

## Binomial distribution:

Perform experiment  $n$  times  
Each trial has prob.  $p$  of success.

The number of successes  $S$  has

$$P(S=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

for  $k=0, \dots, n$

Say  $S$  has a binomial distribution with parameters  $n$  and  $p$ .

## Poisson distribution: $X$ has a

Poisson distribution with parameter  $\lambda > 0$

( $X \sim \text{Poisson}(\lambda)$ ) if

$$P(X=k) = e^{-\lambda} \frac{\lambda^k}{k!} \quad \text{for } k=0, 1, 2, \dots$$

claim: This is a probability function

$$(f) \quad \sum_{k=0}^{\infty} P(X=k) = \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \left( \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \right) \cdot e^{\lambda} = 1 \quad \checkmark$$

Recall:  $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

Thm: (Poisson approximation to the binomial). Suppose  $S_n$  has a binomial distribution with parameters  $n$  and  $p_n$ .

If  $p_n \rightarrow 0$  and  $n p_n \rightarrow \lambda$  as  $n \rightarrow \infty$ ,  
 then  $P(S_n = k) \rightarrow e^{-\lambda} \frac{\lambda^k}{k!}$

"If we have a large number of independent events with small probability then the number that occur has approximately a Poisson dist."

Think: If  $S_n = \text{binomial}(n, p)$  and  $p$  is small, then  $S_n$  is approximately Poisson ( $np$ ).

Ex: Suppose roll two dice 12 times.

$D = \#$  times a double 6 appears.  
Binomial dist  
 $n=12$ ,  $p = 1/36$ ,  $\lambda = np = \frac{1}{3}$

$P(D=k)$  is approximately Poisson  $(\frac{1}{3})$ .

$$P(D=k) \approx e^{-1/3} \frac{(\frac{1}{3})^k}{k!}$$

$k=0$ : exact answer:

$$P(D=0) = \left(\frac{1}{36}\right)^0 \left(\frac{35}{36}\right)^{12} \\ \approx 0.7132.$$

Poisson approx:

$$P(D=0) \approx e^{-1/3} \cdot \frac{1}{1} \approx 0.7165$$

$k=1$ : exact answer:

$$P(D=1) = 12 \left(\frac{1}{36}\right)^1 \left(\frac{35}{36}\right)^{11} \\ \approx 0.2445$$

Poisson approx:

$$P(D=1) \approx e^{-1/3} \left(\frac{1}{3}\right) \approx 0.2388$$

$k=2$ : exact:

$$\begin{aligned} P(D=2) &= \binom{12}{2} \left(\frac{1}{36}\right)^2 \left(\frac{35}{36}\right)^{10} \\ &= \frac{12 \cdot 11}{2} \left(\frac{1}{36}\right)^2 \left(\frac{35}{36}\right)^{10} \end{aligned}$$

Poisson approx:

$$P(D=2) \approx \frac{e^{-1/3} \left(\frac{1}{3}\right)^2}{2} \approx 0.0384$$
$$P(D=2) \approx \frac{e^{-1/3} \left(\frac{1}{3}\right)^2}{2} \approx 0.0398$$

Pf of Thm:

$$\lim_{n \rightarrow \infty} \lambda_n = \lambda$$

Lemma: If  $\lambda_n \rightarrow \lambda$  then

$$\lim_{n \rightarrow \infty} \left(1 - \frac{\lambda_n}{n}\right)^n = e^{-\lambda}.$$

Pf when  $k=0$ :  $P(S_n=0) = (1-p_n)^n$

$$\left. \begin{array}{l} \lambda_n = np_n \\ p_n = \frac{\lambda_n}{n} \end{array} \right\} = \left(1 - \frac{\lambda_n}{n}\right)^n \xrightarrow[\text{by lemma}]{} e^{-\lambda}.$$

pf when  $k > 0$ :

$$P(S_n = k) = \binom{n}{k} p_n^k (1 - p_n)^{n-k}$$

$$= \binom{n}{k} \left(\frac{\lambda_n}{n}\right)^k \left(1 - \frac{\lambda_n}{n}\right)^{n-k}$$

$$= \frac{n(n-1)\dots(n-k+1)}{k!} \frac{\lambda_n^k}{n^k} \left(1 - \frac{\lambda_n}{n}\right)^n \left(1 - \frac{\lambda_n}{n}\right)^{-k}$$

$$= \frac{1}{k!} \underbrace{\frac{n}{n}}_{\rightarrow 1} \underbrace{\frac{(n-1)}{n}}_{\rightarrow 1} \dots \underbrace{\frac{(n-k+1)}{n}}_{\rightarrow 1} \underbrace{\lambda_n^k}_{\rightarrow \lambda^k} \underbrace{\left(1 - \frac{\lambda_n}{n}\right)^n}_{\rightarrow e^{-\lambda}} \underbrace{\left(1 - \frac{\lambda_n}{n}\right)^{-k}}_{\rightarrow 1}$$

So  $\lim_{n \rightarrow \infty} P(S_n = k)$

$$= \frac{1}{k!} \lambda^k e^{-\lambda} = e^{-\lambda} \frac{\lambda^k}{k!}$$