Lagrange Multipliers

Theorem 1. Let g be a continuously differentiable function from $\mathbb{R}^2 \to \mathbb{R}$, and let S be the zero set of g. Suppose $\mathbf{p}$ is a point of S where $\nabla g(\mathbf{p}) \neq \mathbf{0}$. Then there is a rectangle Q centered at $\mathbf{p}$ and a differentiable curve $\varphi\colon \mathbb{R} \to \mathbb{R}^2$ with $\varphi(0) = \mathbf{p}$ and $\varphi'(0) \neq \mathbf{0}$ such that S and the image of φ agree inside Q. That is, a point of Q lies on the zero set S of g if and only if it lies on the image of the curve φ.

Theorem 2. Let f and g be continuously differentiable functions on $\mathbb{R}^2$. Suppose that f attains its maximum or minimum value on the zero set S of g at the point $\mathbf{p}$, and that $\nabla g(\mathbf{p}) \neq \mathbf{0}$. Then there is a number λ such that

$$\nabla f(\mathbf{p}) = \lambda \nabla g(\mathbf{p}).$$

The number λ is called a **Lagrange multiplier**.

This theorem provides a recipe for locating the points $\mathbf{p} = (x, y)$ at which f attains its maximum and minimum values (if any) on the zero set of g (provided they are attained at points where the gradient vector $\nabla g \neq \mathbf{0}$). The vector equation gives two scalar equations in the unknowns x, y, λ, while $g(x, y) = 0$ is a third equation. In principle these three equations can be solved for x, y, λ. Each solution (x, y, λ) gives a candidate (x, y) for an extreme point. We can finally compare the values of f at these candidate points to ascertain where its maximum and minimum values on S are attained.

1 Lagrange Multipliers for $z = f(x, y)$

2 Lagrange Multipliers for $w = f(x, y, z)$

3 More Examples

Example 1. Find the point(s) on the cone $z^2 - x^2 - y^2 = 0$ that are closest to the point $(1, 3, 1)$.

Example 2. $f(x, y) = x^2 + xy + y^2$ subject to $x^2 + y^2 = 8$

$2x + y = \lambda 2x$

$x + 2y = \lambda 2y$

$\lambda \neq 0.$

$2y(2x + y) = 2x(x + 2y)$

$y(2x + y) = x(x + 2y)$

$2xy + y^2 = x^2 + 2xy$
$x^2 = y^2$
$x = \pm y$
If $x = -y$, get $2x^2 = 8$, so $x = 2$ or $x = -2$
If $x = y$, get $(2, 2)$ or $(-2, -2)$