Exponential Growth and Decay

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Last time we discussed exponential functions of the form $f(x) = b^x$ where $b > 0$ and $b \neq 1$. In general, for any constants a and b, an **exponential function** is a function that can be written in the form

$$f(x) = a \cdot b^x,$$

where a is nonzero, b is positive, and $b \neq 1$. The constant a is the *initial value* of f (the value at $x = 0$) and b is the **base**.

For any exponential function $f(x) = a \cdot b^x$, and any real number x,

$$f(x+1) = b \cdot f(x).$$

In other words, as x increases by 1, the function value is multiplied by the case b. If $a > 0$ and $b > 1$, the function f is increasing and is an **exponential growth function**. The base b is its **growth factor**. If $a > 0$ and $0 < b < 1$, f is decreasing and is an **exponential decay function**. The base b is its **decay factor**.

Both an exponential growth and exponential decay function have a **constant percentage change**, which is $r = b - 1$, written as a percent. This percentage is the difference between the base b and 1, and thus $b - 1$ gives us the percent of growth or decay for a unit of time t.

Example 1. Suppose that starting at a certain time, the amount of a certain quantity increases by 1.3% a day. Let $f(t)$ be the amount of the quantity after t days since the starting time. Suppose that the initial amount is $f(0) = a$. Then in one day, the amount increases by 1.3% of a. This is $0.013a$. So after one day, the amount is $a + 0.013a = 1.013a$. Then in one more day, the amount increased by 1.3% of that amount. We have seen that increasing an amount by 1.3% of that amount is the same as multiplying that amount by 1.013. So after t days, the amount is $f(t) = a \cdot 1.013^t$. So there is exponential growth, and the growth factor is 1.013.

Example 2. For each model, P represents population of the city proper and t, time in years since 2010.

- Tell whether the population model of each city is an exponential growth function or exponential decay function, and
- find the growth or decay factor and the constant percentage change.

(a) San Antonio, Texas: $P(t) = 1,326,810 \cdot 1.0168^t$
(b) Detroit, Michigan: $P(t) = 713,956 \cdot 0.9930^t$
SOLUTION:
(a) Because $b = 1.0168$, and this value is greater than 1, P is an exponential growth function with a growth factor of 1.0168 and a constant percentage increase of 1.68% per year.
(b) Because $b = 0.9930$ and $0 < 0.9930 < 1$, P is an exponential decay function with a decay factor of 0.9930 and a constant percentage decrease of 0.70% per year.

Example 3. The graph of an exponential function passes through $(0, 15)$ and $(2, 17)$. What is the function?

SOLUTION: We know that $f(x) = a \cdot b^x$. Since $f(0) = 15$, we get $a \cdot b^0 = 15$, and since $b^0 = 1$, $a = 15$. We know that $f(2) = 17$, so $15 \cdot b^2 = 17$. Then $b^2 = 17/15$, so $b = \sqrt{17/15}$. The function is $f(x) = 10 \cdot \sqrt{\frac{17}{15}}^x$, or approximately $f(x) = 10 \cdot 1.065^x$.