constant $b > 0$, $b \neq 1$

We will define a function $f(x) = b^x$

Ex: $f(x) = 2^x$, $f(x) = \left(\frac{1}{5}\right)^x$.

We will define $b^x$ for any real number $x$.

Steps:

1. $x$ positive integer.

$b^x = \underbrace{b \cdots b}_{x \ b\text{'s}}$

Example:

$2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$
$= 32$

properties:

1) $(b^m)^n = b^{mn}$
2) $b^m \cdot b^n = b^{m+n}$

2. $x = 0$

$b^0 = 1$

$2^0 = 1$

Why? If we want property 1 to hold when m is possibly 0, we would need

want $\rightarrow$ $b^0 \cdot b^1 = b^{0+1}$

$b^0 \cdot b = b$, divide both sides by $b$

$\boxed{b^0 = 1}$

3. $x$ is a negative integer

n pos int.

Want $b^{-n} \cdot b^n = b^{-n+n} = b^0 = 1$

So we define $\boxed{b^{-n} = \frac{1}{b^n}}$

$2^{-3} = \frac{1}{2^3}$
$= \frac{1}{8}$

④ $x = \frac{1}{n}$, n is a nonzero integer

Want: $\left(b^{\frac{1}{n}}\right)^n = b^{\frac{1}{n}\cdot n} = b^1 = b$

Take nth root ↓ $\boxed{b^{\frac{1}{n}} = \sqrt[n]{b}}$ $(n>0)$

Want: $b^{-\frac{1}{n}} = \left(b^{\frac{1}{n}}\right)^{-1} = \frac{1}{b^{1/n}}$

$= \frac{1}{\sqrt[n]{b}}$

⑤ $x = \frac{m}{n}$

Want: $\left(b^{\frac{1}{n}}\right)^m = b^{m/n}$

Define: $\boxed{b^{\frac{m}{n}} = \left(b^{\frac{1}{n}}\right)^m}$

$2^{\frac{1}{2}} = \sqrt{2}$

$2^{\frac{1}{5}} = \sqrt[5]{2}$

$2^{-\frac{1}{3}} = \frac{1}{2^{1/3}}$

$= \frac{1}{\sqrt[3]{2}}$

$2^{5/3} = \left(2^{\frac{1}{3}}\right)^5$

$= \left(\sqrt[3]{2}\right)^5$

$= \left(\sqrt[3]{2}\right)^3 \cdot \left(\sqrt[3]{2}\right)^2$

$= 2 \cdot \sqrt[3]{4}$

$\sqrt[3]{2} \cdot \sqrt[3]{2}$

$= \sqrt[3]{2 \cdot 2}$

⑥ $\pi = 3.14159\ldots$

$\sqrt{2} = 1.414\ldots$

To define $b^{\pi}$, look at is the list

$b^3, b^{3.1}, b^{3.14}, b^{3.141}, b^{3.1415}, \ldots$

$2^{\pi} \to 2^{3.1415}$

$2^{3.14159}$

$\vdots$

↓

approach a number, we call it

$2^{\sqrt{2}}$

↓

$2^{1.414}$

look at what number the numbers in this list are approaching

$2^{1.4142}$

$2^{1.41421}$

$\vdots$

$\downarrow$

approach a number, we call it $2^{\sqrt{2}}$

$2^{\pi}$

This is how we define $f(x) = b^x$. ← Exponential Functions

This way, $f(x)$ has some nice properties:

$$\left.\begin{aligned} b^{m+n} &= b^m \cdot b^n \\ (b^m)^n &= b^{mn} \end{aligned}\right\} \text{For all real numbers } m \text{ and } n.$$

Ex: $(b^{\pi})^{\sqrt{3}} = b^{\pi\sqrt{3}}$

$(b^{\frac{1}{7}})^{-50} = b^{-\frac{50}{7}}$

# Exponential Growth / Decay

$f(x) = b^x$ , where $b > 0$, $b \neq 1$.

$x_{old}$ $\quad$ $x_{new} = x_{old} + 1$

How are $f(x_{old})$ and $f(x_{new})$ related?

$y_{old}$ $\quad$ $y_{new}$

$$b^{x+y} = b^x \cdot b^y$$

where $x$ and $y$ are any real #'s

$$y_{new} = f(x_{new}) = f(x_{old}+1)$$
$$= b^{x_{old}+1}$$
$$= b^{x_{old}} \cdot b^1$$
$$= y_{old} \cdot b$$

~~An~~ When $x$ increases by 1, $y$ gets multiplied by $b$.

Ex: $f(x) = 3^x$

$f(4)$ and $f(5)$

$= 3 \cdot 3 \cdot 3 \cdot 3$ $\quad$ $= 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$

$\cdot 3$

Ex: $f(x) = 5^x$ means that

$f(x)$ gets multiplied by a factor of 5 when $x$ increases by 1.

Ex: $f(x) = \left(\frac{1}{2}\right)^x$

$f(x)$ gets multiplied by a factor of $\frac{1}{2}$ whenever $x$ increases by 1.

gets halved

$f(x) = b^x$ has the property that when $x=0$, $f(x) = 1$.

In general, you have $\boxed{f(x) = a \cdot b^x}$, which is also considered an exponential fn.

This fn has the property that

$$f(x+1) = a \cdot b^{x+1}$$
$$= a \cdot b^x \cdot b$$
$$= b \cdot f(x)$$

Interpret $b$ as the growth factor $(b>1)$ or the decay factor $(b<1)$

Interpret $a$: $f(0) = a \cdot b^0 = a \cdot 1 = a$

Ex: "Sales increased 12% per year since 2000."

$y$

$12\% \rightarrow 0.12$

Each year, $y$ gets increased by 12% of $y$

$$y + 0.12y = 1.12y$$

growth factor is 1.12

If sales at a certain year is $y$, then next year, sales is $1.12y$.

| $t$ (years since 2000) | Year | Sales |
|---|---|---|
| 0 | 2000 | $a$ |
| 1 | 2001 | $1.12a$ |
| 2 | 2002 | $1.12^2a = 1.2544a$ |
| 3 | 2003 | $1.12^3a$ |
| 4 | 2004 | $1.12^4a$ |
| $n$ | $2000+n$ | $1.12^n \cdot a = a(1.12^n)$ |

$\times 1.12$

$\times 1.12$

$\times 1.12$

$\times 1.12$

Suppose $a = 1500$

Then $f(n) = 1500(1.12^n)$

Some quantity increased $x\%$ a year since ____ year. In that year, the quantity was $a$. In $n$ years?

After 1 year $\quad a + \frac{x}{100} \cdot a = \underbrace{\left(1 + \frac{x}{100}\right)}_{\text{amt.}} a$

After $n$ years $\quad f(n) = \boxed{a\left(1 + \frac{x}{100}\right)^n} \leftarrow$ quantity

growth factor is $1 + \frac{x}{100}$

exponential function.

exponential growth

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Ex: Sales decreasing $5\%$ a year.

Initially, sales were \$$a$.

What are sales after $n$ years?

Initial $\quad a$

After 1 year $\quad a - \frac{5}{100} a = \frac{95}{100} a = \left(1 - \frac{5}{100}\right) a$

$\times 0.95$

$\times 0.95$

$\downarrow$ After 2 years $\quad \left(1-\frac{5}{100}\right)\underbrace{\left(1-\frac{5}{100}\right)a}_{\text{amt after 1 year}}$

$x.95 \downarrow$
$\vdots$

After n years, $\quad \underbrace{\left(1-\frac{5}{100}\right)^n a}_{\text{amt after n years}}$

$f(n) \rightarrow \boxed{a\left(1-\frac{x}{100}\right)^n}$ amt after n years

$b = 1-\frac{x}{100}$

Start off with a, decrease by x% a year.

exponential decay with a decay factor of $1-\frac{x}{100}$