Gradient, Divergence, and Curl

1 Gradient

We define the **gradient** of a scalar-valued differentiable function (that is, a scalar field) $f\colon X \subset \mathbb{R}^n \to \mathbb{R}$ to be the vector field $\nabla f\colon X \to \mathbb{R}^n$ defined by

$$\nabla f(\mathbf{a}) = (f_{x_1}(\mathbf{a}), ..., f_{x_n}(\mathbf{a}))$$

for all $\mathbf{a} \in X$.

We can think of ∇ as taking the scalar field f to the vector field ∇f. Something that maps functions to functions is often called an *operator* (this word doesn't have a universal mathematical definition). So ∇ is an operator, called the del operator.

Symbolically, we can write

$$\nabla = \mathbf{e}_1 \frac{\partial}{\partial x_1} + \cdots + \mathbf{e}_n \frac{\partial}{\partial x_n}.$$

You should think of this as meaningless but helpful notation. The notation means that the result of "plugging in" f into the operator ∇ is the function

$$\nabla f = \mathbf{e}_1 \frac{\partial}{\partial x_1} f + \cdots + \mathbf{e}_n \frac{\partial}{\partial x_n} f.$$

Confusingly, some authors refer to the gradient of f as a vector. This would be like calling the derivative of a function a number. The derivative of a function f is a function f'. The function f' takes a number a in the domain of f and spits out the number $f'(a)$. In the same way, the gradient of f is a vector field ∇f. It takes a vector $\mathbf{a}$ in the domain of f and spits out the vector $(\nabla f)(\mathbf{a})$.

2 Divergence

Let $\mathbf{F}\colon X \subset \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable vector field. Then the **divergence** of $\mathbf{F}$, denoted div $\mathbf{F}$, is the scalar field

$$\text{div } F = \frac{\partial F_1}{\partial x_1} + \cdots + \frac{\partial F_n}{\partial x_n},$$

where $F_1, ..., F_n$ are the component functions of $\mathbf{F}$.

Sometimes the divergence of $\mathbf{F}$ is expressed as $\nabla \cdot \mathbf{F}$. This is another meaningless but useful notation. The dot product is defined for vectors. If we think of $\dfrac{\partial}{\partial x_i} F_i$ as a "product" then we get the "dot product" of

$$\nabla = \mathbf{e}_1 \frac{\partial}{\partial x_1} + \cdots + \mathbf{e}_n \frac{\partial}{\partial x_n}$$

and

$$\mathbf{F} = F_1 \mathbf{e}_1 + \cdots + F_n \mathbf{e}_n$$

to be

$$\nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x_1} + \cdots + \frac{\partial F_n}{\partial x_n}.$$

3 Curl

The curl is only defined on vector fields in $\mathbb{R}^3$. Let $\mathbf{F}\colon X \subset \mathbb{R}^3 \to \mathbb{R}^3$ be a differentiable vector field on $\mathbb{R}^3$. The **curl** of $\mathbf{F}$, denoted curl $\mathbf{F}$ or $\nabla \times \mathbf{F}$, is the vector field

$$\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) \mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathbf{k}.$$

It turns out that this vector field, in a very vague sense, measures how much the vector field $\mathbf{F}$ curls.

The curl can be remembered by using the symbolic determinant

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Again, if we pretend that $\dfrac{\partial F_3}{\partial y}\mathbf{i}$, say, means the "product" of $\dfrac{\partial}{\partial y}$, F_3, and $\mathbf{i}$, then the curl of $\mathbf{F}$ would be the determinant above.

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The two statements below relate the concepts of curl, gradient, and divergence. They can be proven easily by a direct computation.

Theorem 1. Let $f\colon X \subset \mathbb{R}^3 \to \mathbb{R}$ be of class C^2. Then $\operatorname{curl}(\operatorname{grad} f) = \mathbf{0}$.

Theorem 2. Let $\mathbf{F}\colon X \subset \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field of class C^2. Then $\operatorname{div}(\operatorname{curl} \mathbf{F}) = 0$.

Notice that the equality of mixed partials is essential to the proof.