Expected Value and Change of Variables

The purpose of this note is to explain the connection between the "undergraduate" and "graduate"-level definitions of expected value. The undergraduate-level definition of the *expected value* of a continuous random variable $X \geq 0$ with density function f is

$$E(X) = \int_0^\infty x f(x)\, dx.$$

The graduate-level definition (see Durrett, *Probability: Theory and Examples*, 5th edition, p.25) is this. Suppose that P is a probability measure. If $X \geq 0$ is a random variable on $(\Omega, \mathcal{M}, P)$, then we define its *expected value* to be

$$E(X) = \int X\, dP.$$

This is an integral over the probability space Ω. The undergraduate expectation is an integral over $\mathbf{R}$. The link is from the change of variables formula. The random variable X induces the measure $P \circ X^{-1}$ on $\mathbf{R}$. A special case of the change of variables formula is that

$$\int X\, dP = \int_{\mathbf{R}} \mathrm{id}_{\mathbf{R}}\, d(P \circ X^{-1}).$$

I am using the notation $\mathrm{id}_{\mathbf{R}}$ for the function $x \mapsto x$, the identity function on $\mathbf{R}$. In our case, since $X \geq 0$, the latter integral is really an integral on $[0, \infty)$:

$$\int X\, dP = \int_0^\infty x\, d(P \circ X^{-1}).$$

Now we need another result.

I use dm to denote Lebesgue measure on $\mathbf{R}$. Suppose that $f\colon \mathbf{R} \to [0, \infty]$ is a measurable function. For each measurable set E, define $\lambda_f(E) =$

$\displaystyle\int_E f\,dm$. Then λ_f is a measure on the collection of Lebesgue measurable sets. Furthermore, for any measurable function $g\colon \mathbf{R} \to [0,\infty]$,

$$\int g\,d\lambda_f = \int fg\,dm.$$

(This is an exercise in Folland's *Real Analysis*, Second Edition, exercise 14 on page 52.)

Suppose that f is the density function of X. This means that for all x,

$$P(X \leq x) = \int_{-\infty}^{x} f\,dm.$$

But $P(X \leq x)$ is the probability notation for $(P \circ X^{-1})([0,x])$. So $P \circ X^{-1}$ and λ_f coincide on the measurable sets $[0,x)$ for all x. Now we let $g\colon \mathbf{R} \to [0,\infty]$ be defined by $g(x) = 0$ if $x < 0$ and $g(x) = x$ if $x \geq 0$. We get

$$\int_0^\infty x\,d(P \circ X^{-1}) = \int_0^\infty x\,d\lambda_f = \int_0^\infty xf(x)\,dx.$$

This proves the equivalence of the two definitions of expected value of a continuous random variable $X \geq 0$. The same argument carries over to the case of a discrete random variable, and in this case, letting $p(x) = P(X = x)$,

$$\int X\,dP = \sum_{x=0}^{\infty} xp(x).$$

July 1, 2025