Multivariable Chain Rule

Theorem 1 (The Chain Rule)**.** Let U and V be open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. If the mappings $F\colon U \to \mathbb{R}^m$ and $G\colon V \to \mathbb{R}^k$ are differentiable at $\mathbf{a} \in U$ and $F(\mathbf{a}) \in V$ respectively, then their composition $H = G \circ F$ is differentiable at $\mathbf{a}$, and

$$dH_{\mathbf{a}} = dG_{F(\mathbf{a})} \circ dF_{\mathbf{a}}.$$

In terms of derivatives, we therefore have

$$DH(\mathbf{a}) = DG(F(\mathbf{a})) \cdot DF(\mathbf{a}).$$

In brief, the differential of the composition is the composition of the differentials; the derivative of the composition is the product of the derivatives.

We shall now apply the chain rule to generalize some of the basic results of single-variable calculus.

Theorem 2. Let U be a connected open subset of $\mathbb{R}^n$. Then the differentiable mapping $F\colon U \to \mathbb{R}^m$ is constant on U if and only if $DF(\mathbf{x}) = 0$ for all $\mathbf{x} \in U$.

Corollary 3. Let F and G be two differentiable mappings of the connected set $U \subset \mathbb{R}^n$ into $\mathbb{R}^m$. If $DF(\mathbf{x}) = DG(\mathbf{x})$ for all $\mathbf{x} \in U$, then there exists $\mathbf{c} \in \mathbb{R}^m$ such that

$$F(\mathbf{x}) = G(\mathbf{x}) + \mathbf{c}$$

for all $\mathbf{x} \in U$. That is, F and G differ only by a constant.

Theorem 4 (Mean Value Theorem)**.**

Next we are going to use the mean value theorem to prove that the second partial derivatives $D_j D_i f$ and $D_i D_j f$ are equal under appropriate conditions.

Theorem 5. Let f be a real-valued function defined on the open set U in $\mathbb{R}^n$. If the first and second partial derivatives of f exist and are continuous on U, then $D_i D_j f = D_j D_i f$ on U.