Lebesgue Measure

In Lebesgue's day, mathematicians had noticed a number of deficiencies in Riemann's way of defining the integral. Many functions with reasonable properties turned out not to possess integrals in Riemann's sense. Moreover, certain limiting procedures, when applied to sequences not of numbers but of functions, behaved in very strange ways as far as integration was concerned. Several mathematicians tried to develop better ways to define the integral, and the best of all was Lebesgue's.

Consider, for example, the function defined by $f(x) = 0$ whenever x is a rational number but $f(x) = 1$ whenever x is irrational. What is a sensible value for

$$\int_0^1 f(x)\,dx?$$

Using Riemann's definition, this function does not possess a well-defined integral. The reason is that within any interval it takes values both 0 and 1, so that it hops wildly up and down between these two values. Unfortunately for this example, Riemann's integral is based on the assumption that over sufficiently small intervals the value of the function changes by only a very small amount.

However, there is a sense in which the rational numbers form a very tiny proportion of the real numbers. In fact, "almost all" real numbers are irrational. Specifically, the set of all rational numbers can be surrounded by a collection of intervals whose total length is as small as is wanted. In a well-defined sense, then, the "length" of the set of rational numbers is zero. There are good reasons why values on a set of zero length ought not to affect the integral of a function. Granted this, if the definition of the function f is changed so that it takes value 1 on the rational numbers instead of 0, its integral should not be altered. However, the resulting function g now takes the form $g(x) = 1$ for all x, and this function does possess a Riemann integral.

1 Measure Zero Sets

Let $S \subset \mathbf{R}$. We say that S *has measure zero* if for every $\epsilon > 0$, there exists a countable family $\{I_n\}$ of open intervals such that $S \subset \cup I_n$ and $\sum |I_n| < \epsilon$, where $|I|$ denotes the length of the interval $|I|$. In other words, a subset S of $\mathbf{R}$ is of measure zero if S can be covered by a countable family of open intervals the sum of whose lengths can be made arbitrarily small.

The set of rational numbers in $[0, 1]$ has Lebesgue measure zero. Indeed, any countable set is of measure zero. For suppose

$$S = \{x_1, x_2, x_3, ...\}$$

is a countable set. For each positive integer n, choose an open interval I_n containing x_n such that $|I_n| < \epsilon/3^n$. Then

$$S \subset \bigcup_{n=1}^{\infty} I_n$$

and

$$\sum_{n=1}^{\infty} |I_n| = \frac{\epsilon}{2} < \epsilon.$$

Therefore, S has measure zero.

2 Outer Measure

Let $P(\mathbf{R})$ denote the power set of $\mathbf{R}$, that is, the set of all subsets of $\mathbf{R}$. We would like there to be a function $m\colon P(\mathbf{R}) \to [0, \infty]$ with the following properties:

- $m(I) = |I|$ for every interval $I \subseteq \mathbf{R}$, and
- For every sequence $S_1, S_2, ...$ of pairwise disjoint subsets of $\mathbf{R}$,

$$m\left(\bigcup_{n\in\mathbb{N}} S_n\right) = \sum_{n\in\mathbb{N}} m(S_n).$$

Unfortunately, such an m doesn't exist. However, it turns out that one can restrict m to a sufficiently small subset of $P(\mathbf{R})$ so that these properties do hold. This subset is called the set of Lebesgue measurable sets. We will define this set.

First, we assign to every subset of $\mathbf{R}$ a number called its outer measure. If $S \subset \mathbf{R}$, the *Lebesgue outer measure* of S is defined to be

$$m^*(S) = \inf\{\sum_{I \in \mathcal{C}} |I|\}$$

where $\mathcal{C}$ is a countable collection of open intervals such that $S \subset \cup\mathcal{C}$.

3 Measurable Sets

A subset $E \subset \mathbf{R}$ is said to be *Lebesgue measurable* if

$$m^*(T \cap E) + m^*(T \cap E^c) = m^*(T)$$

for every subset T of $\mathbf{R}$. In this case, the outer measure $m^*(E)$ of E is called the *Lebesgue measure* of E and is denoted $m(E)$. It can be shown that every measure zero set is Lebesgue measurable.

4 Integration

For a measurable set E, we define χ_E to be the function which is 1 on E but 0 elsewhere. Suppose that f has the form $\sum_{i \in \mathbb{N}} c_i \chi_{E_i}$ for Lebesgue measurable sets E_i and constants c_i. Such a function is called a *simple function*. Then we define the *Lebesgue integral* of f to be $\sum c_i m(E_i)$. For the function defined on $[0, 1]$ which is 0 on rationals and 1 on irrationals, the integral is 0.

Now we extend the definition of integral to a wider class of function. A function $f \colon \mathbf{R} \to [-\infty, \infty]$ is an *extended real measurable function* on $\mathbf{R}$ with respect to Lebesgue measurable sets if for every open set $V \subset [-\infty, \infty]$, $f^{-1}(V)$ is a Lebesgue measurable set.

Suppose that f is an extended real measurable function on $\mathbf{R}$ and that $f(x) \geq 0$ for all x. We define the *Lebesgue integral* of f over $\mathbf{R}$ with respect to Lebesgue measure to be

$$\int_{\mathbf{R}} f\, dm = \sup\left\{\int_{\mathbf{R}} s\, dm\right\}$$

where s is a simple measurable function on $\mathbf{R}$ with $0 \leq s \leq f$.

For any measurable f define

$$f^+ = \max\{f, 0\}, \quad f^- = -\min\{f, 0\}.$$

Then $f = f^+ - f^-$ and $|f| = f^+ + f^-$, and it turns out that f^+ and f^- are measurable. If at least one of $\int_{\mathbf{R}} f^+ \, dm$ or $\int_{\mathbf{R}} f^- \, dm$ is finite, we define

$$\int_{\mathbf{R}} f \, dm = \int_{\mathbf{R}} f^+ \, dm - \int_{\mathbf{R}} f^- \, dm.$$

If both are ∞, then we say that $\int_{\mathbf{R}} f \, dm$ is not defined.

Then we have the following theorem, which shows that the Lebesgue integral is a generalization of the Riemann integral.

Theorem. If f is Riemann integrable on $[a, b]$ (a closed interval in $\mathbf{R}$), then f is Lebesgue integrable on $[a, b]$, and

$$\int_{[a,b]} f \, dm = \int_a^b f(x) \, dx.$$

All of this can be generalized to the concept of a measure space and integration with respect to a measure.