Motivation: infinite sequence of independent trials with probability p (of success)

N = # trials needed to see the first success.

$$P(N = k) = \underline{(1-p)^{k-1}\, p}$$

One way to think about this...

$$X_j = \begin{cases} 1 & \text{if trial } j \text{ is a success} \\ 0 & \text{if} \qquad\qquad\qquad \text{failure} \end{cases}$$

$$P(X_1 = 0, X_2 = 0, \ldots, X_{k-1} = 0, X_k = 1)$$
$$= P(X_1 = 0)\, P(X_2 = 0) \cdots P(X_{k-1} = 0)\, P(X_k = 1)$$
$$= (1-p) \cdots (1-p)\, p.$$

Def'n: Let $0 < p \leq 1$. A random variable X has the <u>geometric distribution</u> <u>with success parameter p</u> if the possible values of X are $\{1, 2, 3, \ldots\}$ and $P(X = k) = (1-p)^{k-1}\, p$.

Rmk: Sometimes define $Y = X - 1$. $\boxed{X \sim \text{Geom}(p)}$
possible values of Y are $0, 1, 2, \ldots$
Y = # of failures before first success.

Ex: What is the probability that it takes more than 7 rolls of a fair die to roll a 6?

Sol'n: N = # of rolls of a fair die until the first 6.

$$P(N > 7) = \sum_{k=8}^{\infty} P(N = k)$$

$$= \sum_{k=8}^{\infty} \left(\frac{5}{6}\right)^{k-1} \left(\frac{1}{6}\right)$$

$$= \frac{1}{6} \sum_{k=8}^{\infty} \left(\frac{5}{6}\right)^{k-1}$$

$$= \frac{1}{6} \; \frac{(5/6)^7}{1 - \frac{5}{6}}$$

$$= \left(\frac{5}{6}\right)^7 .$$

Alternate: want first 7 rolls to be a failure. $\left(\frac{5}{6}\right)^7$

Ex: Roll a pair of fair dice until you get either a sum of 5 or a sum of 7. What is the probability that you get 5 first?

a (under "sum of 5"), b (under "sum of 7")

Soln: A: the event that 5 comes first.

$A_n = \{$ no 5 or 7 in first $n-1$ rolls, and 5 on roll $n \}$.

$A = \bigcup_n A_n$.

$(1-(a+b))^{n-1} \cdot a$

$$P(A) = \sum_n P(A_n).$$

$$P(A_n) = \left[P(\text{no 5 or 7 on a roll})^{n-1} \cdot P(5 \text{ on roll } n) \right]$$

$P(\text{pair of dice gives } 5) = \frac{4}{36} = a = \frac{1}{9}$

$P(\text{pair of dice gives } 7) = \frac{6}{36} = b = \frac{1}{6}$

or.

$$= \left(1 - \frac{10}{36}\right)^{n-1} \left(\frac{4}{36}\right) \quad \leftarrow \text{ where geom dist shows up.}$$

$$P(A) = \frac{1}{9} \sum_n \left(1 - \frac{10}{36}\right)^{n-1} = \frac{1}{9} \sum_{n=1}^{\infty} \left(\frac{13}{18}\right)^{n-1}$$

$$= \frac{1}{9} \cdot \frac{1}{1 \sim \frac{13}{18}} = \boxed{\frac{2}{5}}$$

$$P(A) = \sum_{n=1}^{\infty} \left(1 - (a+b)\right)^{n-1} a$$

$$= a \cdot \frac{1}{1 - (1-(a+b))} = \frac{a}{a+b}$$

In this example, $\dfrac{a}{a+b} = \dfrac{4/36}{\frac{4}{36} + \frac{6}{36}}$

$$= \frac{4}{4+6} = \frac{2}{5} ,$$

Expected Value of Geom. Distribution.

X discrete rv

$$E(X) = \sum_{k} k\, P(X=k) \qquad (q \overset{def}{=} 1-p)$$

$X \sim \text{Geom}(p), \qquad 0 < p < 1 . \checkmark$

$$P(X=k) = (1-p)^{k-1} p = q^{k-1} p$$

$$E(x) = p \sum_{k=1}^{\infty} k q^{k-1}.$$

If $|t| < 1$

$$\underbrace{\sum_{k=1}^{\infty} k t^{k-1}}_{1 + 2t + 3t^2 + \dots} = \sum_{k=0}^{\infty} \frac{d}{dt}(t^k)$$

$$= \frac{d}{dt}\left(\sum_{k=0}^{\infty} t^k\right) \quad \Bigg\} \begin{array}{l} \text{Analysis} \\ \text{fact} \end{array}$$

$$= \frac{d}{dt}\left(\frac{1}{1-t}\right)$$

$$= \frac{1}{(1-t)^2}$$

Set $t = q$

Then $$E(x) = p \sum_{k=1}^{\infty} k q^{k-1} = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \boxed{\frac{1}{p}}.$$