Diagonalization

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In this section, you will

- Determine if a matrix is diagonalizable
- Diagonalize a matrix
- Use a diagonalization of a matrix to calculate powers of a matrix.

Suppose that the population of a certain metropolitan area remains constant but there is a continual movement of people between the city and the suburbs. Suppose that 90% of people currently living in the city will live next year in the city, with the other 10% moving to the suburbs. And suppose that of the people currently in the suburbs, 2% will move to the city next year while the other 98% remain in the suburbs. Suppose that right now, 70% of the population of the metropolitan area live in the city and 30% of the population is in the suburbs. Then in five years, what will the population look like? And what will the population in the long run look like, over the course of many years?

This is a simplified problem in demography, which is the statistical study of human populations. We'll explain how to use matrices to solve this problem.

We know that a diagonal matrix scales the standard basis vectors. For example, the matrix

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

scales $\mathbf{e}_1$ by a factor of 2, $\mathbf{e}_2$ by a factor of 3, and $\mathbf{e}_3$ by a factor of 4. The matrix $\begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix}$ scales the vector $\langle 5, 4 \rangle$ by a factor of 7 and scales the vector $\langle -1, 1 \rangle$ by a factor of -2. So while this matrix is not diagonal, it is "similar" to a diagonal matrix in the sense that it stretches one line by a certain factor and another line by a certain factor. The lines might not be perpendicular.

Diagonalizable Matrices

An $n \times n$ matrix A is said to be **diagonalizable** if there are n linearly independent eigenvectors for the matrix A.

Not every matrix is diagonalizable. For example, the matrix $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable. There are many ways to see why. One way is to observe that 1 is the only eigenvalue of A. If A scales two linearly independent eigenvectors by a factor of 1, then we've seen, in our discussion of the eigenspace, that A has to scale any linear combination of those eigenvectors by a factor of 1. The span of two linearly independent eigenvectors in $\mathbf{R}^2$ is all of $\mathbf{R}^2$, and the only matrix that scales every vector in $\mathbf{R}^2$ by a factor of 1 is the 2 by 2 identity matrix. Since A is not this matrix, A cannot have two linearly independent eigenvectors.

Every diagonal matrix is itself diagonalizable, because a diagonal matrix D can be written as IDI^{-1}.

Suppose that the vectors $\mathbf{v}_1, ..., \mathbf{v}_n$ form a basis for $\mathbf{R}^n$ and are also eigenvectors for A, with eigenvalues $\lambda_1, ..., \lambda_n$. Let P be the matrix whose jth column is $\mathbf{v}_j$ and let D be the diagonal matrix $\text{diag}(\lambda_1, ..., \lambda_n)$. Then $AP = PD$. Since the eigenvectors are linearly independent, P is invertible. Multiplying the equation $AP = PD$ by P^{-1} on the right, we get $A = PDP^{-1}$. This shows that if A has eigenvectors that form a basis for $\mathbf{R}^n$, then A is diagonalizable.

The other direction also holds: An $n \times n$ matrix A is diagonalizable if and only if A has eigenvectors that form a basis for $\mathbf{R}^n$. In fact, if $A = PDP^{-1}$, then the columns of P are eigenvectors for A and the a_{jj} term of D is an eigenvalue for the jth column of P.

For example, suppose that the matrix A has eigenvectors $\langle 1, -1, 1\rangle$ with eigenvalue 1 and eigenvectors $\langle -1, 1, 0\rangle$ and $\langle -1, 0, 1\rangle$ with eigenvalue -2. We can figure out what A must be: It is PDP^{-1} where the columns of P are the linearly independent eigenvectors and D is the diagonal matrix consisting of the eigenvalues:

$$A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1}.$$

We get

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}.$$

Exercise 1. Diagonalizability has nothing to do with invertibility.

(a) Give a 2 by 2 matrix that is diagonalizable but not invertible.

(b) Give a 2 by 2 matrix that is invertible but not diagonalizable.

Going back to the problem before, it's annoying to compute A^n when n is large. However, it's easy if A is diagonalizable. If $A = PDP^{-1}$, then

$$A^2 = PDP^{-1}PDP^{-1}.$$

Then $P^{-1} \cdot P = I$, so $A^2 = PD^2P^{-1}$. To find A^3, we get

$$A^3 = AA^2 = PDP{-}1PD^2P^{-1}$$

and using the same fact that $P^{-1}P = I$, we get

$$A^3 = PD^3P^{-1}.$$

And so on. If $A = PDP^{-1}$, then $A^n = PD^nP^{-1}$. And it's easy to compute D^n because you just raise each diagonal entry to the nth power.

Powers of Diagonalizable Matrices

If A is an $n \times n$ matrix of the form

$$A = PDP^{-1}$$

for an invertible $n \times n$ matrix P and a diagonal $n \times n$ matrix D, then

$$A^k = PD^kP^{-1}.$$

Exercise 2. Find a 2×2 matrix A with eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 3$ and corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$. Your answer should be a single matrix.

Exercise 3. Consider the matrix $A = \begin{bmatrix} -3 & 4 \\ -2 & 3 \end{bmatrix}$.

(a) Write A in the form $A = PDP^{-1}$.
(b) Compute A^{2022}. Your answer should be a single matrix.
(c) Compute A^{2023} Your answer should be a single matrix.

Exercise 4. Decide whether or not the matrix $B = \begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix}$ is diagonalizable. Justify your answer.

Exercise 5. (a) For the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

quickly decide if 1 is an eigenvalue of A.

(b) What are the eigenvalue(s) of RREF(A)?

Lesson: The eigenvalues are usually *not* the pivots of A. Row operations usually change the eigenvalues, so you cannot use the echelon form of A to calculate the eigenvalues of A.

Exercise 6. This problem is about the following matrix:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}$$

(a) Give the eigenvalues of A.
(b) Find a basis for $\mathbf{R}^2$ consisting of eigenvectors for A.
(c) Give a diagonal matrix D and invertible matrix P such that $A = PDP^{-1}$. Do NOT compute P^{-1} explicitly.
(d) Give a simple expression for A^{10} as a product of other matrices (not including A itself). Give the entries of your matrices explicitly and make your expression as simple as possible, without doing excessive arithmetic or finding inverses.

Exercise 7. In this problem, A is the matrix that can be written like this:

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

Decide which of these statements is true or false.

(a) A is diagonalizable.
(b) $\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$ is an eigenvector of A with eigenvalue 3.
(c) $\det(A) = 18$
(d) $\det(2A) = 36$.
(e) A is invertible.
(f) A is also equal to

$$\begin{bmatrix} 0 & 2 & -3 \\ 5 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} `3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 2 & -3 \\ 5 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}^{-1}$$

Exercise 8. Calculate A^5 if $A = PDP^{-1}$ where

$$P = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 4 \\ 0 & 0 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Exercise 9. Find the matrix A that has the given eigenvalues and corresponding eigenvectors: $\lambda_1 = 1; \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$, $\lambda_2 = 0; \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$, $\lambda_2 = 1; \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

Exercise 10. The 3×3 matrix A has eigenvalues $\lambda = 2, 0$, and -1, and the corresponding eigenvectors are $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$, respectively.

(a) Give a diagonal matrix D and invertible matrix P such that $A = PDP^{-1}$. Do NOT compute P^{-1} explicitly.

(b) Is A invertible? Explain in 1 sentence.

(c) Determine the rank of A. Justify your answer.

(d) Give a simple expression for A^{10} as a product of other matrices (not including A itself). Give the entries of your matrices explicitly and make your expression as simple as possible, without doing excessive arithmetic or finding inverses.

Exercise 11. Suppose that a 7×7 diagonalizable matrix has 3 distinct eigenvalues, one with an eigenspace of dimension 1 and another with an eigenspace of dimension 2. What is the dimension of the third eigenspace?

Exercise 12. Suppose that A is a 3 by 3 matrix with eigenvalues 0, 1, and -1. Find all positive integers k such that $A^k = A$.

Exercise 13. Let $A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix}$. Its characteristic polynomial turns out to be

$$p(\lambda) = -(\lambda - 3)^2(\lambda - 5).$$

(Don't bother checking this.)

(a) Give the eigenvalues of A.

(b) Find a basis of $\mathbf{R}^3$ consisting of eigenvectors of A, if such a basis exists. If there is no such basis, explain why.

(c) If it is possible, express A in the form $A = PDP^{-1}$ where P is invertible and D is a diagonal matrix. If it is not possible, explain why.

Exercise 14. (This problem requires familiarity with projections onto a line.) Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be the linear transformation that projects vectors onto the line spanned by the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(a) Find a basis for the kernel of T. It may help to take vectors where one of the coordinates is 0.

(b) Find a basis for the 1-eigenspace, i.e., the set of eigenvectors with eigenvalue 1, together with the zero vector.

(c) If A is the standard matrix of T, express A in the form $A = PDP^{-1}$. Use specific matrices for P and D, but do NOT compute P^{-1}.

(d) Express A^{20} in terms of P and D. How does your answer compare to your answer in part c?

Exercise 15. Let A be the 5 by 5 matrix whose every entry is 8:

$$\begin{bmatrix} 8 & 8 & 8 & 8 & 8 \\ 8 & 8 & 8 & 8 & 8 \\ 8 & 8 & 8 & 8 & 8 \\ 8 & 8 & 8 & 8 & 8 \\ 8 & 8 & 8 & 8 & 8 \end{bmatrix}$$

(a) Find a basis for the nullspace of A.

(b) Show that $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector for A. What is the associated eigenvalue?

(c) Find matrices P and D such that $A = PDP^{-1}$.

Exercise 16. Let A be the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & -3 \\ 3 & 2 & -1 \end{bmatrix}$$

(a) Give all eigenvalues of A.

(b) For each (distinct) eigenvalue λ, give the dimension of the λ-eigenspace (the geometric multiplicity).

(c) Is A diagonalizable? Explain briefly. DO NOT find matrices P, D etc.

Exercise 17. Repeat the problem above for the matrix

$$B = \begin{bmatrix} 3 & -1 & 1 \\ 7 & -5 & 1 \\ 6 & -6 & 2 \end{bmatrix}$$

The characteristic polynomial of B is $-(\lambda - 2)^2(\lambda + 4)$.

Exercise 18. Consider

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & -1 \end{bmatrix}$$

(a) Give the characteristic polynomial of A (Do not expand), and then state the eigenvalues.

(b) For each of the eigenvalues, calculate the dimension of the eigenspace. You do NOT need to find a basis for each eigenspace.

(c) **Using part (b),** decide whether or not A is diagonalizable. Answer YES or NO and explain in one or two sentences. DO NOT find matrices D, P, etc.