$I = [a,b]$, (a,b), $[a,b)$, $(a,b]$

define $\ell(I) = b-a$.

A <u>null set</u> $A \subseteq \mathbb{R}$ is a set that can be covered by a countable sequence of intervals of arbitrarily small length.

More precisely, we say A is null if, for every $\varepsilon > 0$, there is a sequence $I_1, I_2, \ldots$ of intervals such that

$$A \subseteq \bigcup_{n=1}^{\infty} I_n$$

and $\sum_{n=1}^{\infty} \ell(I_n) < \varepsilon$.

<u>Ex</u>: Any one-element set is null.

$$\{x\} \subseteq \underbrace{\left(x - \frac{\varepsilon}{3}, x + \frac{\varepsilon}{3}\right)}_{\text{has length } \frac{2}{3}\varepsilon < \varepsilon.}$$

<u>Ex</u>: Any countable ~~set~~ A is null.

Pf) Enumerate the elements of A as $x_1, x_2, x_3, \ldots$. Let $\varepsilon > 0$.

For each i, let $I_i = \left(x_i - \frac{\varepsilon}{4 \cdot 2^i}, x_i + \frac{\varepsilon}{4 \cdot 2^i}\right)$

Then $A \subseteq \bigcup I_n$ because each $x_i \in I_i$.

and $\sum \ell(I_n) = \sum_{n=1}^{\infty} \frac{\varepsilon}{2 \cdot 2^n} =$

$= \frac{\varepsilon}{2} \cdot \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{\varepsilon}{2} < \varepsilon.$

More generally, if $(N_n)_{n \geq 1}$ is a sequence of null sets, then their union

$$N = \bigcup_{n=1}^{\infty} N_n$$

is also null.

pf) Let $\varepsilon > 0$. Since N_1 is null,

$\exists\ I_k^1$, $k \geq 1$, such that

$$\sum_{k=1}^{\infty} \ell(I_k^1) < \frac{\varepsilon}{2} \quad \text{and} \quad N_1 \subseteq \bigcup_{k=1}^{\infty} I_k^1.$$

$N_2 \ldots$ $\exists\ I_k^2$, $k \geq 1$, s.t.

$$\sum_{k=1}^{\infty} \ell(I_k^2) < \frac{\varepsilon}{4} \quad \text{and} \quad N_2 \subseteq \bigcup_{k=1}^{\infty} I_k^2.$$

in general, N_n, $\exists\ I_k^n$, $k \geq 1$ s.t.

$$\sum_{k=1}^{\infty} \ell(I_k^n) < \frac{\varepsilon}{2^n}.$$

The set $\{ I_k^n : n \geq 1, k \geq 1 \}$

is countable and

$$\sum_{n=1, k=1}^{\infty, \infty} l(I_k^n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \cdots = \varepsilon .$$

and $N \subseteq \bigcup_{n,k} I_k^n$.

Uncountable sets can be null.

<u>Ex:</u> Cantor set

$l(C_0) = 1$ $\quad C_0 = [0, 1]$ Remove the middle third

$l(C_1) = \frac{2}{3}$ $\quad C_1 = \underbrace{[0, \tfrac{1}{3}]}_{I_{11}} \cup \underbrace{[\tfrac{2}{3}, 1]}_{I_{12}}$ Remove the middle third of each of these two intervals

$l(C_2) = \frac{4}{9}$ $\quad C_2 = \underbrace{[0, \tfrac{1}{9}]}_{I_{21}} \cup [\tfrac{2}{9}, \tfrac{3}{9}] \cup [\tfrac{6}{9}, \tfrac{7}{9}] \cup [\tfrac{8}{9}, 1]$

$\vdots$

Keep going $\quad C_1 \supset C_2 \supset C_3 \supset C_4 \supset \ldots$

Let $C = \bigcap_{i=1}^{\infty} C_i$. (Cantor set)

One can show that C is uncountable.

Why is C a null set?

$$\ell(C_n) = \left(\frac{2}{3}\right)^n$$

Given $\varepsilon > 0$, pick n large enough enough so that $\left(\frac{2}{3}\right)^n < \varepsilon$.

$$C \subset \underset{\bigcup I_n}{\underset{\parallel}{C_n}}, \quad \text{and} \quad \sum_n \ell(I_n) < \varepsilon.$$

so C is a null set.