Partial Derivatives

Review of the Derivative of a Function of One (Independent) Variable:

What if you have a function of several variables?

Real-world examples:

- Profit of a company depends on (i) number of workers, (ii) cost of producing goods, (iii) price of items, (iv) money spent on advertising,...
- Your class grade is a function of (i) your level of interest, (ii) the number of hours you study, (iii) number of hours you sleep,...
- The temperature of a point in space depends on x, y, and z.

1 Partial Derivatives at a Point

We want to know the effect of only changing one variable, and keeping all of the other variables constant. This leads to *partial derivatives*.

Given a function $f(x, y)$, and a point (a, b) in the domain of f. Suppose we want to keep x constant and look at the rate of change of f with respect to y only. We can plug in $x = a$ to get

$$f(a, y)$$

which is a function of y by itself. We can think of it like $f(a, -)$ and call it $h(y)$. Then we evaluate the derivative of this function of one variable at b. This is called the *partial derivative* of f *with respect to* y at (a, b).

$$\frac{\partial f}{\partial y}(a, b) = h'(b).$$

Example 1. Find the partial derivative of $f(x, y) = x^3 \ln y$ with respect to y at $(2, e^2)$.

Solution: First keep x at 2 by substituting $x = 2$ into the function:

$$h(y) = f(2, y) = 2^3 \ln y = 8 \ln y.$$

Then evaluate h' at b. In this case, we want $h'(e^2)$:

$$h'(y) = \frac{8}{y},$$

so

$$\frac{\partial f}{\partial y}(1, e^2) = \frac{8}{e^2}.$$

You can also keep y fixed and only vary x. The partial derivative is the partial derivative of f *with respect to* x at (a, b):

$$\frac{\partial f}{\partial x}(a, b).$$

Example 2. Find the partial derivative of $f(x, y) = x^3 \ln y$ with respect to x at $(2, e^2)$.

Solution:

$$h(x) = f(x, e^2) = x^3 \ln(e^2) = 2x^3,$$

so

$$\frac{\partial f}{\partial x}(2, e^2) = h'(2) = 6(2)^2 = 24.$$

2 Partial Derivatives as Functions

Since we can find the partial derivatives at each point in the domain of f, we can think of the partial derivatives of f as functions of the same variables.

Note: The partial derivative $\dfrac{\partial f}{\partial x}$ is often written as f_x, and similarly for y.

Example 3. Find the first partial derivatives of $f(x, y) = xe^y$.

Solution: To find $\dfrac{\partial f}{\partial x}$, keep y fixed; this gives $e^y x$ where you think of y as a constant, so e^y is a constant (like 5) and differentiate with respect to x.:

$$\frac{\partial f}{\partial x} = e^y.$$

To find $\dfrac{\partial f}{\partial y}$, first we keep x fixed, so the function is like $5e^y$. Then we differentiate that with respect to y.

$$\frac{\partial f}{\partial y} = xe^y.$$

Example 4. Find the first partial derivatives of $f(x, y) = 4x^3y^2 + 3x^2y^3 + 10$.

Example 5. Find the first partial derivatives of $f(w,z) = \dfrac{w}{w^2+z^2}$

Example 6. Find the first partial derivatives of $z(x,y) = y^2 \tan(xy)$

3 Limit Definition of Partial Derivatives

Note: We can also define f_x and f_y in terms of limits:

$$f_x(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}.$$

Notice that for the function $f(-, b)$, its derivative at a is

$$\lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h} = f_x(a, b).$$

Similarly,

$$f_y(a, b) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h}.$$

4 Higher-Order Partial Derivatives

Example 7. Compute $f_{xy}(1,1)$ if $f(x,y) = x^5 \ln(x+y)$.

Solution:

$$f_x = x^5 \cdot \frac{1}{x+y} + 5x^4 \ln(x+y)$$

$$f_{xy} = x^5 \cdot \frac{-1}{(x+y)^2} + 5x^4 \cdot \frac{1}{x+y}$$

$$f_{xy}(1,1) = 1^5 \cdot \frac{-1}{(1+1)^2} + 5 \cdot 1^4 \cdot \frac{1}{1+1} = -\frac{1}{4} + \frac{5}{2} = \frac{9}{4}.$$

Theorem. (Clairaut's Theorem) Assume that f is defined on an open set D of $\mathbf{R}^2$, and that f_{xy} and f_{yx} are continuous throughout D. Then $f_{xy} = f_{yx}$ at all points of D. (Similar statement for $f_{xyx} = f_{xxy} = f_{yxx}$, etc.)

5 Partial Derivatives of Functions of Three Variables

Example 8. Find f_x, f_y, and f_z where $f(x,y,z) = e^{-xy} \cos z$.