Differential Forms

To quote Gerald Folland, roughly speaking, "a differential k-form is an object whose mission in life is to be integrated over k-dimensional sets." Thus, 1-forms are designed to be integrated over curves, 2-forms are designed to be integrated over surfaces, and so on.

1 1-Forms

A **differential 1-form** on $\mathbb{R}^n$ is an expression of the form

$$\omega = F_1(x_1, ..., x_n)dx_1 + \cdots + F_n(x_1, ..., x_n)dx_n,$$

where the F_j's are continuous functions. There is an obvious correspondence between the 1-form ω and the vector field $\mathbf{F} = \langle F_1, ..., F_n \rangle$. In particular, in 3 dimensions the correspondence between 1-forms and vector fields takes the form

$$\omega = F dx + G dy + H dz \leftrightarrow \mathbf{F} = F\mathbf{i} + G\mathbf{j} + H\mathbf{k}.$$

One type of 1-form is the differential of a C^1 function,

$$df = (\partial_1 f)dx_1 + \cdots + (\partial_n f)dx_n.$$

2 2-Forms and the Exterior Product

3 Pullbacks and Integrals of 2-Forms

4 3-Forms

A **differential** 3-**form** on $\mathbb{R}^n$ is an expression of the form

$$\omega = \sum_{1 \le i < j < k \le n} C_{ijk}(x_1, ..., x_n) dx_i \wedge dx_j \wedge dx_k.$$

Here, as in the case of 2-forms, one can think of the expressions $dx_i \wedge dx_j \wedge dx_k$ simply as formal basis elements, and one can put the indices i, j, k in an order other than $i < j < k$ with the understanding that whenever one interchanges two of the dx's one introduces a minus sign. The number of terms in the sum is the binomial coefficient $\binom{n}{3}$. When $n = 3$, this number is 1: All 3-forms on $\mathbb{R}^3$ have the form

$$\omega = f(x,y,z)dx \wedge dy \wedge dz$$

and hence can be identified with functions:

$$f(x,y,z)dx \wedge dy \wedge dz \leftrightarrow f(x,y,z).$$

The notion of exterior product extends so as to yield a 3-form as the product of three 1-forms or as the product of a 1-form and a 2-form.

5 The Exterior Derivative

When the operations of gradient, curl, and divergence are expressed in terms of differential forms, they are all instances of a single operation, denoted by d and called the **exterior derivative**, which maps k-forms on $\mathbb{R}^n$ into $(k+1)$-forms on $\mathbb{R}^n$.

Here is how it works. First, a 0-form is, by definition, a function; if f is a 0-form, df is just the differential of f. If we identify 1-forms with vector fields, df becomes ∇f. That is,

The gradient is the exterior derivative on 0-forms.

Now let $k \geq 1$. Any k-form ω is a sum of terms of the form $f\beta$ where f is a function and β is one of the basis elements (dx_i for 1-forms, $dx_i \wedge dx_j$ for 2-forms, etc.). Then $d\omega$ is defined to be the $(k+1)$-form obtained by replacing each such term $f\beta$ by $df \wedge \beta$.

The curl is the exterior derivative on 1-forms in $\mathbb{R}^3$.

The divergence is the exterior derivative on 2-forms in $\mathbb{R}^3$.

6 Stokes's Theorem

Theorem 1 (The General Stokes Theorem)**.** *Let M be a smooth, oriented k-dimensional submanifold of $\mathbb{R}^n$ with a piecewise smooth boundary ∂M, and let ∂M carry the orientation that is (in a suitable sense) compatible with the one on M. If ω is a $(k-1)$-form of class C^1 on an open set containing M, then*

$$\int_{\partial M} \omega = \int_M d\omega.$$

In particular, if M has no boundary, then

$$\int_M d\omega = 0.$$