

Binomial distribution:

Perform experiment n times
Each trial has prob. p of success.

The number of successes S has

$$P(S=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

for $k=0, \dots, n$

Say S has a binomial distribution with parameters n and p .

Poisson distribution : X has a

Poisson distribution with parameter $\lambda > 0$

$(X \sim \text{Poisson}(\lambda))$ if

$$P(X=k) = e^{-\lambda} \frac{\lambda^k}{k!} \quad \text{for } k=0, 1, 2, \dots$$

claim: This is a probability function

$$\text{PF}) \sum_{k=0}^{\infty} P(X=k) = \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \left(\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \right) \cancel{\cdot} e^{\lambda} = 1 \quad \checkmark$$

Recall: $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

Thm: (Poisson approximation to the binomial). Suppose S_n has a binomial distribution with parameters n and p_n .

If $p_n \rightarrow 0$ and $n p_n \rightarrow \lambda$ as $n \rightarrow \infty$,
then $P(S_n = k) \rightarrow e^{-\lambda} \frac{\lambda^k}{k!}$

"If we have a large number of independent events with small probability then the number that occur has approximately a Poisson dist."

Think: If $S_n = \text{binomial}(n, p)$ and p is small, then S_n is approximately Poisson (np).

Ex: Suppose roll two dice 12 times.

$D = \# \text{ times a double 6 appears.}$

Binomial dist
 $n=12$, $p = 1/36$, $\lambda = np = \frac{1}{3}$

$P(D=k)$ is approximately Poisson $(\frac{1}{3})$.

$$P(D=k) \approx e^{-1/3} \frac{\left(\frac{1}{3}\right)^k}{k!}$$

$k=0$: exact answer:

$$P(D=0) = \left(\frac{1}{36}\right)^0 \left(\frac{35}{36}\right)^{12} \approx 0.7132.$$

Poisson approx:

$$P(D=0) \approx e^{-1/3} \cdot \frac{1}{1} \approx 0.7165$$

$k=1$: exact answer:

$$P(D=1) = 12 \left(\frac{1}{36}\right)^1 \left(\frac{35}{36}\right)^{11} \approx 0.2445$$

Poisson approx:

$$P(D=1) \approx e^{-1/3} \left(\frac{1}{3}\right)^1 \approx 0.2388$$

$k=2$: exact:

$$\begin{aligned} P(D=2) &= \binom{12}{2} \left(\frac{1}{36}\right)^2 \left(\frac{35}{36}\right)^{10} \\ &= \frac{12 \cdot 11}{2} \left(\frac{1}{36}\right)^2 \left(\frac{35}{36}\right)^{10} \end{aligned}$$

Poisson approx: ≈ 0.0384

$$P(D=2) \approx e^{-1/3} \frac{\left(\frac{1}{3}\right)^2}{2} \approx 0.0398$$

Pf of Thm:

$$\lim_{n \rightarrow \infty} \lambda_n = \lambda$$

Lemma: If $\lambda_n \rightarrow \lambda$ then

$$\lim_{n \rightarrow \infty} \left(1 - \frac{\lambda_n}{n}\right)^n = e^{-\lambda}.$$

Pf when $k=0$: $P(S_n=0) = (1-p_n)^n$

$$\left. \begin{array}{l} \lambda_n = np_n \\ p_n = \frac{\lambda_n}{n} \end{array} \right\} = \left(1 - \frac{\lambda_n}{n}\right)^n \xrightarrow{\text{by Lemma}} e^{-\lambda}.$$

Pf when $k > 0$:

$$\begin{aligned} P(S_n = k) &= \binom{n}{k} p_n^k (1-p_n)^{n-k} \\ &= \binom{n}{k} \left(\frac{\lambda_n}{n}\right)^k \left(1 - \frac{\lambda_n}{n}\right)^{n-k} \\ &= \underbrace{n(n-1)\cdots(n-k+1)}_{k!} \underbrace{\frac{\lambda_n^k}{n^k} \left(1 - \frac{\lambda_n}{n}\right)^n}_{\left(1 - \frac{\lambda_n}{n}\right)^{-k}} \\ &= \frac{1}{k!} \left(\frac{n}{n}\right) \left(\frac{n-1}{n}\right) \cdots \left(\frac{n-k+1}{n}\right) \lambda_n^k \left(1 - \frac{\lambda_n}{n}\right)^n \left(1 - \frac{\lambda_n}{n}\right)^{-k} \\ &\quad \xrightarrow{n \rightarrow \infty} 1 \quad e^{-\lambda} \quad 1 \end{aligned}$$

$$\text{So } \lim_{n \rightarrow \infty} P(S_n = k)$$

$$= \frac{1}{k!} \lambda^k e^{-\lambda} = e^{-\lambda} \frac{\lambda^k}{k!}$$