50 Counting and Basic Probability Problems

cooking

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Chapter 1

Problems

These are taken from Durrett, The Essentials of Probability. This book is out of print.

1. Two dice are rolled. How many outcomes are in the events (a) the sum is 9, and (b) the sum is 10?
2. How many outcomes are there if we roll three dice?
3. (a) How many outcomes are there if we flip four coins? How many outcomes are in (b) We get one head, (c) We get two heads?
4. A man receives presents from his three children, A, B, and C. To avoid disputes he opens the presents in a random order. What are the possible outcomes?
5. Suppose $\Omega = \{1, 2, 3, 4, 5, 6\}$, $A = \{1, 2\}$, and $B = \{2, 3, 4\}$. Compute $A \cup B$, $A \cap B$, A^c, and $B - A$.
6. Compute the probability that the sum of the numbers on two dice is k for $2 \leq k \leq 12$.
7. Suppose we pick a number at random from the phone book and look at the last digit. (a) What is the set of outcomes and what probability should be assigned to each outcome? (b) Would this model be appropriate if we were looking at the first digit?
8. Two dice are rolled. What is the probability (a) the two numbers will differ by 1 or less, (b) the maximum of the two numbers will be 5 or larger?
9. Two boys are repeatedly playing a game that they each have probability 1/2 of winning. The first person to win five games wins the match. What is the probability that Al will win if (a) he has won 4 games and Bobby has won 3; (b) he leads the score of 3 games to 2?
10. Suppose we roll 3 dice. Compute the probability that the sum is (a) 3, (b) 4, (c) 5, (d) 6, (e) 7, (f) 8.
11. In a group of 320 high school graduates, only 160 went to college but 100 of the 170 men did. How many women did not go to college?

12. Suppose A and B are disjoint with $P(A) = 0.3$ and $P(B) = 0.5$. What is $P(A^c \cap B^c)$?

13. A restaurant offers soup or salad to start, and has 11 entrees to choose from, each of which is served with rice, baked potato, or zucchini. How many meals can you have if you can choose to eat one of their 4 desserts or have no dessert?

14. How many answer sheets are possible for a true/false test with 15 questions?

15. How many different batting orders are possible for 9 baseball players?

16. How many ways can 8 books be put on a shelf?

17. In a horse race, the first three finishers are said to win, place, and show. How many finishes are possible for a race with 11 horses?

18. Five different awards are to be given to a class of 30 students. How many ways can this be done if (a) each student can receive any number of awards, (b) each student can receive at most one award?

19. A restaurant offers 15 possible toppings for its pizzas. How many different pizzas with 4 different toppings can be ordered?

20. We are going to pick 5 cards out of a deck of 52. In how many ways can this be done?

21. Show that (a) $\sum_{m=0}^{n} \binom{n}{m} = 2^m$.

    (b) $\sum_{m=0}^{n} (-1)^m \binom{n}{m} = 0$.

22. Find (a) $(x+2)^5$, (b) $(2x+3)^3$.

23. There are 37 students in a class. In how many ways can a professor give out 3 A's, 4 B's, 5 C's, and 25 F's?

24. A child has 15 blocks: 6 red, 4 yellow, and 5 blue. How many ways can they be put in a line?

25. Four people play a card game in which each gets 13 cards. How many possible deals are there?

26. (a) How many license plates are possible if the first three places are occupied by letters and the last three by numbers?

    (b) Assuming all combinations are equally likely, what is the probability the three letters and the three numbers are different?

27. A basketball team has 5 players over six feet tall and 6 who are under six feet. How many ways can they have their picture taken if the 5 taller players stand in a row behind the 6 shorter players who are sitting on a row of chairs?

28. Seven people sit at a round table. How many ways can this be done if Mr. Jones and Miss Smith (a) must sit next to each other, (b) must not sit next to each other?

29. Suppose that n people are seated at random in a row of n seats. What is the probability Mr. Jones and Miss Smith sit next to each other?

30. How many ways can 4 men and 4 women sit in a row if no two men or two women sit next to each other?

31. Twelve different toys are to be divided among 3 children so that each one gets 4 toys. How many ways can this be done?

32. If seven dice are rolled, what is the probability that each of the six numbers will appear at least once?

33. Suppose three runners from team A and three runners from team B have a race. If all six runners have equal ability, what is the probability that the three runners from team A will finish first, second, and fourth?

34. Indistinguishable particles are said to obey Fermi-Dirac statistics if all arrangements that have at most one particle per box have the same probability. How many ways can we put m of these particles in $n \geq m$ boxes?

35. How many ways can we divide n indistinguishable balls into m groups in such a way that there is at least one ball per group?

36. How many ways can we divide n indistinguishable balls into m groups where the groups can have size 0?

37. How many different partial derivatives of order 3 are there for a function of 3 variables?

38. Suppose we place 14 indistinguishable balls in 7 boxes. What is the probability that there will be at least one ball per box?

39. Suppose we put n indistinguishable balls into m boxes. What is the probability there will be exactly j empty boxes?

40. An elevator starts in the basement with 10 people (not including the elevator operator) and each person gets off at one of the six floors. How many outcomes are there if all people look the same?

41. In the New York State Lotto, you actually pick six numbers and a seventh supplemental number from 1 through 54, and then six numbers are selected in a televised drawing. If your supplemental number and three of your six numbers are selected, you win a share of the fourth prize. What is the probability you win a share of the fourth prize?

42. Two red cards and two black cards are lying face down on the table. You pick two cards and turn them over. What is the probability that the two cards are different colors?

43. You pick 5 cards out of a deck of 52. What is the probability you get exactly 2 spaces?

44. In a carton of 12 eggs, 2 are rotten. If we pick 4 eggs to make an omelet, what is the probability we do not get a rotten egg?

45. A closet contains 8 pairs of shoes. You pick out 5. What is the probability of (a) no pair, (b) exactly one pair, (c) two pairs?

46. Two cards are a blackjack if one is an A and the other is a K, Q, J, or 10. If you pick two cards out of a deck, what is the probability you will get a blackjack?

47. A football team has 16 seniors, 12 juniors, 8 sophomores, and 4 freshmen. If we pick 5 players at random, what is the probability we will get 2 seniors and 1 from each of the other 3 classes?

48. If 6 balls are thrown at random into 10 boxes, what is the probability no box will contain more than 1 ball?

49. Suppose we have an urn with 6 black balls and 6 red balls. Compute the probability of getting 2 black balls and 1 red ball when we pick 3 balls out (a) without replacing them, (b) when we replace each ball after it is drawn.

50. The World Series is won by the first team to win four games. What is the probability the team that wins the first game will win the series?

Chapter 2

Solutions to Problems

Warning: There might be errors. Use at your own risk!

Exercise 1. Two dice are rolled. How many outcomes are in the events (a) the sum is 9, and (b) the sum is 10?

Answer: (a) 4, (b) 3

Exercise 2. How many outcomes are there if we roll three dice?

Answer: 216

Exercise 3. (a) How many outcomes are there if we flip four coins? How many outcomes are in (b) We get one head, (c) We get two heads?

Answer: (a) 16; (b) 4; (c) 6

Exercise 4. A man receives presents from his three children, A, B, and C. To avoid disputes he opens the presents in a random order. What are the possible outcomes?

Answer: $ABC, ACB, BAC, BCA, CAB, CBA$

Exercise 5. Suppose $\Omega = \{1, 2, 3, 4, 5, 6\}$, $A = \{1, 2\}$, and $B = \{2, 3, 4\}$. Compute $A \cup B$, $A \cap B$, A^c, and $B - A$.

Answer: $A \cup B = \{1, 2, 3, 4\}$; $A \cap B = \{2\}$; $A^c = \{3, 4, 5, 6\}$; $B - A = \{3, 4\}$

Exercise 6. Compute the probability that the sum of the numbers on two dice is k for $2 \leq k \leq 12$.

Answer: $P(2) = P(12) = 1/36$, $P(3) = P(11) = 2/36$, $P(4) = P(10) = 3/36$, $P(5) = P(9) = 4/36$, $P(6) = P(8) = 5/36$, $P(7) = 6/36$.

Exercise 7. Suppose we pick a number at random from the phone book and look at the last digit. (a) What is the set of outcomes and what probability should be assigned to each outcome? (b) Would this model be appropriate if we were looking at the first digit?

Answer: (a) 0,1,2,3,4,5,6,7,8,9; probability is 1/10. (b) No, because the first digit can't be 1.

Exercise 8. Two dice are rolled. What is the probability (a) the two numbers will differ by 1 or less, (b) the maximum of the two numbers will be 5 or larger?

Answer: (a) The total possibilities are

$$\{(1,1),(1,2),(2,1),(2,2),(2,3),(3,2),(3,3),(3,4),(4,3),(4,4),(4,5),(5,4),(5,5),(5,6),(6,5),(6,6)\}$$

out of 36. The answer is $16/36 = 4/9$. (b) We need at least one 5 or 6. If the first die is 5, there are 6 possibilities. If the second die is 5, there are 6 possibilities but we overcounted $(5,5)$. If the first die is 6 there are 4 possibilities we haven't counted, and if the second die is 6 there are 4 more possibilities. There is also $(6,6)$. So the answer is $20/36 = 5/9$.

Exercise 9. Two boys are repeatedly playing a game that they each have probability 1/2 of winning. The first person to win five games wins the match. What is the probability that Al will win if (a) he has won 4 games and Bobby has won 3; (b) he leads the score of 3 games to 2?

Answer: (a) $1/2 + 1/2(1/2) = 1/2 + 1/4 = 3/4$. (b) If it's AA, ABA, ABBA, B and then 1/2. So $1/4 + 1/8 + 1/16 + 1/4 = 11/16$.

Exercise 10. Suppose we roll 3 dice. Compute the probability that the sum is (a) 3, (b) 4, (c) 5, (d) 6, (e) 7, (f) 8.

Answer: (a) $1/216$, (b) $3/216 = 1/72$, (c) $6/216 = 1/36$, (d) $10/216 = 5/108$, (e) $15/216$, (f) $21/216 = 7/72$

Exercise 11. In a group of 320 high school graduates, only 160 went to college but 100 of the 170 men did. How many women did not go to college?

Answer: 90. Not sure what this has to do with probability.

Exercise 12. Suppose A and B are disjoint with $P(A) = 0.3$ and $P(B) = 0.5$. What is $P(A^c \cap B^c)$?

Answer: $1 - 0.3 - 0.5 = 0.2$

Exercise 13. A restaurant offers soup or salad to start, and has 11 entrees to choose from, each of which is served with rice, baked potato, or zucchini. How many meals can you have if you can choose to eat one of their 4 desserts or have no dessert?

Answer: $2 \cdot 11 \cdot 3 \cdot 5 = 330$

Exercise 14. How many answer sheets are possible for a true/false test with 15 questions?

Answer: $2^{15} = 32,768$

Exercise 15. How many different batting orders are possible for 9 baseball players?

Answer: $9!$

Exercise 16. How many ways can 8 books be put on a shelf?

Answer: $8!$

Exercise 17. In a horse race, the first three finishers are said to win, place, and show. How many finishes are possible for a race with 11 horses?

Answer: $11 \cdot 10 \cdot 9 = 990$

Exercise 18. Five different awards are to be given to a class of 30 students. How many ways can this be done if (a) each student can receive any number of awards, (b) each student can receive at most one award?

Answer: (a) For each award, there are 30 students it can go to, so the answer is 30^5. (b) $30 \cdot 29 \cdot 28 \cdot 27 \cdot 26$

Exercise 19. A restaurant offers 15 possible toppings for its pizzas. How many different pizzas with 4 different toppings can be ordered?

Answer: We have to choose 4 of the toppings, so the answer is $\binom{15}{4} = 15 \cdot 7 \cdot 13 = 1365$

Exercise 20. We are going to pick 5 cards out of a deck of 52. In how many ways can this be done?

Answer: $\binom{52}{5} = \dfrac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2} = 2598960$

Exercise 21. Show that (a) $\sum_{m=0}^{n} \binom{n}{m} = 2^m$.

Answer: (a) Take $a = b = 1$ in the binomial formula.

(b) $\sum_{m=0}^{n} (-1)^m \binom{n}{m} = 0$.

Answer: Take $a = -1$ and $b = 1$ in the binomial formula.

Exercise 22. Find (a) $(x+2)^5$, (b) $(2x+3)^3$.

Answer: (a) $x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$; (b) $8x^3 + 36x^2 + 54x + 27$

Exercise 23. There are 37 students in a class. In how many ways can a professor give out 3 A's, 4 B's, 5 C's, and 25 F's?

Answer: $\dfrac{37!}{25!3!4!5!}$

Exercise 24. A child has 15 blocks: 6 red, 4 yellow, and 5 blue. How many ways can they be put in a line?

Answer: $\dfrac{15!}{6!4!5!}$

Exercise 25. Four people play a card game in which each gets 13 cards. How many possible deals are there?

Answer: $\binom{52}{13}\binom{39}{13}\binom{26}{13} = \dfrac{52!}{(13!)^4}$.

Exercise 26. (a) How many license plates are possible if the first three places are occupied by letters and the last three by numbers?

Answer: $26^3 \cdot 1000$.

(b) Assuming all combinations are equally likely, what is the probability the three letters and the three numbers are different?

Answer: $\dfrac{26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8}{26^3 \cdot 1000} = \dfrac{25 \cdot 24 \cdot 9 \cdot 8}{26^2 \cdot 100}$

Exercise 27. A basketball team has 5 players over six feet tall and 6 who are under six feet. How many ways can they have their picture taken if the 5 taller players stand in a row behind the 6 shorter players who are sitting on a row of chairs?

Answer: $6!5! = 86400$

Exercise 28. Seven people sit at a round table. How many ways can this be done if Mr. Jones and Miss Smith (a) must sit next to each other, (b) must not sit next to each other?

Answer: (a) There are 7 spots for Mr. Jones and then 2 spots for Miss Smith and then 5! ways to seat the other people, so the answer is $120 \cdot 14 = 1680$.

(b) The total number is 7! so the answer is $7! - 1680 = 5040 - 1680 = 3360$

Exercise 29. Suppose that n people are seated at random in a row of n seats. What is the probability Mr. Jones and Miss Smith sit next to each other?

Answer: If you think of Mr. Jones and Miss Smith in one block, there are $(n-1)!$ ways to arrange the $n-1$ people and then 2 ways for the order for Mr. Jones and Miss Smith, which leaves $2(n-1)!$ arrangements. The total number of arrangements is $n!$ So the answer is $2/n$.

Exercise 30. How many ways can 4 men and 4 women sit in a row if no two men or two women sit next to each other?

Answer: They have to alternate. There are 2 ways for which gender comes first. Then there are 4! ways for the men and 4! for the women so the answer is $2 \cdot 24 \cdot 24 = 1,152$.

Exercise 31. Twelve different toys are to be divided among 3 children so that each one gets 4 toys. How many ways can this be done?

Answer: The first child has $\binom{12}{4}$ ways to choose the 4 toys. Then the second child has $\binom{8}{4}$ ways. The third child has one way. So the answer is

$$\binom{12}{4} \cdot \binom{8}{4} = 34650.$$

Exercise 32. If seven dice are rolled, what is the probability that each of the six numbers will appear at least once?

Answer: There are 6^7 total outcomes. For the successes, there are 6 choices for the number to show up twice and $\binom{7}{2}$ ways for the number of places for that number to show up. Then there are 5! ways to arrange the other numbers. So the answer is

$$7! \cdot 6/(2 \cdot 6^7) \approx 0.054$$

Exercise 33. Suppose three runners from team A and three runners from team B have a race. If all six runners have equal ability, what is the probability that the three runners from team A will finish first, second, and fourth?

Answer: There are 6 ways for the A's and 6 ways for the B's, out of a total of 6! ways. So the answer is

$$\frac{36}{6!} = \frac{1}{20}$$

Exercise 34. Indistinguishable particles are said to obey Fermi-Dirac statistics if all arrangements that have at most one particle per box have the same probability. How many ways can we put m of these particles in $n \geq m$ boxes?

Answer: We have to choose m slots that have a particle, so the answer is $\binom{n}{m}$.

Exercise 35. How many ways can we divide n indistinguishable balls into m groups in such a way that there is at least one ball per group?

Answer: We imagine a line of n white balls and $m-1$ pieces of cardboard to place between them to indicate the boundaries of the groups. Since we must pick $m-1$ of the $n-1$ spaces to put our cardboard dividers into, there are $\binom{n-1}{m-1}$ possibilities.

Exercise 36. How many ways can we divide n indistinguishable balls into m groups where the groups can have size 0?

Answer: We imagine having n white balls and $m-1$ black balls that will indicate the boundaries between groups. Each possible division of n indistinguishable balls into m boxes corresponds to one arrangement of the n white balls and $m-1$ black balls, so there are $\binom{n+m-1}{m-1}$ outcomes.

Exercise 37. How many different partial derivatives of order 3 are there for a function of 3 variables?

Answer: Here $n = 3$, $m = 3$, so our formula says the answer is $\binom{5}{2} = 10$.

Exercise 38. Suppose we place 14 indistinguishable balls in 7 boxes. What is the probability that there will be at least one ball per box?

Answer: There are $\binom{20}{6}$ ways to place them and $\binom{13}{6}$ ways to get a success. So the answer is

$$\frac{\binom{13}{6}}{\binom{20}{6}} \approx 0.04427.$$

Exercise 39. Suppose we put n indistinguishable balls into m boxes. What is the probability there will be exactly j empty boxes?

Answer: We have to pick j boxes to be empty. There are $C_{m,j}$ possibilities. So there will be $m-j$ boxes with at least one ball in it. Then there are $C_{n-m+j,m-j+1}$ possibilities. So there are $C_{m,j}C_{n-m+j,m-j+1}$ possibilities out of $C_{n+m-1,m-1}$ possibilities. The answer is

$$\frac{\binom{m}{j}\binom{n-m+j}{m-j+1}}{\binom{n+m-1}{m-1}}$$

Exercise 40. An elevator starts in the basement with 10 people (not including the elevator operator) and each person gets off at one of the six floors. How many outcomes are there if all people look the same?

Answer: That's the number of ways of putting 10 indistinguishable items into 6 boxes, which is $\binom{15}{5}$.

Exercise 41. In the New York State Lotto, you actually pick six numbers and a seventh supplemental number from 1 through 54, and then six numbers are selected in a televised drawing. If your supplemental number and three of your six numbers are selected, you win a share of the fourth prize. What is the probability you win a share of the fourth prize?

Answer: We need the probability that the televised numbers match the supplemental number and exactly three of the other six numbers. There are $\binom{54}{6}$ ways for drawing the numbers. There is one choice for the supplemental number, and then $\binom{6}{3}$ ways to choose 3 of the picked numbers and then $\binom{47}{2}$ ways to pick the remaining two numbers. The answer is

$$\frac{\binom{6}{3}\binom{47}{2}}{\binom{54}{6}}.$$

Exercise 42. Two red cards and two black cards are lying face down on the table. You pick two cards and turn them over. What is the probability that the two cards are different colors?

Answer: There are $\binom{4}{2}$ ways of picking the cards, and 4 ways for them to be different colors, so the answer is $2/3$.

Exercise 43. You pick 5 cards out of a deck of 52. What is the probability you get exactly 2 spaces?

Answer: There are $\binom{52}{5}$ ways to choose the cards. There are $\binom{13}{2}$ ways to pick the 2 spaces and $\binom{39}{3}$ ways to pick the non-spade cards. The answer is

$$\frac{\binom{13}{2}\binom{39}{3}}{\binom{52}{5}} \approx 0.2743.$$

Exercise 44. In a carton of 12 eggs, 2 are rotten. If we pick 4 eggs to make an omelet, what is the probability we do not get a rotten egg?

Answer: There are $\binom{12}{4}$ ways to pick the eggs. There are $\binom{10}{4}$ ways to pick non-rotten eggs. The answer is $\binom{10}{4}/\binom{12}{4} = 14/33$.

Exercise 45. A closet contains 8 pairs of shoes. You pick out 5. What is the probability of (a) no pair, (b) exactly one pair, (c) two pairs?

Answer: (a) There are $16 \cdot 15 \cdot 14 \cdot 13 \cdot 12$ ways to pick the 5 shoes, counting the order. There are 16 possibilities for the first shoe, then 14 for the second, 12 for the third, 10 for the fourth, and 8 for the fifth. So the answer is

$$\binom{16 \cdot 14 \cdot 12 \cdot 10 \cdot 8}{16 \cdot 15 \cdot 14 \cdot 13 \cdot 12} \approx 0.41.$$

(b) There are 8 choices for the type that is picked in a pair. Then there are $\binom{7}{3}$ ways to pick the other three types, and 8 ways to pick the shoes. So the answer is

$$\frac{8 \cdot \frac{7}{3} \cdot 8}{\binom{16}{5}} \approx 0.5128.$$

(c) There are $\binom{8}{2}$ ways to pick the two pairs to be repeated and 12 choices for the remaining shoe. So the answer is

$$\frac{12 \cdot \binom{8}{2}}{\binom{16}{5}} \approx 0.0769.$$

Exercise 46. Two cards are a blackjack if one is an A and the other is a K, Q, J, or 10. If you pick two cards out of a deck, what is the probability you will get a blackjack?

Answer: There are $\binom{52}{2}$ ways to pick the cards. There are 4 ways to get an A and 16 ways to get the other ones. So the answer is

$$\frac{4 \cdot 16}{\binom{52}{2}} \approx 0.0483.$$

Exercise 47. A football team has 16 seniors, 12 juniors, 8 sophomores, and 4 freshmen. If we pick 5 players at random, what is the probability we will get 2 seniors and 1 from each of the other 3 classes?

Answer: There are $\binom{40}{5}$ ways to pick the players. There are $\binom{16}{2}$ ways to pick the seniors, and then $12 \cdot 8 \cdot 4$ ways to pick the others. The answer is

$$\frac{\binom{16}{2} \cdot 12 \cdot 8 \cdot 4}{\binom{40}{5}}.$$

Exercise 48. If 6 balls are thrown at random into 10 boxes, what is the probability no box will contain more than 1 ball?

Answer: Once one ball in put into a box, there are 9 choices for the second ball, then 8 for the third, 7 for the fourth, 6 for the fifth, and 5 for the sixth. So the answer is

$$\frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{10^6} = 0.1512.$$

Exercise 49. Suppose we have an urn with 6 black balls and 6 red balls. Compute the probability of getting 2 black balls and 1 red ball when we pick 3 balls out (a) without replacing them, (b) when we replace each ball after it is drawn.

Answer: (a) It's like having a sequence of B's and R's where the first three letters are an arrangement of BBR. The answer is

$$\frac{3 \cdot \binom{9}{4}}{\binom{12}{6}} = \frac{9}{22}.$$

(b) We can pick the first black ball in 6 ways, the second black ball in 6 ways, and the red ball in 6 ways. Then these can be arranged in 3 ways. The answer is

$$\frac{3 \cdot 6 \cdot 6 \cdot 6}{12^3} = \frac{3}{8}.$$

Exercise 50. The World Series is won by the first team to win four games. What is the probability the team that wins the first game will win the series?

Answer: Say A is the team that won the first game. To win in m games, A must win the last game and 2 of the first $m-1$, which can be done in $\binom{m-1}{2}$ ways. Each possibility has probability $\frac{1}{2^m}$, so we need

$$\sum_{m=3}^{6} 2^{-m}\binom{m-1}{2} = \frac{21}{32}.$$