Tangent Planes and Linear Approximation Examples

Example: Let
$$F(x, y, z) = x^2/9 + y^2/25 + z^2 - 1 = 0$$
.

(a) Find an equation of the plane tangent to the ellipsoid at $(0, 4, 3/5)$.
(b) At what points on the ellipsoid is the tangent plane horizontal?

Solution: $F_x = 2x/9, F_y = 2y/25, F_z = 2z$
$\nabla F(0, 4, 3/5) = \langle 0, 8/25, 6/25 \rangle$
Equation of tangent plane:
$0(x - 0) + 8/25(y - 4) + 6/5(z - 3/5) = 0$
$4y + 15z = 25$
(b) When $F_x = F_y = 0$
$x = y = 0$.
Plug into equation of ellipsoid
get $z = \pm 1$
so (0,0,1) and (0,0,-1).

Your turn: Find an equation of the plane tangent to the surface $x^2 + y^3 + z^4 = 2$ at the point (-1,0,1).

Now consider the specific case of a surface $z = f(x, y)$ can be viewed as the 0-level set of a function of 3 variables, $F(x, y, z) = f(x, y) - z$.

We have seen that, at a point (a,b,c) on a level set, the gradient of a function $F(x, y, z)$ evaluated at (a,b,c), is orthogonal to the level set through (a,b,c).

So if (a,b,c) is on the surface $z = f(x, y)$, then $\nabla F(a, b, c)$ is orthogonal to the surface at (a,b,c). This is a normal vector to the surface.

This vector is orthogonal to the tangent plane to the surface at (a,b,c) (the plane that most closely emulates the surface at (a,b,c)).

Using the equation for a plane, the consequence is that we know an equation for the tangent plane to the surface $z = f(x, y)$. Define a new function of three variables, $F(x, y, z)$, by $F(x, y, z) = z - f(x, y)$.

It is

$$F_x(a, b, c)(x - a) + F_y(a, b, c)(y - b) + F_z(a, b, c)(z - c) = 0.$$

Actually, in this case F_z is always just 1, $F_x(a, b, c)$ is always just $f_x(a, b)$, and $F_y(a, b, c)$ is always just $f_y(a, b)$. In addition, c is just $f(a, b)$. So the equation of the tangent plane is

$$f_x(a,b)(x-a)+f_y(a,b)(y-b)-(z-f(a,b))=0$$

You can also rearrange the terms to get

$$z=f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)$$

Example: Let $w(x,y,z)=x^3-2y^2+z^2$. Let S be the surface defined by $w(x,y,z)=25$.
(a) Find a vector perpendicular to the surface S at the point (3,1,0).
(b) Find an equation of the tangent plane to S at (3,1,0).

Solution:
(a) The surface is the level surface $w=25$. Important fact: The gradient of w at P is orthogonal to the surface at P.
$\nabla w=\langle 3x^2,-4y,2z\rangle$
$\nabla w(P)=\langle 27,-4,0\rangle$.
(b) $27(x-3)-4(y-1)=0$.

Example: Find the equation of the tangent plane to the surface

$$2xy+ze^y=0$$

at the point $(e,1,-2)$.
Solution: see solutions online.

Example: Calculate the tangent plane to the graph of the function $f(x,y)=2x^2y+x^2$ at the point (1,2,5).

Solution:
The graph of $f(x,y)$ is the level surface $z=2x^2y+x^2$. This is the same as the level surface $w=0$, where $w=2x^2y+x^2-z$.
$w=2x^2y+x^2-z$
$\nabla w=\langle 4xy+2x,2x^2,-1\rangle$
$\nabla w(1,2,5)=\langle 10,2,-1\rangle$
tangent plane:
$10(x-1)+2(y-2)-(z-5)=0$
Example: Find an equation of the plane tangent to the paraboloid $z=f(x,y)=32-3x^2-4y^2$ at (2,1,16).

Solution: $f_x=-6x$, $f_y=-8y$
$f_x(2,1)=-12$ and $f_y(2,1)=-8$.
So an equation of the tangent plane is
$z=16-12(x-2)-8(y-1)$

Your turn: $z=x^2e^{x-y};(2,2,4)$
Example: Find the linear approximation to the function $f(x,y)=2x^2y+x^2$ at the point (1,2).

Solution:

You could use the answer from the previous problem. From the tangent plane equation, solve for z:

$z = 10(x-1) + 2(y-2) + 5$.

Write it as a function of x and y:

$L(x, y) = 10(x-1) + 2(y-2) + 5$.

(you could also use the formula $L(x, y) = f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b)$)

Example: Let $f(x, y) = (x+y)e^{xy}$.

(a) Find the linear approximation to the function f at $(2, 0)$.

(b) Estimate $f(1.95, 0.05)$ (and simplify your answer).

Solution:

(a) $f_x = (x+y)e^{xy}y + e^{xy}$

$f_y = (x+y)e^{xy}x + e^{xy}$

$f_x(2, 0) = 1$

$f_y(2, 0) = 5$

$f(2, 0) = 2$

$L(x, y) = 2 + (x-2) + 5y$

(b) $L(1.95, 0.05) = 2 + (-0.05) + .25 = 1.95 + .25 = 2.2$.

Example: Use the linear approximation to the function

$$f(x, y) = x^3 - 2y^2$$

at $(1, 1)$ to estimate $f(1.05, 0.9)$.

Solution: see online solutions

Example: Let $f(x, y) = \dfrac{5}{x^2 + y^2}$.

a. Find the linear approximation to the function at (-1,2,1)

b. Estimate $f(-1.05, 2.1)$.

Solution: $f_x = -\dfrac{10x}{(x^2+y^2)^2}$ and $f_y = -\dfrac{10y}{(x^2+y^2)^2}$.

So

$f_x(-1, 2) = 2/5$ and $f_y(-1, 2) = -4/5$.

So the linear approximation at (-1,2,1) is

$$L(x, y) = 1 + 2/5(x+1) - 4/5(y-2)$$

$$= 2/5x - 4/5y + 3$$

Your turn:

$$f(x, y) = \sqrt{x^2 + y^2}$$

(a) Find the linear approximation to f at (3,-4)

(b) Use part (a) to estimate $f(3, 06, -3.92)$

Example: Let $f(x, y) = \dfrac{5}{x^2 + y^2}$. Approximate the change in z when the independent variables change from (-1,2) to (-0.93,1.94).

Solution: $\Delta x = 0.07$ and $\Delta y = -0.06$. So Δz is about $f_x(-1, 2)\Delta x + f_y(-1, 2)\Delta y$ $= 0.076$.

Your turn: Approximate the change in z for the given changes in the independent variables.

$z = 2x - 3y - 2xy$ when (x, y) changes from (1,4) to (1.1,3.9)