Cylindrical Coordinates

Visualize cylinders centered at the z-axis of various radii. Each point (x, y, z) other than the origin is on exactly one cylinder, say of radius r. Let θ be the angle made from the positive x-axis to the vector $\langle x, y\rangle$. The **cylindrical coordinates** of the point (x, y, z) are (r, θ, z).

There is a way to convert:

$$r = \sqrt{x^2 + y^2}, \quad x = r\cos\theta, \quad y = r\sin\theta.$$

When the parameter θ does not appear, the equation is independent of θ. Suppose that a point (x_0, y_0, z_0) has coordinates (r_0, θ_1, z_0). Then for all θ, the point (r_0, θ, z_0) also satisfies the equation. The collection of points with a fixed r_0 and z_0 is the intersection of the plane $z = z_0$ and the sphere $r = r_0$, that is, a circle. So if a point satisfies an equation that does not have θ in it, so does every point in $\mathbb{R}^3$ obtained by rotating that point about the z-axis. Such a surface is called a **surface of revolution** (about the z-axis).

Example 1. Graph the surface having equation $z = 2r$ in cylindrical coordinates.

SOLUTION: We observe that θ does not appear in the equation. So to graph the surface, we can just fix a specific $\theta = \theta_0$ and look at the points of the surface that lie in this half-plane. Then we can revolve the resulting curve about the z-axis. To keep things simple, we take $\theta = 0$, that is, $x \geq 0$ and $y = 0$. Converting to Cartesian coordinates, we get

$$z = 2r = 2\sqrt{x^2 + y^2} = 2\sqrt{x^2 + 0} = 2|x| = 2x$$

the last equation holding because we are only considering $x \geq 0$. The graph is just a line with slope 2. Now we rotate and that gives us a cone.

Example 2. Graph $(r - 2)^2 + z^2 = 1$.

SOLUTION: Again we note that this is an equation in cylindrical coordinates where θ does not appear. So we consider the intersection of the surface with the half-plane $y = 0$, $x \geq 0$, so that $r = x$. We get

$$(x - 2)^2 + z^2 = 1.$$

On the xz-plane, this is the circle of radius 1 centered at $(2, 0)$. So we draw this circle and then rotate it about the z-axis. The resulting surface is called a **torus**.

We can graph *solids of revolution* using the same technique.

Example 3. Graph the solid given by

$$r^2 - 1 \leq z \leq 5 - r^2.$$

SOLUTION: Since θ does not appear, we set $r = x$ and graph

$$x^2 - 1 \leq z \leq 5 - x^2$$

on the xz-plane (the part where $x \geq 0$). This is the region between two parabolas. We rotate this region about the z-axis to get the solid.