Let $a_1=1$. Let $a_{n+1}=\sqrt{a_n+3}$ for $n=1,2,3,\dots$.
Prove that (a_n) converges and find the limit.

$a_1=1, \quad a_2=\sqrt{1+3}=2, \quad a_3=\sqrt{2+3}=\sqrt{5}, \dots$

Claim 1: (a_n) is bounded above by 3.

pf) WTS for each n, $a_n \le 3$.

We'll use induction.

Base case: $n=1$ $\quad a_1=1\le 3$ ✓.

Inductive Step: Suppose $a_k \le 3$. $(k\ge 1)$

WTS $a_{k+1}\le 3$.

We have $a_{k+1}=\sqrt{a_k+3}\le\sqrt{3+3}=\sqrt{6}\le 3$.

This proves the inductive step.

Claim 2: The sequence (a_n) is increasing,

i.e. $a_1\le a_2\le a_3\le a_4\le\dots$

pf) We'll prove this by induction.

Base case: Show $a_1\le a_2$.

$a_1=1, \; a_2=2$, so $a_1\le a_2$ ✓.

Inductive Step: Assume that

$a_k\le a_{k+1}$. WTS $a_{k+1}\le a_{k+2}$.

We have

$$a_{k+2} = \sqrt{a_{k+1}+3}$$

We're assuming:

$$\sqrt{a_{k-1}+3} = a_k \leq a_{k+1} = \sqrt{a_k+3}$$

wts $\quad a_{k+1} = \sqrt{a_k+3} \leq \sqrt{a_{k+1}+3}$

Since $a_k \leq a_{k+1}$, $\quad a_k+3 \leq a_{k+1}+3$,

and since $a_i \geq 0$ for all i,

$$a_{k+1} = \sqrt{a_k+3} \leq \sqrt{a_{k+1}+3} = a_{k+2}$$

By the Monotone Convergence theorem,

(a_n) converges.

Let $L = \lim_{n\to\infty} a_n$.

$$a_{n+1} = \sqrt{a_n+3} \quad \text{for all } n \geq 1.$$

So $\displaystyle \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{a_n+3}$

$$= \sqrt{\left(\lim_{n\to\infty} a_n\right)+3}$$

So $\quad L = \sqrt{L+3}$

$$\Rightarrow L = \frac{1-\sqrt{13}}{2} \text{ or } \frac{1+\sqrt{13}}{2}$$

Since $a_i \geq 0$, $L \geq 0$, so

$$\boxed{L = \frac{1+\sqrt{13}}{2}}.$$