Determinants

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In this chapter, we will assign to every square matrix a number, called its determinant. Determinants will be needed in the next chapter. They also show up in the change of variables formula in multivariable calculus, and in the second derivative test in multivariable calculus. Sometimes properties of a matrix are easy to verify by calculating determinants.

1 Definition

In this section, you will

- Compute determinants using a cofactor expansion

Determinants can be defined in at least three different ways. We will discuss two different ways to define determinants. One way involves row reduction and out of the three ways it is most closely tied to what we have done, so we will start with it. Here are the steps to calculate the determinant of a square matrix A.

Determinants:

- Using only elementary row operations of type 1 (swapping rows) and type 3 (replacing a row by that row plus a multiple of a different row), bring A into upper triangular form (echelon form). (In other words, we reduce A to echelon form *without* scaling any rows.)
- Multiply the entries on the diagonal of this upper triangular matrix. The result is a number, call it c, which, like any real number, is either positive, negative, or 0 (it's 0 only if one of the diagonal entries is 0).
- Count the number of row exchanges that were applied. If the number of row exchanges is odd, multiply c by -1; if the number of row exchanges is even, keep c the same. (An equivalent way of saying this: Let k be the number of row exchanges. Calculate $(-1)^k c$.)
- The resulting number is called the **determinant** of A. The determinant of A is written as $\det A$.

Example 1. The determinant of I_n is 1 – it's already in echelon form, so we just multiply the diagonal entries. If k is a scalar, the determinant of kI_n is k^n.

Example 2. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Compute $\det A$.

We row reduce A until it is in echelon form. In this case, only one step is needed: replacing row 2 by row 2 minus 3 times row 1:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$$

Then we multiply the diagonal entries of the echelon form matrix: $1(-2) = -2$. The determinant of A is -2. $\square$

Another common notation for the determinant of a matrix uses a pair of vertical lines in place of brackets. So

$$\det A = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2.$$

The way I've defined determinants is actually not the most common way of defining determinants, and there is a reason for this. There is something sneaky I left out. We know that every matrix A can be put into row echelon form. However, we also know that a matrix has more than one row echelon form, even if you do not perform the elementary row operation of scaling rows. For example, suppose that A was this matrix:

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$$

The top left entry is 0, so the algorithm says: swap two rows so that the number at the top is not zero. But there are two ways to do this: you could either swap rows 1 and 2, or swap rows 1 and 3. Suppose that you swap rows 1 and 2 and then row reduce until the matrix is in echelon form, while your friend swaps rows 1 and 3 and then row reduces until their matrix is in echelon form. The matrix you get will be different from the matrix your friend gets. Now you both calculate the number of row exchanges and multiply the pivots. You will get a number, and your friend will get a number. The question is: how do you know that the determinant you get is the same as the determinant your friend gets?

In order for the definition of determinant we gave above to make sense, we have to know that two different row echelon forms of the same original matrix don't somehow end up giving you different determinants. For example, suppose we decided to make up a new term, the "peterminant" of A, to be this: Row reduce A until it is in echelon form. The peterminant of A is defined to be the top-left entry (in row 1, column 1). What would be wrong with this? What's wrong with this is that you get different values for the peterminant depending on how you did the row operations. In the case of the matrix A above, if you swapped rows 1 and 2, the peterminant of A would be 1, but if you swapped rows 1 and 3, the peterminant would be 2. Because of this, it wouldn't make sense to refer to "the peterminant of A."

We will show that this issue does not arise with the determinant. To do so, we will give another formula for the determinant which doesn't involve row exchanges at all – it only involves the entries of the matrix. Then we will show that this definition matches the previous one. This will prove that there are no issues with how we defined the determinant before.

In most books I've seen, the determinant of a square matrix is defined by a messy formula. The formula is given recursively. First we define the determinant of a 1 by 1 matrix and a 2 by 2 matrix, and then the determinant of a 3 by 3 matrix is defined in terms of determinants of 2 by 2 matrices, the determinant of a 4 by 4 matrix is defined in terms of determinants of 3 by 3 matrices, and so on. To state this formula, let A be an n by n matrix. If we delete a row (say, row i) and a column (say, column j), we get an $n-1$ by $n-1$ matrix. This matrix is called the (i,j)-**minor** of A, and we call it A_{ij}. The number

$$(-1)^{i+j} \det A_{ij}$$

is called the (i,j)-**cofactor** of A.

Cofactor Expansion of the Determinant

Suppose you have a square matrix, and pick any row out of the matrix, say row i. Then

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}.$$

As you can see, this formula is a mess. It's not so bad for 2 by 2 and 3 by 3 matrices. But to calculate the determinant of an 8 by 8 matrix using the cofactor expansion formula, you need to calculate the determinants of a bunch of 7 by 7 matrices, and for *each* of those determinants you need determinants of a bunch of 6 by 6 matrices, and for *each* of those 6 by 6 matrices you need to calculate several 5 by 5 determinants, and so on. Using row reduction is much faster. Even faster, enter the 64 numbers into a computer program.

Exercise 1. Calculate the determinant of the matrix

$$\begin{bmatrix} a-x & b \\ c & d-x \end{bmatrix}$$

where $a,b,c,d \in \mathbf{R}$.

Exercise 2. Let φ, θ be real numbers, $r > 0$. Calculate the determinant of the matrix

$$\begin{bmatrix} \sin\varphi\cos\theta & r\cos\varphi\cos\theta & -r\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & r\cos\varphi\sin\theta & r\sin\varphi\cos\theta \\ \cos\varphi & -r\sin\varphi & 0 \end{bmatrix}$$

This determinant shows up in integration using spherical coordinates in multivariable calculus.

Exercise 3. Let $\mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} c \\ 0 \end{bmatrix}$ where $a, b, c > 0$ (for simplicity). Compute the area of the parallelogram determined by $\mathbf{u}, \mathbf{v}, \mathbf{u}+\mathbf{v}$, and $\mathbf{0}$, and compute the determinants of the matrix whose first column is $\mathbf{u}$ and second column is $\mathbf{v}$, and the matrix whose first column is $\mathbf{v}$ and second column is $\mathbf{u}$. Draw a picture and explain what you find.

Exercise 4. Compute $\begin{vmatrix} 1 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -1 & -8 & 0 & 3 \\ 0 & 5 & 0 & 6 \end{vmatrix}$.

2 Equivalence of the Two Definitions

We have given two very different definitions of the determinant. One is by row reduction, and we still have not seen that it is a good definition because we have not yet ruled out the possibility that two different echelon forms of a matrix A will result in two different values for $\det A$. The second definition, given recursively, only involves the entries of the matrix, but it has the drawback of being complicated. In this...

Linearity property...

Suppose that $A = \begin{bmatrix} 1 & 2 & 3 \\ b_1 + kc_1 & b_2 + kc_2 & b_3 + kc_3 \\ 4 & 5 & 6 \end{bmatrix}$. Let B be the matrix obtained from A by replacing row 2 of A by $\mathbf{u}$. Let C be the matrix obtained from A by replacing row 2 of A by $\mathbf{v}$. So

$$B = \begin{bmatrix} 1 & 2 & 3 \\ b_1 & b_2 & b_3 \\ 4 & 5 & 6 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & 3 \\ c_1 & c_2 & c_3 \\ 4 & 5 & 6 \end{bmatrix}$$

We must prove that $\det(A) = \det(B) + k\det(C)$. Then

$$\tilde{A}_{11} = \begin{bmatrix} b_2 + kc_2 & b_3 + kc_3 \\ 5 & 6 \end{bmatrix}, \quad \tilde{A}_{12} = \begin{bmatrix} b_1 + kc_1 & b_3 + kc_3 \\ 4 & 6 \end{bmatrix}, \quad \tilde{A}_{13} = \begin{bmatrix} b_1 + kc_1 & b_2 + kc_2 \\ 4 & 5 \end{bmatrix}.$$

Also,

$$\tilde{B}_{11} = \begin{bmatrix} b_2 & b_3 \\ 5 & 6 \end{bmatrix}, \quad \tilde{B}_{12} = \begin{bmatrix} b_1 & b_3 \\ 4 & 6 \end{bmatrix}, \quad \tilde{B}_{13} = \begin{bmatrix} b_1 & b_2 \\ 4 & 5 \end{bmatrix}$$

and

$$\tilde{C}_{11} = ...$$

Theorem 1. *Let B be an $n \times n$ matrix, where $n \geq 2$. If row i of B equals $\mathbf{e}_k$ for some k ($1 \leq k \leq n$), then $\det(B) = (-1)^{i+k} \det(\tilde{B}_{ik})$.*

To get an idea of the proof, let's use induction and prove the theorem for a specific matrix B, with $n = 4$ and $i = 2$ and $k = 3$. Suppose

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ 0 & 0 & 1 & 0 \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}$$

For each $j \neq 3$, let C_{2j} be the 2×2 matrix obtained from B by deleting rows 1 and 2 and columns j and 3. Thus,

$$C_{21} = \begin{bmatrix} b_{32} & b_{34} \\ b_{42} & b_{44} \end{bmatrix}, \quad C_{22} = \begin{bmatrix} b_{31} & b_{34} \\ b_{41} & b_{44} \end{bmatrix}, \quad C_{24} = \begin{bmatrix} b_{31} & b_{32} \\ b_{41} & b_{42} \end{bmatrix}$$

For each j, row 1 of $\tilde{B}_{1j}$ is the following vector:

$$j = 1 : \langle 0, 1, 0 \rangle, \quad j = 2 : \langle 0, 1, 0 \rangle, \quad j = 3 : \langle 0, 0, 0 \rangle, \quad j = 4 : \langle 0, 0, 1 \rangle.$$

We have

$$\tilde{B}_{11} = \begin{bmatrix} 0 & 1 & 0 \\ b_{32} & b_{33} & b_{34} \\ b_{42} & b_{43} & b_{44} \end{bmatrix}, \quad \tilde{B}_{12} = \begin{bmatrix} 0 & 1 & 0 \\ b_{31} & b_{33} & b_{34} \\ b_{41} & b_{43} & b_{44} \end{bmatrix}, \quad \tilde{B}_{13} = \begin{bmatrix} 0 & 0 & 0 \\ b_{31} & b_{32} & b_{34} \\ b_{41} & b_{42} & b_{44} \end{bmatrix}, \quad \tilde{B}_{14} = \begin{bmatrix} 0 & 0 & 1 \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{bmatrix}$$

Therefore, by the inductive hypothesis,

$$\det(\tilde{B}_{11}) = -\det C_{21}, \quad \det(\tilde{B}_{12}) = -\det C_{22}, \quad \det(\tilde{B}_{13}) = 0, \quad \det(\tilde{B}_{14}) = \det C_{24}.$$

Therefore,

$$\begin{aligned}\det(B) &= b_{11}\det(\tilde{B}_{11}) - b_{12}\det(\tilde{B}_{12}) + b_{13}\det(\tilde{B}_{13}) - b_{14}\det(\tilde{B}_{14}) \\ &= -b_{11}\det C_{21} + b_{12}\det C_{22} - b_{14}\det C_{24}\end{aligned}$$

Note that

$$\det(\tilde{B}_{23}) = \det\begin{bmatrix} b_{11} & b_{12} & b_{14} \\ b_{31} & b_{32} & b_{34} \\ b_{41} & b_{42} & b_{44}\end{bmatrix} = b_{11}\det C_{21} - b_{12}\det C_{22} + b_{14}\det C_{24}$$

so the result follows.

3 Properties of Determinants

In this section, you will

- State, prove, and apply determinant properties, including determinant of a product, inverse, transpose, and diagonal matrix.
- Use the determinant to determine whether a matrix is singular or nonsingular.
- Use the determinant of a coefficient matrix to determine whether a system of equations has a unique solution.

There are three key properties of determinants. We will see that they *characterize* the determinant. This means that the determinant is the only function that has these properties. All the other properties of determinants will be consequences of these properties.

The determinant function satisfies these properties.

Key Properties of Determinants:

- The determinant of the identity matrix is 1:

$$\det(I_n) = 1.$$

- If B is the matrix obtained by exchanging any two rows of A, then $\det B = -\det A$.
- Given i, the function $\det A$ is linear as a function of the ith row alone. This means the following.

  3a) If B is obtained from A by multiplying any one row by a scalar k, then $|B| = k|A|$. (Elementary row operation of type (2) multiplies the determinant by k.)

  3b) Suppose that A_1 and A_2 are matrices that are the same except for one row, say row i. Let B be the matrix whose rows are the same as those of rows A_1 and A_2 except for row i, which is the sum of row i of A_1 and row i of A_2. Then $|B| = |A_1| + |A_2|$.

Exercise 5. Show directly that for all a, b, x_1, x_2, y_1, y_2,

$$\begin{vmatrix} a & b \\ x_1 & x_2 \end{vmatrix} + \begin{vmatrix} a & b \\ y_1 & y_2 \end{vmatrix} = \begin{vmatrix} a & b \\ x_1 + y_1 & x_2 + y_2 \end{vmatrix}$$

Exercise 6. Show directly that for all $a, b, c, d, e, f, x_1, x_2, x_3, y_1, y_2, y_3$,

$$\begin{vmatrix} a & b & c \\ d & e & f \\ x_1 & x_2 & x_3 \end{vmatrix} + \begin{vmatrix} a & b & c \\ d & e & f \\ y_1 & y_2 & y_3 \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ x_1 + y_1 & x_2 + y_2 & x_3 + y_3 \end{vmatrix}$$

Using the three Key Properties, one can determine how the elementary row operations affect the value of the determinant.

How Elementary Row Operations Affect the Determinant

Let A be an n by n matrix.
(a) If E is the elementary matrix corresponding to the operation that exchanges rows i_1 and i_2, then $\det(EA) = -\det A$.
(b) If E' is the elementary matrix corresponding to the operation that replaces row i_1 of A by itself plus c times row i_2, then $\det(E'A) = \det A$.
(c) If E'' is the elementary matrix corresponding to the operation that multiplies row i of A by the non-zero scalar λ, then $\det(E''A) = \lambda(\det A)$.

The determinant function is uniquely characterized by its three Key Properties because the calculation of $\det A$ just given uses only the three Key Properties.

Example 3. Let $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$. Compute $|A|$.

We have

$$\det \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = 2 \det \begin{bmatrix} 1 & 3/2 \\ 4 & 5 \end{bmatrix} = 2 \det \begin{bmatrix} 1 & 3/2 \\ 0 & -1 \end{bmatrix} = 2(1)(-1) = -2.$$

Notice that we didn't have to scale the first row by $1/2$; we could have also done one step in the elimination process and then multiplied the pivots. $\square$

Exercise 7. Let A be a 5×5 matrix and let the rows of A be $\mathbf{a}_1, ..., \mathbf{a}_5$. Let B be the matrix in which the rows are $\mathbf{a}_5, ..., \mathbf{a}_1$. Calculate $\det(B)$ in terms of $\det(A)$.

Exercise 8. Suppose that $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$. Find the determinants.

(a) $\begin{vmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{vmatrix}$ (b) $\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix}$ (c) $\begin{vmatrix} a & b & c \\ 2d + a & 2e + b & 2f + c \\ g & h & i \end{vmatrix}$ (d) $\begin{vmatrix} a + d & b + e & c + f \\ d & e & f \\ g & h & i \end{vmatrix}$

Properties of Determinants

(a) A square matrix A is invertible if and only if $\det(A) \neq 0$.
(b) For any two n by n matrices A and B,

$$\det(AB) = \det(A)\det(B).$$

(c) The determinant of any upper triangular matrix (or lower triangular matrix) is just the product of the diagonal entries.
(d) For an $n \times n$ matrix A, and any scalar c,

$$\det(cA) = c^n \det(A).$$

(e) $\det(A^T) = \det(A)$

It is *not* true that $\det(A+B) = \det(A) + \det(B)$. For example, take $A = I$ and $B = 2I$.

Exercise 9. If the determinant of A is 5, what is the determinant of A^2? What is the determinant of $2A^2$?

Exercise 10. Show that if $A = A^T$, then $\det A$ is either 1 or -1.

Exercise 11. Show that if $A = A^2$, then $\det A$ is either 0 or 1.

Exercise 12. (a) Suppose that A is an $n \times n$ matrix. Prove that $\det(A^T A) \geq 0$.
(b) Make up a 2×3 matrix A and calculate $\det(A^T A)$. Check that $\det(A^T A) \geq 0$. It turns out that for any $m \times n$ matrix A (with real entries), $\det(A^T A) \geq 0$. Why doesn't the argument from part (a) apply to prove this more general fact?

Exercise 13. How are $\det(A)$ and $\det(-A)$ related?

Exercise 14. Show that if n is odd and A is an $n \times n$ matrix, then $A^2 \neq -I_n$. (Hint: determinants)

Exercise 15. Suppose that A is a square matrix and the entries of each row of A add to 0. Show that $\det(A) = 0$.

Exercise 16. Suppose that $\det(A) = 10$. Show that A^{-1} cannot have all integer entries.

Exercise 17. Suppose that B is a square matrix with $\det(B) = -2$. Find each of the following.
(a) $\det(B^4)$
(b) $\det(B^3)$
(c) $\det(BB^T)$
(d) $\det((B^{-1})^5)$

Exercise 18. If $A^T = A^{-1}$, what are the possibilities for $\det A$?

Exercise 19. Suppose that A is a 5 by 5 matrix whose determinant is 12. Give all possibilities for the RREF of A.

Exercise 20. Find the determinant of A given below. Is A invertible? Explain in one sentence.

$$A = \begin{bmatrix} 2 & 1 & -1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 1 & -1 & 2 \\ 1 & 0 & 0 & 3 \end{bmatrix}$$

4 Determinant as a Scaling Factor

In this section, you will

- Use determinants to calculate the area of a parallelogram, or the volume of a parallelepiped.

What does the determinant of a matrix tells you about the matrix?

We now discuss n-dimensional hypervolume - the generalization to higher dimensions of area in $\mathbf{R}^2$ and volume in $\mathbf{R}^3$. Actually, mathematicians just call the general n-dimensional concept volume. In 2 dimensions, two vectors determine a parallelogram, and in 3 dimensions three vectors determine a parallelepiped.

Similarly, n vectors in $\mathbf{R}^n$ are said to determine an n-dimensional parallelepiped.

> **Determinants and Volume**
>
> Let the rows of A be the direction vectors of parallelepiped P. Then the volume of P is the absolute value of the determinant of A:
> $$\text{Volume}(P) = |\det(A)|.$$

This interprets the absolute value of the determinant in terms of the volume of an object. Let $T\colon \mathbf{R}^n \to \mathbf{R}^n$ be an invertible linear map. Then $T(\mathbf{e}_1), ..., T(\mathbf{e}_n)$ are linearly independent. The parallelepiped spanned by those vectors is the image of the n-cube spanned by the standard basis vectors, and $|\det(A)|$ is the volume of the parallelepiped spanned by $T(\mathbf{e}_1), ..., T(\mathbf{e}_n)$. The n-cube spanned by $\mathbf{e}_1, ..., \mathbf{e}_n$ has hypervolume 1, since the matrix with rows/columns $\mathbf{e}_1, ..., \mathbf{e}_n$ is I_n. Therefore,

$$|\det(T(\mathbf{e}_1), ..., T(\mathbf{e}_n))|$$

can be viewed as a *volume scaling factor*. The volume of the image of the n-cube is $|\det(A)|$ times the volume of the n-cube: the map scaled the volume by a factor of $|\det(A)|$. Let R denote an n-rectangle in $\mathbf{R}^n$. Then:

$$V(T(R)) = |\det(A)|V(R).$$

This only interprets the *absolute value* of the determinant. What about if the determinant is negative or positive?...

Determinants will be used to calculate eigenvalues and eigenvectors of a square matrix, the topic of the next chapter.

Exercise 21. Find the area of the parallelogram whose vertices are listed.
(a) $(0,0),(5,2),(6,4),(11,6)$
(b) $(0,0),(-1,3),(4,-5),(3,-2)$
(c) $(-1,0),(0,5),(1,-4),(2,1)$

Exercise 22. Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices at $(1,0,-2),(1,2,4)$, and $(7,1,0)$.

Exercise 23. Suppose that

$$\begin{vmatrix} a & b & c \\ 0 & 1 & 2 \\ d & e & f \end{vmatrix} = 5, \quad \begin{vmatrix} c & b & a \\ 3 & 4 & 5 \\ f & e & d \end{vmatrix} = 4.$$

What is

$$\begin{vmatrix} a & b & c \\ d & e & f \\ 5 & 6 & 7 \end{vmatrix}?$$

Exercise 24. True or False?

(a) The area of the parallelogram with edges $\begin{pmatrix}2\\3\end{pmatrix}$ and $\begin{pmatrix}4\\5\end{pmatrix}$ is 22.

(b) For any $n \times n$ matrix A, the determinant of kA is $k \cdot \det(A)$.

(c) If A and B are 2×2 matrices, then $\det(AB) = \det(B) \cdot \det(A)$.

(d) There is a 3×3 matrix with rank 2 and determinant 1.

(e) If A and B are $n \times n$ matrices, $\det(AB) = \det(BA)$.

Exercise 25. Let

$$A = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 1 & 1 \\ a & 2 & -1 \end{bmatrix}.$$

(a) Without expanding the determinant, find an a such that $\det A = 0$. [*Hint:* Add the first two rows.]

(b) Is there more than one possibility for a? Justify your answer. (Try doing this without calculating the determinant.)

Exercise 26. Let W be the set of all vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ that make the determinant of this matrix equal to 0:

$$\begin{bmatrix} 1 & 2 & a \\ 3 & 4 & b \\ 5 & 6 & c \end{bmatrix}$$

How is W related to the first two column vectors of this matrix?

Exercise 27. Suppose that

$$\begin{vmatrix} a & b & c \\ 0 & 1 & 2 \\ d & e & f \end{vmatrix} = 4 \quad \text{and} \begin{vmatrix} a & b & c \\ 2 & 1 & 0 \\ d & e & f \end{vmatrix} = 6.$$

What is the value of $\begin{vmatrix} d & e & f \\ 2 & 2 & 2 \\ a & b & c \end{vmatrix}$? Show all work.

Exercise 28. Suppose that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are vectors in $\mathbf{R}^3$ that form a parallelepiped with volume 10 units^3. Find the volume of the parallelepiped formed by $3\mathbf{v}_1, 4\mathbf{v}_2$, and $\mathbf{v}_3 + 2\mathbf{v}_1$. Show all work.

Exercise 29. Suppose that

$$\begin{vmatrix} a & b & c \\ 0 & 1 & 2 \\ d & e & f \end{vmatrix} = 4 \quad \text{and} \quad \begin{vmatrix} a & b & c \\ 2 & 1 & 0 \\ d & e & f \end{vmatrix} = 6.$$

What is the volume of the parallelepiped formed by the vectors $\begin{bmatrix} 1 \\ a \\ d \end{bmatrix}, \begin{bmatrix} 1 \\ b \\ e \end{bmatrix}, \begin{bmatrix} 1 \\ c \\ f \end{bmatrix}$? Show all work.