Vectors and Analytic Geometry

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Contents

1 Introduction

This chapter introduces vectors and three-dimensional coordinate systems for space. Just as the coordinate plane is the natural place to study functions of a single variable, coordinate space is the place to study functions of two variables. We establish coordinates in space by adding an axis that measures distance above and below the xy-plane. This builds on what we already know without forcing us to start over again.

2 Vectors in $\mathbf{R}^n$

In this section you will

- Add, subtract, and scale vectors in $\mathbf{R}^n$.
- Calculate lengths of vectors in $\mathbf{R}^n$.
- Calculate the angle between vectors.
- Identify if vectors are orthogonal.
- Identify if vectors are parallel.
- Find parametric equations of lines using vector arithmetic.
- Find equations of planes using vector arithmetic.
- Determine if two planes are parallel.
- Find the line of intersection of two intersecting planes.

2.1 Vectors in $\mathbf{R}^2$

An ordered pair of real numbers is a pair of real numbers with a definite choice of which one comes first and which one comes second. A **2-vector** is defined to be an ordered pair of real numbers. We will also simply call 2-vectors "vectors." We typically write out a vector horizontally with angled brackets, like this:

$$\langle 2, 3\rangle$$

or vertically with rectangular brackets, like this:

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

So

$$\langle 5, 2\rangle = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

When we discuss matrices we'll use the vertical notation a lot more, but we'll often use the horizontal notation because the vertical notation takes up a lot of space.

The **first component** of a 2-vector is the first number in the ordered pair, and the **second component** is the second number in the ordered pair. In the case of $\langle 5, 2\rangle$, the first component is 5 and the second component is 2.

The set of all 2-vectors is called $\mathbf{R}^2$. So an element of the set $\mathbf{R}^2$ is a 2-vector.

Two vectors in $\mathbf{R}^2$ are **equal** if their corresponding components are equal. Thus $\langle 4, 7\rangle$ and $\langle 7, 4\rangle$ are not equal, because the first components of these vectors do not match. The only vector that is equal to $\langle 4, 7\rangle$ is $\langle 4, 7\rangle$.

Given two vectors $\mathbf{u} = \langle a_1, a_2\rangle$ and $\mathbf{v} = \langle b_1, b_2\rangle$ in $\mathbf{R}^2$, the **sum** of $\mathbf{u}$ and $\mathbf{v}$ is defined to be the 2-vector $\langle a_1 + b_1, a_2 + b_2\rangle$. The sum of $\mathbf{u}$ and $\mathbf{v}$ is, naturally, written as $\mathbf{u} + \mathbf{v}$. Notice that $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$. This property comes from the commutativity of addition of real numbers.

The vector $\langle 0, 0\rangle$ is important – it is called the **zero vector**. It is often written as $\mathbf{0}$. It is the only vector with the property that, when added to any vector $\mathbf{u}$, the sum is $\mathbf{u}$. In other words,

$$\mathbf{0} + \mathbf{u} = \mathbf{u},$$

for any $\mathbf{u} \in \mathbf{R}^2$.

Given a vector $\mathbf{a} = \langle a_1, a_2 \rangle$ and a scalar c, we define the **scalar multiple** $c\mathbf{a}$ to be the 2-vector $\langle ca_1, ca_2 \rangle$.

On the xy-plane, each point is an ordered pair of numbers. We write a typical point as (a, b). The vector $\langle a, b \rangle$ is also an ordered pair, so strictly speaking there is no difference between 2-vectors and points on the xy-plane. So we may regard $\mathbf{R}^2$ as the set of all points in the plane. However, we make a mental distinction between vectors and points based on what we do with them. We add two vectors to get another vector, but we don't think of adding two points on the plane to get another point on the plane. We scale a vector by a certain factor, but we don't think of doing the same thing to points.

We can visualize a vector $\langle a, b \rangle$ as a directed line segment (an arrow) from the origin $(0, 0)$ to the point (a, b).

The sum of two vectors has a useful geometric interpretation. If $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{R}^2$ are represented as points in the plane, then $\mathbf{u}+\mathbf{v}$ corresponds to the fourth vertex of the parallelogram whose other vertices are $\mathbf{u}$, $\mathbf{0}$, and $\mathbf{v}$.

2.2 $\mathbf{R}^3$

Vectors in $\mathbf{R}^3$ are lists of 3 numbers. They are represented geometrically by points in a three-dimensional coordinate space, with arrows from the origin sometimes included for visual clarity.

The spaces $\mathbf{R}^1, \mathbf{R}^2$, and $\mathbf{R}^3$ have simple geometric interpretations which will help us to understand them. To picture $\mathbf{R}^3$, for instance, we represent the vector $\langle a_1, a_2, a_3 \rangle$ by the directed line segment in space having its initial point at the origin and its end point at the point with coordinates (a_1, a_2, a_3). Hence our geometric interpretation of the 3-tuple $\langle a_1, a_2, a_3 \rangle$ is the directed line segment which is commonly denoted by $a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ in physics.

Notice that we define $\mathbf{R}^2$ to consist of vectors that have two components and $\mathbf{R}^3$ to consist of vectors with three components. They are different "spaces": $\mathbf{R}^2$ is not a subset of $\mathbf{R}^3$. The xy-plane in $\mathbf{R}^3$ is not considered the same as $\mathbf{R}^2$. The xy-plane in $\mathbf{R}^3$ consists of points $(x, y, 0)$.

It is usual in physics and in vector calculus to allow directed line segments to lie anywhere in space. This we shall not do; instead we shall consider only those directed line segments whose initial points lie at the origin. This will be more convenient for our purpose, which is to use directed line segments as a means of picturing $\mathbf{R}^3$.

Also, note that we are *not* thinking of a vector as "something with magnitude and direction."

We make a mental distinction between the triple $\langle a_1, a_2, a_3 \rangle$ which we call a *vector*, and the triple (a_1, a_2, a_3) which represents a *point*. The reason is that we *do* different things with them. We add two triples of the first kind, for instance, but we certainly do not add two points together. On the other hand, we speak of the distance between two points but not of the distance between two vectors.

2.3 Geometric Addition - The Parallelogram Law

If we view vectors in $\mathbf{R}^2$ and $\mathbf{R}^3$ geometrically, two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ can be added geometrically by drawing $\mathbf{v}_1$ from O to, say P, and then a translation of $\mathbf{v}_2$ starting from the terminal point P of $\mathbf{v}_1$. The sum $\mathbf{v}_1 + \mathbf{v}_2$ is then the vector from O to the terminal point C of $\mathbf{v}_2$. This description of addition is sometimes called the **Parallelogram Law** of addition because $\mathbf{v}_1 + \mathbf{v}_2$ is given by the diagonal of the parallelogram determined by $\mathbf{v}_1$ and $\mathbf{v}_2$.

2.4 $\mathbf{R}^n$

Let $\mathbf{R}$ denote the set of real numbers. We are ready for our first definition:

Definition. Let n be a positive integer. An **n-vector of real numbers** is a list of n real numbers.

There are different notations for n-vectors. For example, n-vectors can be written as enclosed by brackets, as in $\langle 3,4,9\rangle$. Sometimes the numbers are enclosed by parentheses, like this: $(3,4,9)$. However, in this book, we usually write a vector as a column enclosed by rectangular brackets, like this:

$$\begin{bmatrix} 3 \\ 4 \\ 9 \end{bmatrix}$$

You will see why when we learn about matrices later.

Throughout this book, we will use this convention. If we want to write the vector horizontally, we will use the $\langle,\rangle$ brackets, and if we want to write the same vector vertically, we will use rectangular brackets. So

$$\langle 2,5,6\rangle = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$$

For each positive integer n, let $\mathbf{R}^n$ (pronounced "r n") be the set of all n-vectors $\langle a_1, ..., a_n\rangle$ of real numbers. The elements of $\mathbf{R}^n$ are simply called **vectors**. We often denote a vector $\langle a_1, ..., a_n\rangle$ by the symbol $\mathbf{v}$; the numbers $a_1, ..., a_n$ are called the **components** of $\mathbf{v}$.

If $n \neq m$, then the vectors in $\mathbf{R}^n$ don't "interact" with the vectors in $\mathbf{R}^m$ in any way. You should think of each $\mathbf{R}^n$ as a separate "space."

$$\mathbf{R}^1, \quad \mathbf{R}^2, \quad \mathbf{R}^3, \quad \mathbf{R}^4, \quad ...$$

2.5 Lists versus Sets

Sets and lists are different. A set is just a collection of objects and the order in which the objects are written doesn't matter. For example,

$$\{A,B,C\} \quad \text{and} \quad \{B,C,A\}$$

are the same set. But in a list, the order in which the elements appear matters. The lists

$$A,B,C \quad \text{and} \quad B,C,A$$

are different, because in the first list, A comes first but in the second list B comes first. The list

$$\text{Fall, Spring, Summer, Winter}$$

lists the seasons in alphabetical order. The list

$$\text{Spring, Summer, Fall, Winter}$$

lists the seasons in chronological order starting from Spring. They are not the same list! Since vectors are ordered,

$$\langle 2,4,5\rangle \quad \text{and} \quad \langle 5,2,4\rangle$$

are different vectors, in the same way that $(3,5)$ and $(5,3)$ are different points on the xy-plane.

Given any two n-vectors

$$\mathbf{a} = \langle a_1, ..., a_n\rangle, \mathbf{b} = \langle b_1, ..., b_n\rangle,$$

we define $\mathbf{a}+\mathbf{b}$ to be the vector defined by the equation

$$\mathbf{a}+\mathbf{b}=\langle a_1+b_1,...,a_n+b_n\rangle.$$

This vector is called the **sum** of $\mathbf{a}$ and $\mathbf{b}$. Given a scalar c (a real number), we define $c\cdot\mathbf{a}$, or simply $c\mathbf{a}$, to be the vector

$$c\mathbf{a}=\langle ca_1,...,ca_n\rangle.$$

The function

$$+\colon \mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}^n$$

defined by $(\mathbf{a},\mathbf{b})\mapsto\mathbf{a}+\mathbf{b}$ is called **addition of vectors**. The function

$$\cdot\colon \mathbf{R}\times\mathbf{R}^n\to\mathbf{R}^n$$

defined by $(c,\mathbf{a})\mapsto c\cdot\mathbf{a}$ is called **scalar multiplication**. We define

$$\mathbf{a}-\mathbf{b}=\langle a_1-b_1,...,a_n-b_n\rangle.$$

Note that the addition symbol $+$ is now being used in two different ways. We are used to using the $+$ symbol for real numbers, and now we are using $+$ for addition of vectors. This shouldn't cause any confusion, but it is always a good idea to ask yourself, "Are vectors being added, or numbers?" Later we will learn about matrices and how to add them, and we will use the $+$ symbol for that, too.

Addition of vectors and scalar multiplication of a vector by a scalar have the following properties.

1. $\mathbf{a}+(\mathbf{b}+\mathbf{c})=(\mathbf{a}+\mathbf{b})+\mathbf{c}$ (associativity).
2. $\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$ (commutativity)
3. There is a unique vector $\mathbf{0}$ such that $\mathbf{a}+\mathbf{0}=\mathbf{a}$ for every $\mathbf{a}$.
4. $\mathbf{a}+(-1)\mathbf{a}=\mathbf{0}$.
5. $c(d\mathbf{a})=(cd)\mathbf{a}$ (associativity).
6. $(c+d)\mathbf{a}=c\mathbf{a}+d\mathbf{a}$ (distributivity).
7. $c(\mathbf{a}+\mathbf{b})=c\mathbf{a}+c\mathbf{b}$ (distributivity).
8. $1\mathbf{a}=\mathbf{a}$.

Verification of the properties is routine. For instance, to check (1), one first applies the definition of vector addition to compute

$$\begin{aligned}\mathbf{a}+(\mathbf{b}+\mathbf{c})&=\langle a_1+(b_1+c_1),...,a_n+(b_n+c_n)\rangle,\\(\mathbf{a}+\mathbf{b})+\mathbf{c}&=\langle(a_1+b_1)+c_1,...,(a_n+b_n)+c_n\rangle.\end{aligned}$$

Then one notes that $a_i+(b_i+c_i)=(a_i+b_i)+c_i$ for $i=1,...,n$; this is one of the familiar properties of the addition operation for real numbers. It shows that the above two n-tuples have all their components equal, and hence are equal.

3 Definition of a Vector Space

The first eight properties discussed above, which involve only addition of vectors and scalar multiplication, are called the *vector space properties*; *any* set of objects having two operations which possess these properties is called a **vector space**. (More precisely, it is called a vector space over the real numbers, since we take the scalars to be real numbers. One could instead take the scalars to be complex numbers, but this we will not do.)

A **vector space** is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition* and *multiplication by scalars*, subject to the ten axioms (or rules) listed below. The axioms must hold for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d.

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u}+\mathbf{v}$, is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
4. There is a vector $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$.

4 Dot Product

4.1 Dot product in $\mathbf{R}^2$

We have discussed vector addition. This is a way of taking two vectors and combining them to produce another vector. In this section, we will discuss a way of taking two vectors and producing a scalar. The **dot product** of two vectors $\mathbf{a}=\langle a_1,a_2\rangle$ and $\mathbf{b}=\langle b_1,b_2\rangle$ is defined to be

$$\mathbf{a}\cdot\mathbf{b}=a_1b_1+a_2b_2.$$

Let θ be the (smaller) angle formed by $\mathbf{a}$ and $\mathbf{b}$. Then

$$\mathbf{a}\cdot\mathbf{b}=||\mathbf{a}||||\mathbf{b}||\cos(\theta).$$

If $||\mathbf{a}||$ and $||\mathbf{b}||$ are non-zero, then we can divide both sides of the equation above to get

$$\cos\theta=\frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{a}||||\mathbf{b}||}.$$

An important case is when two vectors make a 90° angle. Since $\cos(90^\circ)=0$, this happens precisely when $\mathbf{a}\cdot\mathbf{b}=0$.

We say that $\mathbf{a}$ and $\mathbf{b}$ are **orthogonal** if $\mathbf{a}\cdot\mathbf{b}=0$. In this case, the two vectors form a 90 degree angle.

Theorem 1. *Suppose that* $\mathbf{u}$ *and* $\mathbf{v}$ *are two vectors in* $\boldsymbol{R}^2$ *and that* $\mathbf{v} \neq \mathbf{0}$*. Then there is a vector* $\boldsymbol{w}_1$ *parallel to* $\mathbf{v}$ *and a vector* $\boldsymbol{w}_2$ *orthogonal to* $\mathbf{v}$ *such that* $\boldsymbol{u} = \boldsymbol{w}_1 + \boldsymbol{w}_2$*.*

The length, or **magnitude**, of a vector $\langle a_1, a_2\rangle$ is defined to be the length of a line segment from $(0,0)$ to (a_1, a_2). To find this length, notice that it is the hypotenuse of a right triangle with vertices $(0,0)$, $(a_1, 0)$, and (a_1, a_2). The base of the triangle is $|a_1|$ and the height is $|a_2|$, so the hypotenuse is $\sqrt{|a_1|^2 + |a_2|^2} = \sqrt{a_1^2 + a_2^2}$.

The length of a vector $\mathbf{a}$ is related to the dot product of $\mathbf{a}$ with itself:

$$\mathbf{a} \cdot \mathbf{a} = \langle a_1, a_2\rangle \cdot \langle a_1, a_2\rangle = a_1^2 + a_2^2,$$

so

$$||\mathbf{a}|| = \sqrt{\mathbf{a} \cdot \mathbf{a}}.$$

For example, the length of the vector $\langle 1, 1\rangle$ is $\sqrt{1^2 + 1^2} = \sqrt{2}$.

A vector is called a **unit vector** if its length is 1. The vector $\langle 1, 1\rangle$ is not a unit vector because its length is not 1. The vector $\langle 3/5, 4/5\rangle$ has length

$$\sqrt{(3/5)^2 + (4/5)^2} = \sqrt{16/25 + 9/25} = \sqrt{1} = 1,$$

so it is a unit vector.

The length of a vector can be modified by multiplying the vector by a scalar. The **scalar multiple** of a vector $\mathbf{u}$ by a scalar c is the vector $c\mathbf{u}$ obtained by multiplying each entry by c. For example, if $\mathbf{u} = \langle 3, 4\rangle$, then $2\mathbf{u} = \langle 2(3), 2(4)\rangle = \langle 6, 8\rangle$ and $-3\mathbf{u} = \langle -9, -12\rangle$. The length of $\mathbf{u}$ is $\sqrt{3^2 + 4^2} = 5$. What about the length of $2\mathbf{u}$? This length is equal to $\sqrt{6^2 + 8^2} = 10$, which is double the length of $\mathbf{u}$. The vector $-3\mathbf{u}$ has length $\sqrt{(-9)^2 + (-12)^2} = \sqrt{81 + 144} = \sqrt{225} = 15$, which is triple the length of $\mathbf{u}$.

In general, if $\mathbf{u} = \langle u_1, u_2\rangle$, then $c\mathbf{u} = \langle cu_1, cu_2\rangle$, and this has length

$$\sqrt{(cu_1)^2 + (cu_2)^2} = \sqrt{c^2u_1^2 + c_2u_2^2} = \sqrt{c^2(u_1^2 + u_2^2)} = \sqrt{c^2}\sqrt{u_1^2 + u_2^2} = |c|||\mathbf{u}||.$$

So multiplying $\mathbf{u}$ by the scalar c changes the length of $\mathbf{u}$ by a factor of $|c|$. If $c > 0$, then $|c| = c$, of course.

How are $\mathbf{u}$ and $-\mathbf{u}$ related? The vector $-\mathbf{u}$ is obtained by multiplying each component of $\mathbf{u}$ by -1; this has the effect of reversing the direction of $\mathbf{u}$ while maintaining its length.

Multiplying a vector $\mathbf{u}$ by $-c$ (where c is positive) can be thought of as multiplying $\mathbf{u}$ by -1 first and then by c. So $-c\mathbf{u}$ is the result of reversing the direction of $\mathbf{u}$ to get $-\mathbf{u}$ and then stretching $-\mathbf{u}$ by a factor of c.

4.2 The Dot Product in $\mathbf{R}^3$

Exercise 1. Find the length of $\langle 1, -2, 3\rangle$.

Exercise 2. Find the distance between $P_1(2, 1, 5)$ and $P_2(-2, 3, 0)$.

Exercise 3. Find a unit vector $\mathbf{u}$ in the direction of the vector from $P_1(1, 0, 1)$ to $P(2(3, 2, 0)$.

Exercise 4. Find a unit vector 6 units long in the direction of $2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

4.3 Dot Product in $\mathbf{R}^n$

Let n be a positive integer. We define the **dot product** of two n-vectors $\mathbf{a} = \langle a_1, ..., a_n\rangle$ and $\mathbf{b} = \langle b_1, ..., b_n\rangle$ to be the scalar

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + \cdots + a_nb_n.$$

Sometimes in physics this is called the *scalar product*, but we will use the term "dot product."

The dot product has the following properties:

1. $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$.
2. $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = (\mathbf{a}\cdot\mathbf{b}) + (\mathbf{a}\cdot\mathbf{c})$.
3. $(c\mathbf{a})\cdot\mathbf{b} = c(\mathbf{a}\cdot\mathbf{b})$.
4. $\mathbf{a}\cdot\mathbf{a} > 0$ unless $\mathbf{a} = \mathbf{0}$.

The first three can be checked easily and follow from the distributive property and commutativity of multiplication. The last one follows from the fact that if $\mathbf{a} = \langle a_1, ..., a_n\rangle$, then

$$\mathbf{a}\cdot\mathbf{a} = a_1^2 + \cdots + a_n^2,$$

and since $a_i^2 \geq 0$ for all i, the only way that $a_1^2+\cdots+a_n^2$ can equal 0 if when $a_1 = 0$, $a_2 = 0$,...,$a_n = 0$.

The dot product can be used to define the **length** of a vector $\mathbf{v}$:

$$|\mathbf{v}| = \sqrt{\mathbf{v}\cdot\mathbf{v}}.$$

In the case when $\mathbf{v} = \langle a_1, a_2\rangle \in \mathbf{R}^2$ or $\mathbf{v} = \langle a_1, a_2, a_3\rangle \in \mathbf{R}^3$, we have

$$|\mathbf{v}| = \sqrt{a_1^2 + a_2^2} \quad \text{or} \quad |\mathbf{v}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

which coincides with the length of the line segment from the origin to the point (a_1, a_2) (respectively, (a_1, a_2, a_3)).

The dot product can also be used to define the angle between two nonzero vectors. The **angle** between two nonzero vectors $\mathbf{v}$ and $\mathbf{w}$ is

$$\theta = \cos^{-1}\left(\frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{v}||\mathbf{w}|}\right).$$

Actually, in order for this to make sense, we need to know that $\dfrac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{v}||\mathbf{w}|}$ is between -1 and 1. This turns out to be true – see the section on orthogonality.

This also generalizes the case of $\mathbf{R}^2$ and $\mathbf{R}^3$. For the case of $\mathbf{R}^3$, let $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$. Then $\mathbf{b} - \mathbf{a} = \langle b_1 - a_1, b_2 - a_2, b_3 - a_3\rangle$ and the length of this vector is the length of the line segment opposite the angle θ formed by $\mathbf{a}$ and $\mathbf{b}$. We apply the law of cosines to the triangle with vertices $(0, 0, 0)$, (a_1, a_2, a_3), and (b_1, b_2, b_3) and obtain

$$|\mathbf{b}-\mathbf{a}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta,$$

from which we get

$$|\mathbf{a}||\mathbf{b}|\cos\theta = \frac{|\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{b}-\mathbf{a}|^2}{2}.$$

The right side simplifies to

$$a_1b_1 + a_2b_2 + a_3b_3,$$

which is $\mathbf{a}\cdot\mathbf{b}$. So in the case of $\mathbf{R}^3$ (the case of $\mathbf{R}^2$ is similar), $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$, where θ measures the angle ($0 \le \theta \le \pi$) made by vectors $\mathbf{a}$ and $\mathbf{b}$.

Two vectors whose dot product is zero are said to be **orthogonal**. The zero vector $\mathbf{0}$ is orthogonal to every vector because its dot product with every vector is zero. When neither $|\mathbf{a}|$ nor $|\mathbf{b}|$ is zero, the equation $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$ tells us that $\mathbf{a}\cdot\mathbf{b} = 0$ if and only if $\cos\theta$ is zero, that is, when θ equals $\pi/2$. So two non-zero vectors in $\mathbf{R}^2$ or $\mathbf{R}^3$ are orthogonal if and only if they are perpendicular. In some scientific context in which the word "orthogonal" is used, there is no geometric interpretation.

Exercise 5. Find the distance between $\mathbf{x} = \begin{bmatrix} 3 \\ 10 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$.

Exercise 6. Let $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$. Describe the set H of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ that are orthogonal to $\mathbf{v}$. [*Hint*: Consider $\mathbf{v} = \mathbf{0}$ and $\mathbf{v} \neq \mathbf{0}$.]

Exercise 7. Show that if $\mathbf{v}_1, \mathbf{v}_2$ are orthogonal, then so are $c_1\mathbf{v}_1$ and $c_2\mathbf{v}_2$ for any scalars c_1, c_2.

5 Cross Product

5.1 Determinants of 2 by 2 and 3 by 3 Matrices

A 2 by 2 matrix is a 2 by 2 array of numbers. The **determinant** of the general 2 by 2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is defined to be $ad - bc$.

The determinant of the general 3 by 3 matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

is defined to be

$$a_{11}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}.$$

5.2 The Cross Product

Given two vectors $\mathbf{u}, \mathbf{v} \in \mathbf{R}^3$, the **cross product** of $\mathbf{u}$ and $\mathbf{v}$, written $\mathbf{u}\times\mathbf{v}$, is another vector in $\mathbf{R}^3$. It is defined to be the vector

$$\mathbf{u}\times\mathbf{v} = \mathbf{i}\det\begin{bmatrix} u_2 & u_3 \\ v_2 & v_3 \end{bmatrix} - \mathbf{j}\det\begin{bmatrix} u_1 & u_3 \\ v_1 & v_3 \end{bmatrix} + \mathbf{k}\det\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}$$

The cross product $\mathbf{u}\times\mathbf{v}$ has the property that it is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

Let $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$. The **cross product** of $\mathbf{a}$ and $\mathbf{b}$ is defined to be the vector...

A useful way to remember this is the "symbolic determinant"

...

(This isn't a determinant since the top entries are not numbers; it is just a way to remember it.)

The cross product has the property that it is orthogonal to both **a** and **b**. We check this by taking the dot product of the cross product with **a** and with **b**:

...

Exercise 8. (a) Find a vector perpendicular to the plane of $P(1,-1,0)$, $Q(2,1,-1)$, and $R(-1,1,2)$.

(b) Find the area of the triangle with vertices $P(1,-1,0)$, $Q(2,1,-1)$, and $R(-1,1,2)$.

(c) Find a *unit* vector perpendicular to the plane of $P(1,-1,0)$, $Q(2,1,-1)$, and $R(-1,1,2)$.

6 Lines

In $\mathbf{R}^2$, lines can be described as the set of points that satisfy an equation of the form $ax+by=c$. As long as $b \neq 0$, this equation can be rewritten as $y = -a/bx + c/a$. Therefore, as long as $b \neq 0$, the line $ax+by=c$ has a slope of $-a/b$.

Lines can also be described by vectors. Let $\mathbf{v} \in \mathbf{R}^2$. The set of all scalar multiples of $\mathbf{v}$ is a line through the origin, provided that $\mathbf{v} \neq \mathbf{0}$. The set of all scalar multiples of $\mathbf{v}$ is $\{t\mathbf{v} : t \in \mathbf{R}\}$.

We can also describe lines that do not go through the origin. Let $\mathbf{p} \in \mathbf{R}^2$. The set $\{\mathbf{p}+t\mathbf{v} : t \in \mathbf{R}\}$ is a line that goes through the point corresponding to $\mathbf{p}$ and is parallel to the line that contains $\mathbf{v}$ that goes through the origin.

This way of describing lines by vectors carries over to $\mathbf{R}^n$. For lines in $\mathbf{R}^3$, there is no notion of slope. Suppose we are given a point $P = (x_0, y_0, z_0)$ and $\mathbf{p} = \langle x_0, y_0, z_0 \rangle$ and a vector $\mathbf{v} = \langle a, b, c \rangle$. Consider the set of points of the form $\mathbf{p} + t\mathbf{v}$. The points trace out a shape that resembles what we typically think of as a line. We use this to give a precise definition of a line in $\mathbf{R}^n$.

Definition. Let $\mathbf{p} \in \mathbf{R}^n$ and let $\mathbf{v} \in \mathbb{R}^n$. The **line** through $\mathbf{p}$ and with direction vector $\mathbf{v}$ is the set of all points of the form $\mathbf{p} + t\mathbf{v}$, for $t \in \mathbf{R}$.

7 Planes in $\mathbf{R}^3$

Let $\langle a, b, c \rangle$ be a non-zero vector. The collection of all vectors orthogonal to this vector is a plane through the origin. Suppose that $P = (x, y, z)$ is a point in $\mathbf{R}^3$. Then the vector $\overrightarrow{OP} = \langle x, y, z \rangle$ is orthogonal to $\langle a, b, c \rangle$ if and only if

$$ax + by + cz = 0.$$

Therefore the plane that passes the origin and is orthogonal to $\langle a, b, c \rangle$ can be described as

$$\{(x, y, z) : ax + by + cz = 0\}.$$

Definition. A subset $\Pi \subset \mathbf{R}^3$ is called a **plane** in $\mathbf{R}^3$ if there exist real numbers a, b, c, and d such that Π is the set of points (x, y, z) that satisfy the equation

$$ax + by + cz = d.$$

Exercise 9. Sketch the graph of $\mathbf{r}(t) = \langle 1+t, 2-t \rangle = \langle 1, 2 \rangle + t\langle 1, -1 \rangle$.

Exercise 10. Give two non-zero and non-parallel vectors that are both orthogonal to the vector $\langle 2, 4, -2 \rangle$.

Exercise 11. Find a parametric equation for the line that is parallel to $\mathbf{r}(t) = \langle 3-t, -2+2t, -1+4t \rangle$ that passes through the point $(1, 2, 3)$.

Exercise 12. Find the point in which the line

$$x = \frac{8}{3} + 2t, \quad y = -2t, \quad z = 1 + t$$

meets the plane $3x + 2y + 6z = 6$.

Exercise 13. Find a vector parallel to the line of intersection of the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

Exercise 14. Find parametric equations for the line in which the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$ intersect.