$Q = [a_1, b_1] \times [a_2, b_2] \times \dots \times \underbrace{[a_n, b_n]}_{\text{component interval}}$

rectangle in $\mathbb{R}^n$.

<u>width</u> of Q : $\max(b_1 - a_1, b_2 - a_2, \dots, b_n - a_n)$.

The product $(b_1 - a_1) \cdots (b_n - a_n) = v(Q)$ is called the <u>volume of Q</u>.

<u>Defn</u>: $[a, b]$ closed interval.

A <u>partition</u> of $[a, b]$ is a finite collection P of points of $[a, b]$ that includes a and b.

$a = t_0 < t_1 < \dots < t_k = b$.

$[t_{i-1}, t_i]$ a subinterval determined by P.

More generally, a <u>partition</u> P of Q is an n-tuple $(P_1, \dots, P_n)$ where each P_j is a partition of $[a_j, b_j]$ for each j.

If, for each j, I_j is one of the subintervals determined by P_j of $[a_j, b_j]$, then the rectangle

$R = I_1 \times \dots \times I_n$

is a <u>subrectangle</u> <u>determined by P</u>, of Q.

The maximum width of these subrectangles is called the <u>mesh</u> of P.

<u>Def'n:</u> $f: Q \longrightarrow \mathbb{R}$ bounded.

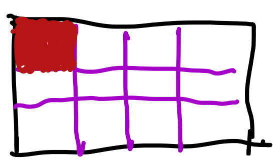
For each subrectangle R determined by P, let

$$m_R(f) = \inf\{f(x) \mid x \in R\}$$
$$M_R(f) = \sup\{f(x) \mid x \in R\}.$$

We define the <u>lower sum</u> and <u>upper sum</u> of f, determined by P, to be

$$L(f, P) = \sum_R m_R(f) \cdot v(R)$$

$$U(f, P) = \sum_R M_R(f) \cdot v(R).$$

Q 

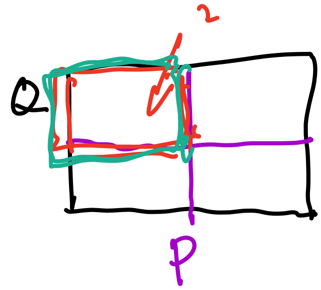
$P = (P_1, \dots, P_n)$

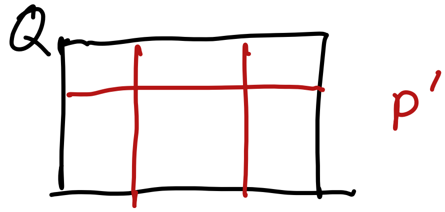
If P'' is a partition of Q obtained from P by adjoining additional points to some or all of the partitions $P_1, \dots, P_n$, then P'' is called a <u>refinement</u> of P.

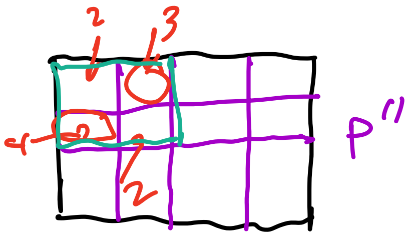
Given two partitions P and $P' = (P_1', \ldots, P_n')$ of Q, the partition

$$P'' = (P_1 \cup P_1', \ldots, P_n \cup P_n')$$

is called the <u>common refinement</u> of P and P'.







common refinement of P and P'.

<u>Lemma</u>. Let P be a partition of Q, $f: Q \to \mathbb{R}$ bounded fn.

If P'' is a refinement of P, then

$$L(f, P) \le L(f, P''),$$

$$U(f, P'') \le U(f, P).$$

<u>Lemma</u>: If P and P' are two partitions of Q, then $L(f, P) \leq U(f, P')$.

Pf) Take the common refinement P'' of P and P'.

$$L(f, P) \leq L(f, P'') \leq U(f, P'') \leq U(f, P')$$

<u>Def'n</u>: $\underline{\int_Q} f = \sup_P \{L(f, P)\}$ lower integral of f over Q

$\overline{\int_Q} f = \inf_P \{U(f, P)\}$. upper integral of f over Q.

In general,

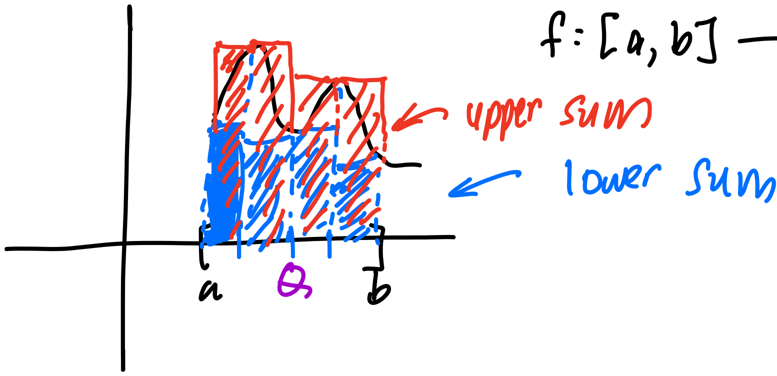
$$\underline{\int_Q} f \leq \overline{\int_Q} f.$$

If $\underline{\int_Q} f = \overline{\int_Q} f$, we say that f is <u>integrable over Q</u>, and we define the <u>integral of f over Q</u> to be this common value.

We write $\int_Q f$.

Ex:

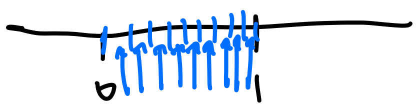
$f: [a,b] \to \mathbb{R}$



Ex: $I = [0,1]$.

$f: I \to \mathbb{R}$

$$x \mapsto \begin{cases} 0 & \text{if } x \text{ rational} \\ 1 & \text{if } x \text{ irrational.} \end{cases}$$



each subrectangle contains rational and irrational numbers

Every lower sum is 0

Every upper sum is 1.

So f is not integrable over I.

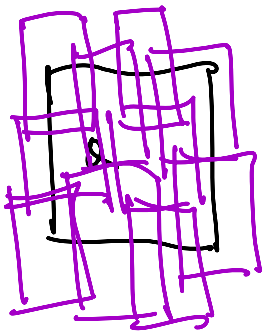
Thm (Riemann condition) f is integrable $\iff$ $\forall \varepsilon > 0$, $\exists$ partition P of Q for which $U(f,P) - L(f,P) < \varepsilon$.

Thm: Every constant function $f(x) = c$ is integrable, and

$$\int_Q c = c \cdot v(Q).$$

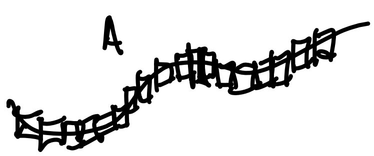
Cor: Let $\{Q_1, \dots, Q_k\}$ be a finite collection of rectangles that covers Q. Then

$$v(Q) \leq \sum_{i=1}^{k} v(Q_i).$$



Sets of Measure zero.

Def'n: Let $A \subset \mathbb{R}^n$. We say A has measure zero if $\forall \varepsilon > 0$, $\exists$ covering $Q_1, Q_2, \dots,$ of A by countably many rectangles such that $\sum_{i=1}^{\infty} v(Q_i) < \varepsilon$.



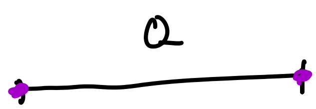
Thm:

(a) If $B \subset A$ and A has measure 0, then B has measure 0.

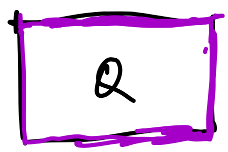
(b) If $A_1, A_2, \ldots$ is a countable collection of sets of measure 0, then so is $\bigcup_{i=1}^{\infty} A_i$.

(c) A set A has measure 0 $\iff$ $\forall \varepsilon > 0$, $\exists$ ctble covering of A by open rectangles $\operatorname{Int} Q_1, \operatorname{Int} Q_2, \ldots$ s.t. $\sum_{i=1}^{\infty} v(Q_i) < \varepsilon$.

(d) If Q is a rectangle in $\mathbb{R}^n$, then $\operatorname{Bd} Q$ has measure 0 in $\mathbb{R}^n$ but Q does not.



Bd Q has measure 0 in $\mathbb{R}$.



Bd Q has measure 0 in $\mathbb{R}^2$.

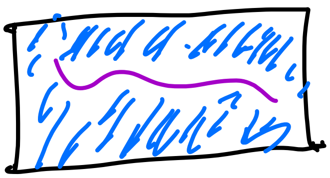
Q rectangle in $\mathbb{R}^n$, $f: Q \to \mathbb{R}$ bounded.

Thm: Let D be the set of points of Q at which f fails to be continuous. Then

$$\int_Q f \text{ exists}$$
$$\Updownarrow$$
$$D \text{ has measure zero in } \mathbb{R}^n.$$

Thm: Suppose $f: Q \to \mathbb{R}$ integrable.

(a) If f vanishes except on a set of measure 0, then $\int_Q f = 0$.



(b) If $f \geq 0$ and $\int_Q f = 0$, then f vanishes except on a set of measure 0.