Limits and Continuity

1. **Example:** *Evaluate*

$$\lim_{(x,y)\to(2,2)} \frac{y^2-4}{xy-2x}$$

Solution:

If I plug in $x = 2$ and $y = 2$ into the function I get $0/0$ so I should simplify as much as possible. Factor:

$$\lim_{(x,y)\to(2,2)} \frac{y^2-4}{xy-2x} = \lim_{(x,y)\to(2,2)} \frac{(y+2)(y-2)}{x(y-2)}$$

$$= \lim_{(x,y)\to(2,2)} \frac{y+2}{x}$$

$$= \frac{2+2}{2} = 4.$$

Your turn: Evaluate

$$\lim_{(x,y)\to(-1,4)} \frac{x^2-9}{xy+3y}$$

2. **Example:** *Evaluate*

$$\lim_{(x,y)\to(4,5)} \frac{\sqrt{x+y}-3}{x+y-9}$$

Solution: I get $0/0$ so I should simplify as much as possible. Multiply and divide by the conjugate:

$$\lim_{(x,y)\to(4,5)} \frac{\sqrt{x+y}-3}{x+y-9} = \lim_{(x,y)\to(4,5)} [\frac{\sqrt{x+y}-3}{x+y-9} \cdot \frac{\sqrt{x+y}+3}{\sqrt{x+y}+3}]$$

$$= \lim_{(x,y)\to(4,5)} \frac{x+y-9}{(x+y-9)(\sqrt{x+y}+3)}$$

$$= \lim_{(x,y)\to(4,5)} \frac{1}{\sqrt{x+y}+3}$$

$$= \frac{1}{6}$$

Your turn: Evaluate

$$\lim_{(x,y)\to(1,2)} \frac{\sqrt{y}-\sqrt{x+1}}{y-x-1}$$

3. **Example:** *Evaluate or show the limit doesn't exist.*

$$\lim_{(x,y)\to(0,0)} \frac{y^4-2x^2}{y^4+x^2}$$

Solution:

Along $x = 0$

$$\lim_{y\to 0} \frac{-2x^2}{x^2} = -2$$

but along $y = 0$

$$\lim_{x\to 0} \frac{y^4}{y^4} = 1$$

since the two limits are not equal, the limit does not exist.

Your turn:

$$\lim_{(x,y)\to(0,0)} \frac{x^3-y^2}{x^3+y^2}$$

4. **Example:** *Evaluate the limits or explain why the limit fails to exist.*

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$$

Solution:

One way:

Along $y = mx$, get

$$\lim_{x\to 0} \frac{x^2}{x^2+(mx)^2}$$

$$= \lim_{x\to 0} \frac{x^2}{(m+1)x^2}$$

$$= \lim_{x\to 0} \frac{1}{m+1}$$

$$= \frac{1}{m+1}$$

Since this depends on m, the limit doesn't exist.

Another way:

Along $x = 0$, get

$$\lim_{y\to 0} \frac{0}{0+y^2} = \lim_{y\to 0} 0 = 0,$$

but

along $y = 0$ get

$$\lim_{x\to 0} \frac{x^2}{x^2+0} = \lim_{y\to 0} 1 = 1.$$

Since the two limits are not equal, the limit does not exist.

Your turn: Use the Two-Path Test to prove that the following limit does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{xy^2}$$

5. Evaluate the limits or explain why the limit fails to exist.

   (a)

$$\lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x^2+y^2}$$

   (b)

$$\lim_{(x,y)\to(0,0)} \frac{2x^2+y^2}{x^2+y^2}$$

   (d)

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}.$$

(e)

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$$

(f)

$$\lim_{(x,y)\to(0,0)} \frac{x^{30}y^{10}}{x^{60}+y^{20}}$$

(g) Let

$$f(x,y) = \frac{x^2-2x-y^2+1}{x^2-2x+y^2+1}.$$

Compute

$$\lim_{(x,y)\to(1,0)} f(x,y)$$

or show that it does not exist.

(h) Let $f(x,y) = \dfrac{2x(y+1)}{4x^2+5(y+1)^2}$.

Evaluate $\lim_{(x,y)\to(0,-1)} f(x,y)$ or show it doesn't exist.

(i) Evaluate

$$\lim_{(x,y)\to(2,2)} \frac{y^2-4}{xy-2x}$$

(j) Evaluate

$$\lim_{(x,y)\to(4,5)} \frac{\sqrt{x+y}-3}{x+y-9}$$