The Spectral Theorem

In this note all matrices have real entries. This note proves a theorem called the spectral theorem. There is a version for complex numbers, but the proof is essentially the same as the proof we give below. We say that a matrix is *orthogonally diagonalizable* if it is diagonalizable and the matrix whose columns are eigenvectors can be chosen to be orthogonal. In other words: a matrix A is orthogonally diagonalizable if there exist an orthogonal matrix Q and a diagonal matrix Λ such that $A = Q\Lambda Q^{-1}$.

Theorem 1. Every orthogonally diagonalizable matrix with real entries is symmetric.

This follows from the following computation:

$$(Q\Lambda Q^{-1})^T = (Q^{-1})^T \Lambda^T Q^T = Q\Lambda Q^{-1}.$$

The spectral theorem asserts that the converse is true:

Theorem 2 (Spectral Theorem)**.** Every real symmetric matrix is orthgoonally diagonalizable.

We will prove this theorem in this note. This proof appears in many books, but, annoyingly, many elementary linear algebra books state the theorem but do not prove it. First, we recall the concept of the orthogonal complement of a subspace. Suppose that W is a subspace of $\mathbf{R}^n$. Then the *orthogonal complement* of W, denoted by $W^\perp$, consists of the collection of vectors that are orthogonal to every vector in W. We have

$$\mathbf{R}^n = W \oplus W^\perp.$$

Now, suppose that T is a linear transformation $\mathbf{R}^n \to \mathbf{R}^n$. The eigenvalues of T are real (I'm assuming that this fact is known.) Let λ_1 be a real eigenvalue and let $\mathbf{v}_1$ be an eigenvector of T with eigenvalue λ. Let W be the real span of $\mathbf{v}_1$. Then

$$\mathbf{R}^n = W \oplus W^\perp.$$

The key fact we will prove is that if $W^\perp$ is T-invariant when T is a *symmetric* linear map, that is, if its standard matrix is symmetric.

Theorem 3. Suppose that $\mathbf{v}_1$ is an eigenvector of a symmetric linear map T, with eigenvalue λ. Let W be the span of $\mathbf{v}_1$. Then $W^\perp$ is T-invariant.

Proof. Let $\mathbf{x} \in W^\perp$. We want to show that $T(\mathbf{x}) \in W^\perp$. So, let $\mathbf{w} \in W$. We want to show that $\mathbf{w} \cdot T(\mathbf{x}) = 0$. Let A be the standard matrix of T. We have

$$\mathbf{w} \cdot T(\mathbf{x}) = \mathbf{w} \cdot A\mathbf{x} = \mathbf{w}^T A\mathbf{x}.$$

Since A is symmetric,

$$\mathbf{w}^T A\mathbf{x} = \mathbf{w}^T A^T\mathbf{x} = (A\mathbf{w})^T\mathbf{x} = A\mathbf{w} \cdot \mathbf{x}.$$

Now, $A\mathbf{w} = \lambda\mathbf{w}$, so

$$(A\mathbf{w}) \cdot \mathbf{x} = (\lambda\mathbf{w}) \cdot \mathbf{x} = \lambda\mathbf{w} \cdot \mathbf{x} = 0,$$

the last equality holding because $\mathbf{x} \in W^\perp$. This completes the proof. $\square$

Theorem 4. If $T\colon \mathbf{R}^n \to \mathbf{R}^n$ is a symmetric linear map, then T has an orthonormal basis of eigenvectors.

Proof. We proceed by induction on n. The case $n = 1$ is clear. Now suppose that the result holds for $n - 1$. We take an eigenvector $\mathbf{v}_1$ of T. Let W be the span of $\mathbf{v}_1$. Then $W^\perp$ has dimension $n - 1$, and by induction, there is an orthonormal basis $\{\mathbf{v}_2, ..., \mathbf{v}_n\}$ of $W^\perp$. Then $\mathbf{v}_1$ is orthogonal to each of $\mathbf{v}_2, ..., \mathbf{v}_n$, and so the set $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ is an orthonormal basis for $\mathbf{R}^n$ consisting of eigenvectors of T. $\square$

The spectral theorem we have given above is the "matrix version" of the result above. (It is a repeat of Theorem 2.)

Theorem 5 (Spectral Theorem)**.** Let A be a symmetric matrix. Then there exists an orthogonal matrix Q and a diagonal matrix Λ such that $A = Q\Lambda Q^{-1}$.

Proof. Let T_A be the matrix transformation defined by $T_A(\mathbf{x}) = A\mathbf{x}$. By the result above, there is an orthonormal basis $\mathcal{B} = \{\mathbf{v}_1, ..., \mathbf{v}_n\}$ for $\mathbf{R}^n$ consisting of eigenvectors of A. Let Λ be the matrix of T_A relative to $\mathcal{B}$. Then Λ is a diagonal matrix. Let Q be the matrix whose jth column is $\mathbf{v}_j$. Then Q is an orthogonal matrix and $A = Q\Lambda Q^{-1}$. $\square$

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