VECTOR FIELDS

1. Vector fields

Reference: Varadarajan 1974, p.5

Let $X(x \mapsto X_x)$ be any assignment such that $X_x \in T_{xc}(M)$ for all $x \in M$. Then for any function $f \in C^\infty(M)$, the function $Xf : x \mapsto X_x(\mathbf{f}_x)$ is well defined on M, $\mathbf{f}_x$ being the germ at x defined by f.

We say that X is a **vector field** on M if $Xf \in C^\infty(M)$ for all $f \in C^\infty(M)$. A vector field X is said to be **real** if $X_x \in T_x(M)$ for all $x \in M$.

Then X is real if and only if Xf is real for all real $f \in C^\infty(M)$.

Given a vector field X, the mapping $f \mapsto Xf$ is a derivation of the algebra $C^\infty(M)$; i.e., for all f and $g \in C^\infty(M)$,

$$X(fg) = f \cdot Xg + g \cdot Xf.$$

This correspondence between vector fields and derivations is one to one and maps the set of all vector fields onto the set of all derivations of $C^\infty(M)$.

Notation: Denote by $\mathfrak{T}(M)$ the set of all vector fields on M.

If $X \in \mathfrak{T}(M)$ and $f \in C^\infty(M)$, then $fX : x \mapsto f(x)X_x$ is also a vector field. In this way, $\mathfrak{T}(M)$ becomes a module over $C^\infty(M)$. We make in general no distinction between a vector field and the corresponding derivation of $C^\infty(M)$.

We now define a Lie algebra structure. Let X and Y be two vector fields. Then $X \circ Y - Y \circ X$ is an endomorphism of $C^\infty(M)$ which is easily verified to be an endomorphism. The associated vector field is denoted by $[X, Y]$ and is called the **Lie bracket** of X with Y. The map

$$(X, Y) \mapsto [X, Y]$$

is bilinear and possesses the following easily verified properties:

(Lie algebra axioms).

2. Definition

If M is a smooth manifold with or without boundary, a **vector field on** M is a section of the map $\pi : TM \to M$. More concretely, a vector field is a continuous map $X : M \to TM$, usually written $p \mapsto X_p$, with the property that

$$\pi \circ X = \mathrm{Id}_M,$$

or equivalently, $X_p \in T_pM$ for each $p \in M$. (We write the value of X at p as X_p instead of $X(p)$ to be consistent with our notation for elements of the tangent bundle. You should visualize a vector field on M in the same way as you visualize vector fields in Euclidean space: as an arrow attached to each point of M, chosen to be tangent to M and to vary continuously from point to point.

Date: October 2020.

3. Vector fields as derivations of $C^\infty(M)$

An essential property of vector fields is that they define operators on the space of smooth real-valued functions. If $X \in \mathfrak{X}(M)$ and f is a smooth real-valued function defined on an open subset $U \subseteq M$, we obtain a new function $Xf : U \to \mathbb{R}$, defined by

$$(Xf)(p) = X_p f.$$

A smooth vector field X defines a map from $C^\infty(M)$ to itself by $f \mapsto Xf$. This map is clearly linear over $\mathbb{R}$. Moreover, the product rule for tangent vectors translates into the following product rule for vector fields:

$$X(fg) = fXg + gXf,$$

as you can easily check by evaluating both sides at an arbitrary point $p \in M$. In general, a map $X : C^\infty(M) \to C^\infty(M)$ is called a **derivation** if it is linear over $\mathbb{R}$ and satisfies

$$X(fg) = fXg + gXf$$

for all $f, g \in C^\infty(M)$.

The next proposition shows that derivations of $C^\infty(M)$ can be identified with smooth vector fields.

Proposition 3.1. *Let M be a smooth manifold. A map $D\colon C^\infty(M) \to C^\infty(M)$ is a derivation if and only if it is of the form $Df = Xf$ for some smooth vector field $X \in \mathcal{X}(M)$.*

See Lee, Proposition 8.15.

Because of this result, we sometimes *identify* smooth vector fields on M with derivations of $C^\infty(M)$, using the same letter for both the vector field (thought of as a smooth map from M to TM) and the derivation (thought of as a linear map from $C^\infty(M)$ to itself).

4. Lie brackets

In this section we introduce an important way of combining two smooth vector fields to obtain another vector field.

Let X and Y be smooth vector fields on a smooth manifold M. Given a smooth function $f : M \to \mathbb{R}$, we can apply X to f and obtain another smooth function Xf. In turn, we can apply Y to this function, and obtain yet another smooth function $YXf = Y(Xf)$. The operation $f \mapsto YXf$, however, does not in general satisfy the product rule and thus cannot be a vector field.

We can also apply the same two vector fields in the opposite order, obtaining a (usually different) function XYf. Applying both of these operators to f and subtracting, we obtain an operator $[X, Y] : C^\infty(M) \to C^\infty(M)$, called the **Lie bracket** of X and Y, defined by

$$[X, Y]f = XYf - YXf.$$

The key fact is that this operator *is* a vector field.

Lemma. The Lie bracket of any pair of smooth vector fields is a smooth vector field.