X: random variable

$E[X^k]$ is called the k^{th} moment of X.

$$= \begin{cases} \sum_{j=0}^{\infty} x^j P(X=j) \\ \int x^k f(x)\,dx \end{cases}$$

$\phi_X(t) = E[e^{tX}]$ is called the

<u>moment generating function of X</u>.

<u>Thm:</u> Suppose $\phi_X(t) < \infty$ for

$t \in (-t_0, t_0)$ for some $t_0 > 0$.

Then $\phi^{(n)}(0) = E[X^n]$.

n^{th} derivative

<u>Ex:</u> $X \sim Exp(\lambda)$ density $\lambda e^{-\lambda x}$.

$$E[e^{tX}] = \int_0^\infty e^{tx} \lambda e^{-\lambda x}\,dx$$

$$= \lambda \int_0^\infty e^{-(\lambda-t)x} dx$$

when $t < \lambda$

$$= -\frac{\lambda}{\lambda-t} e^{-(\lambda-t)x} \Big|_0^\infty$$

$$= 0 + \frac{\lambda}{\lambda-t} e^{-0}$$

$$= \frac{\lambda}{\lambda-t} \leftarrow \text{mgf of } X.$$

E<u>x</u>: More generally, if $X \sim \text{Gamma}(n, \lambda)$,

$$E[e^{tX}] = \int_0^\infty e^{tx} \frac{\lambda^n x^{n-1}}{(n-1)!} e^{-\lambda x} dx$$

$$= \frac{\lambda^n}{(\lambda-t)^n} \underbrace{\int_0^\infty \frac{(\lambda-t)^n x^{n-1}}{(n-1)!} e^{-(\lambda-t)x} dx}_{1}$$

Gamma $(n, \lambda-t)$

$$\phi_X(t) = \frac{\lambda^n}{(\lambda-t)^n} \quad \text{mgf of Gamma}(n,\lambda).$$

Moments of Gamma(n,λ) dist.

$$\phi(t) = \lambda^n (\lambda - t)^{-n}$$

$$\phi'(t) = n\lambda^n (\lambda - t)^{-(n+1)}$$

$$\phi''(t) = n(n+1)\lambda^n (\lambda - t)^{-(n+2)}$$

$$\vdots$$

$$\phi^{(k)}(t) = n(n+1)\cdots(n+k-1)\lambda^n(\lambda-t)^{-(n+k)}$$

so $E[X^k] = \phi^{(k)}(0) =$

$$\boxed{\frac{n(n+1)\cdots(n+k-1)}{\lambda^k}}$$

In particular, if $X \sim Exp(\lambda)$

$$E[X^k] = \frac{k!}{\lambda^k}$$

<u>Ex</u>: Mgf of Poisson dist. Poisson(λ)

$$P(X=k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k=0,1,2,\ldots$$

$$E[e^{tX}] = \sum_{k=0}^{\infty} e^{tk} e^{-\lambda}\frac{\lambda^k}{k!}$$

$$= \sum_{k=0}^{\infty} e^{-\lambda}\frac{(\lambda e^t)^k}{k!}$$

$$= e^{-\lambda}\sum_{k=0}^{\infty}\frac{(\lambda e^t)^k}{k!}$$

$$= e^{-\lambda} e^{\lambda e^t}$$

$$= e^{\lambda e^t - \lambda}$$

$$E[X] = \phi'(0), \quad E[X^2] = \phi''(0)$$

$$\phi' = e^{\lambda e^t - \lambda}(\lambda e^t)$$

$$\phi'(t) = \lambda e^{\lambda e^t - \lambda + t}$$

$$\phi''(t) = \lambda e^{\lambda e^t - \lambda + t}(\lambda e^t + 1)$$

$$E[X] = \phi'(0) = \lambda e^{\lambda - \lambda + 0} = \lambda.$$

$$E[X^2] = \phi''(0) = \lambda e^{\lambda - \lambda + 0}(\lambda e^0 + 1)$$

$$= \lambda(\lambda+1) = \lambda^2 + \lambda$$

$$\mathrm{var}(X) = E[X^2] - (EX)^2$$

$$= \lambda^2 + \lambda - \lambda^2 = \lambda$$

When X is a non-negative integer-valued rv, the <u>generating function of X</u> is

$$\gamma_X(z) = E[z^X] = \sum_{k=0}^{\infty} z^k P(X=k).$$

(Use convention $0^0 = 1$).

<u>Rmk:</u> If we set $z = e^t$, get

$$E[e^{tX}] = mgf.$$

<u>Ex:</u> Poisson dist. (λ)

$$P(X=k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

$$\gamma(z) = E[z^X] = \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} z^k$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda z)^k}{k!}$$

$$= e^{-\lambda} e^{\lambda z} = e^{\lambda(z-1)}$$

<u>Thm:</u> let X be a nonneg. integer-valued r.v.

Then $$\gamma_X^{(n)}(1) = E[X(X-1)\ldots(X-n+1)].$$

In particular, $\gamma_X'(1) = E[X]$

and $\gamma_X''(1) = E[X^2 - X] = E[X^2] - E[X]$.

<u>Ex:</u> Poisson(λ)

$$\gamma(z) = e^{\lambda(z-1)}$$

$$\gamma'(z) = \lambda e^{\lambda(z-1)} \Rightarrow E[X] = \gamma'(1) = \lambda$$

$$\gamma''(z) = \lambda^2 e^{\lambda(z-1)} \Rightarrow E[X^2 - X] = \lambda^2$$

$\vdots$

$$\gamma^{(n)}(z) = \lambda^n e^{\lambda(z-1)}$$

$$\Rightarrow E[X(X-1)\dots(X-n+1)] = \lambda^n.$$

$$\mathrm{var}(X) = E[X^2] - (EX)^2$$

$$E(X^2) - E(X) = \lambda^2 \Rightarrow E[X^2] = \lambda^2 + \lambda.$$

$$\Rightarrow \mathrm{var}(X) = \lambda^2 + \lambda - \lambda^2 = \lambda,$$

<u>Ex:</u> $X \sim \mathrm{Geom}(p)$

$$P(X=k) = (1-p)^{k-1} p.$$

Calculate generating fn and find EX.

$$\gamma(z) = E[z^X] = \sum_{k=1}^{\infty} z^k (1-p)^{k-1} p$$

$$= \frac{p}{1-p} \sum_{k=1}^{\infty} [z(1-p)]^k$$

$$= \frac{p}{\cancel{1-p}} \cdot \frac{z\cancel{(1-p)}}{1-z(1-p)}$$

$$\gamma(z) = \frac{pz}{1-z(1-p)}$$

$$\gamma'(z) = p\left[\frac{1-z(1-p) - z[-(1-p)]}{(1-z(1-p))^2}\right]$$

$$\gamma'(1) = p\left[\frac{1-(1-p) - (-(1-p))}{[1-(1-p)]^2}\right]$$

$$= p\left[\frac{p-(-1+p)}{p^2}\right]$$

$$= \frac{p}{p^2} = \frac{1}{p}$$