Metric Spaces (following Rosenlicht, Introduction to Analysis)

Def'n: A metric space is a set E, together with a rule that associates with each pair $p,q \in E$ a real number $d(p,q)$ such that

(1) $d(p,q) \geq 0 \quad \forall p,q \in E$

(2) $d(p,q) = 0 \iff p = q$

(3) $d(p,q) = d(q,p) \quad \forall p,q \in E$

(4) (Triangle inequality) $\forall p,q,r \in E$,

$$d(p,r) \leq d(p,q) + d(q,r).$$

The function $d : E \times E \to \mathbb{R}$ is called the metric.

Ex: (1) $E = \mathbb{R}$, $d(p,q) = |p-q| = \sqrt{(p-q)^2}$.

Check that d is a metric:

(1)–(3) easy

(4): $d(p,r) = |p-r| = |p-q+q-r|$

$$\leq |p-q| + |q-r|$$

$$= d(p,q) + d(q,r).$$

(2) more generally, for any positive integer n, $E = \mathbb{R}^n$ together with the metric

$$\underset{\substack{\| \\ (x_1, \ldots, x_n)}}{d(p,}\ \underset{\substack{\| \\ (y_1, \ldots, y_n)}}{q)} \overset{\text{def}}{=} \sqrt{(x_1-y_1)^2 + \cdots + (x_n-y_n)^2}.$$

Then d is a metric (to be proved...)

(3) If (E, d_E) is a metric space, and $E_1 \subseteq E$, then E_1 can be made into a metric space by defining (for $p, q \in E_1$)

$d(p,q)$ to be the distance between p and q as elements of E.

i.e. $d_{E_1}(p,q) = d_E(p,q)$.

(4) Let E be an arbitrary set.

For $p, q \in E$, define

$$d(p,q) = \begin{cases} 0 & \text{if } p = q \\ 1 & \text{if } p \neq q \end{cases}$$

This is called the <u>discrete metric</u>

Proof that $\mathbb{R}^n$ is a metric space:
(triangle inequality).

<u>Prop</u>: (Cauchy-Schwarz inequality).
For any real numbers $a_1, \dots, a_n$, $b_1, \dots, b_n$,

$$|a_1 b_1 + \dots + a_n b_n| \leq \sqrt{a_1^2 + \dots + a_n^2} \cdot \sqrt{b_1^2 + \dots + b_n^2}.$$

$$(\Leftrightarrow (a_1 b_1 + \dots + a_n b_n)^2 \leq (a_1^2 + \dots + a_n^2)(b_1^2 + \dots + b_n^2)),$$

Pf) $\forall \alpha, \beta \in \mathbb{R}$,

$$0 \leq (\alpha a_1 - \beta b_1)^2 + (\alpha a_2 - \beta b_2)^2 + \dots + (\alpha a_n - \beta b_n)^2$$

$$= \alpha^2 (a_1^2 + a_2^2 + \dots + a_n^2) - 2\alpha\beta (a_1 b_1 + a_2 b_2 + \dots + a_n b_n) + \beta^2 (b_1^2 + b_2^2 + \dots + b_n^2).$$

Set $\alpha = \sqrt{b_1^2 + b_2^2 + \dots + b_n^2}$

$\beta = \pm \sqrt{a_1^2 + a_2^2 + \dots + a_n^2}$.

Get $0 \le (a_1^2+\dots+a_n^2)(b_1^2+\dots+b_n^2)$

$$\mp 2\sqrt{a_1^2+\dots+a_n^2}\,\sqrt{b_1^2+\dots+b_n^2}\,(a_1b_1+\dots+a_nb_n)$$

$$+ (a_1^2+\dots+a_n^2)(b_1^2+\dots+b_n^2).$$

$$0 \le 2(a_1^2+\dots+a_n^2)(b_1^2+\dots+b_n^2)$$

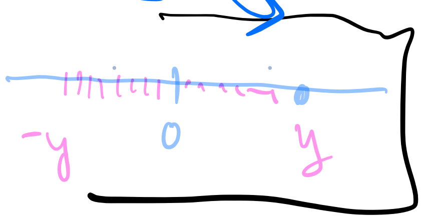
$$\mp 2\sqrt{a_1^2+\dots+a_n^2}\,\sqrt{b_1^2+\dots+b_n^2}\,(a_1b_1+\dots+a_nb_n)$$

$$\Rightarrow \pm \left[\sqrt{a_1^2+\dots+a_n^2}\,\sqrt{b_1^2+\dots+b_n^2}\,(a_1b_1+\dots+a_nb_n)\right]$$

$$\le (a_1^2+\dots+a_n^2)(b_1^2+\dots+b_n^2)$$

$$\Rightarrow \sqrt{a_1^2+\dots+a_n^2}\,\sqrt{b_1^2+\dots+b_n^2}\,|a_1b_1+\dots+a_nb_n|$$

$$\le (a_1^2+\dots+a_n^2)(b_1^2+\dots+b_n^2)$$



If either $\sqrt{a_1^2+\dots+a_n^2}$ or $\sqrt{b_1^2+\dots+b_n^2}=0$,

then either $a_1=\dots=a_n=0$ or $b_1=\dots=b_n=0$,

in which case C-S follows.

If $\sqrt{a_1^2+\dots+a_n^2}\neq 0$ and $\sqrt{b_1^2+\dots+b_n^2}\neq 0$, then get

$$\boxed{|a_1b_1+\cdots+a_nb_n| \le \sqrt{a_1^2+\cdots+a_n^2}\cdot\sqrt{b_1^2+\cdots+b_n^2}.}$$

$\square$

Corollary: For any real numbers $a_1,\dots,a_n, b_1,\dots,b_n$,

$$\sqrt{(a_1+b_1)^2+\cdots+(a_n+b_n)^2} \le \sqrt{a_1^2+\cdots+a_n^2}+\sqrt{b_1^2+\cdots+b_n^2}.$$

pf)

$$\begin{aligned}
(a_1+b_1)^2+\cdots+(a_n+b_n)^2 &= a_1^2+\cdots+a_n^2+2(a_1b_1+\cdots+a_nb_n) \\
&\quad + b_1^2+\cdots+b_n^2. \\
&\le a_1^2+\cdots+a_n^2+2|a_1b_1+\cdots+a_nb_n| \\
&\quad + b_1^2+\cdots+b_n^2 \qquad \text{by C-S} \\
&\le a_1^2+\cdots+a_n^2+2\sqrt{a_1^2+\cdots+a_n^2}\sqrt{b_1^2+\cdots+b_n^2} \\
&\quad + b_1^2+\cdots+b_n^2 \\
&= \left(\sqrt{a_1^2+\cdots+a_n^2}+\sqrt{b_1^2+\cdots+b_n^2}\right)^2
\end{aligned}$$

$\square$.

Let $p = (x_1, \dots, x_n)$,
$q = (y_1, \dots, y_n)$,
$r = (z_1, \dots, z_n)$.

Then

$$d(p,r) = \sqrt{(x_1 - z_1)^2 + \dots + (x_n - z_n)^2}$$

$$= \sqrt{(x_1 - y_1 + y_1 - z_1)^2 + \dots + (x_n - y_n + y_n - z_n)^2}$$

by corollary

$$\leq \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2} + \sqrt{(y_1 - z_1)^2 + \dots + (y_n - z_n)^2}$$

$$= d(p,q) + d(q,r).$$

Last time... metric spaces.

Open and Closed Sets.

Def'n: E: metric space.

$p_0 \in E$, $r > 0$ a real number.

Then the open ball in E of center p_0 and radius r is

$$B_r(p_0) \overset{\text{def}}{=} \{p \in E : d(p_0, p) < r\}.$$

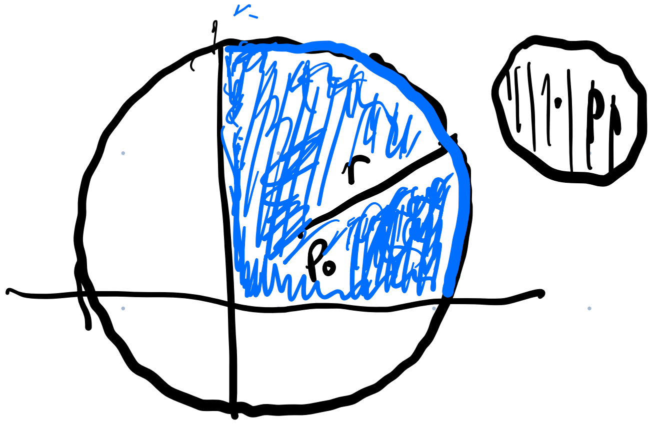
The closed ball in E of center p_0 and radius r is

$$\{p \in E : d(p_0, p) \leq r\}.$$

Ex: In $\mathbb{R}^2$,

$E = \{(x,y) \in \mathbb{R}^2 : x > 0, y > 0\}$

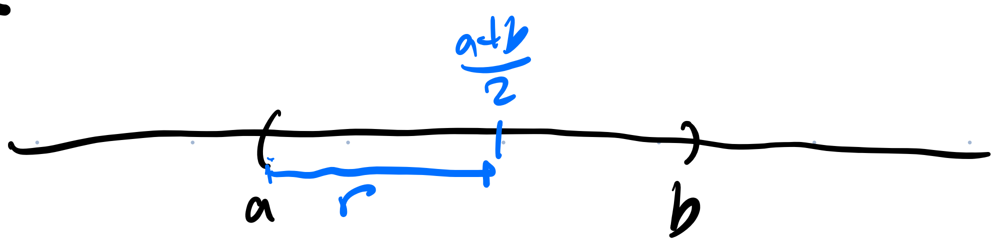
Ex:



In general, if $S \subseteq E$ with the induced metric,

$$\underbrace{B_r(p_0)}_{\text{in } S} = \underbrace{B_r(p_0)}_{\text{in } E} \cap S$$

Ex: In $\mathbb{R}$, (a,b) is an open ball



because $(a,b) = B_r(p_0)$ where

$$p_0 = \frac{a+b}{2} \quad \text{and} \quad r = \frac{a+b}{2} - a = \frac{b-a}{2}$$

Def'n: A subset S of a metric space E is open if, for each $p \in S$, S contains some open ball of center p.
in E

Prop. For any metric space E,

(1) ϕ is open, (in E)

(2) E is open,

(3) the union of any collection of open subsets of E is open,

(4) the intersection of a finite number of open subsets is open.

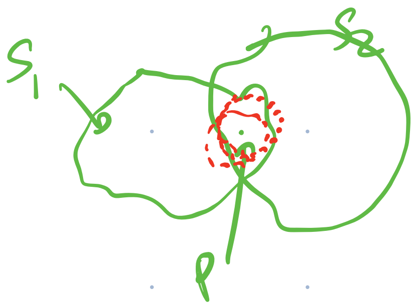
Pf) (1) Trivially true.

(2) Let $p \in E$. Take any $r > 0$. Then $B_r(p) \subset E$.

(3) Let $\{U_\alpha\}_{\alpha \in A}$ be a collection of open subsets of E. Let $p \in \bigcup_{\alpha \in A} U_\alpha$. Then $\exists\, \alpha \in A$ s.t. $p \in U_\alpha$. Since U_α is open, $\exists\, r > 0$ s.t. $B_r(p) \subset U_\alpha$, so

$B_r(p) \subset \bigcup_{\alpha \in A} U_\alpha$.

(4) Let $S_1, \ldots, S_n$ be open subsets of E and let $p \in S_1 \cap \ldots \cap S_n$.



For $i = 1, \ldots, n$, each S_i is open, so $\exists\, r_i > 0$ s.t. $B_{r_i}(p) \subset S_i$.

Let $r = \min\{r_1, \ldots, r_n\}$.

Then $\underline{B_r(p)} \subset \underline{B_{r_i}(p)}$ $\forall i = 1, \ldots, n$

$\Rightarrow B_r(p) \subset S_1 \cap \ldots \cap S_n$.

Prop: In any metric space E, an open ball is an open set.

Pf) Let $p \in E$, $r > 0$.

WTS $B_r(p)$ is an open set.

Let $q \in B_r(p)$. WTS $\exists\, r' > 0$ s.t. $B_{r'}(q) \subset B_r(p)$.

Let $r' > 0$ be such that $r' < r - d(p,q)$.

Let $q' \in B_{r'}(q)$. Then

$$d(q', q) < r'$$

WTS $d(q', p) < r$.

We have

$$\begin{aligned} d(q',p) &\le d(q',q) + d(q,p) \\ &< r' + d(p,q) \\ &< r - d(p,q) + d(p,q) \\ &= r. \end{aligned}$$

The open subsets of E are precisely the unions of open balls of E. Why?

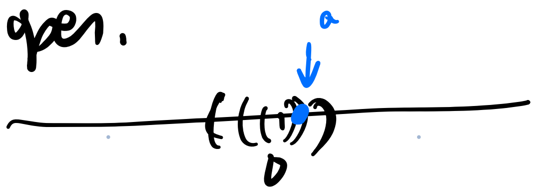
Let U be an open subset of E.

Then $U = \bigcup_{p \in U} \{p\} = \bigcup_{p \in U} \boxed{B_{r_p}(p)}$.

for some $r_p > 0$, for every $p \in U$.

Rmk: An arbitrary intersection of open sets need not be open.

Ex: In $\mathbb{R}$,

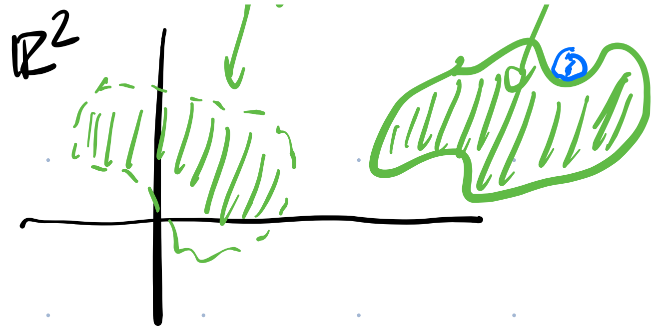


$$\bigcap_{n \in \mathbb{Z}_{>0}} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}.$$

Def'n: A subset S of a metric space E is **closed** if its complement S^c is open.

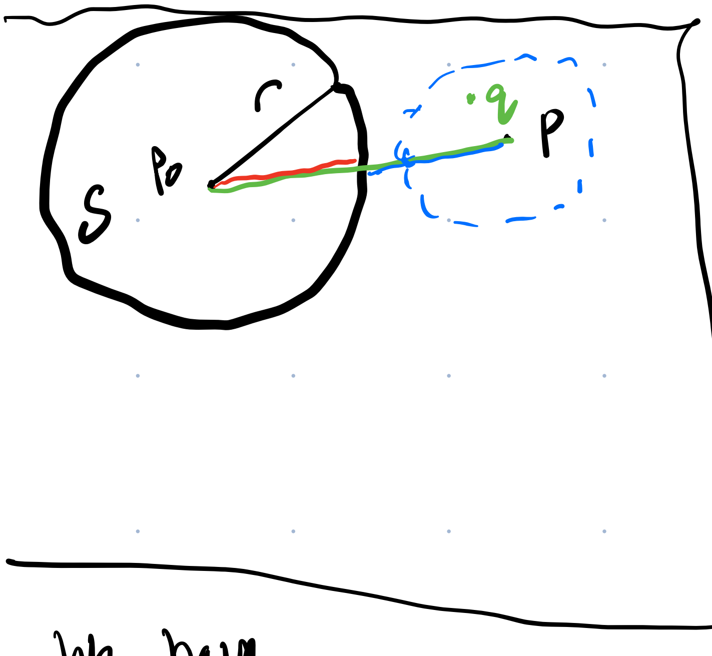
$(S^c = \{x \in E : x \notin S\})$.

open closed



Prop: In any metric space, a closed ball is a closed set.

Pf) Let S be the closed ball of center $p_0 \in E$ and radius r. Let $p \in S^c$.



Let r' be such that

$r' < d(p_0, p) - r.$

Claim:

$B_{r'}(p) \subset S^c$.

Let $q \in B_{r'}(p)$.

Then

$d(q,p) < r'$

$\Rightarrow -d(p,q) > -r'$

We have

$$d(p_0, q) = \underbrace{d(p_0,q) + d(q,p)} - d(p,q)$$

$$\geq d(p_0, p) - d(p,q)$$

$$> \underline{r' + r} - d(p,q)$$

$> r.$

Prop: For any metric space E,

(1) E is closed

(2) ϕ is closed.

(3) The intersection of any collection of closed subsets of E is closed.

(4) The union of a finite number of closed subsets of E is closed.

Pf) (1), (2) ✓

(3): $\left(\bigcap_{\alpha \in A} S_\alpha\right)^c = \bigcup_{\alpha \in A} S_\alpha^c$

so if each S_α is closed then S_α^c is open, so $\left(\bigcap_{\alpha \in A} S_\alpha\right)^c$ is open.

(4): $(S_1 \cup \dots \cup S_n)^c = S_1^c \cap \dots \cap S_n^c$,

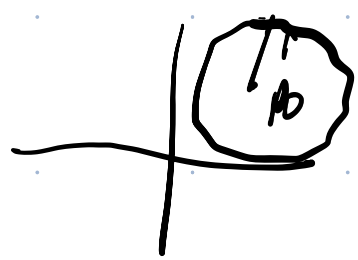
so if each S_i is closed, then $(S_1 \cup \dots \cup S_n)^c$ is open

Exer: Show that any finite subset of a metric space is closed.

If $p \in E$, then $\{p\}$ is closed.

pf) Let $q \in \{p\}^c$. Let $r < d(p,q)$. Then $p \notin B_r(q)$, so $\{p\}^c$ is open, so $\{p\}$ is closed. Since finite unions of closed sets are closed, any finite subset of E is closed.

Ex: Let $p_0 \in E$, and $r > 0$. Then $\{x \in E : d(p_0, x) = r\}$ is closed.



pf)

$\{x \in E : d(p_0, x) < r\}$ is open,

Also, $\{x \in E : d(p_0, x) \leq r\}$ is closed,

so $\{x \in E : d(p_0, x) \leq r\}^c$ is open

$\shortparallel$

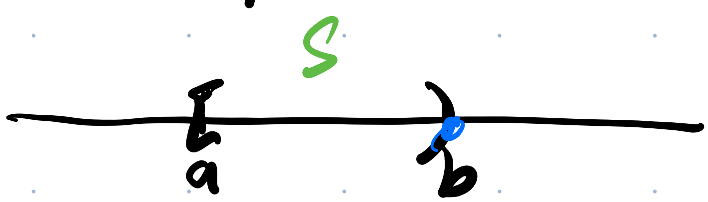
$\{x \in E : d(p_0, x) > r\}$

So $\{x \in E : d(p_0, x) = r\}$ is closed, because $\{x \in E : d(p_0, x) > r\}$ $\cup \{x \in E : d(p_0, x) < r\}$ is open.

If $E = \mathbb{R}$, $a < b$,

$$[a, b) = \{x \in \mathbb{R} : a \le x < b\}$$

is neither open nor closed.

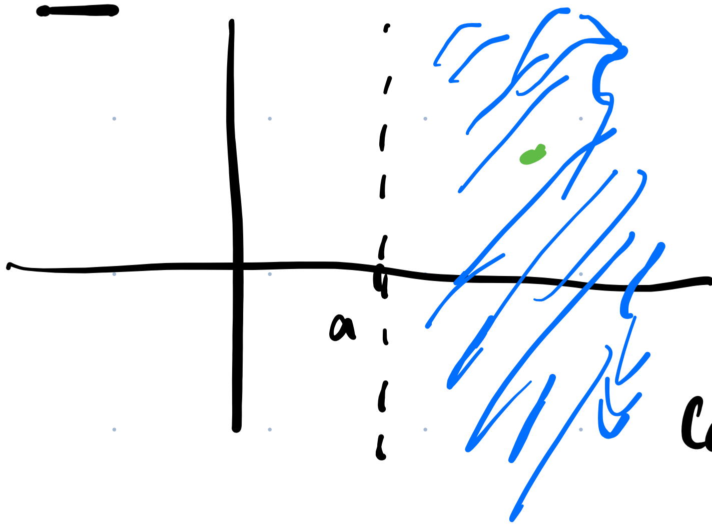
15) 

This set is not open, because every neighborhood centered at a contains numbers less than a.

The complement of $[a, b)$ is not open, because $b \in [a, b)^c$, but there is no interval around b that is a subset of S^c, because every neighborhood of b contains elements in $[a,b)$.

Claim:

Ex: In $\mathbb{R}^2$ $\quad S = \{(x_1, x_2) : x_1 > a\}$. is open.



Pf) Let $p = (x_1, x_2) \in S$. Consider the open ball of center p and radius $x_1 - a$. If $q = (y_1, y_2)$ is in this ball, then

$$|x_1 - y_1| \leq \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$$

$$= d(p,q) < x_1 - a.$$

So,

$$
\begin{aligned}
y_1 &= x_1 - (x_1 - y_1) \\
&\geq x_1 - |x_1 - y_1| \\
&> x_1 - (x_1 - a) \\
&= a.
\end{aligned}
$$

Therefore, S is open.