Partial Fractions

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Before proving the general theorem, we give an example. Let $Q(t) = (t-1)^2(t-2)$, and let V be the real vector space of all rational functions of the form $P(t)/Q(t)$ where $P(t)$ is a polynomial of degree at most 2. Let

$$x_1(t) = 1/Q(t), \quad x_2(t) = t/Q(t), \quad x_3 = t^2/Q(t);$$

$$y_1(t) = 1/(t-1)^2, \quad y_2(t) = 1/(t-1), \quad y_3(t) = 1/(t-2).$$

Then y_1, y_2, and y_3 are elements of V. We show that they are linearly independent. Suppose that

$$c_1 y_1(t) + c_2 y_2(t) + c_3 y_3(t) = 0.$$

Multiplying both sides by $(t-1)^2$ gives

$$c_1 + c_2(t-1) + c_3(t-1)/(t-2) = 0.$$

Take the limit as $t \to 1$ of both sides to get

$$c_1 = 0.$$

Therefore,

$$c_2 y_2(t) + c_3 y_3(t) = 0.$$

Multiply both sides by $(t-1)$ and take the limit as $t \to 1$ to get $c_2 = 0$. Finally, multiply both sides of

$$c_3 y_3(t) = 0$$

by $(t-2)$ to get $c_3 = 0$.

Since x_1, x_2, and x_3 form a basis for V, V is three-dimensional. Hence y_1, y_2, y_3 form a basis also. This fact guarantees that $P(t)/Q(t)$ can be written as a sum of partial fractions in a unique way:

$$\frac{P(t)}{Q(t)} = \frac{A}{(t-1)^2} + \frac{B}{(t-1)} + \frac{C}{t-2}.$$

Theorem 1. *Let $Q(t) = (t-\gamma_1)^{n_1} \dots (t-\gamma_k)^{n_k}$, where $\gamma_1, \dots, \gamma_k$ are distinct complex numbers. Let $N = n_1 + \cdots + n_k$ and let V be the complex vector space consisting of all rational functions of the form $P(t)/Q(t)$, where P is a polynomial of degree strictly less than N with complex coefficients. Let*

$$y_{i,j}(t) = 1/(t-\gamma_i)^j \quad \text{for } i = 1, 2, ..., k; j = 1, 2, ..., n_i.$$

The $y_{i,j}$'s form a basis for V.

Proof. The functions $1/Q(t)$, $t/Q(t)$,...,$t^{N-1}/Q(t)$ form a basis for the complex vector space V. Therefore, V has dimension N. The argument as above implies that the $y_{i,j}$'s are linearly independent, and there are N elements. Therefore, the $y_{i,j}$'s form a basis for V. $\square$

It follows from the theorem that every such rational function $P(t)/Q(t)$ has a unique expression in the form

$$\frac{c_{11}}{(t-\gamma_1)}+\cdots+\frac{c_{1n_1}}{(t-\gamma_1)^{n_1}}+\frac{c_{21}}{(t-\gamma_2)}+\cdots+\frac{c_{2n_2}}{(t-\gamma_2)^{n_2}}+\cdots+\frac{c_{k1}}{(t-\gamma_k)}+\cdots+\frac{c_{kn_k}}{(t-\gamma_k)^{n_k}}.$$

Now suppose that the coefficients of $P(t)$ and $Q(t)$ are real. If γ_i is a complex root then $\overline{\gamma}_i$ is also.

Lemma 2 (Bezout's Identity)**.** *Suppose that f and g are polynomials, and let d be the gcd of f and g. There are polynomials r and s such that*

$$d = rf + sg.$$

Proof. The same proof works as in the analogous proof for integers. $\square$

Lemma 3. *Let $a(t)$ and $b(t)$ be relatively prime polynomials, and let $g(t) = a(t)b(t)$. Suppose that $f(t)$ is a polynomial with $\deg f < \deg g$. Then there are unique polynomials $r(t), s(t)$ with $\deg r < \deg a$, $\deg s < \deg b$, such that*

$$\frac{f(t)}{g(t)} = \frac{r(t)}{a(t)} + \frac{s(t)}{b(t)}.$$

Proof. We know that there are polynomials t and w such that $f = tb + wa$. Write $t = aq + r$ with $\deg r < \deg a$. Then $f = (aq + r)b + wa = rb + (qb + w)a$. Therefore, $\deg(rb) < \deg(ab) = \deg g$. Since $\deg f < \deg g$, we must also have $\deg(f - rb) < \deg g$. Therefore, letting $s = qb + w$, we have $\deg(sa) < \deg g$. Since $\deg(sa) = \deg s + \deg a$ and $\deg g = \deg b + \deg a$, we have $\deg s < \deg b$. Thus we have

$$f = rb + sa,$$

with $\deg r < \deg a$ and $\deg s < \deg b$. Now dividing both sides by $g = ab$, we get

$$\frac{f}{g} = \frac{r}{a} + \frac{s}{b}.$$

This proves existence. To prove uniqueness, suppose that

$$\frac{r_1}{a} + \frac{s_1}{b} = \frac{r}{a} + \frac{s}{b}$$

where $\deg r_1 < \deg a$ and $\deg s_1 < \deg b$. We need to show that $r = r_1$ and $s = s_1$. We have

$$r_1 b + s_1 a = rb + sa,$$

so that

$$(r_1 - r)b = (s - s_1)a.$$

The left side is a polynomial multiple of b, and since a and b are relatively prime, b divides $s - s_1$. But $\deg s - s_1 < \deg b$, and so $s - s_1 = 0$. Thus $(r_1 - r)b = 0$, and so $r_1 = r$. This proves the uniqueness. $\square$

By the lemma, we may write

$$\frac{f}{g} = \frac{r}{p_1^{e_1} \dots p_k^{e_k}} + \frac{s}{p_{k+1}^{e_{k+1}}}.$$

By induction, there are unique polynomials $h_1, ..., h_k$ such that

$$\frac{r}{p_1^{e_1} \dots p_k^{e_k}} = \frac{h_1}{p_1^{e_1}} + \dots + \frac{h_k}{p_k^{e_k}}.$$

Then we set $h_r = s$.

We apply this to the case of $\mathbb{R}$. Every polynomial with real coefficients factors into a product of linear and irreducible quadratic polynomials.

Suppose we have an expression of the form

$$\frac{f(t)}{(t^2 + bt + c)^e}$$

where $\deg f < 2e$. Let $p(t) = t^2 + bt + c$. We can write

$$f(t) = p(t)q_0(t) + r_0(t)$$

with $\deg r_0 < \deg p$; then divide p into the quotients, successively. We get

$$q_0 = pq_1 + r_1$$

with $\deg r_1 < \deg p$. Then

$$f = p(pq_1 + r_1) + r_0 = r_0 + r_1 p + q_1 p^2.$$

Since the process eventually stops, we can write

$$f = r_0 + r_1 p + \dots + r_k p^k$$

for some k. Since the quotient and remainder in the division algorithm are unique, this representation is unique. This leads us to write

$$\frac{f}{p^e} = \frac{r_k}{p^{e-k}} + \dots + \frac{r_0}{p^e}$$

Since $\deg r_i < 2$, we can write

$$\frac{f}{(t^2 + bt + c)^e} = \frac{A_1 t + B_1}{t^2 + bt + c} + \dots + \frac{A_e t + B_e}{(t^2 + bt + c)^e}.$$