Jensen's Inequality

φ : convex function (concave up)

$\varphi''(x) \geq 0$. Ex: $\varphi(x) = x^2$,

$\varphi(x) = 1/x$, $x > 0$

$\varphi(x) = x$

X : random variable $\varphi \circ X$

Jensen's Inequality

$$\boxed{\varphi(EX) \leq E[\varphi(X)]}$$

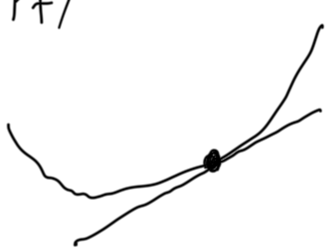
μ : probability measure

Suppose f and $\underset{\varphi \circ f}{\underset{\wr\wr}{\varphi(f)}}$ are integrable.

Jensen's inequality :

$$\boxed{\varphi\left(\int f \, d\mu\right) \leq \int (\varphi \circ f) \, d\mu.}$$

Pf)



Let $c = \int f \, d\mu$.

Let l be the linearization of φ at c.

Then l has the form $l(x) = ax + b$,

with $l(c) = \varphi(c)$ and $\boxed{\varphi(x) \geq l(x)}$.

where convexity is used

$$\boxed{\int (\varphi \circ f)\, d\mu} \geq \int (\ell \circ f)\, d\mu$$

$$= \int (af + b)\, d\mu$$

$$= a \int f\, d\mu + \int b\, d\mu.$$

μ is a probability measure

$$= a \int f\, d\mu + b \cdot 1$$

$$= a \underbrace{\int f\, d\mu}_{c} + b$$

$$= \ell(c)$$

$$= \varphi(c) = \boxed{\varphi\left(\int f\, d\mu\right)}$$

$$EX = \int X\, d\mu.$$

JI: $\varphi\left(\int X\, d\mu\right) \leq \int (\varphi \circ X)\, d\mu$

ie, $\varphi(EX) \leq E[\varphi(X)]$.

Ex: $\varphi(x) = x^2$.

$(EX)^2 \leq E(X^2)$

$(EX) = E(X)$

So $\operatorname{var}(X) = E(X^2) - (EX)^2 \geq 0$.