Eigenvalues and Eigenvectors

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1 Definition

Eigenvalues and eigenvectors have new information about a square matrix. Properties of a matrix are reflected in the properties of the eigenvalues and eigenvectors. In this section, you will

- Determine whether a vector is an eigenvector.
- Relate the eigenvalues of A to the eigenvalues of A^{-1} and powers of A.

Eigenvalues and Eigenvectors

Let A be a square matrix. An **eigenvector** is a non-zero vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$ for some scalar λ. The scalar λ is called an **eigenvalue**.

Geometrically, an eigenvector is a vector that A scales. Notice that λ can be 0, but an eigenvector must be a non-zero vector. For example, the $n \times n$ identity matrix I_n fixes every vector of $\mathbf{R}^n$, so every non-zero vector in $\mathbf{R}^n$ is an eigenvector for I_n, and 1 is the only eigenvalue.

As another example, the vector $\begin{bmatrix} 6 \\ -5 \end{bmatrix}$ is an eigenvector with eigenvalue -4 for $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, because $A\begin{bmatrix} 6 \\ 5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4\begin{bmatrix} 6 \\ -5 \end{bmatrix}$, but $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ is not an eigenvector because $A\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix}$, which is not a scalar multiple of $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$. It is easy to tell whether or not a non-zero vector is an eigenvector for a particular matrix – just multiply the matrix by the vector and see if the result is a scalar multiple of the vector. If it is, then the vector is an eigenvector; if not, it is not.

Exercise 1. Review the definition of *eigenvector* and *eigenvalue*. Can 0 be an eigenvalue? Can $\mathbf{0}$ be an eigenvector?

Exercise 2. (a) What are the eigenvectors/eigenvalues of the identity matrix?

(b) What are the eigenvectors/eigenvalues of the (square) zero matrix?

Exercise 3. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

(a) If you are given a vector $\mathbf{v}$ in $\mathbf{R}^2$ and want to know if it is an eigenvector of A, what are the steps to do so?

(b) Check that $\mathbf{v} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ is an eigenvector of A. What is the corresponding eigenvalue?

(c) Check that $\mathbf{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not an eigenvector of A.

(d) How many distinct eigenvalues does it have? Answer: 0; the only eigenvalue is 1.

Exercise 4. Let $A = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}$.

(a) Determine whether or not $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an eigenvector for A. If it is, give the eigenvalue corresponding to this vector.

(b) Determine whether or not $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is an eigenvector for A. If it is, give the eigenvalue corresponding to this vector.

If $\mathbf{v}$ is an eigenvector for A with eigenvalue λ, then so is $c\mathbf{v}$ for any non-zero scalar c. If $A\mathbf{v} = \lambda v$, then by the linearity of matrix-vector multiplication, $A(c\mathbf{v}) = c(A\mathbf{v})$. Thus

$$A(c\mathbf{v}) = c(A\mathbf{v}) = c(\lambda\mathbf{v}) = \lambda(c\mathbf{v})$$

Therefore, $c\mathbf{v}$ is an eigenvector for A with eigenvalue λ. Geometrically, what this says is that if A scales a vector $\mathbf{v}$ by a factor of λ, then it stretches the line spanned by $\mathbf{v}$ by a factor of λ.

The sum of two eigenvectors for A with the same eigenvalue λ is also an eigenvector for A (as long as the sum is non-zero), with the same eigenvalue λ. This also is an immediate consequence of the linearity of matrix-vector multiplication. Suppose that $\mathbf{v}$ and $\mathbf{w}$ are eigenvectors for A with eigenvalue λ. Then

$$A(\mathbf{v} + \mathbf{w}) = A\mathbf{v} + A\mathbf{w} = \lambda\mathbf{v} + \lambda\mathbf{w} = \lambda(\mathbf{v} + \mathbf{w}).$$

Therefore, if $\mathbf{v}$ and $\mathbf{w}$ are scaled by a factor of λ, then so is $\mathbf{v} + \mathbf{w}$.

Eigenspace

Let A be an $n \times n$ matrix and λ an eigenvalue of A. The set of eigenvectors with eigenvalue λ, together with the zero vector, is a subspace of $\mathbf{R}^n$. This space is called the **λ-eigenspace**.

If $\mathbf{v}$ and $\mathbf{w}$ are eigenvectors for a matrix A with different eigenvalues, then the sum $\mathbf{v} + \mathbf{w}$ might not be an eigenvector of A. For example, if $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ then $\mathbf{e}_1$ is an eigenvector with eigenvalue 2 and $\mathbf{e}_2$ is an eigenvector with eigenvalue 3, but $A(\mathbf{e}_1 + \mathbf{e}_2) = 2\mathbf{e}_1 + 3\mathbf{e}_2$, which is not a scalar multiple of $\mathbf{e}_1 + \mathbf{e}_2$. Therefore $\mathbf{e}_1 + \mathbf{e}_2$ is not an eigenvector of A.

Exercise 5. For square matrices A, what is another word for the 0-eigenspace of A?

In general, row operations change the eigenvalue and eigenvectors of a matrix. The topic of eigenvalues and eigenvectors is not very related to row reduction. You can find the determinant of a matrix by row reduction, but not the eigenvalues – an echelon form of a matrix A usually does *not* display the eigenvalues of A.

Exercise 6. If A is an invertible matrix, how are the eigenvalues of A^{-1} related to those of A?

Exercise 7. Suppose that $\mathbf{v}$ is an eigenvector for a 3×3 matrix A and $\mathbf{w}$ is an eigenvector for a 3×3 matrix B. Is it true that $\mathbf{v} + \mathbf{w}$ is an eigenvector for $A + B$?

Exercise 8. Suppose that $\mathbf{v}$ is an eigenvector for a 3×3 matrix A with eigenvalue 2 and $\mathbf{w}$ is an eigenvector for a 3×3 matrix B with eigenvalue 2. Is it true that $\mathbf{v} + \mathbf{w}$ is an eigenvector for $A + B$?

Exercise 9. Suppose that $\mathbf{v}$ and $\mathbf{w}$ are eigenvectors for a square matrix A. Is it true that $\mathbf{v} + \mathbf{w}$ is an eigenvector of A?

Exercise 10. Suppose that $\mathbf{v}$ and $\mathbf{w}$ are eigenvectors for a square matrix A and that they have the same eigenvalue. Is it true that $\mathbf{v} + \mathbf{w}$ is an eigenvector of A?

Exercise 11. Suppose that A and P are $n \times n$ matrices, and P is invertible. Show that if $\mathbf{v}$ is an eigenvector of A with eigenvalue λ, then $P\mathbf{v}$ is an eigenvector of PAP^{-1} with eigenvalue λ.

Exercise 12. Let A be a square matrix and c be a real number. How do the eigenvalues of A relate to those of $A + cI$?

Exercise 13. (a) Show that $A^3 = 0$ (the zero matrix) if

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) Suppose that A (not the same matrix as in part (a)) is a square matrix that has a non-zero eigenvalue. Explain why A^k is never the zero matrix (k is any positive integer).

Exercise 14. Let A be a square matrix and let $\mathbf{v}$ be an eigenvector of A with a non-zero eigenvalue. Show that $\mathbf{v}$ is in the column space of A.

2 The Characteristic Polynomial

In this section you will

- Calculate the characteristic polynomial of a matrix.
- Calculate eigenvalues and eigenvectors of a matrix by using the characteristic polynomial.
- Calculate the eigenspaces of a matrix.
- Relate the characteristic polynomial of a matrix to its determinant.

Now we will discuss how to find the eigenvalues and eigenvectors of a matrix. Suppose that $A\mathbf{v} = \lambda\mathbf{v}$. The key observation is that we can view this equation as a *homogeneous* equation. Subtract both sides by $\lambda\mathbf{v}$:

$$A\mathbf{v} - \lambda\mathbf{v} = \mathbf{0}.$$

This is the same as saying

$$A\mathbf{v} - \lambda I\mathbf{v} = \mathbf{0}.$$

This factors as

$$(A - \lambda I)\mathbf{v} = \mathbf{0}.$$

This is a homogeneous equation with coefficient matrix $A - \lambda I$. We conclude the following. Let $\mathbf{v}$ be a non-zero vector. Then $\mathbf{v}$ is an eigenvector of A with eigenvalue λ if and only if $\mathbf{v}$ is an element of the nullspace of $A - \lambda I$. Put differently: The λ-eigenspace of A is precisely the nullspace of $A - \lambda I$.

Therefore, λ is an eigenvalue for A if and only if $A - \lambda I$ has a non-trivial nullspace. Since $A - \lambda I$ is a square matrix, this is equivalent to the condition that the determinant of $A - \lambda I$ is 0: The eigenvalues of a square matrix A are the scalars λ such that $\det(A - \lambda I) = 0$. We call $p(\lambda) = \det(A - \lambda I)$ the **characteristic polynomial** of A. The eigenvalues of λ are the roots of $p(\lambda)$.

The Characteristic Polynomial of A

Suppose that A is an $n \times n$ matrix. We call $p(\lambda) = \det(A - \lambda I)$ the **characteristic polynomial** of A. It is a degree n polynomial.
The eigenvalues of A are the roots of the characteristic polynomial $p(\lambda)$.

Some authors define the characteristic polynomial of A to be $\det(\lambda I - A)$ instead. This is equal to $(-1)^n \det(A - \lambda I)$.

As an example, we will calculate the characteristic polynomial, eigenvalues, and eigenvectors for the matrix $A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$. The characteristic polynomial is $\lambda^2 - 3\lambda + 2$. Since $\lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1)$, the eigenvalues of A are 2 and 1. Finally, for the eigenvectors of each eigenvalue, we calculate a basis for the nullspace of $A - \lambda I$ (Review the section on the nullspace if you forgot how to do this!).

For $\lambda_1 = 2$, $A - 2I = \begin{bmatrix} 2 & -3 \\ 2 & -3 \end{bmatrix} \sim \begin{bmatrix} 2 & -3 \\ 0 & 0 \end{bmatrix}$. The nullspace of $A - 2I$ is 1-dimensional, spanned by $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$. For $\lambda_2 = 1$, $A - I = \begin{bmatrix} 3 & -3 \\ 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 3 & -3 \\ 0 & 0 \end{bmatrix}$. The nullspace of $A - I$ is 1-dimensional, spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Now we have a good idea of how the matrix A acts on $\mathbf{R}^2$: It scales the line spanned by $\langle 3, 2 \rangle$ by a factor of 2, and it fixes the vectors spanned by $\langle 1, 1 \rangle$. Notice that this description is simpler than the description we get from just looking at A. The columns of A tell us what A does to $\mathbf{e}_1$ and $\mathbf{e}_2$: A sends $\mathbf{e}_1$ to $\langle 4, 2 \rangle$ and $\mathbf{e}_2$ to $\langle -3, 1 \rangle$. This isn't as nice of a description.

To compute the eigenvalues of the matrix $A = \begin{bmatrix} 1 & -4 \\ 4 & 2 \end{bmatrix}$, we first find the characteristic polynomial of A to be $\lambda^2 - 3\lambda + 18$. From the quadratic formula, we see that the eigenvalues of A are

$$\lambda = \frac{3 \pm \sqrt{-63}}{2}.$$

The matrix A has no real eigenvalues. Instead, it has two *complex* eigenvalues.

Exercise 15. Review the definition of *characteristic polynomial.* The characteristic polynomial of a matrix A is defined to be [fill in the blank]. The roots of the characteristic polynomial are the [fill in the blank] of A.

Exercise 16. If $p(t)$ is the characteristic polynomial of A, then $p(0)$ is the [fill in the blank] of A.

Exercise 17. Suppose that 0 is a root of the characteristic polynomial of a matrix A. What does that say about the nullspace of A? Can A be invertible?

Exercise 18. Compute the characteristic polynomial of the matrix

$$A = \begin{bmatrix} 7 & -2 \\ 2 & 3 \end{bmatrix}$$

Exercise 19. For this matrix A

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 1 & 4 & 0 \\ -6 & 7 & -1 \end{bmatrix}$$

(a) Find the characteristic polynomial for the matrix A.
(b) Find the real eigenvalues for the matrix A.
(c) Find a basis for each eigenspace for the matrix A.

Exercise 20. Repeat for the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Exercise 21. Gregg was working on this problem: Find the eigenvalues of the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & -3 \\ 3 & 2 & -1 \end{bmatrix}.$$

Here's what Gregg did. He changed row 2 to row 2 minus 3 times row 3:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & -3 \\ 3 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ -7 & -2 & 0 \\ 3 & 2 & -1 \end{bmatrix}.$$

Gregg knew that this is a triangular matrix and the eigenvalues are the diagonal entries, so he said that the eigenvalues are 1, -2, and -1. What do you think of Gregg's work? If it's wrong, what is the correct answer?

Exercise 22. Find a basis for the eigenspace of A associated with the given eigenvalue λ:

$$A = \begin{bmatrix} 9 & -4 & 12 \\ 8 & 1 & 8 \\ 8 & -4 & 13 \end{bmatrix}, \quad \lambda = 5.$$

Exercise 23. Show, using the characteristic polynomial, that a square matrix has the same eigenvalues as its transpose.

Exercise 24. Suppose that A is a square matrix with characteristic polynomial

$$(\lambda - 2)^2(\lambda - 6)^4(\lambda + 1).$$

(a) What are the dimensions of A? (Give n such that the dimensions are $n \times n$.)
(b) What are the eigenvalues of A?
(c) Is A invertible?
(d) What is the largest possible dimension for an eigenspace of A?

Exercise 25. Suppose that the characteristic polynomial of a matrix A is $p(\lambda) = \lambda^2(1-\lambda)(2-\lambda)$.
(a) Give the value of $\det(A - 5I)$.
(b) Is A invertible?

Exercise 26. Give an example of two different 2 by 2 matrices that have a characteristic polynomial of $\lambda^2 - 6\lambda + 8$.

Exercise 27. A certain 6×6 matrix has characteristic polynomial $\lambda^6 - 4\lambda^5 - 12\lambda^4$. Find the eigenvalues of the matrix. For each of the eigenvalues, state its (algebraic) multiplicity.

Exercise 28. Consider the matrix $A = \begin{bmatrix} 3 & -1 & 1 \\ 7 & -5 & 1 \\ 6 & -6 & 2 \end{bmatrix}$.

(a) Show that 2 is an eigenvalue of B, without computing the characteristic polynomial. (Hint: The eigenvalues are the numbers λ such that $A - \lambda I$ have a certain property.)

(b) Find a basis for the 2-eigenspace of B.

Exercise 29. Let λ be a scalar and let A and P be invertible $n \times n$ matrices. Let $B = PAP^{-1}$. Show that $\det(A - \lambda I) = \det(B - \lambda I)$. This shows that A and PAP^{-1} have the same (*fill in the blank*) polynomial.

Exercise 30. (Note: this problem requires the fact that the determinant is the product of the eigenvalues)

Let A be a 4 by 4 matrix with the following properties:

- two of the eigenvalues of A are 3 and 2.
- 3 is an eigenvalue of the matrix $A + 2I_4$.
- $\det(A) = 12$.

(a) What are all of the eigenvalues of A?

(b) Give the characteristic polynomial of A.

(c) Give the characteristic polynomial of A^{-1}.