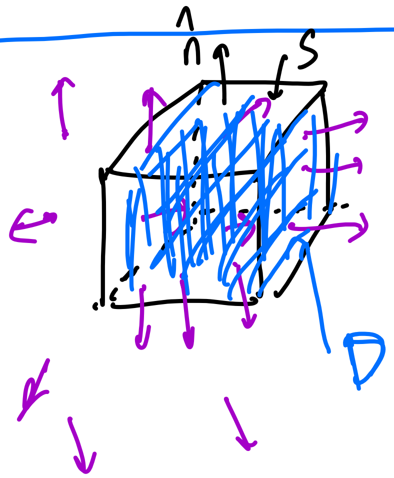
Find the net outward flux of the field $\vec{F} = \langle xyz, xyz, xyz \rangle$ across the boundaries of the cube $D = \{(x,y,z) : 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$.



$$\iint_S \underbrace{\vec{F} \cdot \hat{n}}_{\text{tangential component of } \vec{F} \text{ on } S} \, dS$$

outward flux

Divergence Theorem

$$= \iiint_D \operatorname{div}(\vec{F}) \, dV$$

$$\vec{F} = \langle F_1, F_2, F_3 \rangle$$

$$\operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\text{"} \vec{\nabla} \cdot \vec{F} \text{"} = \text{"} \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle F_1, F_2, F_3 \rangle \text{"}$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$F_1 = xyz, \quad F_2 = xyz, \quad F_3 = xyz$$

$$\operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = yz + xz + xy$$

net outward flux $= \int_0^1\int_0^1\int_0^1 (yz + xz + xy)\,dz\,dy\,dx$

inner: $\int_0^1 (yz + xz + xy)\,dz = \frac{y}{2}z^2 + \frac{x}{2}z^2 + xyz\Big|_0^1$

$= \frac{y}{2} + \frac{x}{2} + xy$

middle: $\int_0^1 \left(\frac{y}{2} + \frac{x}{2} + xy\right) dy = \frac{y^2}{4} + \frac{xy}{2} + \frac{xy^2}{2}\Big|_0^1$

$= \frac{1}{4} + \frac{x}{2} + \frac{x}{2} = x + \frac{1}{4}$

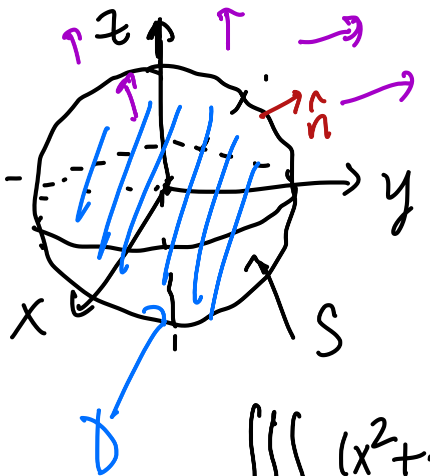
outer: $\int_0^1 \left(x + \frac{1}{4}\right) dx = \frac{1}{2}x^2 + \frac{1}{4}x\Big|_0^1$

$= \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$.

Ex: For $F = \langle xz^2, \frac{y^3}{3}, x^2 z \rangle$, evaluate

$$\iint \underbrace{\vec{F}\cdot d\vec{S}}\, = \iint (\vec{F}\cdot\hat{n})\, dS$$

where S is the sphere of radius 2 centered at the origin. Orient the surface with the outward pointing normal vector.



divergence theorem says:

$$\iint \vec{F}\cdot\hat{n}\, dS = \iiint_D \operatorname{div}\vec{F}\, dV$$

$$\operatorname{div}\vec{F} = z^2 + y^2 + x^2$$

$$\iiint_D (x^2+y^2+z^2)\, dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{2} \underbrace{\rho^2\, \rho^2 \sin\varphi}_{\rho^4 \sin\varphi}\, d\rho\, d\varphi\, d\theta$$

inner: $\displaystyle\int_0^2 \rho^4 \sin\varphi\, d\rho = \sin\varphi \int_0^2 \rho^4\, d\rho$

$$= \sin\varphi \left.\frac{\rho^5}{5}\right|_0^2$$

$$= \sin\varphi\left(\frac{32}{5} - 0\right)$$

$$= \frac{32}{5}\sin\varphi$$

middle: $\frac{32}{5}\int_0^{\pi} \sin\varphi \, d\varphi = -\frac{32}{5}\cos\varphi \Big|_0^{\pi}$

$= -\frac{32}{5}\left[\cos\pi - \cos 0\right]$

$= -\frac{32}{5}(-1-1)$

$= \frac{64}{5}$

outer: $\int_0^{2\pi} \frac{64}{5}\, d\theta = \frac{64}{5}(2\pi) = \frac{128\pi}{5}$