Systems of Linear Equations

September 2025

1 What is a System of Linear Equations?

In various places throughout we will need to solve a system of linear equations. Many problems in mathematics boil down to solving systems of linear equations. In this chapter, we will discuss a general procedure for solving these systems. It is called Gaussian elimination.

First, we need to know what a system of linear equations is, exactly. Let $x_1, ..., x_n$ be an (ordered) list of symbols. A **linear equation in** $x_1, ..., x_n$ is an equation of the form

$$a_1x_1 + \cdots + a_nx_n = b,$$

where $a_1, ..., a_n$, and b are real numbers. The symbols are called **unknowns**. A **linear system in** $x_1, ..., x_n$ is a set of linear equations in $x_1, ..., x_n$.

For example, the system

$$\begin{aligned} x + y + z &= 13 \\ 4x + 4y - 3z &= 3 \\ 5x + y + 2z &= 13 \end{aligned}$$

is a system of 3 equations in 3 unknowns. A **solution** to the system above is an ordered triple (d_1, d_2, d_3) of real numbers such that, when d_1 is substituted for x, d_2 is substituted for y, and d_3 is substituted for z, all the equations are satisfied.

A **solution** of the system is a vector $\langle d_1, ..., d_n\rangle$ of $\mathbf{R}^n$ such that when d_1 is substituted for x_1, d_2 is substituted for x_2, and so on, all the equations are satisfied. To **solve** a linear system means to find all of its solutions. A linear system that has one or more solutions is said to be **consistent**. Otherwise it is **inconsistent**. The **general solution** to the system is any description of *all* the solutions of the system.

Example 1. Determine if $(2, 3)$ is a solution to the system

$$\begin{aligned} x + 4y &= 1 \\ x - y &= 0 \end{aligned}$$

We plug in $x = 2$ and $y = 3$ into the equations to see if they hold: $2 + 4(3) \neq 1$ because $2 + 4(3) = 14$. So $(2, 3)$ is not a solution to the system.

Example 2. Determine if $(1, -1)$ is a solution to the system

$$\begin{aligned} 3x + 2y &= 1 \\ x - y &= 2 \end{aligned}$$

We plug in $x = 2$ and $y = 3$ into the equations to see if they hold: $3(1)+2(-1) = 1$ and $1-(-1) = 2$. So $(1, -1)$ is a solution to the system.

A crucial thing to notice about the definition above is that a solution is defined to be "something that works," and *not* "something you find by 'solving."' That is, if it acts like a solution, it's a solution - it doesn't matter where you got it. When we verified that $x = 1$, $y = -1$ is a solution to the linear system

$$\begin{aligned} 3x + 2y &= 1 \\ x - y &= 2, \end{aligned}$$

all we needed to do was to plug in $x = 1, y = -1$ and see that the equations are both true. You *do not* need to solve the system and then observe that what you found is that $x = 1$, $y = -1$. You could have solved the system – it's just that you would be wasting your time.

Some systems of linear equations have a solution, and some don't. *Our first task is to give an algorithm that, given a system of linear equations, will determine whether or not the system has any solution, and, if so, what its solutions are.* This is the main goal of this chapter.

Exercise 1. Is $(3, 4, -2)$ a solution of the following system?

$$\begin{aligned} 5x_1 - x_2 + 2x_3 &= 7 \\ -2x_1 + 6x_2 + 9x_3 &= 0 \\ -7x_1 + 5x_2 - 3x_3 &= -7 \end{aligned}$$

Answer: Although the first two equations are satisfied, the third is not, so $(3, 4, -2)$ is not a solution to the system.

(4 min in)

2 Matrices and Elementary Row Operations

You are probably used to solving systems of equations with two unknowns or three unknowns. In the rest of this chapter, we are going to learn a general method for solving any system of linear equations, in any number of variables and any number of equations. A **matrix**, the word for a rectangular array of real numbers, helps to organize the relevant information. A matrix that has m rows and n columns is said to be an m by n matrix. The numbers in the matrix are called the **entries** of the matrix.

How are matrices related to systems of equations? Every system of linear equations corresponds to a matrix. For example, the system

$$\begin{aligned} x + y + z &= 13 \\ 4x + 4y - 3z &= 3 \\ 5x + y + 2z &= 13 \end{aligned}$$

corresponds to the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 13 \\ 4 & 4 & -3 & 3 \\ 5 & 1 & 2 & 13 \end{bmatrix}$$

This matrix is called the **augmented matrix** of the system. To see how to use the matrix to solve the system, notice that there are a few operations we can perform on the equations to get a new system that has the same solutions. One of these is to swap two equations. If we change the position of the first two equations in the system above, we get

$$\begin{aligned} 4x + 4y - 3z &= 3 \\ x + y + z &= 13 \\ 5x + y + 2z &= 13 \end{aligned}$$

Changing the order in which the equations are written down does not affect the solutions. Another operation we can perform is to multiply an equation by a non-zero constant. For example, we could multiply both sides of the first equation by 5, giving us

$$\begin{aligned} 5x + 5y + 5z &= 65 \\ 4x + 4y - 3z &= 3 \\ 5x + y + 2z &= 13 \end{aligned}$$

This new system has the same solutions as the previous one.

The third operation we can perform is to replace one equation by that equation plus a multiple of a different equation. For example if we replace the second equation by the second equation plus -1 times the first equation we get

$$\begin{aligned} x + y + z &= 13 \\ 3x + 3y - 4z &= -10 \\ 5x + y + 2z &= 13 \end{aligned}$$

Hopefully this fact is believable, from your experience working with equations with two variables, but details are included in the notes in the back.

Inspired by the three operations listed above, we define the three elementary row operations on a matrix.

Elementary Row Operations

The **elementary row operations** on a matrix come in three types. They are
(type 1) Swap two rows, say the ith and jth rows.
(type 2) Multiply every entry in a given row, say the ith row, by a non-zero scalar c.
(type 3) Replace a given row, say the ith row, by itself plus a scalar multiple d of some other row, say the jth row.

Each of these operations, if applied to a matrix, results in a new matrix. Although we can perform any of the three ERO's without changing the solution set, we want to apply ERO's to make the system easier to solve. We will discuss this simpler form in the next section.

Exercise 2. The augmented matrix of a linear system has been transformed by row operations into the form below. Determine if the system has any solutions.

$$\begin{bmatrix} 1 & 5 & 2 & -6 \\ 0 & 4 & -7 & 2 \\ 0 & 0 & 5 & 0 \end{bmatrix}$$

Exercise 3. Construct three different augmented matrices for linear systems whose solution set is $x_1 = -2$, $x_2 = 1$, $x_3 = 0$.

Exercise 4. For what values of h and k is the following system consistent?

$$\begin{aligned} 2x_1 - x_2 &= h \\ -6x_1 + 3x_2 &= k \end{aligned}$$

Exercise 5. Suppose the system below is consistent for all possible values of f and g. What can you say about the coefficients c and d? Justify your answer.

$$\begin{aligned} x_1 + 3x_2 &= f \\ cx_1 + dx_2 &= g \end{aligned}$$

Exercise 6. Find the elementary row operation that transforms the first matrix into the second, and then find the reverse row operation that transforms the second matrix into the first.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -3 & -2 \\ 0 & -3 & 9 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

3 Row Echelon Form

In the last section, we introduced the elementary row operations on a matrix. We are interested in solving systems of linear equations. First, we have seen that we can convey all the necessary information by forming the augmented matrix of the system. Then we have seen that by applying elementary row operations to that augmented matrix, we can get a new augmented matrix, whose corresponding system of equations has the same solution set. Our goal is to apply elementary row operations to get a simpler system of equations that we can easily solve.

In this section, you will

- Identify if a matrix is in echelon form.
- Use the row reduction algorithm to put a matrix into row echelon form.

What will this simpler system of equations look like? It is easier to explain this in terms of the augmented matrix. The form we want the matrix to be in is called row echelon form, or simply echelon form. To explain what this is, we will first introduce a mathematical term. The first non-zero entry of a row of a matrix (if it exists) is called the **leading entry** of that row. With this term at hand, we can now say what it means for a matrix to be in row echelon form.

Row Echelon Form

A matrix is said to be in **row echelon form**, or simply **echelon form**, if

- the leading entry of each non-zero row appears to the left of the leading entries of the rows below it, and
- any zero rows (rows where all the entries are 0) appear below the non-zero rows (rows with at least one non-zero entry).

Fortunately, every matrix can be put into echelon form by row operations. The process of applying row operations to a matrix A to produce a matrix in row echelon form is called "putting

A into row echelon form." To see how this works, we will give an algorithm that puts any matrix into echelon form. This algorithm consists of a series of elementary row operations.

Algorithm for finding an echelon form of a matrix A

1. Find the first nonzero column of A, say column p. If the top entry of the column is 0, bring a nonzero entry to the top of the array by swapping two rows.
2, Then apply a type 3 operation several times, replacing each of the rows after the first by itself plus an appropriate scalar multiple of the first row, where the scalar is chosen so that the resulting matrix has the form

$$\begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ 0 & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & b_{m2} & \dots & b_{mn} \end{bmatrix}$$

3. Now consider the submatrix obtained by removing the first column and row, and start the reduction process over again, applying it now to this smaller matrix.
The reduction process eventually stops, because the smaller matrices get smaller at each step. At the end of the process the matrix will be in row echelon form. Note that up to now we have used only the elementary row operations (1) and (3).

Example 3. We will reduce this matrix to echelon form.

$$A = \begin{bmatrix} 1 & 3 & 2 & 13 \\ 4 & 4 & -3 & 3 \\ 5 & 1 & 2 & 13 \end{bmatrix}$$

We start with the first column. The top left entry is 1. We want the 4 and 5 below the 1 to be 0. To get 0 in place of the 4 we multiply row 1, $R1$, by 4 and take the result away from row 2, $R2$. We write $R2 \to R2 - 4R1$.

$$\begin{bmatrix} 1 & 3 & 2 & 13 \\ 4 & 4 & -3 & 3 \\ 5 & 1 & 2 & 13 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 2 & 13 \\ 0 & -8 & -11 & -49 \\ 5 & 1 & 2 & 13 \end{bmatrix}$$

To get 0 in place of 5 in the bottom row we multiply row 1 by 5 and take the result away from row 3. So $R3 \to R3 - 5R1$. Doing the two row operations, we have

$$\begin{bmatrix} 1 & 3 & 2 & 13 \\ 0 & -8 & -11 & -49 \\ 5 & 1 & 2 & 13 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 2 & 13 \\ 0 & -8 & -11 & -49 \\ 0 & -14 & -8 & -52 \end{bmatrix}$$

Now the first column is done. We ignore the first row and column and work again with the smaller two by three matrix. The -8 is not zero. We want to get the -14 below the -8 to be 0. We replace row 3 by row 3 minus 14/8 times row 2. This gives us

$$\begin{bmatrix} 1 & 3 & 2 & 13 \\ 0 & -8 & -11 & -49 \\ 0 & -14 & -8 & -52 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 2 & 13 \\ 0 & -8 & -11 & -49 \\ 0 & 0 & 45/4 & 135/4 \end{bmatrix}$$

The matrix is now in echelon form. If you want, you can scale any of the rows by any non-zero scalar. For example, you can multiply the third row by 4 to get rid of the fractions. □

Any matrix has many echelon forms because, for example, you can always scale a row. Because of this, it is technically incorrect to refer to “the” echelon form of a matrix, but we will do it anyway. When we write REF(A), we refer to any row echelon form of A.

The matrix

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}$$

is called the **coefficient matrix** of the system. When we take the coefficient matrix of the system and adjoin to it an extra column consisting of the constants of the system, the resulting matrix

$$\begin{bmatrix} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{bmatrix}$$

is called the **augmented matrix** of the system.

To solve a system of linear equations using the augmented matrix, we first row reduce the augmented matrix until the coefficient matrix is in echelon form. The easiest situation to handle is when the coefficient matrix has no free columns, so we will discuss this first. In this case, we can solve the system by solving for the last variable and then substituting this into the second to last equation to find the second-to-last variable. Then we can continue the process until we find all the variables.

In the previous example, now that we have a matrix in echelon form, we can write out the corresponding equations:

$$\begin{aligned} x + 3y + 2z &= 13 \\ -8y - 11z &= -49 \\ 45/4z &= 135/4 \end{aligned}$$

We multiply $4/45$ on each side of the third equation to get $z = 3$. Then we can plug $z = 3$ into the second equation, giving $y = 2$. Then we plug $z = 3$ and $y = 2$ into the first equation and solve for x, giving $x = 1$. So the solution is $x = 1, y = 2, z = 3$.

Exercise 7. Determine if the matrix is in echelon form.

$$\begin{bmatrix} 1 & -3 & -5 & 0 \\ 0 & 2 & 1 & 3 \end{bmatrix}$$

Exercise 8. Using the algorithm discussed in this section, row reduce the matrix to echelon form.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix}$$

4 Reduced Row Echelon Form

Once a matrix is in row echelon form, it can be further put into another form called reduced row echelon form (reduced row echelon form). In this section you will:

- Identify matrices in reduced row echelon form;

- Use elementary row operations to put a matrix into reduced row echelon form.

Reduced row echelon form of a matrix:

A matrix is in **reduced row echelon form** if
(i) it is in row echelon form;
(ii) its pivots are all 1; and
(iii) the numbers above and below the 1's in the pivot columns are all 0's.

Every matrix is row equivalent to exactly one matrix that is in reduced row echelon form. In other words, every matrix has a unique reduced row echelon form. The reduced row echelon form of a matrix A is denoted by reduced row echelon form(A). Any matrix in row echelon form can be put into reduced row echelon form by these steps.

Before giving the algorithm, we need a few more definitions. Suppose that A is a matrix in echelon form. The first *non-zero* number in a non-zero row is called a **pivot** of A

Note: For us, a pivot is always a non-zero number. I will also often say, a matrix has 3 pivots, when I mean that the echelon form has 3 pivots.

With these definitions at hand, we can give an algorithm for putting a matrix into reduced row echelon form.

Algorithm for finding the reduced row echelon form of a matrix A:

1. To find the reduced row echelon form of a matrix A, first put it into row echelon form using the previous algorithm.

2. Next, by several applications of operations of type (2), bring the matrix to the form where each leading entry (i.e., each pivot) is equal to 1.

3. Finally, going from right to left, multiply each pivot row by appropriate scalars and add it to the rows lying *above* it (using type (3) elementary row operations), so that each entry which lies directly above a pivot becomes 0. The matrix is now in reduced row echelon form.

The free columns can have any number in the entries (as long as there are enough 0's at the bottom to keep the matrix in echelon form).

For example, the reduced row echelon form of the matrix from a previous example:

$$\begin{bmatrix} 1 & 3 & 2 & 13 \\ 0 & -8 & -11 & -49 \\ 0 & 0 & 45/4 & 135/4 \end{bmatrix}$$

is

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Now the solution can be read off the matrix. The first row tells us that $x = 1$, the second row tells us that $y = 2$, and the third row tells us that $z = 3$.

With the aid of the reduced row echelon form, we can solve the system.

(20 min in)

5 Systems with a Free Column

Suppose that A is a matrix in echelon form. A **pivot column** of A is a column of A that has a pivot of A as one of its entries; any other column of A is called a **free column**.

Suppose you row reduce an augmented matrix and its coefficient submatrix has at least one free column. The goal of this section is to figure out what a free column in the coefficient matrix tells you about the solutions to the corresponding linear system.

Example 4. Suppose the augmented matrix is

$$\begin{bmatrix} 1 & 3 & 0 & 5 \\ 0 & 1 & 2 & 4 \end{bmatrix}$$

The system of equations that corresponds to this matrix is

$$\begin{aligned} x + 3y &= 5 \\ y + 2z &= 4 \end{aligned}$$

First we look at each equation and identify the leftmost variables (whose coefficient is not 0). These are called **leading variables** or **basic variables**. All other variables are called **free variables**. In this case, x and y are leading variables and z is free. Suppose we subtract any free variables and get only leading variables on the left side of the equations. This gives us

$$\begin{aligned} x + 3y &= 5 \\ 0x + y &= 4 - 2z \end{aligned}$$

This format shows that no matter what z is, we can solve for y using the second equation and then solve for x using the first equation. More precisely, suppose that t is any real number. There is a solution (x, y, z) where $z = t$. Just let $y = 4-2t$ and let $x = 5-3y$, so that $x = 5-3(4-2t) = 6t-7$. By the way we chose x and y, the two equations above are satisfied. Thus for any real number t, $(6t - 7, 4 - 2t, t)$ is a solution to the system.

But is it possible for there to be a solution that doesn't have that form? The answer is no. To see why, suppose (x, y, z) is a solution to the system. Then whatever z is, the two equations show that y must be $4-2z$ and that x must be $6z-7$. So the system has an infinite number of solutions, one for each real number t: For each real number t, $(6t-7, 4-2t, t)$ is a solution and every solution has this form. Using set notation, we can express the solution as

$$\{(6t - 7, 4 - 2t, t) : t \in \mathbf{R}\}.$$

This means: all ordered triples of the form $(6t - 7, 4 - 2t, t)$ where t is a real number. $\square$

The example above illustrates the general procedure for handling free variables: We are free to choose them to be any real number. Then there will be exactly one solution corresponding to each of those choices.

Example 5. Suppose we want to solve for (x, y, z) for this system:

$$\begin{aligned} x + 2y &= 0 \\ x - 2y &= 2 \end{aligned}$$

We are looking for the intersection of the planes $x + 2y = 0$ and $x - 2y = 2$. Since z is a variable, the equations are really $x + 2y + 0z = 0$ and $x - 2y + 0z = 2$. The augmented matrix of the system is

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & -2 & 0 & 2 \end{bmatrix}$$

Now we row reduce:

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -4 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & -1/2 \end{bmatrix}$$

The system of equations we get is

$$\begin{aligned} x + 2y &= 0 \\ y &= -1/2 \end{aligned}$$

In this case the leading variables are x and y, and z is free (remember that every variable is either a leading variable or a free variable). So $y = -1/2$ and substituting into the first equation we get $x = 1$, and z is free to be any real number, say t. So the solution is

$$\{(1, -1/2, t) : t \in \mathbf{R}\}$$

$\square$

Example 6. Find the general solution of the system whose augmented matrix is $\begin{bmatrix} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{bmatrix}$.

Solution: First, we row reduce the augmented matrix until the coefficient matrix is in reduced row echelon form:

$$\begin{bmatrix} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & -5 & -15 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Now we write out the corresponding system of equations:

$$\begin{aligned} x_1 + 3x_2 \quad &= 5 \\ x_3 &= 3 \end{aligned}$$

To solve this, we first identify the basic variables. To do this, we look at each equation and identify the first variable that appears. These are x_1 and x_3. All other variables are free. In this case, x_2 is free, so we set it equal to a parameter t. Then we solve for the basic variables in terms of any free variables. In this case, we get

$$\begin{aligned} x_1 &= -5 - 3t \\ x_2 &= t \\ x_3 &= 3 \end{aligned}$$

To get this into parametric vector form, we first add 0 and $0t$ as needed to get all unknowns in the form $a + bt$:

$$\begin{aligned} x_1 &= -5 - 3t \\ x_2 &= 0 + 0t \\ x_3 &= 3 + 0t \end{aligned}$$

so

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}$$

□

The reason this method works is that elementary row operations don't change the solutions to the corresponding linear system. That is because elementary row operations are reversible. Suppose that $[A|\mathbf{b}]$ were reduced to $[\tilde{A}|\tilde{\mathbf{b}}]$ and this latter system were solved. By performing the opposite elementary row operation, we deduce that it must be a solution to $[A|b]$.

Warning! We need to be aware of how many variables we are working with.

Example 7. The set of solutions to $y = x$ is a line we are talking about points in $\mathbf{R}^2$; however, if we are talking about points $(x, y, z) \in \mathbf{R}^3$, the solutions fill out a plane. □

Example 8. In the system of equations

$$\begin{aligned} x_1 + x_2 &= 0 \\ 2x_1 + 3x_2 &= 2 \end{aligned}$$

if there are only two variables involved, the solution is $(-2, 2)$. However, if this system is in the context of (x_1, x_2, x_3), then there are infinitely many solutions: $(-2, 2, x_3)$, where x_3 can be any real number. Notice how the matrix detects this. In the first case, the coefficient matrix is

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}.$$

The echelon form of this matrix has no free columns. In the second case, the coefficient matrix is

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 0 \end{bmatrix}.$$

The echelon form of this matrix has one free column. □

Example 9. To solve this system

$$\begin{aligned} x + y + z &= 1 \\ x - y + z &= 2 \\ 2x + 0y + 2z &= 3 \end{aligned}$$

we first set up the augmented matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 2 \\ 2 & 0 & 2 & 4 \end{bmatrix}$$

and row reduce:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 2 \\ 2 & 0 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 1 \\ 0 & -2 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This matrix is in row echelon form. The third equation reads:

$$0x + 0y + 0z = 1.$$

This equation has no solutions. Since row reduction does not change the solution set, we see that the original system of equations above does not have any solutions. □

Exercise 9. Find the general solution of the linear system whose augmented matrix is

$$\begin{bmatrix} 1 & -3 & -5 & 0 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

Exercise 10. Find the general solution of the system

$$\begin{aligned} x_1 - 2x_2 - x_3 + 3x_4 &= 0 \\ -2x_1 + 4x_2 + 5x_3 - 5x_4 &= 3 \\ 3x_1 - 6x_2 - 6x_3 + 8x_4 &= 2 \end{aligned}$$

6 Solving a Linear System

In this section you will

- Solve systems of linear equations using the row reduction algorithm.

This section summarizes solving a linear system of equations using matrices.

Algorithm for solving a system of linear equations:

1. First write the augmented matrix M of the linear system. Using the algorithm to put a matrix into reduced row echelon form, apply elementary row operations until the coefficient matrix A is in reduced row echelon form.
2. Go along each row, from top to bottom, and identify the leftmost non-zero entry of the coefficient matrix. The variables corresponding to the columns of these entries are *leading variables. All* other variables are *free variables.*
3. Set each free variable equal to a parameter, and solve for any leading variables in terms of the free variables. This is the general solution to the system. The free variables can be any real number. The leading variables are in terms of the free variables, or they are constant.

Example 10. It is a common misconception that if you get $0 = 0$, then the system has an infinite number of solutions. Getting $0 = 0$ by itself doesn't mean anything. For example,

$$\begin{aligned} x + y &= 2 \\ x - y &= 5 \\ 3x + 3y &= 6 \end{aligned}$$

gives a bottom row of all zeros but has a unique solution. And

$$\begin{aligned} x + y &= 2 \\ x + y &= 3 \\ 3x + 3y &= 6 \end{aligned}$$

gives a bottom row of zeros but has no solution. □

Exercise 11. We want to find an equation for a plane that contains the points $(1, -2, 0)$, $(3, 1, 4)$, and $(0, -1, 2)$. Suppose the equation of the plane is $ax + by + cz = d$. We want to find a, b, c, and d. Set up a system of 3 equations that allow you to solve for a, b, c, and d. Then solve the system to find an equation for the plane.

Exercise 12. Suppose experimental data are represented by a set of points in the plane. An **interpolating polynomial** for the data is a polynomial whose graph passes through every point. In scientific work, such a polynomial can be used, for example, to estimate values between the known data points. Another use is to create curves for graphical images on a computer screen. One method for finding an interpolating polynomial is to solve a system of linear equations.

Find the interpolating polynomial $p(t) = a_0 + a_1 t + a_2 t^2$ for the data $(1, 12)$, $(2, 15)$, $(3, 16)$.

7 Determining the Number of Solutions

We can determine the number of solutions that a system of linear equations has by forming the augmented matrix and performing row operations on it until the coefficient matrix is in row echelon form. In this section you will

- Use the echelon form of an augmented matrix to determine the number of solutions to a linear system.

The Number of Solutions to $A\mathbf{x} = \mathbf{b}$:

- If $REF(A)$ has a zero row but the entry in the vector of constants in that row is nonzero, the system has no solution. Otherwise, the system does have at least one solution.
- Suppose the system has at least one solution. If $REF(A)$ has no free columns, the system has exactly one solution.
- Suppose the system has at least one solution. If $REF(A)$ has at least one free column, the system has infinitely many solutions.

In this chapter we introduced matrices and used them to solve systems of linear equations. The columns of a matrix play an important role in understanding systems of equations. This is the topic of the next chapter.

Exercise 13. What would you have to know about the pivot columns in an augmented matrix in order to know that the linear system is consistent and has a unique solution?

Exercise 14. Row reduce the augmented matrix. Using the echelon form, give the values of h and k such that the system has **(a)** no solution, **(b)** a unique solution, and **(c)** infinitely many solutions.

$$\begin{aligned} x_1 + hx_2 &= 2 \\ 4x_1 + 8x_2 &= k \end{aligned}$$

Exercise 15. Row reduce the augmented matrix. Using the echelon form, give the values of h and k such that the system has **(a)** no solution, **(b)** a unique solution, and **(c)** infinitely many solutions.

$$\begin{aligned} x_1 + x_2 + x_3 &= h \\ 4x_1 + 8x_2 - kx_3 &= 0 \end{aligned}$$

Exercise 16. Find an equation involving g, h, and k that makes this augmented matrix correspond to a consistent system:

$$\begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{bmatrix}$$

Exercise 17. Suppose the coefficient matrix of a linear system of three equations in three variables has a pivot position in each column. Explain why this system has a unique solution.