§8 of Analysis on Manifolds by Munkres

$A \overset{\text{open}}{\subset} \mathbb{R}^n$

$f: A \to \mathbb{R}^n$ C^∞.

f has a diff'ble inv $\Rightarrow f \circ f^{-1} = id$

$\Rightarrow Df \cdot D(f^{-1}) = I_n$

($Df(\vec{a}) \Rightarrow$ f is locally 1-1) $\Rightarrow Df$ is invertible.

<u>Lemma</u>. $A \overset{\text{open}}{\subset} \mathbb{R}^n$, $f: A \to \mathbb{R}^n$ of class C^1.

If $Df(\vec{a})$ is non-singular, then $\exists\, \alpha > 0$ s.t.

$$\| f(x_0) - f(x_1) \| \geq \alpha \| x_0 - x_1 \|$$

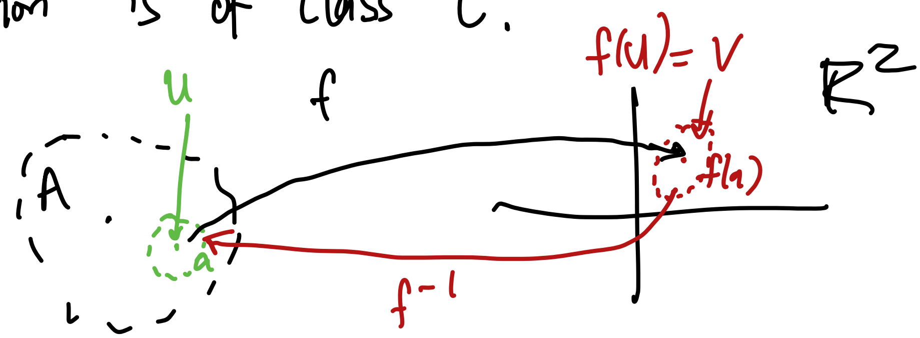
for all x_0, x_1 in some open cube centered at a. In particular, f is 1-1 on this open cube.

Brouwer thm on invariance of domain

<u>Thm</u>: If A is open in $\mathbb{R}^n$, and $f: A \to \mathbb{R}^n$ is continuous and 1-1, then $f(A)$ is open and the inverse $f^{-1}: f(A) \to A$ is continuous.

<u>Thm</u>: If, in addition, f is of class C^r and $Df(x)$ is non-singular $\forall\, x \in A$, then the inverse fn $f^{-1}: f(A) \to A$ is of class C^r.

Thm (Inverse function thm). If $Df(x)$ is nonsingular $\forall a \in A$, then $\exists$ nbd U of a s.t. f carries U in a 1-1 fashion onto an open set V of $\mathbb{R}^n$ and the inverse function is of class C^r.



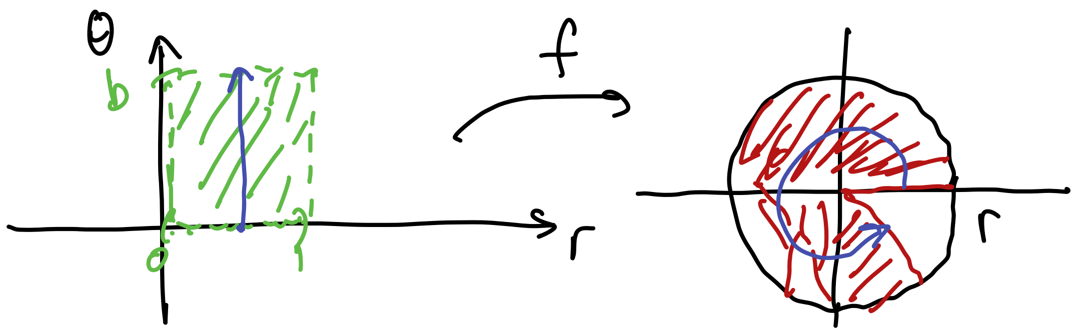
Note: f will not, in general, be 1-1.

Ex: $f: \mathbb{R}^2 \to \mathbb{R}^2$

$$(r, \theta) \longmapsto (r\cos\theta, r\sin\theta)$$

$$Df(r,\theta) = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$$\det Df(r,\theta) = r.$$



If $b > 2\pi$, f is not 1-1.