Cavilieri's Principle

Theorem 1 (Cavilieri's Principle)**.** Let A be a contented subset of $\mathbb{R}^{n+1}$ with $A \subset \mathbb{R} \times [a, b]$ where $R \subset \mathbb{R}^n$ and $[a, b] \subset \mathbb{R}$ are intervals. Suppose

$$A_t := \{\mathbf{x} \in \mathbb{R}^n : (\mathbf{x}, t) \in A\} \subset \mathbb{R}^n$$

is contented for each $t \in [a, b]$, and write $A(t) = v(A_t)$. Then

$$v(A) = \int_a^b A(t)\, dt.$$