Nullspace, Column Space, Row Space

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1 The Nullspace

The last sections concerned subspaces of $\mathbf{R}^n$ and bases for the subspaces. In this section and the next couple of sections, we will examine subspaces associated with a matrix. Every matrix A has three natural subspaces associated with it: the nullspace, the column space, and the row space. This section is about the nullspace.

Nullspace of a Matrix

The **nullspace** of a matrix A is defined to be the set $N(A)$ of solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Example 1. The vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is an element of the nullspace of $\begin{bmatrix} 3 & -2 \\ 6 & -4 \end{bmatrix}$ because $\begin{bmatrix} 3 & -2 \\ 6 & -4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. However, $\begin{bmatrix} 3 & -2 \\ 6 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ so $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not an element of the nullspace of $\begin{bmatrix} 3 & -2 \\ 6 & -4 \end{bmatrix}$. $\square$

The nullspace of an $m \times n$ matrix A is a subspace of $\mathbf{R}^n$. The set of vectors obtained from the parametric vector form of the solutions to $A\mathbf{x} = \mathbf{0}$ is a basis for the nullspace of A.

Since the dimension of a subspace is just the number of elements in a basis for the subspace, the dimension of the nullspace of a matrix it is the number of free columns in the echelon form of the matrix.

Example 2. To find a basis for $A = \begin{bmatrix} 3 & -2 \\ 6 & -4 \end{bmatrix}$, we solve $A\mathbf{x} = \mathbf{0}$: We row reduce A to get $\begin{bmatrix} 3 & -2 \\ 0 & 0 \end{bmatrix}$. Then letting the variables be x_1 and x_2, we get $3x_1 = 2x_2$ so $x_1 = 2/3x_2$.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}.$$

Thus, a basis for $N(A)$ is $\left\{ \begin{bmatrix} 2/3 \\ 1 \end{bmatrix} \right\}$. The nullspace of A is 1-dimensional.

We could also have done this by using the following reasoning. The nullspace is a subspace of $\mathbf{R}^2$, so it must be either 0-dimensional, 1-dimensional, or 2-dimensional. It cannot by 0-dimensional because A is not invertible and it cannot be 2-dimensional, so it must be 1-dimensional. So it is spanned by one vector. We can see that $\langle 2, 3 \rangle$ is orthogonal to the first row of A. $\square$

Example 3. What is the nullspace of $\begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}$? We can see the answer by inspection. I see that the columns are linearly independent, so the nullspace is 0-dimensional; it is $\{\mathbf{0}\}$. $\square$

Example 4. The matrix A and its reduced row echelon form are given below:

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \\ 5 & 16 & 20 \end{bmatrix} \qquad reducedrowechelonform(A) = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

What is the nullspace of A? From this I see two free columns (the second and third), so the nullspace is a 2-dimensional subspace of $\mathbf{R}^3$ – a plane. We get $x_1 = -3s - 4t$, so

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}.$$

The vectors $\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$ span $N(A)$ and are linearly independent. Therefore, $\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} \right\}$ is one basis for $N(A)$. □

Example 5. What is the nullspace of $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$? This one is easy: This matrix takes every vector in $\mathbf{R}^3$ to $\mathbf{0}$, so the nullspace is $\mathbf{R}^3$. □

Exercise 1. Let

$$A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}.$$

Determine if $\mathbf{u} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$ belongs to the null space of A. This is equivalent to asking if $(x_1, x_2, x_3) = (5, 3, -2)$ is a solution to which system of linear equations?

Exercise 2. True or False:

a.) The matrices $\begin{bmatrix} 45 & 92 & 78 \end{bmatrix}$ and $\begin{bmatrix} 45 & 92 & 78 \\ 90 & 184 & 156 \end{bmatrix}$ have the same nullspace.

b.) The nullspace of an $m \times n$ matrix is a subspace of $\mathbf{R}^n$.

Exercise 3. Let $A = [\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\ \mathbf{a}_4]$, and suppose that $\mathbf{x} = \langle 7, -3, 2, 1 \rangle$ is in $N(A)$. Write $\mathbf{a}_4$ as a linear combination of the other three vectors.

Exercise 4. Suppose that $\langle 1, 2, 4 \rangle$ is in the nullspace of an unknown matrix A. Find a and b such that $\langle a, b, 1 \rangle$ must also be in the nullspace of A.

Exercise 5. The set

$$W = \{(x, y, z, w) \in \mathbf{R}^4 | x + y + z + w = 0\}$$

is a subspace of $\mathbf{R}^4$. Find a basis for W.

Exercise 6. This question is about the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & 4 \\ 3 & 6 & 3 & 9 \end{bmatrix}$$

(a) Find the reduced row echelon form RREF(A).

(b) What is the rank of this matrix A? (Your answer should be a number.)

(c) Find a basis for the nullspace of A.

(d) If the vector $\mathbf{b}$ is the sum of the four columns of A, write down the complete solution to $A\mathbf{x} = \mathbf{b}$.

2 The Column Space

In this section you will:

- Find a basis for the column space or row space and the rank of a matrix

Besides the nullspace of a matrix, we can form a subspace of a matrix by examining its columns. The **column space** of a matrix A is the span of its columns. The column space of A, denoted by $C(A)$, is a subset of $\mathbf{R}^m$. It is in fact a *subspace* of $\mathbf{R}^m$, being the span of vectors in $\mathbf{R}^m$.

Column Space of a Matrix

The **column space** of a matrix A is the span of its columns. It is in a *subspace* of $\mathbf{R}^m$, being the span of vectors in $\mathbf{R}^m$.

Example 6. The column space of

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

is a subspace of $\mathbf{R}^3$. It is the span of $\mathbf{e}_1$ and $\mathbf{e}_2$, so it is the xy-plane. The equation of this plane is $z = 0$.

Example 7. The column space of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

is $\mathbf{R}^2$. I see that the first two columns are linearly independent, so the span of the first two columns is $\mathbf{R}^2$. So the span of the three columns is also $\mathbf{R}^2$.

The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is in the column space of A. After all, by definition, $\mathbf{b}$ is in the column space only if it is a linear combination of the columns of A.

How do you find a basis for the column space of a matrix? Let's work with a matrix in echelon form. One standard procedure for finding a basis for the column spaces of a matrix is based on two facts: The first fact is: *The pivot columns are linearly independent.* To check this, write out the equations to the homogeneous equation and check that the only solution is the trivial solution.

The second fact is: *In an echelon form matrix, every free column is a linear combination of the pivot columns to the left of that column.* The best way to understand this is to just work with an example. The idea is this. Take a free column. If it weren't a linear combination of the pivot columns, then there would have to be a non-zero number in a row where the component of the pivot column is 0, but then that would make the free column a pivot column.

The pivot columns of a matrix in echelon form span the column space of that matrix.

Putting that together with the fact that the pivot columns are linearly independent, we can conclude that the set of pivot columns of a matrix in echelon form is a basis for the column space of the matrix.

Example 8. Find a basis for the column space of the matrix

$$B = \begin{bmatrix} 1 & 0 & -3 & 5 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Columns 1, 2, and 5 will do. Notice that they are linearly independent and that columns 3 and 4 are linear combinations of columns 1 and 2. □

What happens if the matrix A is not in echelon form? How do we find a basis for the column space of A?

Elementary Row Operations and the Column Space
Elementary row operations might change the column space. However, they do not change the linear relations among the columns.

This follows from the fact that when a matrix A is row reduced to echelon form B, the equations $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ have the same set of solutions. Thus, the ways to combine the columns of A to produce $\mathbf{0}$ are the same as the ways to combine the columns of B to product $\mathbf{0}$. That is, the columns of A have exactly the same linear dependence relationships as the columns of B. Therefore, the set of columns of A that correspond to the pivot columns of $\mathrm{REF}(A)$ is a basis for the column space of A.

Therefore: Let $A \in M_{m,n}(\mathbf{R})$. The dimension of $\mathrm{Col}A$ is the rank of A.

Example 9. Let

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 & -9 \\ -2 & -2 & 2 & -8 & 2 \\ 2 & 3 & 0 & 7 & 1 \\ 3 & 4 & -1 & 11 & -8 \end{bmatrix}.$$

Its reduced row echelon form turns out to be B in the previous example. Therefore, the first, second, and fifth columns of A form a basis for $C(A)$:

$$\text{A basis for } C(A) \text{ is } \left\{ \begin{bmatrix} 1 \\ -2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -9 \\ 2 \\ 1 \\ -8 \end{bmatrix} \right\}.$$ □

Be careful to use pivot columns of A itself for the basis of $C(A)$. The columns of an echelon form B are often not in the column space of A.

Exercise 7. Find a basis for $\mathrm{Span}\{\langle 1,1,2,1\rangle, \langle 2,4,4,1\rangle, \langle 0,2,0,-1\rangle\}$. (Hint: Write down a matrix whose column space is this span. Then use the technique for finding a basis for the column space.)

Exercise 8. If Q is a 4×4 matrix and $C(Q) = \mathbf{R}^4$, what can you say about solutions of equations of the form $Q\mathbf{x} = \mathbf{b}$ for $\mathbf{b}$ in $\mathbf{R}^4$?

Exercise 9. Let the reduced row echelon form of A be

$$\begin{bmatrix} 1 & 0 & 2 & 0 & -2 \\ 0 & 1 & -5 & 0 & -3 \\ 0 & 0 & 0 & 1 & 6 \end{bmatrix}$$

Determine A if the first, second, and fourth columns of A are

$$\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix},$$

respectively.

Exercise 10. Let the reduced row echelon form of A be

$$\begin{bmatrix} 1 & 0 & 2 & 0 & -2 \\ 0 & 1 & -5 & 0 & -3 \\ 0 & 0 & 0 & 1 & 6 \end{bmatrix}$$

Determine A if the first, second, and *fifth* columns of A are

$$\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix},$$

respectively.

Exercise 11. The matrix A and its reduced row echelon form are given below:

$$A = \begin{bmatrix} 5 & 9 & a & 2 \\ -2 & 3 & b & 5 \\ 4 & 1 & c & 7 \end{bmatrix}, \qquad RREF(A) = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) Find the values of $a, b,$ and c. (Hint: Row reducing A is not a good idea. How else could you find column 3 using the information given?)

(b) Let T_A be the linear transformation associated with the matrix A. That is, $T_A(\mathbf{x}) = A\mathbf{x}$ for all vectors $\mathbf{x} \in \mathbf{R}^4$.

Choose the correct options: The function T_A is (1-1 / onto) (and / but not) (1-1 / onto).

(c) Let B be the matrix whose columns are columns 1, 2, and 4 of A:

$$B = \begin{bmatrix} 5 & 9 & 2 \\ -2 & 3 & 5 \\ 4 & 1 & 7 \end{bmatrix}$$

Find the matrix C such that $BC = A$.

Exercise 12. Let

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Decide whether each set below is a basis for the column space of A.

(a) the set consisting of columns 1, 2, and 3

(b) the set consisting of columns 1, 3, and 4

(c) the set consisting of columns 2, 3, and 4

(d) the set consisting of all four columns

3 The Row Space

Just as any matrix has a column space, any matrix has a row space. The **row space** of a matrix is the span of its rows. In this section, you will

- Calculate a basis for the row space of a matrix.

Example 10. The row space of the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is the span of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. This is the plane $y = 0$ in $\mathbf{R}^3$.

Example 11. The row space of the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \\ 4 & 8 & 12 \end{bmatrix}$$

is the span of $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$, and $\begin{bmatrix} 4 \\ 8 \\ 12 \end{bmatrix}$. Since these are all scalar multiples of each other, the span is equal to the span of $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. So the row space of this matrix is a line, spanned by $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. $\square$

Warning: In general, the row space of a matrix and the column space of a matrix are NOT the same!

Example 12. Let's look at the two examples we just went over. The column space of the first matrix is the span of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Since we see two linearly independent vectors in $\mathbf{R}^2$, the span is $\mathbf{R}^2$. So it's not the same as the row space. In this example, the row space is a subspace of $\mathbf{R}^3$, whereas the column space is a subspace of $\mathbf{R}^2$, so they certainly are not the same.

Example 13. What about if A is a square matrix? Even then the row space and column space are not necessarily the same. It's easy to come up with an example. Look at the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$. The row space of A is spanned by $\langle 1, 2 \rangle$, whereas the column space of A is spanned by $\langle 1, 3 \rangle$. They are not the same!

Example 14. In the second example, the row space and column space are subspaces of the same space, because $m = n$. Note that the three columns are scalar multiples of each other, so the column space is the span of $\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$. The column space is a line in $\mathbf{R}^3$. The row space of this matrix is also a line in $\mathbf{R}^3$, but it is a different line.

The row space of an m by n matrix is a subspace of $\mathbf{R}^n$, while the column space is a subspace of $\mathbf{R}^m$, so the row space and column space "live" in different places. However, we will see that they both have something important in common. The first two examples we did give us a clue. In the first example, the row and column spaces were both planes. In the second example, the row and column spaces were both lines.

To see what is going on, we want to figure out what the dimension of the row space of a matrix is. As is typical, we will proceed in two steps. The first step is to figure out how the dimension of the row space changes when row reduction steps are applied. The second step is to figure out the dimension of the row space of a matrix in row echelon form.

Elementary Row Operations and the Row Space

Elementary row operations do not change the row space. In other words: Let B be a matrix obtained by applying an elementary row operation to the matrix A. Then the row space of B is the same as the row space of A.

A detailed proof of this is included in the notes.

Since Gaussian elimination is just doing elementary row operations over and over again, *A and $REF(A)$ have the same row space.*

If a matrix B is in row-echelon form, its nonzero rows form a basis for its row space.

Now we know how to find a basis for the row space of any matrix A. We apply elementary row operations to the matrix A until we reduce it to a matrix B in row-echelon form. For such a matrix B, the nonzero rows of B automatically form a basis for the row space of B. Hence they are a basis for the row space of A as well, since A and B have the same row space.

The **rank** of a matrix is the number of pivots in its REF. Therefore the dimension of the row space of a matrix A is the rank of A. It equals the number of nonzero rows of $REF(A)$.

Row Space of a Matrix

The **row space** of a matrix is the span of its rows. The dimension of the row space of a matrix A is the rank of A. It equals the number of nonzero rows of $REF(A)$.

It's an easy step to give a procedure for finding a basis for the subspace W of $\mathbf{R}^n$ spanned by a given set of vectors $\mathbf{v}_1, ..., \mathbf{v}_k$. We form the matrix A whose rows are the vectors $\mathbf{v}_1, ..., \mathbf{v}_k$, and then we apply elementary row operations to the matrix A until we reduce it to a matrix B in a row echelon form. For such a matrix B, the nonzero rows of B automatically form a basis for the row space of B. hence they are a basis for the row space of A as well, since A and B have the same row space.

Exercise 13. Sketch the row space, null space, and column space of the matrix

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

Exercise 14. True or False?

a.) The row space of a matrix is equal to its column space.

b.) The matrices $\begin{bmatrix} 45 & 92 & 78 \end{bmatrix}$ and $\begin{bmatrix} 45 & 92 & 78 \\ 90 & 184 & 156 \end{bmatrix}$ have the same row space.

Exercise 15. Consider the matrices

$$\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 4 & 0 \\ 3 & 2 & 0 \end{bmatrix}$$

Compare their row, column, and null spaces.

Exercise 16. Give an example of a 2 by 2 matrix whose row space and column space are different.

Exercise 17. What are the vectors in both the row space and null space of a matrix A?

Exercise 18. Find the values of a for which this matrix

$$\begin{bmatrix} 1 & 1 & a \\ 1 & a & 1 \\ a & 1 & 1 \end{bmatrix}$$

has

(a) rank 1
(b) rank 2
(c) rank 3

Exercise 19. Suppose that $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ are vectors in $\mathbf{R}^5$ that are linearly independent (more precisely, the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent). Let A be the matrix whose columns are $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$. Which of the following statements about the row space of A is correct?

(i) The row space of A is never all of $\mathbf{R}^3$.
(ii) The row space of A is sometimes all of $\mathbf{R}^3$ and sometimes not, depending on $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.
(iii) The row space of A must be all of $\mathbf{R}^3$.

Exercise 20. In this problem, A is the matrix

$$\begin{bmatrix} 10 & 2 & 1 & 1 \\ 41 & 6 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) Give the dimension of the row space of A.
(b) Find ALL values of k for which the set

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 10 \\ 41 \\ k \end{bmatrix} \right\}$$

is a basis for the column space of A.

4 The Rank-Nullity Theorem

The number of pivot columns is equal to the number of pivot rows (which is equal to the number of pivots!). Let A be any $m \times n$ matrix. We have seen that the pivot rows of REF(A) form a basis for the row space of A, and the columns of A corresponding to the pivots columns of REF(A) form a basis for the column space of A. Therefore, the row space and the column space of any matrix have the same dimension. This dimension is the rank of the matrix.

Another way to think of the rank: The rank of a matrix A is the number of rows/columns of A in a maximal subset of linearly independent rows/columns.

The rank plus the nullity of A is the number of columns of A. This is the Rank-Nullity Theorem.

Rank-Nullity Theorem

If a matrix A has n columns, then rank $A +$ dim Nul $A = n$.

The reason this is true: Bring the matrix to echelon form. The rank is the number of pivot columns and the nullity is the number of free columns, and every column is either a pivot column or a free column.

The rank-nullity theorem provides us with a good way to summarize a matrix A. The rank of A tells us how many "essential" columns and rows the matrix A has ("essential" meaning *linearly independent*). The number of linearly independent rows is actually equal to the number of linearly independent columns – it's the number of pivots of A. The other columns of A are linear combinations of the pivot columns. Now, these are certainly not vectors in the nullspace of A, but if we write out the equation $A\mathbf{x} = \mathbf{0}$ and assign parameters to the non-pivot variables (the free variables), we can solve $A\mathbf{x} = \mathbf{0}$ and the vectors we get form a basis for the nullspace of A. Since the number of free variables is equal to the number of non-pivot columns, the rank and nullity add up to the number of columns of the matrix.

It is also helpful to phrase this theorem in terms of linear maps. Suppose that T is a linear mapping from $\mathbf{R}^n$ to $\mathbf{R}^m$. Then the dimension of the kernel of T plus the dimension of the image of T is equal to n.

Exercise 21. Let $T(x_1, x_2, x_3) = (0, x_2, x_3)$.

(a) Describe the kernel of T explicitly. Do this directly from the definition of T.

(b) Describe the image of T explicitly. Do this from the definition of T.

(c) Write down the standard matrix of T and then calculate the nullspace and column space.

So far, we have been working with general m by n matrices – the number of rows and columns are arbitrary. In the remainder of this book, we will work with square matrices. These have particular properties that non-square matrices don't. In particular, we can think of an $n \times n$ matrix as mapping vectors in $\mathbf{R}^n$ to vectors in $\mathbf{R}^n$, or in other words, mapping $\mathbf{R}^n$ to itself. This leads us to the topic of eigenvectors, which are discussed in the last chapter. Before that, we will discuss determinants, because they are used to calculate eigenvectors and have important applications in multivariable calculus and probability.

Exercise 22. Let A be the 3×5 matrix

$$A = \begin{bmatrix} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) Find a basis for the column space of A. Give the dimension of this space.

(b) Find a basis for the nullspace of A. Give the dimension of this space.

(c) Verify that the Rank-Nullity Theorem holds for A.

Exercise 23. If A is a 3×5 matrix, what inequality does the nullity of A satisfy?

Exercise 24. If A is a 5×4 matrix whose rank is 2, what is the dimension of the nullspace of A?

Exercise 25. Suppose that A is an $m \times n$ matrix, with rank 3, nullity 4, and $C(A)$ a subspace of $\mathbf{R}^5$. What are the dimensions of A?

Exercise 26. Give an example of the following or explain why it is not possible.

A 2 by 3 matrix whose nullspace is the span of the vector $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

Exercise 27. Give an example of the following or explain why it is not possible. A 2 by 2 matrix A and a 2 by 2 matrix B that *are not equal*, but that have the *same column space* and the *same nullspace*, which is the span of $\begin{pmatrix}1\\2\end{pmatrix}$:

$$C(A) = N(A) = t\begin{pmatrix}1\\2\end{pmatrix}, \quad C(B) = N(B) = t\begin{pmatrix}1\\2\end{pmatrix}$$

Exercise 28. Give an example of the following or explain why it is not possible. 3 linearly independent vectors that span this subset of $\mathbf{R}^5$:

$$Q = \{(x_1, x_2, x_3, x_4, x_5) : x_1 - 2x_2 + x_3 + x_5 = 0\}$$

Exercise 29. Give an example of a 4 by 3 matrix that satisfies both of the conditions, or write NOT POSSIBLE if it is not possible.

- condition 1: The general solution to the equation $A\mathbf{x} = \begin{bmatrix}1\\2\\0\\0\end{bmatrix}$ is $\begin{bmatrix}1\\0\\0\end{bmatrix}$, and
- condition 2: The general solution to the equation $A\mathbf{x} = \begin{bmatrix}1\\3\\0\\0\end{bmatrix}$ is $\begin{bmatrix}0\\1\\0\end{bmatrix}$.

Exercise 30. If Q is a 4×4 matrix and $C(Q) = \mathbf{R}^4$, what can you say about solutions of equations of the form $Q\mathbf{x} = \mathbf{b}$ for $\mathbf{b}$ in $\mathbf{R}^4$?

Exercise 31. If F is a 5×5 matrix whose column space is not equal to $\mathbf{R}^5$, what can you say about $N(F)$?

Exercise 32. What can you say about the shape of an $m \times n$ matrix A when the columns of A form a basis for $\mathbf{R}^m$?

Exercise 33. If R is a 6×6 matrix and $N(R)$ is *not* the zero subspace, what can you say about $C(R)$?

Solution: Its dimension is at most 5.

Exercise 34. Give an example of a 2 by 2 matrix A and a 2 by 2 matrix B that have *different column spaces*, but that have the *same nullspace*, which is equal to the span of $\begin{pmatrix}1\\1\end{pmatrix}$:

$$N(A) = c\begin{pmatrix}1\\1\end{pmatrix} \text{ and } N(B) = c\begin{pmatrix}1\\1\end{pmatrix}.$$

Exercise 35. Suppose that A is a 5 by 3 matrix (5 rows, 3 columns), and that there are infinitely many solutions to the equation

$$A\mathbf{x} = \begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}$$

Give ALL possible values for the rank of A.

Exercise 36. Suppose that A is a 5 by 3 matrix (5 rows, 3 columns), and that there is exactly one solution to the equation

$$A\mathbf{x} = \begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}$$

Give ALL possible values for the rank of A.

Exercise 37. Suppose that A is a 5 by 3 matrix (5 rows, 3 columns), and that there is no solution to the equation

$$A\mathbf{x} = \begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}$$

Give ALL possible values for the rank of A.

Exercise 38. Let A be a 3×5 matrix. Form a 3×6 matrix, B, by making the first 5 columns of B equal to the 5 columns of A, and making the 6th column of B equal to the sum of the columns of A. Suppose that the nullity of A is 3. Give:

(a) the nullity of B
(b) the nullity of A^T
(c) the nullity of B^T

Exercise 39. Suppose A is a matrix whose first column is $\begin{bmatrix}1\\0\\3\\-2\\8\end{bmatrix}$. Suppose that $\begin{bmatrix}3\\0\\-1\end{bmatrix}$ and $\begin{bmatrix}1\\4\\-3\end{bmatrix}$ are in $N(A)$. Write down the matrix A.

Exercise 40. Suppose A is a matrix such that the complete solution to $A\mathbf{x} = \begin{bmatrix}1\\4\\1\\1\end{bmatrix}$ is

$$\mathbf{x} = \begin{bmatrix}0\\1\\1\end{bmatrix} + c\begin{bmatrix}0\\2\\1\end{bmatrix}.$$

What can you say about the columns of A? Identify as many columns of A as you can. For any columns you can't identify, give a column vector that it could be and a column vector it can't be.

Exercise 41. For a certain matrix A, the complete solution to the equation $A\mathbf{x} = \begin{bmatrix}3\\4\\5\\6\end{bmatrix}$ is

$$\mathbf{x} = \begin{bmatrix}0\\1\\0\end{bmatrix} + s\begin{bmatrix}1\\-1\\0\end{bmatrix} + t\begin{bmatrix}2\\0\\-1\end{bmatrix}$$

(a) Give the dimension of the row space of A, with a brief justification (two sentences maximum).
(b) Find the matrix A.

Exercise 42. For this problem, A is an $m \times n$ matrix (m rows, n columns) with rank r. For each part, **choose the symbol** $\leq, <, >, \geq$, or $=$, that correctly relates m, n, and r. Choose the most restrictive option that must be correct. For example, if m must be strictly greater than n and you pick "$m \geq n$," that will be counted as incorrect because "$m > n$" would be the correct option. Similarly, if m must equal n, then "$m \leq n$" and "$m \geq n$" will be counted as incorrect.

(a) Suppose that for *every* vector $\mathbf{b}$ in $\mathbf{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has *infinitely many* solutions. In this case,

m _____ n and r _____ m

(b) Suppose that instead, the equation $A\mathbf{x} = \mathbf{b}$ has *infinitely many solutions for some (but not all)* vectors $\mathbf{b}$ in $\mathbf{R}^m$ and *no solutions* for all of the other vectors $\mathbf{b}$ in $\mathbf{R}^m$. In this case,

r _____ m and r _____ n

(c) If, for *every* vector $\mathbf{b}$ in $\mathbf{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has *exactly one* solution, then we can conclude that

r _____ m and r _____ n

Exercise 43. Rank 1 matrices are important in some computer algorithms. It can be shown that an $m \times n$ matrix A has rank 1 if and only if it is an outer product that is $A = \mathbf{u}\mathbf{v}^T$ for some $\mathbf{u}$ in $\mathbf{R}^m$ and $\mathbf{v}$ in $\mathbf{R}^n$. These exercises suggest why this property is true.

(a) Verify that the rank of $\mathbf{u}\mathbf{v}^T \leq 1$ if $\mathbf{u} = \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

(b) Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Find $\mathbf{v}$ in $\mathbf{R}^3$ such that $\begin{bmatrix} 1 & -3 & 4 \\ 2 & -6 & 8 \end{bmatrix} = \mathbf{u}\mathbf{v}^T$.

(c) Let A be any 2 by 3 matrix such that rank $A = 1$. Let $\mathbf{u}$ be the first column of A, and suppose that $\mathbf{u} \neq \mathbf{0}$. Explain why there is a vector $\mathbf{v}$ in $\mathbf{R}^3$ such that $A = \mathbf{u}\mathbf{v}^T$. How could this construction be modified if the first column of A were zero?

Exercise 44. Suppose that A is $m \times n$ and B is $n \times p$.

(a) Explain why the column space of AB is a subset of the column space of A.
(b) Deduce that the rank of AB is less than or equal to the rank of A.
(c) Recall that $(AB)^T = B^T A^T$. Deduce that the rank of AB is less than or equal to the rank of B.
(d) Now suppose that P is $m \times m$ and invertible. Using part (a), deduce that $\text{rank}(PA) \leq \text{rank}(A)$. Since $A = P^{-1}PA$, deduce that $\text{rank}(A) \leq \text{rank}(PA)$. Therefore $\text{rank}(PA) = \text{rank}(A)$.
(e) Show that if Q is $n \times n$ and invertible, then $\text{rank}(AQ) = \text{rank}(A)$.
(f) How is the column space of AQ related to the column space of A? Justify your answer.