Linear Independence

September 2025

In this section you will

- Determine whether a set of vectors is linearly independent.
- Construct linear dependence relations.

We have seen in the previous section that understanding the general solution (that is, finding *all* the solutions) to $A\mathbf{x} = \mathbf{b}$ splits into two parts. One part is finding *one* solution to $A\mathbf{x} = \mathbf{b}$. The second part is finding *all* the solutions to the homogeneous system $A\mathbf{x} = \mathbf{0}$. Adding one solution to $A\mathbf{x} = \mathbf{b}$ to all the solutions to $A\mathbf{x} = \mathbf{0}$ gives us all of the solutions to $A\mathbf{x} = \mathbf{b}$. So it is important to know what the general solution to $A\mathbf{x} = \mathbf{0}$ looks like. This leads us to a very important topic: linear independence and linear dependence.

Recall that multiplying a matrix A by a vector $\mathbf{x}$ amounts to calculating a linear combination of the column vectors of A. The weights of the vectors are provided by the components of the vector $\mathbf{x}$. Let $\mathbf{v}_1, ..., \mathbf{v}_k$ be vectors in $\mathbf{R}^n$ and consider the expression $c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$. Certainly, if each $c_i = 0$, then $c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$. Depending on $\mathbf{v}_1, ..., \mathbf{v}_k$, this may be the only way to take a linear combination of $\mathbf{v}_1, ..., \mathbf{v}_k$ to produce $\mathbf{0}$. If this is the case, the set $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ is said to be linearly independent.

Linear Independence

A set of vectors $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ in $\mathbf{R}^n$ is called **linearly independent** if the only linear combination of $\mathbf{v}_1, ...\mathbf{v}_k$ that produces the zero vector is when all the coefficients are 0:

$$c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{0} \implies c_1 = 0, c_2 = 0, ..., c_k = 0.$$

A set of vectors in $\mathbf{R}^n$ that is not linearly independent is said to be **linearly dependent**.

Spelling this out, a set $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ of vectors in $\mathbf{R}^n$ is said to be linearly *dependent* if there exist scalars $c_1, ..., c_k$, *not all* 0, such that

$$c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{0}.$$

Note that most of the c_i could be 0. We just need at least one of the scalars c_i to not be 0.

Example 1. If

$$\mathbf{v}_1 = \langle 2, 3, -1\rangle, \mathbf{v}_2 = \langle 0, 0, 2\rangle, \mathbf{v}_3 = \langle 4, 6, 0\rangle, \mathbf{v}_4 = \langle -6, 3, 1\rangle$$

then the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly dependent because

$$2\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3 + 0\mathbf{v}_4 = \langle 0, 0, 0\rangle.$$

□

Example 2. The set
$$\mathbf{v}_1 = \langle 1,0,0\rangle, \mathbf{v}_2 = \langle 0,1,0\rangle, \mathbf{v}_3 = \langle 0,0,1\rangle$$
is linearly independent. To check this, we suppose that
$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \langle 0,0,0\rangle.$$
We need to show that c_1, c_2, and c_3 are all equal to 0. We have
$$\begin{aligned} c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 &= c_1\langle 1,0,0\rangle + c_2\langle 0,1,0\rangle + c_3\langle 0,0,1\rangle \\ &= \langle c_1,0,0\rangle + \langle 0,c_2,0\rangle + \langle 0,0,c_3\rangle \\ &= \langle c_1,c_2,c_3\rangle. \end{aligned}$$
If this equals $\langle 0,0,0\rangle$, then $c_1 = 0$, $c_2 = 0$, and $c_3 = 0$. This shows that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent. □

Linear dependence and independence are properties of sets of vectors; it is customary, however, to apply the adjectives to vectors themselves, and thus we shall sometimes say "a set of linearly independent vectors" instead of "a linearly independent set of vectors." This is very sloppy. Saying that "the vectors are linearly independent" is not meant in the sense of, "the pencils are yellow." When we say that the pencils are yellow, we mean that each of the pencils is yellow. This is certainly not what we mean when we say that "the vectors are linearly independent." What we mean is that the set of vectors is linearly independent – linear independence is not a property of a vector. However, as long as you understand this, then it's fine to refer to "linearly independent vectors."

Note: We *never* say, "The matrix is linearly independent." The adjective *linearly independent* goes with the noun *set of vectors*. It is OK to say, "A matrix whose columns are linearly independent," or, "A matrix with linearly independent columns," if that is what you mean.

The condition for a set of vectors to be linearly independent can be phrased in terms of matrices.

Theorem 1. *Let $\{\boldsymbol{v}_1, ..., \boldsymbol{v}_k\}$ be a set of distinct vectors in $\boldsymbol{R}^n$. Let A be the $n \times k$ matrix whose columns are $\boldsymbol{v}_1, ..., \boldsymbol{v}_k$. The following are equivalent:*

- *The set $\{\boldsymbol{v}_1, ..., \boldsymbol{v}_k\}$ is linearly independent.*
- *The homogeneous system $A\boldsymbol{x} = \boldsymbol{0}$ has only the trivial solution $\boldsymbol{0}$.*
- *For every $\boldsymbol{b} \in \boldsymbol{R}^m$, $A\boldsymbol{x} = \boldsymbol{b}$ has at most one solution.*
- *REF(A) has no free columns, that is, if every column is a pivot column.*
- *Every vector in Span$\{\boldsymbol{v}_1, ..., \boldsymbol{v}_k\}$ is a linear combination of $\boldsymbol{v}_1, ..., \boldsymbol{v}_k$ in a unique way.*

That leads to the general way to tell if a set of vectors is linearly independent

Determining Linear Independence

Form the matrix A whose columns are the vectors $\mathbf{v}_1, ..., \mathbf{v}_k$. The set is linearly independent precisely when REF(A) has a pivot in every column.

Notice that in order to determine whether or not a set of vectors is linearly independent, you *do not* have to solve a vector equation. For determining linear independence/dependence, we want to know *whether or not* the vector equation

$$c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

has a nontrivial solution for $(c_1, ..., c_k)$, not what the solutions $(c_1, ..., c_k)$ actually are.

Example 3. Determine if the vectors

$$\mathbf{a} = \begin{bmatrix} 16 \\ 2 \\ 8 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 22 \\ 4 \\ 4 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 18 \\ 0 \\ 4 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 18 \\ 2 \\ 6 \end{bmatrix}$$

are linearly independent.

Answer: We form the matrix whose columns are $\mathbf{a}, \mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ and then row reduce it:

$$\begin{bmatrix} 16 & 22 & 18 & 18 \\ 2 & 4 & 0 & 2 \\ 8 & 4 & 4 & 6 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & 0 & 2 \\ 0 & -10 & 18 & 2 \\ 0 & 0 & 4 & 1 \end{bmatrix}$$

and see that column 4 has no pivot. So the set $\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$ is linearly dependent.

Actually, any set of 4 vectors in $\mathbf{R}^3$ is linearly dependent. Can you see why?

Exercise 1. Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}.$$

Is the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent, or linearly dependent? After doing the computations, your answer should be justified using complete sentences.

Exercise 2. For these vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 9 \\ -6 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 5 \\ -7 \\ h \end{bmatrix}$$

(a) for what values of h is $\mathbf{v}_3$ in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$?
(b) for what values of h is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly *dependent*?

Sometimes we can tell whether or not a set of vectors is linearly independent by inspection (in other words, just by looking at the vectors).

Suppose that $\mathbf{v} \in \mathbf{R}^n$. When is the set $\{\mathbf{v}\}$ linearly independent?

Linear Independence of $\{\mathbf{v}\}$:
A set consisting of exactly one vector $\{\mathbf{v}\}$ is linearly dependent if and only if $\mathbf{v}$ is the zero vector.

Make sure you understand the two claims that are being made in the boxed statement above. It is saying that $\{\mathbf{0}\}$ is linearly dependent. And it is saying that if $\mathbf{v} \neq \mathbf{0}$, then $\{\mathbf{v}\}$ is linearly independent.

How can we show that $\{\mathbf{0}\}$ is linearly dependent? First, we think, what does a set have to do to be linearly dependent? From the definition, we have to express $\mathbf{0}$ as a non-trivial linear combination of the vectors in the set. In this case, the set is just $\{\mathbf{0}\}$. What are the linear combinations of a single vector? They are just scalar multiples of that vector. So now we have figured out that we have to find a non-zero scalar multiple of the vector in our set (the vector is $\mathbf{0}$) that is equal to $\mathbf{0}$. We can do this – just take $c = 1$:

$$1\mathbf{0} = \mathbf{0}.$$

Therefore, the set $\{\mathbf{0}\}$ is linearly dependent.

Now suppose that $\mathbf{v} \neq \mathbf{0}$. We will show that $\{\mathbf{v}\}$ is linearly independent. To do this, we let c be a scalar for which

$$c\mathbf{v} = \mathbf{0}$$

and then we have to show that $c = 0$. We will give several proofs of this fact.

First proof: Let A be the $n \times 1$ matrix whose column is $\mathbf{v}$. Since $\mathbf{v} \neq \mathbf{0}$, at least one of the entries is non-zero, so $\mathrm{REF}(A)$ has a non-zero entry at the top. Therefore, every column of A (the only column) has a pivot, so $\{\mathbf{v}\}$ is linearly independent.

Second proof: Suppose that $c\mathbf{v} = \mathbf{0}$. We want to show that $c = 0$. Let $\mathbf{v} = \langle v_1, ..., v_n\rangle$. Then

$$c\mathbf{v} = \langle cv_1, ..., cv_n\rangle.$$

In order for this vector to be $\mathbf{0}$, we need $cv_1 = 0$, ..., $cv_n = 0$. Since $\mathbf{v} \neq \mathbf{0}$, one of the $\mathbf{v}_i$'s is non-zero. For this i, we can divide both sides of the equation $cv_i = 0$ by v_i to get $c = 0$.

Third proof: Suppose that $c\mathbf{v} = \mathbf{0}$. Taking lengths of both sides, we get

$$|c|||\mathbf{v}|| = 0.$$

Since $\mathbf{v} \neq \mathbf{0}$, its length $||\mathbf{v}||$ is also non-zero. Therefore, we may divide both sides of the equation by $||\mathbf{v}||$. This gives

$$|c| = 0,$$

so $c = 0$.

Fourth proof: Suppose that $c\mathbf{v} = \mathbf{0}$. We want to show that $c = 0$. Suppose, towards a contradiction, that $c \neq 0$. Then we can divide both sides of the equation $c\mathbf{v} = \mathbf{0}$ to get $\mathbf{v} = \mathbf{0}$, a contradiction to the assumption that $\mathbf{v} \neq \mathbf{0}$.

What about a set of two vectors? Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be distinct vectors in $\mathbf{R}^n$. The set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if and only if one of $\mathbf{v}_1, \mathbf{v}_2$ is a scalar multiple of the other.

Example 4. Determine whether or not $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\5\\6\end{bmatrix}$ is linearly independent. Since the second vector is not a multiple of the first vector, the vectors are linearly independent. A much more complicated way to see this is to row reduce:

$$\begin{bmatrix}1 & 2\\2 & 5\\3 & 6\end{bmatrix} \to \begin{bmatrix}1 & 2\\0 & 1\\0 & 0\end{bmatrix}.$$

The second matrix is in echelon form and both columns have a pivot. So the vectors are linearly independent. □

Linear Dependence of a Set containing 0:
Any finite set of distinct vectors in $\mathbf{R}^n$ that contains the zero vector is linearly dependent.

Reason: Suppose our set is $\{\mathbf{v}_1, ..., \mathbf{v}_k, \mathbf{0}\}$ (this is a set of $k+1$ vectors). Then

$$0\mathbf{v}_1 + \cdots + 0\mathbf{v}_k + 1\mathbf{0} = \mathbf{0}.$$

expresses $\mathbf{0}$ as a non-trivial linear combination of the vectors, so the set is linearly dependent.

Linear Dependence of a Set Containing More Than n Vectors in $\mathbf{R}^n$:
If $k > n$, then any set of k vectors in $\mathbf{R}^n$ is linearly dependent.

To see why, suppose that $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ is a set of vectors in $\mathbf{R}^n$ where $k > n$. Form the matrix A whose columns are the k vectors. Since $k > n$, there are more columns than rows. Therefore, there must be at least one free column. So the set is linearly dependent.

That concludes our brief detour into some quick ways of telling whether or not a set is linearly dependent. Of course, most of the time, things will not be so simple and you'll have to row reduce a matrix.

The next theorem relates linear independence and span.

Linear Dependence of a Set of At Least Two Vectors:

If $k \geq 2$, a set $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ of vectors in $\mathbf{R}^n$ is linearly dependent when at least one of the vectors in the set is a linear combination of the others. In other words, at least one of the vectors is in the span of the other vectors.

Suppose the set $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ is linearly dependent. There are constants $c_1, ..., c_k$, not all 0, such that

$$c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{0}.$$

Suppose that $i \in \{1, ..., k\}$ satisfies $c_i \neq 0$. Then

$$\mathbf{v}_i = -\frac{1}{c_i}(c_1\mathbf{v}_1 + \cdots + c_{i-1}\mathbf{v}_{i-1} + c_{i+1}\mathbf{v}_{i+1} + \cdots + c_k\mathbf{v}_k).$$

Conversely, suppose that $\mathbf{v}_i$ is a linear combination of the other vectors in the set, say

$$\mathbf{v}_i = a_1\mathbf{v}_1 + \cdots + a_{i-1}\mathbf{v}_{i-1} + a_{i+1}\mathbf{v}_{i+1} + \cdots + a_k\mathbf{v}_k.$$

Then

$$a_1\mathbf{v}_1 + \cdots + a_{i-1}\mathbf{v}_{i-1} - \mathbf{v}_i + a_{i+1}\mathbf{v}_{i+1} + \cdots + a_k\mathbf{v}_k$$

is a non-trivial linear combination of $\mathbf{v}_1, ..., \mathbf{v}_k$ that equals $\mathbf{0}$.

This is the "intuitive" way of thinking about linear dependence and independence. Basically, a set of vectors is linearly independent when there is no "redundancy": Every time you include a new vector into the set, the span of the vectors gets bigger.

Example 5. Suppose $\mathbf{v}_1 \in \mathbf{R}^2$.

- If $\mathbf{v}_1 = \mathbf{0}$ then the span of $\mathbf{v}_1$ is just $\{\mathbf{0}\}$.
- If $\mathbf{v}_1 \neq \mathbf{0}$, then the span of $\mathbf{v}_1$ consists of the line passing through $\mathbf{v}_1$.

Now assume that $\mathbf{v}_1 \neq \mathbf{0}$ and suppose $\mathbf{v}_2 \in \mathbf{R}^2$.

- If $\mathbf{v}_2$ is a scalar multiple of $\mathbf{v}_1$ then $\mathbf{v}_2 \in \text{Span}\{\mathbf{v}_1\}$ and the span of $\mathbf{v}_1$ and $\mathbf{v}_2$ is the line containing $\mathbf{v}_1$ and $\mathbf{v}_2$.
- If $\mathbf{v}_2$ is not a scalar multiple of $\mathbf{v}_1$ then the span of $\mathbf{v}_1$ and $\mathbf{v}_2$ is $\mathbf{R}^2$.
- Let $\mathbf{v}_3 \in \mathbf{R}^2$. If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ lie on the same line, the span of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ is the line. Otherwise the span is $\mathbf{R}^2$, and so on.

Example 6. Suppose $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are in $\mathbf{R}^3$.

- If $\mathbf{v}_1 = \mathbf{0}$, the span of $\mathbf{v}_1$ is just $\{\mathbf{0}\}$.
- If $\mathbf{v}_1 \neq \mathbf{0}$, the span of $\mathbf{v}_1$ is the line through $\mathbf{v}_1$.
- Suppose that $\mathbf{v}_1 \neq \mathbf{0}$. Now we bring in $\mathbf{v}_2$. If $\mathbf{v}_2$ is a scalar multiple of $\mathbf{v}_1$, then the span of $\mathbf{v}_1$ and $\mathbf{v}_2$ is still just the line spanned by $\mathbf{v}_1$.
- If $\mathbf{v}_1$ and $\mathbf{v}_2$ are both not the zero vector, and $\mathbf{v}_2$ is not a scalar multiple of $\mathbf{v}_1$ (this is when $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent), then the span of $\mathbf{v}_1$ and $\mathbf{v}_2$ is the plane that contains $\mathbf{v}_1$ and $\mathbf{v}_2$.
- Now we bring in $\mathbf{v}_3$, assuming that $\mathbf{v}_1$ and $\mathbf{v}_3$ are both non-zero. If $\mathbf{v}_2$ and $\mathbf{v}_3$ are both scalar multiples of $\mathbf{v}_1$, then $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ all lie on a line, and the span of $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ is that line.
- If $\mathbf{v}_1$ and $\mathbf{v}_2$ lie on the same line but $\mathbf{v}_3$ is not a scalar multiple of $\mathbf{v}_1$ or $\mathbf{v}_2$, then the span of $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ is the plane that contains $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.
- If $\mathbf{v}_1$ and $\mathbf{v}_2$ do not lie on the same line and $\mathbf{v}_3$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, then there is a unique plane that contains $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$, and the span of $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ is that plane.
- If $\mathbf{v}_2$ is not a linear combination of $\mathbf{v}_1$, and $\mathbf{v}_3$ is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ (this is when $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent), then the span of $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ is $\mathbf{R}^3$.

For a square matrix, there is a pivot in every row precisely when there is a pivot in every column.

n **Vectors in** $\mathbf{R}^n$

A collection of n vectors in $\mathbf{R}^n$ span $\mathbf{R}^n$ precisely when they are linearly independent.

We think of the $\{\mathbf{0}\}$ as being 0-dimensional, a line as 1-dimensional, a plane as 2-dimensional, and $\mathbf{R}^3$ as 3-dimensional. Every time we include a vector that is not a linear combination of the previous ones, the dimension of the span increases by 1:

- If $\{\mathbf{v}_1\}$ is linearly independent, $\text{Span}\{\mathbf{v}_1\}$ is a line, which is 1-dimensional.
- If $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent, then $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is a plane, which is 2-dimensional.
- If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, then $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is 3-dimensional.

This gives us another way of understanding why if $k > n$, then k distinct vectors in $\mathbf{R}^n$ must be linearly dependent: If the first n vectors are linearly independent, their span must be $\mathbf{R}^n$. Then when you include the next vector (the $n+1$st vector), that vector must be in $\mathbf{R}^n$, which is the span of the previous n vectors. The span of k linearly independent vectors would have a dimension that is larger than n, the number of "available" dimensions. We don't have a precise definition of dimension yet – we will later. Still, we can use the fact that a line is 1-dimensional and a plane is 2-dimensional, and $\mathbf{R}^3$ is 3-dimensional.

Linear Independence and the Dimension of the Span

The span of a collection of k *linearly independent* vectors in $\mathbf{R}^n$ is k-dimensional.

To recap where we are: We have learned about the concept of linear combinations and linear independence. We've seen that asking for the solutions to a linear system amounts to solving a matrix-vector equation $A\mathbf{x} = \mathbf{b}$, where A is the coefficient matrix and $\mathbf{b}$ is the vector of constants. We can think of $A\mathbf{x}$ as the product of the matrix A and the vector $\mathbf{x}$. The result is a vector $\mathbf{b}$, and we want to know what all the vectors $\mathbf{x}$ are that, when multiplied by the matrix A on the left, is equal to $\mathbf{b}$. Thinking of A as a function that maps x to the vector $\mathbf{b}$ gives us a new way of thinking about matrix-vector multiplication, and it is the subject of the next chapter.

Exercise 3. (a) Is $\{\mathbf{0}\}$ linearly independent or linearly dependent?

(b) If $\mathbf{v}$ is a non-zero vector, is $\{\mathbf{v}\}$ linearly independent or linearly dependent?

Exercise 4. Give an example of the following.

(a) Three vectors $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ in $\mathbf{R}^2$ such that

$$\{\mathbf{v}_1, \mathbf{v}_2\}, \{\mathbf{v}_1, \mathbf{v}_3\}, \text{ and } \{\mathbf{v}_2, \mathbf{v}_3\}$$

are all linearly independent sets. Is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly dependent set?

(b) Four vectors in $\mathbf{R}^4$ that are linearly dependent.

Exercise 5. Suppose that $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ are linearly independent vectors in $\mathbf{R}^3$. Let A be the 3×3 matrix whose columns are $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$. What is RREF(A)? Give the matrix (with entries). Justify your answer. Hint: Think about the pivots.

Exercise 6. The adjective "linearly independent" refers to a set of vectors, not a matrix, so we should not say, "The matrix A is linearly dependent." For this matrix: $A = \begin{bmatrix} 1 & -2 & 4 & -1 & 7 \\ 0 & 0 & 1 & 5 & 8 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

(a) Prove that the rows of A form a linearly independent set.

(b) Notice that A is in row echelon form. Do you think that the rows of any matrix in row echelon form are linearly independent? If not, give a counterexample.

(c) Explain how we know that the columns of A form a linearly dependent set. This question doesn't require any computations – there's a fact you can use.

(d) Do you think that the columns of every matrix in row echelon form are linearly dependent? If not, give a counterexample.

Exercise 7. (a) How many pivot columns must a 7×5 matrix have if its columns are linearly independent? Why?

(b) How many pivot columns must a 5×7 matrix have if its columns span $\mathbf{R}^5$? Why?

Exercise 8. (a) Given $A = \begin{bmatrix} 2 & 3 & 5 \\ -5 & 1 & -4 \\ -3 & -1 & -4 \\ 1 & 0 & 1 \end{bmatrix}$, observe that the third column is the sum of the first two columns. Find a nontrivial solution of $A\mathbf{x} = \mathbf{0}$.

(b) Given $A = \begin{bmatrix} 4 & 1 & 6 \\ -7 & 5 & 3 \\ 9 & -3 & 3 \end{bmatrix}$, observe that the first column plus twice the second column equals the third column. Find a nontrivial solution of $A\mathbf{x} = \mathbf{0}$.

Exercise 9. Is there a matrix A such that the only solution to $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is $\begin{bmatrix} 2 \\ 0 \\ 2 \\ 4 \end{bmatrix}$? If so, write down an example. If not, explain why not.

Exercise 10. Is there a matrix A such that the only solution to $A\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 4 \end{bmatrix}$ is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$? If so, write down an example. If not, explain why not.

Exercise 11. Show that if $\mathbf{v}_1, ..., \mathbf{v}_4$ are in $\mathbf{R}^4$ and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly dependent.

Exercise 12. Show that if $\mathbf{v}_1, ..., \mathbf{v}_4$ are linearly independent vectors in $\mathbf{R}^4$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is also linearly independent.

Exercise 13. Let

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 & -5 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Is this matrix in echelon form?
(b) Is this matrix in reduced row echelon form?
(c) How many pivot rows does A have? How many pivot columns?
(d) Determine whether or not the pivot rows are linearly independent.
(e) Determine whether or not the pivot columns are linearly independent.

Exercise 14. Suppose that A is a 3 by 3 matrix and the matrix equation $A\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ has a unique solution. Give all possibilities for the RREF of A.

Exercise 15. Suppose that A is a 3 by 2 matrix and the matrix equation $A\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ has a unique solution. Give all possibilities for the RREF of A.

Exercise 16. Suppose that A is an $m \times n$ matrix and the matrix equation $A\mathbf{x} = \mathbf{0}$ has a unique solution. Give all possibilities for the RREF of A.