Curvature

Reference: Colley, Vector Calculus; Pressley, Elementary Differential Geometry

The curvature of a curve measures the extent to which a curve is not contained in a straight line (so that straight lines have zero curvature).

We will work with plane curves. Since a straight line should have zero curvature, a measure of the curvature of a plane curve at a point p of the curve should be its deviation from the tangent line at p.

Suppose that γ is a unit-speed curve in $\mathbb{R}^2$. Its **curvature** $\kappa(t)$ at the point $\gamma(t)$ is defined to be $||\gamma''(t)||$.

If γ is any regular curve, then γ has a unit-speed parametrization $\tilde{\gamma}$, say, and we define the curvature of γ to be that of $\tilde{\gamma}$. This can be shown to be independent of the parametrization $\tilde{\gamma}$ (the chain rule is used).

Thus the curvature is an intrinsic property of the curve, in the sense that it is independent of the choice of parametrization. In other words, suppose we are given a set which is the image of a parametrization. Let p be a point on this curve. Take any unit-speed parametrization $\gamma(t)$ and let t_0 be such that $\gamma(t_0) = p$. We define the **curvature** of the curve at p to be $||\gamma''(t)||$, and curvature is well-defined.

The curvature is the starting point of *differential geometry*, which uses calculus to study curves and surfaces.