Interior Points

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Definition. Let S be a subset of the metric space E. A point $p \in S$ is called an **interior point of** S if there is an open ball in E of center p which is contained in S. The set of interior points of S is called the **interior of** S and is denoted $\mathrm{Int}(S)$.

Notice that by definition, an interior point of S is an element of S, so $\mathrm{Int}(S)$ is a subset of S. In order for a point p to be an interior point of S, it is not enough for p to be an element of S – there needs to be at least one ϵ-neighborhood centered at p that is contained in S.

Example 1. Let $r > 0$, and let $S \subset \mathbb{R}^n$ be the closed ball of radius r centered at 0. We will show that $\mathrm{Int}(S)$ is the open ball of radius r centered at 0. Suppose that $p \in S$ is such that $d(p, 0) < r$. Then the ball of center p of radius $(r - d(p, 0))/2$ is contained in S. Therefore, $\{p \in S : d(p, 0) < r\}$ is in the interior of S. Let $x \in S$ be such that $d(x, 0) = r$. Let $\epsilon > 0$. We will show that the open ball of radius ϵ centered at x is not contained in S, that is, there is an element in this ball that is not an element of S. The element $(r + \epsilon/2)x/r$ is not in S because its distance from 0 is $r + \frac{\epsilon}{2}$. But it is in the open ball of radius ϵ centered at x because $d(x, (r + \frac{\epsilon}{2})x) = \frac{\epsilon}{2}$.

Example 2. If S is the sphere of radius r centered at a in $\mathbb{R}^n$, then $\mathrm{Int}(S)$ is the empty set.

Example 3. Suppose that $S = \mathbb{Z}$, a subset of $\mathbb{R}$. The interior of S consists of all points $p \in S$ such that there is $\epsilon > 0$ such that every point x with $d(x, p) < \epsilon$ is also in S. We claim that $\mathrm{Int}(\mathbb{Z})$ is the empty set. Let $p \in \mathbb{Z}$. Let $\epsilon > 0$. The open ball in $\mathbb{R}$ contains only p. If $\epsilon > 1/2$ then it contains $p + 0.1$ which is not an integer. Otherwise, any element in the open neighborhood other than p is not in S. Similarly, if $S = \mathbb{Z}^n \subset \mathbb{R}^n$, then $\mathrm{Int}(S)$ is the empty set.

Example 4. Suppose that S is a subset of any metric space with the discrete metric. Then $\mathrm{Int}(S) = S$.

Notice that what makes a point in S an interior point depends on E. It is not an inherent property of the set. For example, consider $\mathbb{Z}$ as a subset of $\mathbb{R}$. The open sets in $\mathbb{Z}$ are the subsets of $\mathbb{Z}$. Also, consider $\mathbb{Z}$ with the discrete metric. The open sets in $\mathbb{Z}$ are also the subsets of $\mathbb{Z}$. However, if $\mathbb{Z}$ is viewed as a subset of $\mathbb{R}$, then $\mathrm{Int}(\mathbb{Z}) = \emptyset$, and if $\mathbb{Z}$ has the discrete metric, $\mathrm{Int}(\mathbb{Z}) = \mathbb{Z}$.

Example 5. Let E be a metric space and let $S = E$. Then $\mathrm{Int}(S) = E$.

The basic properties of interiors are the following:

Proposition 1. *Let S be a subset of the metric space E.*

(1) $Int(S)$ is an open subset of E.

(2) $Int(S)$ is the union of all open subsets of E that are contained in S. (This is often expressed by saying that the interior of S is the largest open subset of S)

(3) S is open if and only if $Int(S) = S$;

Proof. (1) Let $p \in \operatorname{Int}(S)$. We want to show that there is a ball of center p completely contained in $\operatorname{Int}(S)$. We know there is $\epsilon > 0$ such that $B(p;\epsilon) \subset S$. We will show that this ball is in $\operatorname{Int}(S)$. Let $q \in B(p;\epsilon)$. Then there is an open ball centered at q that is contained in $B(p;\epsilon)$ and hence in S.

(2) Let $p \in \operatorname{Int}(S)$. Then $B(p;\epsilon) \subset S$, and since an open ball is an open set, p is contained in an open set contained in S. Conversely, suppose that p is contained in the union of all open subsets of E contained in S. Then p is contained in one such open set, say U. There is $\epsilon > 0$ such that $B(p;\epsilon) \subset U$, and since $U \subset S$, $B(p;\epsilon) \subset S$. Therefore $p \in \operatorname{Int}(S)$.

(3) Suppose S is open in E. Then S is an open set of E that contains S, and so $S \subset \operatorname{Int}(S)$. Conversely, if $S = \operatorname{Int}(S)$, then S is open because $\operatorname{Int}(S)$ is. $\square$

Exercise 1. Let A and B be two subsets of a metric space X, and prove the following:

(a) $\operatorname{Int}(A) \cup \operatorname{Int}(B) \subseteq \operatorname{Int}(A \cup B)$;

(b) $\operatorname{Int}(A) \cap \operatorname{Int}(B) = \operatorname{Int}(A \cap B)$.

Give an example of two subsets A and B of the real line such that $\operatorname{Int}(A) \cup \operatorname{Int}(B) \neq \operatorname{Int}(A \cup B)$.