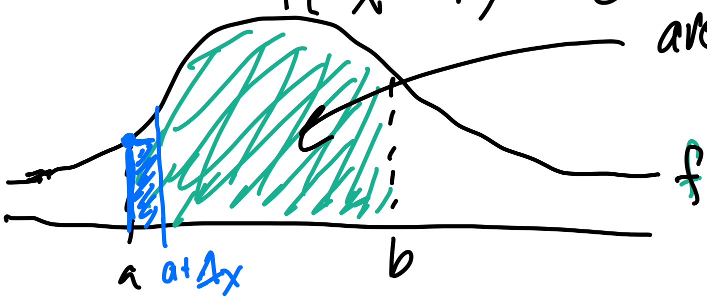
Continuous random variable. X

$P(X = k) = 0$



A rv X has a continuous dist. with density function f if for all $a \le b$, we have

$$P(a \le X \le b) = \int_a^b f(x)\,dx.$$

Intuitively, can think of $f(x)$ as $P(X = x)$.

Reason:

$$P(a \le X \le a + \Delta x) = \int_a^{a+\Delta x} f(x)\,dx$$

$$\approx f(a)\,\Delta x$$

As $\Delta x \to 0$, get 0 on both sides, but

$$f(a) \approx \frac{P(a \le X \le a+\Delta x)}{\Delta x}$$

$\approx$ area of rectangle

Need

$$\underbrace{f(x) \geq 0 \quad \text{and} \quad \int_{-\infty}^{\infty} f(x)\,dx = 1.}_{}$$

Any fn satisfying this is called a density function.

Ex: uniform distribution on (a,b).

$$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise.} \end{cases}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_a^b \frac{1}{b-a}\,dx = \frac{1}{b-a}(b-a)$$

$$= 1.$$

Let $p > 1$.

Ex: $f(x) = \begin{cases} (p-1)x^{-p}, & x \geq 1 \\ 0 & \text{otherwise} \end{cases}$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{1}^{\infty} (p-1)x^{-p}\,dx$$

$$= (p-1)\frac{x^{-p+1}}{-p+1}\Big|_{1}^{\infty}$$

$$= 0 - \frac{(p-1)1^{-p+1}}{-p+1} \qquad -p+1 = -(p-1)$$

$$= 1$$

Ex: X has an exponential distribution with parameter $\lambda > 0$ if

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{\infty} \lambda e^{-\lambda x}\,dx =$$

$$-e^{-\lambda x}\Big|_0^{\infty} = 0-(-e^{-0})$$
$$= 1.$$

Ex: Standard normal distribution

$$f(x) = (2\pi)^{-1/2} e^{-x^2/2}$$
$$= \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Check:

$$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

Consider $\int_{-\infty}^{\infty} f(x)\,dx \int_{-\infty}^{\infty} f(y)\,dy$

$$\frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{\infty} e^{-x^2/2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2/2}\,dy\right)$$

$$= \frac{1}{2\pi} \iint_{\mathbb{R}^2} e^{-(x^2+y^2)/2}\,dx\,dy$$

use polar coordinates.

$$= \frac{1}{2\pi} \cdot 2\pi = 1$$

$$\lim_{R\to\infty} \int_0^{2\pi} \int_0^{R} e^{-r^2/2}\, r\, dr\, d\theta$$

inner: $\int_0^R re^{-r^2/2}\, dr$

$u = r^2/2$

$du = r\,dr$

$$= \int_0^{R^2/2} e^{-u}\, du$$

$$= -e^{-u}\Big|_0^{R^2/2}$$

$$= -e^{-R^2/2} + 1$$

outer: $2\pi\left(1 - e^{-R^2/2}\right)$

$$\lim_{R\to\infty} \left(2\pi\left(1 - e^{-R^2/2}\right)\right) = 2\pi$$

So $\int_{-\infty}^{\infty} f(x)\, dx = 1$.