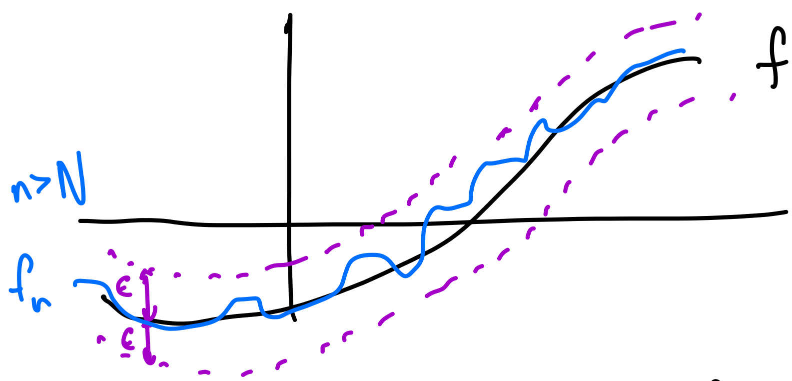
<u>Recall</u>: Suppose $f_1, f_2, f_3, \ldots$ is a sequence of functions from the metric space E to the metric space E'. Suppose $f_1, f_2, f_3 \ldots$ converges to f. Then we say the sequence $f_1, f_2, f_3, \ldots$ <u>converges to f uniformly</u> if $\forall \varepsilon > 0$, $\exists N$ s.t. $\forall n > N$, $\forall p \in E$, $d'(f(p), f_n(p)) < \varepsilon$.

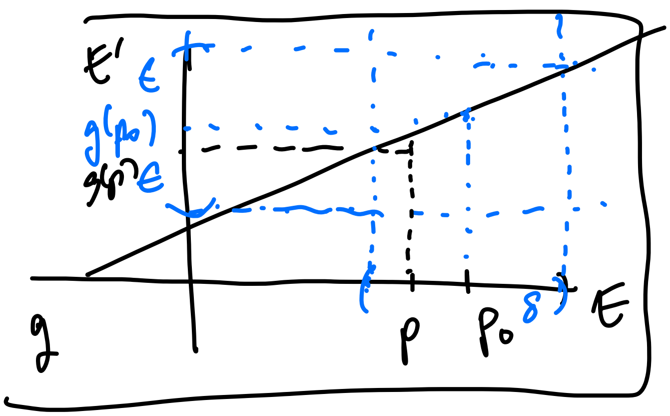


"The distance between f and f_n is $< \varepsilon$."

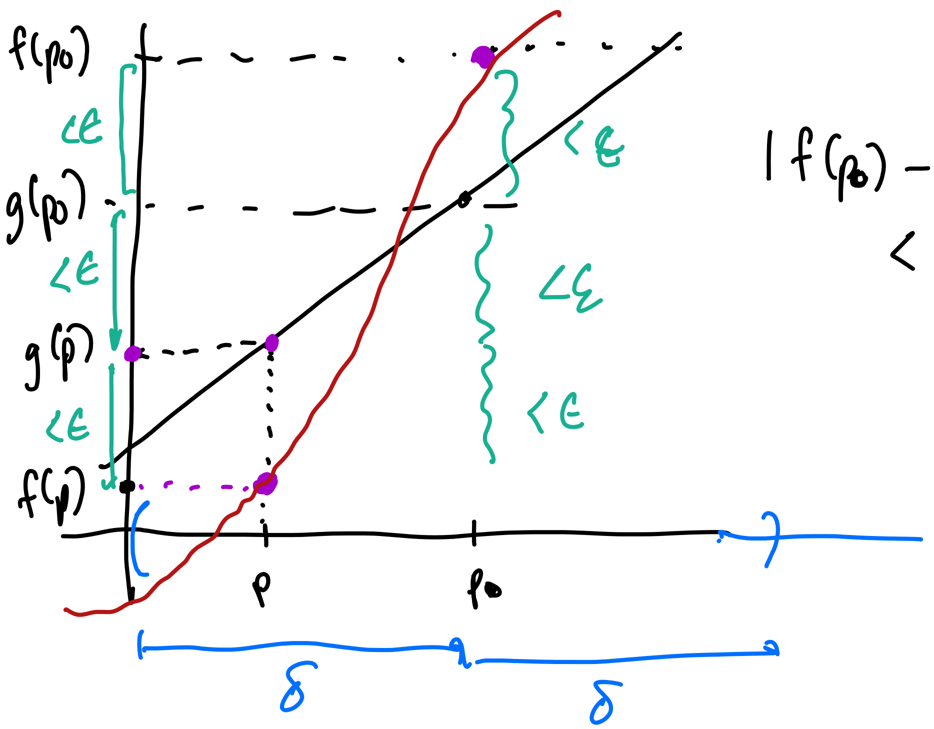
<u>Thm</u>: If $f_1, f_2, f_3 \ldots$ is a sequence of continuous functions that converges uniformly to f, then f must be continuous.

<u>Lemma</u>. Let $f: E \to E'$. Suppose that

$\forall \varepsilon > 0$, $\exists$ continuous $g : E \to E'$ s.t. $\forall p \in E$, $d'(f(p), g(p)) < \varepsilon$. Then f is continuous.



pf) Let $p_0 \in E$.
Let $\varepsilon > 0$.
$\exists\, \delta > 0$ s.t. $\forall p \in B(p_0; \delta)$, $d'(g(p), g(p_0)) < \frac{\varepsilon}{3}$.



$|f(p_0) - f(p)| < 3\varepsilon$

pf) In addition, we have

$$d'(f(p), g(p)) < \varepsilon/3$$

and $d'(f(p_0), g(p_0)) < \mathcal{E}/3$

Therefore,

$$d'(f(p), f(p_0)) \leq d'(f(p), g(p)) + d'(g(p), g(p_0)) + d'(g(p_0), f(p_0)) < \epsilon.$$

Pf) Let $p_0 \in E$. Let $\mathcal{E} > 0$.
of
Thm $\exists N > 0$ s.t. $\forall n > N$, $\forall p \in E$,

$$d'(f(p), f_n(p)) < \epsilon.$$

Fix any $n > N$, say $n = N+1$.

We apply the Lemma above with $g = f_n$.