Linear Maps from Subspaces

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1 Linear Maps from Subspaces

In this section we revisit what we have learned about linear mappings. Before, in our introduction to linear mappings, we proceeded as follows:

1. We saw how any matrix gives a linear map. The $m \times n$ matrix A defines a linear map $T\colon \mathbf{R}^n \to \mathbf{R}^m$ defined by $T(\mathbf{x}) = A\mathbf{x}$ (where $\mathbf{x} \in \mathbf{R}^n$).
2. We saw that, conversely, every linear map $T\colon \mathbf{R}^n \to \mathbf{R}^m$ "comes from" an $m \times n$ matrix. More precisely, given any linear map $T\colon \mathbf{R}^n \to \mathbf{R}^m$, there is a unique $m \times n$ matrix A such that for all $x \in \mathbf{R}^n$, $T(\mathbf{x}) = A\mathbf{x}$. The way A is defined is: The jth column of A is $T(\mathbf{e}_j)$.

The linear maps we examined before have domain $\mathbf{R}^n$ and codomain $\mathbf{R}^m$. Now, we will look at a more general case, in which the domain is a subspace of $\mathbf{R}^n$ and the codomain is a subspace of $\mathbf{R}^m$.

Suppose that V is a subspace of $\mathbf{R}^N$ and W is a subspace of $\mathbf{R}^M$. A **linear mapping** T from V to W is a function from V to W with the properties:

- $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in V$; and
- $T(c\mathbf{v}) = cT(\mathbf{v})$ for all $\mathbf{v} \in V$ and $c \in \mathbf{R}$.

A few words on the definition we have just given. First, notice that this is pretty much exactly the same definition as we have given before. Also, I let V be a subspace of $\mathbf{R}^N$ and W a subspace of $\mathbf{R}^M$ instead of n and m. That is because I want to think of V as n-dimensional and W as m-dimensional. Finally, if V is a subspace of $\mathbf{R}^N$, then a linear mapping with domain V does not have to be a linear mapping on all of $\mathbf{R}^N$. For example, suppose that V is a plane through the origin in $\mathbf{R}^3$. Now we can talk about linear mappings that are defined on just this plane V instead of needing the linear mapping to be defined on all of $\mathbf{R}^3$. This is sort of like how when we talk about functions, we might originally restrict ourselves to functions defined on all of $\mathbf{R}$, but later we learn about functions like $f(x) = \sqrt{x}$ that are defined on only part of $\mathbf{R}$.

However, with this definition, V might be all of $\mathbf{R}^N$, and W might be all of $\mathbf{R}^M$, so this more general definition of linear mapping does include our previous definition as a special example.

2 Coordinate Systems

Theorem 1. *Let $\mathcal{B} = \{\boldsymbol{v}_1, ..., \boldsymbol{v}_n\}$ be a basis for a subspace V of $\boldsymbol{R}^N$. Then, for each $\boldsymbol{x}$ in V, there exists a unique set of scalars $c_1, ..., c_n$ such that*

$$\boldsymbol{x} = c_1\boldsymbol{v}_1 + \cdots + c_n\boldsymbol{v}_n.$$

Proof. Existence follows from the fact that $\mathcal{B}$ spans V. For uniqueness, suppose that $\mathbf{x} = \sum_{i=1}^{n} c_i \mathbf{v}_i$ and $\mathbf{x} = \sum_{i=1}^{n} d_i \mathbf{v}_i$. Then $\sum_{i=1}^{n} (c_i - d_i)\mathbf{v}_i = \mathbf{0}$. Since $\mathcal{B}$ is a linearly independent set, $c_i - d_i = 0$ for all i, i.e., $c_i = d_i$ for all i. This completes the proof. $\square$

Theorem 2. *Let $\mathcal{B}$ be a basis for an n-dimensional subspace V. Then the coordinate mapping $\boldsymbol{x} \mapsto [\boldsymbol{x}]_{\mathcal{B}}$ is a one-to-one linear transformation from V onto $\boldsymbol{R}^n$.*

The coordinate mapping is an important example of an *isomorphism* from V onto $\mathbf{R}^n$. In general, a one-to-one linear mapping from a vector space V onto a vector space W is called an **isomorphism** from V onto W. The notation and terminology for V and W may differ, but the two spaces are indistinguishable as vector spaces.

3 Change of Basis

Theorem 3. *Let $\mathcal{B}$ and $\mathcal{C}$ be bases of an n-dimensional subspace V. Then there is a unique $n \times n$ matrix P_{CB} such that*

$$[\boldsymbol{x}]_{\mathcal{C}} = P_{CB}[\boldsymbol{x}]_{\mathcal{B}}$$

4 The Matrix of a Linear Mapping

Let V be an n-dimensional vector space, W an m-dimensional vector space, and T any linear map from V to W. To associate a matrix with T, choose ordered bases $\mathcal{B}$ and $\mathcal{C}$ for V and W, respectively.

We will give an $m \times n$ matrix M associated with T, and these two bases $\mathcal{B}$ and $\mathcal{C}$. We will define it by specifying each column. Consider the vector $\mathbf{b}_1$, a vector in V. The vector $T(\mathbf{b}_1)$ is a vector in W. Since $\mathcal{C}$ is a basis for W, there are unique scalars $d_1, ..., d_m$ such that

$$T(\mathbf{b}_1) = d_1\mathbf{c}_1 + \cdots + d_m\mathbf{c}_m.$$

We make the first column of M the vector

$$\begin{bmatrix} d_1 \\ \vdots \\ d_m \end{bmatrix}$$

Using the terminology of coordinate vectors, this vector is just the coordinate vector of $T(\mathbf{b}_1)$ relative to the basis $\mathcal{C}$.

In general, we define the jth column of M to be the coordinate vector of $T(\mathbf{b}_j)$ relative to $\mathcal{C}$: $[T(\mathbf{b}_j)]_{\mathcal{C}}$.

This matrix M is called the **matrix of T relative to the bases $\mathcal{B}$ and $\mathcal{C}$.**

In the common case when $W = V$ and $\mathcal{B} = \mathcal{C}$, the matrix M is called the **matrix for T relative to $\mathcal{B}$.**

Exercise 1. Let $V = \{x + y + z = 0\}$ and let $W = \{x - y - z = 0\}$. Let $T\colon V \to W$ be defined by $T(x, y, z) = (2x, -2y, -2z)$.

(a) Show that T does in fact map vectors in V to vectors in W.

(b) Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$ and $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$. Find the matrix of T relative to $\mathcal{B}$ and $\mathcal{C}$.

Theorem 4. *Suppose $A = PDP^{-1}$, where D is a diagonal $n \times n$ matrix. If $\mathcal{B}$ is the basis for $\boldsymbol{R}^n$ formed from the columns of P, then D is the matrix for the linear map T_A relative to the basis $\mathcal{B}$.*

Actually, the proof did not use the information that D was diagonal. Here is the general theorem that we have proven.

Theorem 5. *(i) Suppose that C is an $n \times n$ matrix and that A is similar to a matrix C, with $A = PCP^{-1}$. Let $\mathcal{B}$ be the basis formed from the columns of P. Then C is the matrix of T_A relative to the basis $\mathcal{B}$.*

(ii) Let A be any $n \times n$ matrix, and consider the linear mapping $T_A \colon \boldsymbol{R}^n \to \boldsymbol{R}^n$. Suppose that $\mathcal{B}$ is any basis for $\boldsymbol{R}^n$. Let P be the matrix whose columns come from the vectors in $\mathcal{B}$. Then $A = P[T]_{\mathcal{B}}P^{-1}$.