Subspaces

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In linear algebra, we are often interested not just in $\mathbf{R}^n$ but in particular subsets of $\mathbf{R}^n$. However, some subsets are of more interest than others. In particular, we are often interested in non-empty subsets W of $\mathbf{R}^n$ that satisfy these two conditions:

- Whenever two vectors in the set W are added, the sum remains in the set W, and
- Whenever a vector in W is scaled, the resulting vector also remains in the set W.

Subsets that satisfy these two conditions are called *subspaces*. They are the topic of this chapter. After discussing the general concept of a subspace, we discuss subspaces associated with a matrix: its nullspace, its column space, and its row space. We also define the dimension of a subspace and discuss linear maps from one subspace to another. This provides us with a more flexible way of understanding linear maps.

This section focuses on important sets of vectors in $\mathbf{R}^n$ called subspaces. Often subspaces arise in connection with some matrix A, and they provide useful information about the equation $A\mathbf{x} = \mathbf{b}$.

Subspaces of $\mathbf{R}^n$ are examples of vector spaces. This chapter can really be thought of as an introduction to vector spaces, though we don't use that terminology. It provides useful intuition for when you learn about more general subspaces.

1 Definition

The goal of this section is to understand what a subspace is, and how to tell if a subset is a subspace or not. In this section you will:

- Determine whether or not a given subset is a subspace of $\mathbf{R}^n$

What does a subset have to do in order to qualify as a subspace? It has to have two properties. It needs to be closed under addition, and it needs to be closed under scalar multiplication. We will explain what that means.

A subset S of $\mathbf{R}^n$ is said to be **closed under addition** if *for all* vectors $\mathbf{v}$ and $\mathbf{w}$ in S, the sum $\mathbf{v} + \mathbf{w}$ is also an element of S. Thus, a subset S of $\mathbf{R}^n$ is *not* closed under addition if *there exist* two vectors $\mathbf{v}$ and $\mathbf{w}$ in S such that the sum, $\mathbf{v} + \mathbf{w}$, is *not* an element of S.

Example 1. Let's look at $\mathbf{R}^1 = \mathbf{R}$. Let $S = \{2, 4, 6, 8, ...\}$. Is S closed under addition? The answer is yes: the sum of any two positive even integers is also a positive even integer, so the sum of any two elements of S is also in S.

Example 2. Again, take $\mathbf{R}$ and consider $S_1 = \{2, 4, 6, 8\}$. Is S_1 closed under addition? The answer is no: for example, 4 and 6 are in S, but the sum, 10, is not in S. In order for a set to contain 4 and 6 and also be closed under addition, it must contain 10.

A subset S of $\mathbf{R}^n$ is said to be **closed under scalar multiplication** if *for every* vector $\mathbf{v}$ in S and *for all* scalars $c \in \mathbf{R}$, the vector $c\mathbf{v}$ is also an element of S. Thus, a subset S of $\mathbf{R}^n$ is *not* closed under scalar multiplication if *there exist* a vector $\mathbf{v}$ in S and a scalar c such that $c\mathbf{v}$ is *not* an element of S.

Notice that in the definition, we can take c to be 0. Since $0\mathbf{v} = \mathbf{0}$, *any (non-empty) subset of $\mathbf{R}^n$ that is closed under scalar multiplication must contain $\mathbf{0}$*. Also, in the condition of closure under addition, we can take $\mathbf{w}$ to be $-\mathbf{v}$. Then $\mathbf{v} + \mathbf{w} = \mathbf{0}$. Therefore, *a (non-empty) subset of $\mathbf{R}^n$ that is closed under addition must contain $\mathbf{0}$.*

Subspaces

A non-empty subset W of $\mathbf{R}^n$ is said to be a **subspace** of $\mathbf{R}^n$ if it is both closed under addition and closed under scalar multiplication. This means the following:

- Whenever two vectors in the set W are added, the sum remains in the set W, and
- Whenever a vector $\mathbf{v}$ in W is multiplied by a scalar c, the resulting vector, $c\mathbf{v}$, also remains in the set W.

Example 3. $\mathbf{R}^n$ is itself a subspace of $\mathbf{R}^n$.

Example 4. The set $\{\mathbf{0}\}$ consisting of the zero vector in $\mathbf{R}^n$ is a subspace of $\mathbf{R}^n$.

Thus, a subset W of $\mathbf{R}^n$ is *not* a subspace if W is either *not* closed under addition, *or not* closed under scalar multiplication (or both). (The empty set is declared to not be a subspace. We will not worry about the empty set.) So to show that a subset is not a subspace, you must do one of two things:

(i) Find two vectors $\mathbf{v}$ and $\mathbf{w}$ that are elements of S but whose sum, $\mathbf{v} + \mathbf{w}$, is not an element of S; or

(ii) Find a vector $\mathbf{v}$ that is in S and a scalar c such that $c\mathbf{v}$ is not in S. This may take some trial and error.

We have seen that a subspace of $\mathbf{R}^n$ must contain $\mathbf{0}$, so any subset of $\mathbf{R}^n$ that does not contain $\mathbf{0}$ is certainly not a subspace. Of course, just containing $\mathbf{0}$ is not sufficient for a set to be a subspace. Take the subset $S = \{\mathbf{0}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\} \subset \mathbf{R}^2$. The subset S is neither closed under addition nor under scalar multiplication.

Sometimes a subset will be presented using set notation. For example, take

$$S = \{(x, y, z) : xy = z\}.$$

This reads: the set of (x, y, z) such that $z = 1$. The fact that there are 3 components tells me that S is a subset of $\mathbf{R}^3$. It consists of vectors such that the product of the first two components equals the third component.

Subspaces and Spans

The span of any finite set of vectors in $\mathbf{R}^n$ is a subspace of $\mathbf{R}^n$. If W is a subspace of $\mathbf{R}^n$, there are vectors $\mathbf{v}_1, ..., \mathbf{v}_k \in W$ such that $W = \text{Span}\{\mathbf{v}_1, ..., \mathbf{v}_k\}$.

In $\mathbf{R}^n$, the set consisting only of the zero vector is also a subspace. This set is called the **zero subspace**.

Exercise 1. Let V be the first quadrant in the xy-plane: that is, let

$$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0, y \geq 0 \right\}.$$

(a) If $\mathbf{u}$ and $\mathbf{v}$ are in V, must $\mathbf{u} + \mathbf{v}$ be in V? Why?
(b) Find a specific vector $\mathbf{u}$ in V and a specific scalar c such that $c\mathbf{u}$ is *not* in V.
(c) Is V a subspace of $\mathbf{R}^2$? Justify your answer.

Exercise 2. Let W be the set

$$\{\langle x, y \rangle : xy \geq 0\}.$$

(a) Let $\mathbf{u} = \langle -1, -3 \rangle$ and $\mathbf{v} = \langle 5, 6 \rangle$. Check that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} + \mathbf{v}$ are elements of W. Why doesn't this show that W is closed under addition?
(b) Is W closed under vector addition? Justify.
(c) Is W closed under scalar multiplication? Justify.

Exercise 3. Show that a plane in $\mathbf{R}^3$ *not* through the origin is not a subspace of $\mathbf{R}^3$. Remember that when we refer to a plane as a subspace, what we really mean is the set of position vectors from the origin to a point on the plane.

Exercise 4. Which of the following subsets of $\mathbf{R}^4$ are subspaces? Justify your answers.
(a) $\{\mathbf{x} \in \mathbf{R}^4 : x_1 + x_2 = x_3 + x_4\}$
(b) $\{\mathbf{x} \in \mathbf{R}^4 : x_1 = 1\}$
(c) $\{\mathbf{x} \in \mathbf{R}^4 : x_1 = x_3 = 0\}$
(d) $\{\mathbf{x} \in \mathbf{R}^4 : x_1 = (x_2)^2\}$

Exercise 5. Let $W = \{(x, y, z) \in \mathbf{R}^3 : x^2 = z^2\}$.
(a) Is W closed under vector addition?
(b) Is W closed under scalar multiplication?
(c) Is W a subspace of $\mathbf{R}^3$?

Exercise 6. Let W be the set of all vectors of the form $\begin{bmatrix} 2t \\ 0 \\ -t \end{bmatrix}$. Find a vector $\mathbf{v}$ in $\mathbf{R}^3$ such that $W = \operatorname{Span}(\mathbf{v})$. Why does this show that W is a subspace of $\mathbf{R}^3$?

Exercise 7. Is $W = \{(a, b, c, d) : d \geq 0\}$ a subspace of $\mathbf{R}^4$? Justify your answer.

Exercise 8. Let W be the set of all vectors of the form shown, where a, b, and c represent arbitrary real numbers. In each case, either find a set S of vectors that spans W or give an example to show that W is *not* a subspace.

a.) $\begin{bmatrix} 3a + b \\ 4 \\ a - 5b \end{bmatrix}$ b.) $\begin{bmatrix} -a + 1 \\ a - 6b \\ 2b + a \end{bmatrix}$ c.) $\begin{bmatrix} a - b \\ b - c \\ c - a \\ b \end{bmatrix}$ d.) $\begin{bmatrix} 4a + 3b \\ 0 \\ a + b + c \\ c - 2a \end{bmatrix}$

Exercise 9. Give an example in $\mathbf{R}^2$ to show that the union of two subspaces is not, in general, a subspace.

Exercise 10. Choose the correct option: The set of vectors (x_1, x_2, x_3, x_4) in $\mathbf{R}^4$ such that $x_1x_2 - x_3x_4 = 0$

(i) is closed under vector addition but not under scalar multiplication.
(ii) is closed under scalar multiplication but not under vector addition.
(iii) is neither closed under vector addition nor under scalar multiplication.
(iv) is a subspace.

Exercise 11. (a) Explain why, if H and K are subsets of $\mathbf{R}^n$ that are closed under vector addition, then so is $H \cap K$.

(b) Explain why, if H and K are closed under scalar multiplication, then so is $H \cap K$.
(c) Using parts a and b, explain why, if H and K are subspaces of $\mathbf{R}^n$, then so is $H \cap K$.

Exercise 12. Suppose that A is a 4×4 matrix. Let W be the set of vectors $\mathbf{b}$ in $\mathbf{R}^4$ for which $A\mathbf{x} = \mathbf{b}$ has a solution.

(a) Is W closed under vector addition?
(b) Is W closed under scalar multiplication?
(c) Is W a subspace?

Exercise 13. Suppose that A is a 4×4 matrix, and that $\mathbf{b}$ is a vector in $\mathbf{R}^4$ that is not $\langle 0, 0, 0, 0\rangle$. Let W be the set of solutions to $A\mathbf{x} = \mathbf{b}$.

(a) Is W closed under vector addition?
(b) Is W closed under scalar multiplication?
(c) Is W a subspace?

Exercise 14. If $\mathbf{v}$ and $\mathbf{w}$ are vectors in a subspace W, explain why $\text{Span}\{\mathbf{v}, \mathbf{w}\}$ is a subset of W (which might be equal to W).

Exercise 15. Let A be an $n \times n$ matrix and let c be a constant. Show that the set of vectors $\mathbf{x}$ such that $A\mathbf{x} = c\mathbf{x}$ is a subspace of $\mathbf{R}^n$.

2 Basis and Dimension

Because a subspace typically contains an infinite number of vectors, some problems involving a subspace are handled best by working with a small finite set of vectors that span the subspace. The smaller the set, the better.

> **Basis for a Subspace**
>
> A **basis** for a non-zero subspace W of $\mathbf{R}^n$ is a set of vectors in W that is linearly independent and that spans W.

A typical example of a basis for $\mathbf{R}^n$ is the **standard basis** $\{\mathbf{e}_1, ..., \mathbf{e}_n\}$.
The significance of a basis is:

Theorem 1. *Suppose that $\{\boldsymbol{v}_1, ..., \boldsymbol{v}_k\}$ is a basis for a subspace $W \subset \boldsymbol{R}^n$. Then every vector $\boldsymbol{v} \in W$ can be written uniquely as a linear combination of the basis vectors.*

To see why, first note that since the basis vectors span W, every $\mathbf{v} \in W$ is certainly a linear combination of the basis vectors. For uniqueness: Suppose that

$$\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k, \quad \text{and} \quad \mathbf{v} = d_1\mathbf{v}_1 + \cdots + d_k\mathbf{v}_k.$$

Subtracting both equations, we get $(c_1 - d_1)\mathbf{v}_1 + \cdots + (c_k - d_k)\mathbf{v}_k = \mathbf{0}$. Since the set $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ is linearly independent, $c_i - d_i = 0$ for all i, i.e., $c_i = d_i$.

Every subspace of $\mathbf{R}^n$ other than $\{\mathbf{0}\}$ has an infinite number of bases. For if $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ is a basis for a subspace W, so is $\{c_1\mathbf{v}_1, ..., c_n\mathbf{v}_n\}$ for any non-zero scalars $c_1, ..., c_n$.

Actually, the notion of a basis for the zero subspace is tricky. By the definition we have given, it would have to be a set whose span is $\{0\}$. The only possibility is $\{0\}$. But this set is not linearly independent. So there is no subset of $\{0\}$ that spans $\{0\}$ and is linearly independent. We *define* a basis for $\{0\}$ to be the empty set. The number of elements in this basis is 0. We *define* the dimension of $\{0\}$ to be 0.

Exercise 16. Explain why $\mathbf{0}$ can never be in a basis of a subspace.

Theorem 2. *If a vector space V has a basis with exactly n vectors, then any set in V containing more than n vectors must be linearly independent.*

Theorem 3. *If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.*

This leads us to make the following definition: The number of vectors in any basis for a nonzero subspace $W \subset \mathbf{R}^n$ is called the **dimension** of the subspace. By the result above, the dimension doesn't depend on the choice of basis. The dimension of the zero subspace $\{0\}$ is defined to be zero.

Dimension of a Subspace

If W is a subspace of $\mathbf{R}^n$, the number of vectors in *any* basis for W is called the **dimension** of the subspace. This definition makes sense because any two bases for W, while possibly being very different, must have the same *number* of elements.

Example 5. The standard basis $\{\mathbf{e}_1, ..., \mathbf{e}_n\}$ is a basis for $\mathbf{R}^n$. This shows that $\mathbf{R}^n$ is n-dimensional, as we expect. The span of 1 non-zero vector is 1-dimensional. The span of 2 linearly independent vectors is 2-dimensional, etc.

Some authors describe vectors in $\mathbf{R}^n$ as "n-dimensional vectors." Now you can see why I don't like this description. This is a different usage of the word *dimension*. In this book, dimension is something a subspace has, not something that individual vectors have. I prefer to call vectors in $\mathbf{R}^n$ vectors with n components.

Any set that spans a subspace W is called a **generating set** or a **spanning set** for W. A basis can be thought of as a *minimal* generating set. For a generating set that is not a basis, at least one vector can be removed while still retaining the same span. But if a single vector from a basis is removed, the span of the vectors that remain will be a lower-dimensional subspace. This leads to another way to think of the dimension: *The dimension of a subspace is the smallest number of vectors that are needed to span the subspace.*

Theorem 4. *Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also, H is finite-dimensional and*

$$\mathit{dim}\ H \leq \mathit{dim}\ V.$$

Exercise 17. Let $V = \mathbf{R}^3$ and H be the plane $x - 2y + 3z = 0$. Explain why $\left\{ \begin{bmatrix} 2 \\ 0 \\ -2/3 \end{bmatrix} \right\}$ is a linearly independent set in H and expand it to a basis for H.

Exercise 18. Let $V = H = \mathbf{R}^3$. Extend the set you got in the previous exercise to a basis for H.