Sequences of Functions

Def'n: Let E, E' be metric spaces. For $n = 1, 2, 3, \dots$, let $f_n : E \to E'$ be a function. If $p \in E$, say the sequence $f_1, f_2, \dots$ converges at p if $f_1(p), f_2(p), \dots$ converges in E'.

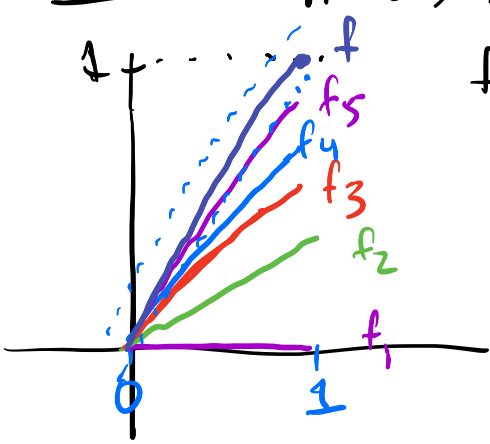
We say the (pointwise) sequence of functions $f_1, f_2, f_3, \dots$ converges (on E) if $\forall p \in E$, $f_1(p), f_2(p), f_3(p), \dots$ converges. In this case, define

$$f(p) = \lim_{n \to \infty} f_n(p).$$

We call f the limit function of $f_1, f_2, f_3 \dots$. Say $f_1, f_2, f_3 \dots$ converges to f.

Write $f = \lim_{n \to \infty} f_n$.

Ex 1: $f_n : [0,1] \to \mathbb{R}$ defined by



$$f_n(x) = \left(1 - \frac{1}{n}\right)x = \left(\frac{n-1}{n}\right)x$$

$f_1(x) = 0$

$f_2(x) = \frac{x}{2}$

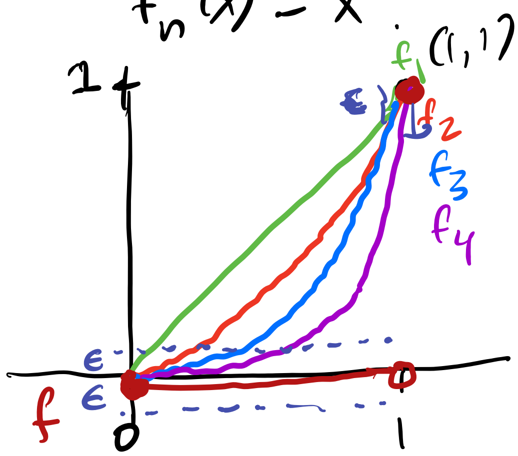
$f_3(x) = \frac{2}{3}x$ $\quad \dots \quad f_n(x) = \frac{n-1}{n}x$

$f_4(x) = \frac{3}{4}x$

$f_n \to f$, where $f(x) = x$.

Ex 2: $f_n : [0,1] \to \mathbb{R}$

$f_n(x) = x^n$



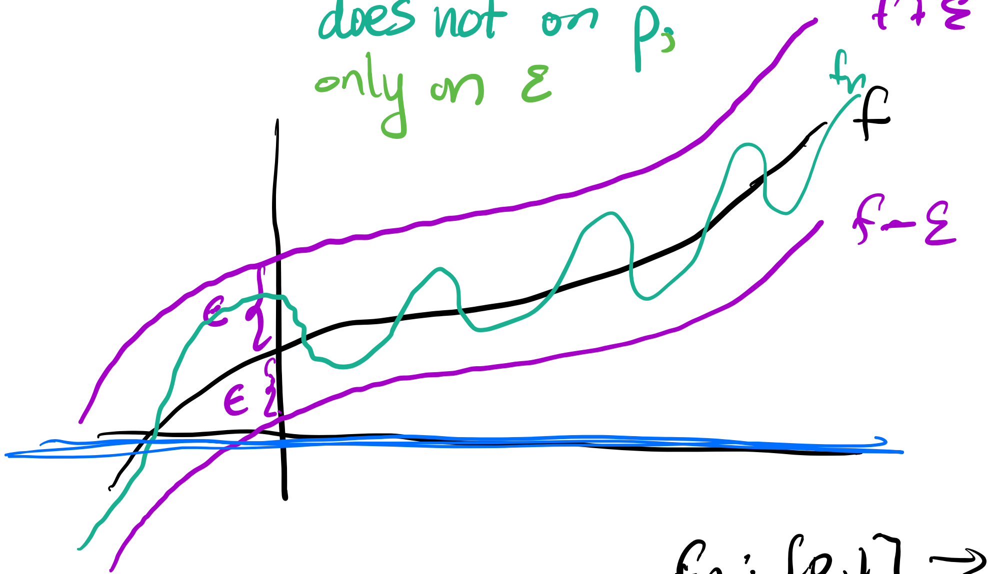
$f_1(x) = x$
$f_2(x) = x^2$
$f_3(x) = x^3$
$\vdots$

$$f(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1 \end{cases}$$

$f = \lim_{n\to\infty} f_n.$

Defn: $f_1, f_2, f_3 \ldots$ converges uniformly to f if, given any $\varepsilon > 0$, there is some pos. int. N s.t. $d'(f(p), f_n(p)) < \varepsilon$ whenever $n > N$, and for all $p \in E$.

does not on p,
only on ε



$f_n : [0,1] \to \mathbb{R}$

Going back to Ex 1: $f_n(x) = \left(1 - \frac{1}{n}\right) x$.

Claim: $f_1, f_2, f_3 \to f$ uniformly on $[0,1]$.

Pf) Let $\varepsilon > 0$. For each $n \in \mathbb{N}$, $\forall x \in [0,1]$

$$|f_n(x) - f(x)| = \left| \left(1 - \frac{1}{n}\right)x - x \right|$$
$$= \left| -\frac{1}{n} x \right| = \frac{1}{n}|x| \leq \frac{1}{n}.$$

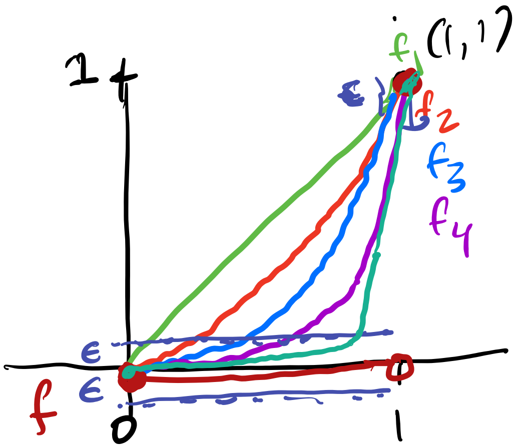
So if $N > \frac{1}{\varepsilon}$, and $n > N$, then $\forall x \in [0,1]$,

$$|f_n(x) - f(x)| < \frac{1}{\;} < \varepsilon.$$

<u>Claim:</u> In Ex 2, $f_n : [0,1] \to \mathbb{R}$

$$f_n(x) = x^n,$$

$f_1, f_2, f_3, \ldots$ does not converge uniformly to f.



Unif conv on $[0,1]$ means:

$\forall \varepsilon > 0$, $\exists N$ s.t. $\forall n$,

$\forall x \in [0,1]$,

$$|x^n - f(x)| < \varepsilon.$$

In particular, $\forall \varepsilon > 0$, $\exists N$ s.t. $\forall n > N$, $\forall x \in (0,1)$,

$$|x^n - 0| < \varepsilon, \text{ ie.}$$

$$x^n < \varepsilon$$

Pf) Let $\epsilon = \frac{1}{2}$. Then $\forall N$, $\exists x \in (0,1)$ s.t. $x^N > \frac{1}{2}$ (because $x^N \to 1$ as $x \to 1$).

So $f_1, f_2, f_3, \ldots$ does not converge uniformly on $(0,1)$, hence not on $[0,1]$.

<u>Exer:</u> Show $f_1, f_2, f_3, \ldots$ does converge uniformly on $[0,a]$ if $0 \leq a < 1$.