of f at (x, y) is the vector-valued function

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle.$$

So we have

$$D_{\mathbf{u}} f(a, b) = \nabla f(a, b) \cdot \mathbf{u}.$$

Example 2. Find $\nabla f(2, 3)$ for $f(x, y) = x^3 - 2xy + y^2$.

Example 3. Let $f(x,y) = 2 - \dfrac{x^2}{5} + \dfrac{x^2 y}{10}$.

a. Compute $\nabla f(1,2)$.

b. Compute $D_{\mathbf{u}} f(1,2)$, where $\mathbf{u} = \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle$

c. Compute the directional derivative of f at $(1,2)$ in the direction of the vector $\langle -3, 4 \rangle$.

3 Interpretation of the Gradient

Example 4. Suppose $z = f(x, y) = 5 + x^2 + 4y^2$.

a. If you are located on the graph of the function at the point $(4, 3, 2)$, in which direction should you move in order to ascend on the surface at the maximum rate? What is the rate of change?

b. In which direction should you move in order to descend on the surface at the maximum rate? What is the rate of change?

c. At this point, in which direction is there no (instantaneous) change in the function values?

Example 5. Let $f(x, y) = 3x^2 - 4y^2$.

a. Compute $\nabla f(x, y)$ and $\nabla f(2, 1)$.

b. Let $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$ be a unit vector. At $(2, 1)$, for what values of θ does the directional derivative have its maximum and minimum values? What are those values?