Linear Maps from $\mathbf{R}^n$ to $\mathbf{R}^m$

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Suppose A is an m by n matrix, and x is any n-vector. When A multiplies x, we can think of it as ***transforming the vector x into a new vector Ax.*** This happens at every point x of the space $\mathbf{R}^n$, so that the whole space is transformed or "mapped" by the matrix A into the new space $\mathbf{R}^m$. We can think of matrix-vector multiplication as a function. An $m \times n$ matrix takes vectors with n components as inputs and outputs vectors with m components. So to every matrix A, there is a function from $\mathbf{R}^n$ into $\mathbf{R}^m$. Call this function T_A. In this chapter we discuss linear maps and how they are related to matrices. We also discuss matrix operations and how they are related to linear maps.

1 Matrix maps

This chapter is all about functions, so before continuing, here are some things about functions. If X and Y are sets, by a **function from X to Y** (or a **function from X into Y**, or a **function on X with values in Y**) is meant a rule which associates with each element of X a definite element of Y. (The word *mapping*, or *map*, is often used instead of *function*.)

The statement "f is a function from X to Y" is often written

$$f\colon X \to Y.$$

The set X is called the **domain** of f and Y is called the **codomain** of f.

If $f\colon X \to Y$ is a function and $X' \subset X$, then the subset of Y given by

$$f(X') = \{f(x) : x \in X'\}$$

(where the last symbol is shorthand for $\{y : \text{there exists } x \in X' \text{ such that } y = f(x)\}$) is called the **image of X' under f**, or simply the **image of X'**, if there is no danger of confusion. The image of X, that is, $f(X)$, is simply called the **image of f**.

Let A be an $m \times n$ matrix. We will distinguish between a matrix (a rectangular array of numbers) and the function that sends the input $\mathbf{x}$ to the output $A\mathbf{x}$. We call this function T_A. So, each matrix A leads to a function, T_A, which is defined by the rule

$$T_A(\mathbf{x}) = A\mathbf{x}.$$

Notice that on the left, we are using the function notation $f(x)$. The function is T_A and we are evaluating the function at the vector $\mathbf{x}$. On the right, there is a matrix-vector multiplication.

The function T_A defined by $\mathbf{x} \mapsto A\mathbf{x}$ is called a **matrix transformation**.

Suppose that A is an $m \times n$ matrix. What are the domain and codomain of T_A? In order for $A\mathbf{x}$ to be defined, $\mathbf{x}$ must be an n-vector, so the domain of T_A is $\mathbf{R}^n$. The output, $A\mathbf{x}$, is an m-vector, so the codomain of T_A is $\mathbf{R}^m$. What about the range, or image, of T_A? Each image $T_A(\mathbf{x})$ is of the form $A\mathbf{x}$. The range, or image, of T is the set of all linear combinations of the columns of A.

Exercise 1. Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, and define $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$. Find the images under T of $\mathbf{u} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$.

Exercise 2. Let $A = \begin{bmatrix} .5 & 0 & 0 \\ 0 & .5 & 0 \\ 0 & 0 & .5 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Define $T\colon \mathbf{R}^3 \to \mathbf{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$. Find $T(\mathbf{u})$ and $T(\mathbf{v})$.

Exercise 3. With T defined by $T(\mathbf{x}) = A\mathbf{x}$, find a vector $\mathbf{x}$ whose image under T is $\mathbf{b}$, and determine whether $\mathbf{x}$ is unique.

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 6 \\ 3 & -2 & -5 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1 \\ 7 \\ -3 \end{bmatrix}$$

Exercise 4. (a) Let A be a 6×5 matrix. What must a and b be in order to define $T\colon \mathbf{R}^a \to \mathbf{R}^b$ by $T(\mathbf{x}) = A\mathbf{x}$?

(b) How many rows and columns must a matrix A have in order to define a mapping from $\mathbf{R}^4$ to $\mathbf{R}^5$ by the rule $T(\mathbf{x}) = A\mathbf{x}$?

Exercise 5. For the matrix

$$A = \begin{bmatrix} 1 & -4 & 7 & -5 \\ 0 & 1 & -4 & 3 \\ 2 & -6 & 6 & -4 \end{bmatrix}$$

(a) Find all $\mathbf{x}$ in $\mathbf{R}^4$ that are mapped into the zero vector by the transformation $\mathbf{x} \mapsto A\mathbf{x}$ for the given matrix A.

(b) Let $\mathbf{b} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. Is $\mathbf{b}$ in the range of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$? Why or why not?

The rest of this section focuses on mappings associated with matrix multiplication.

Example 1. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. This matrix takes $\mathbf{R}^3$ to $\mathbf{R}^3$. To see what it does to an arbitrary vector $\mathbf{x} = \langle x_1, x_2, x_3 \rangle$, multiply A by this vector:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}.$$

Therefore, T_A maps $\mathbf{x} = \langle x_1, x_2, x_3 \rangle$ to $\langle x_1, x_2, 0 \rangle$. A typical way to write this is

$$T_A(x_1, x_2, x_3) = (x_1, x_2, 0).$$

We often think of matrix transformations as sending points to other points, which explains the use of coordinates. Also, to avoid clutter, we write $T_A(x_1, x_2, x_3)$ instead of $T_A((x_1, x_2, x_3))$, which is more correct, since the inside parentheses are from the notation for the point, and the outside parentheses is from function notation.

We say that this matrix *projects* vectors onto the x_1x_2-plane.

Example 2. Let $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$. This maps (x, y) to $(x+3y, y)$. In other words, $T_A(x, y) = (x+3y, y)$.

Example 3. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. We can give a geometric description of the map $\mathbf{x} \mapsto A\mathbf{x}$. Plot some random points (vectors) on graph paper to see what happens. A point such as $(4, 1)$ maps into $(4, -1)$. The map $\mathbf{x} \mapsto A\mathbf{x}$ reflects points through the $x-$axis (or x_1-axis).

Example 4. A square matrix $A = (a_{ij})$ is called a **diagonal** matrix if $a_{ij} = 0$ whenever $i \neq j$. The diagonal entries a_{ii} might be 0. The identity matrix I_n is a diagonal matrix. How do diagonal matrices act on $\mathbf{R}^n$? The vector e_i gets scaled by a factor of a_{ii}.

Example 5. Let θ be a real number. We will look at the matrix

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

To understand what this does to the plane, it helps to represent each non-origin point (x, y) in polar coordinates, as $(r\cos\alpha, r\sin\alpha)$. Then

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} r\cos\alpha \\ r\sin\alpha \end{bmatrix} = \begin{bmatrix} r\cos\theta\cos\alpha - r\sin\theta\sin\alpha \\ r\sin\theta\cos\alpha + r\cos\theta\sin\alpha \end{bmatrix} = \begin{bmatrix} r\cos(\theta+\alpha) \\ r\sin(\theta+\alpha) \end{bmatrix}.$$

So the matrix takes the point $(r\cos\alpha, r\sin\alpha)$ to $(r\cos(\alpha+\theta), r\cos(\alpha+\theta))$. This new point has the same distance from the origin as the old point (they are both r units from the origin) but the angle changed from α to $\alpha+\theta$. If $\theta > 0$, then the angle increased by θ degrees, and if $\theta < 0$, the angle decreased by $|\theta|$ degrees. So the matrix *rotates* the plane by $|\theta|$ degrees counterclockwise if $\theta > 0$ and $|\theta|$ degrees clockwise if $\theta < 0$.

Notice that we have shown that rotation by θ degrees is a linear map, because it is represented by the matrix A above.

Exercise 6. Write down the matrix that rotates the plane counterclockwise by 270°.

Example 6. Here is a matrix

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

This matrix exchanges the x and y coordinates of every point, so that (3,2) is transformed into (2,3) and vice versa. The points come in pairs, except for those like (4,4), which stay fixed when multiplied by P - they are paired with themselves. To visualize how these point-to-point maps produce a map of the whole plane, imagine the line $x = y$ acting as a mirror. The matrix P gives a ***reflection*** which takes every point to its image on the opposite of the mirror. The points on the mirror stay fixed.

Exercise 7. Use a rectangular coordinate system to plot $\mathbf{u} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$ and their images under the given transformation T. Describe geometrically what T does to each vector $\mathbf{x}$ in $\mathbf{R}^2$.

(a) $T(\mathbf{x}) = \begin{bmatrix} .5 & 0 \\ 0 & .5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

(b) $T(\mathbf{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

2 Linear maps

In this section you will:

- Verify whether a mapping is linear.

There are many ways a space could be transformed. Every point could go into the origin (this is achieved by the zero matrix). No matrix can shift a point a unit distance in some direction because it cannot violate $A0 = 0$.

Every matrix leads to a linear map.

The function T_A is forced to satisfy certain properties because of the properties of matrix-vector multiplication. We know that matrix-vector multiplication has the properties

- $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y}$ and
- $A(c\mathbf{v}) = cA\mathbf{v}$.

Expressing this in terms of T_A, we get

- $T_A(\mathbf{x} + \mathbf{y}) = T_A(\mathbf{x}) + T_A(\mathbf{y})$ and
- $T_A(c\mathbf{v}) = cT_A(\mathbf{v})$.

This leads to the idea of a linear map. We define a linear map to be any function that satisfies these two conditions. Here is the precise definition.

Linear Maps from $\mathbf{R}^n$ to $\mathbf{R}^m$:

A **linear map from $\mathbf{R}^n$ into $\mathbf{R}^m$** is a function $T\colon \mathbf{R}^n \to \mathbf{R}^m$ such that
(i) for all vectors $\mathbf{v}, \mathbf{w} \in \mathbf{R}^n$,

$$T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$$

and (ii) for all scalars c and all vectors $\mathbf{v} \in \mathbf{R}^n$,

$$T(c\mathbf{v}) = cT(\mathbf{v}).$$

This definition takes some time to understand. First, let's be clear that in the definition, $\mathbf{v}$ and $\mathbf{w}$ don't have to be different. They can be the same vector. So from the first criterion, if you take $\mathbf{v}$ and $\mathbf{w}$ to be the same vector, you get $T(2\mathbf{v}) = T(\mathbf{v}) + T(\mathbf{v})$, and this says that $T(2\mathbf{v}) = 2T(\mathbf{v})$. This is included in the second condition when $c = 2$. So there is some overlap in the two conditions. However, the second condition is not just part of the first condition, because in the second condition, c can be any real number. So, for example, the second condition implies that $T(2.532\mathbf{v}) = 2.532T(\mathbf{v})$ and $T(-\pi\mathbf{v}) = -\pi T(\mathbf{v})$. This is not something that follows from the first condition.

From the definition, we see that T is *not* a linear map if at least one of the following hold:
(i) There are vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbf{R}^n$ such that $T(\mathbf{v} + \mathbf{w}) \neq T(\mathbf{v}) + T(\mathbf{w})$; or
(ii) There is a vector $\mathbf{v} \in \mathbf{R}^n$ and a scalar c such that $T(c\mathbf{v}) \neq cT(\mathbf{v})$.

Now let's look at some properties of linear mappings. First, take $c = 0$ above. Then $c\mathbf{v} = 0\mathbf{v} = \mathbf{0}$ and $cT(\mathbf{v}) = 0T(\mathbf{v}) = \mathbf{0}$.

Linear Maps take the Zero Vector to the Zero Vector

Every linear map $T\colon \mathbf{R}^n \to \mathbf{R}^m$ satisfies

$$T(\mathbf{0}) = \mathbf{0} \tag{1}$$

Every linear map $T\colon \mathbf{R}^n \to \mathbf{R}^m$ maps the zero vector to the zero vector.

Keep in mind that since T is a linear map from $\mathbf{R}^n$ into $\mathbf{R}^m$, we mean that T takes the zero vector in $\mathbf{R}^n$ to the zero vector in $\mathbf{R}^m$. In our notation, we are using $\mathbf{0}$ to mean the zero vector, but this exact vector depends on the context. Unfortunately, to make it clear, the notation would get cluttered:

$$T(\mathbf{0}_{\mathbf{R}^n}) = \mathbf{0}_{\mathbf{R}^m}.$$

Because of this, we avoid using this notation.

Nobody is saying that if a function sends the zero vector to the zero vector, then it must be a linear map. That is just something that all linear maps must do. There are lots of functions that send the zero vector to the zero vector that are not linear maps.

Example 7. Let $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ be defined by

$$T(x, y) = (x^2, \sin xy).$$

Then $T(0, 0) = (0^2, \sin(0 \cdot 0)) = (0, 0)$. However, $T(1, 0) = (1, 0)$ and $T(0, 1) = (0, 0)$, but $(1, 0) + (0, 1) = (1, 1)$ and $T(1, 1) = (1, \sin 1) \neq (1, 0) + (0, 0)$. Therefore, T is not a linear mapping.

It's often useful to combine the conditions (i) and (ii) together to a new condition:

Condition (iii): For all vectors $\mathbf{v}, \mathbf{w} \in \mathbf{R}^n$ and all $c, d \in \mathbf{R}$,

$$T(c\mathbf{v} + d\mathbf{w}) = cT(\mathbf{v}) + dT(\mathbf{w}). \tag{2}$$

If a map satisfies conditions (i) and (ii), then it must satisfy (iii). To see why this is true, notice that by property (i),

$$T(c\mathbf{v} + d\mathbf{w}) = T(c\mathbf{v}) + T(d\mathbf{w}).$$

By property (ii), $T(c\mathbf{v}) = cT(\mathbf{v})$ and $T(d\mathbf{w}) = dT(\mathbf{w})$, so

$$T(c\mathbf{v}) + T(d\mathbf{w}) = cT(\mathbf{v}) + dT(\mathbf{w}).$$

Conversely, if a map satisfies condition (iii), then by taking $c = 1$ and $d = 1$, we see that the map satisfies condition (i). By taking $d = 0$, we get

$$T(c\mathbf{v}) = T(c\mathbf{v} + 0\mathbf{w}) = cT(\mathbf{v}) + 0T(\mathbf{w}) = cT(\mathbf{v}),$$

so condition (ii) holds. So conditions (i) and (ii) both hold if and only if condition (iii) holds. Therefore, we can give an alternate definition of a linear map:

Alternate Definition of a Linear Map

A **linear map** $T\colon \mathbf{R}^n \to \mathbf{R}^m$ is any function with the property that for all vectors $\mathbf{v}, \mathbf{w} \in \mathbf{R}^n$ and all $c, d \in \mathbf{R}$,

$$T(c\mathbf{v} + d\mathbf{w}) = cT(\mathbf{v}) + dT(\mathbf{w}). \tag{3}$$

Notice that $c\mathbf{v}+d\mathbf{w}$ is a linear combination of $\mathbf{v}$ and $\mathbf{w}$. This is the input on the left side of the equation, and $T(c\mathbf{v}+d\mathbf{w})$ is the output. The equation says that this output is c times the output of $\mathbf{v}$, plus d times the output of $\mathbf{w}$. Roughly speaking, a linear map is a function with the following property: *Any linear combination of two inputs must go to the corresponding linear combination of the two outputs.*

Applying Equation 3 over and over, we see that

Alternate Definition of a Linear Map

For all scalars $c_1, ..., c_k$ and all vectors $\mathbf{v}_1, ..., \mathbf{v}_k \in \mathbf{R}^n$,

$$T(c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k) = c_1T(\mathbf{v}_1) + \cdots + c_kT(\mathbf{v}_k). \tag{4}$$

In engineering and physics, equation 4 is referred to as a *superposition principle.* Think of $\mathbf{v}_1, ..., \mathbf{v}_k$ as signals that go into a system and $T(\mathbf{v}_1), ...T(\mathbf{v}_k)$ as the responses of that system to the signals. The system satisfies the superposition principle if whenever an input is expressed as a linear combination of such signals, the system's response is the same linear combination of the responses to the individual signals. For more information about this in physics, see "Physics for Scientists and Engineers" by Giancoli, section 15-6.

Later on, we will learn a theorem that shows that all linear maps from $\mathbf{R}^n$ into $\mathbf{R}^m$ have a specific form.

Exercise 8. Let $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ be a linear transformation that maps $\mathbf{u} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ into $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and maps $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ into $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Use the fact that T is linear to find the images under T of $3\mathbf{u}$, $2\mathbf{v}$, and $3\mathbf{u}+2\mathbf{v}$.

Exercise 9. Let $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Let $\mathbf{y}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and $\mathbf{y}_2 = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$, and let $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ be a linear map that maps $\mathbf{e}_1$ into $\mathbf{y}_1$ and $\mathbf{e}_2$ into $\mathbf{y}_2$. Find the images of $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

The next result says that the composition of two linear maps is a linear map. Let $T\colon \mathbf{R}^n \to \mathbf{R}^m$ and $S\colon \mathbf{R}^m \to \mathbf{R}^k$ be linear maps. Then $S \circ T$ is a linear map.

NOTE: It is important to keep in mind that a linear map *is* a function. So why don't we call them linear functions? Even though a function technically can take any domain to any target, the word "function" usually denotes a function whose outputs are scalars, like real numbers or complex numbers, not vectors. Other commonly used terms for "linear map" are "linear *map*" and "linear *mapping*." Actually, a linear map from $\mathbf{R}^n$ to $\mathbf{R}$ is often called a *linear functional*.

Also, a function $f(x) = mx + b$ is called a *linear function* but if $b \neq 0$, then f is not a linear map from $\mathbf{R}$ to $\mathbf{R}$.

Linear maps are often called linear *transformations*. This is just another word for the same thing. This is an old-fashioned word to describe a map from one "space" to another "space." I decided not to use the term "transformation" because the word makes it seem as though linear maps are some sort of process that takes time, like some object transforming into gold. However, the term "linear transformation" is still used in most linear algebra books.

Example 8. Let $m, b \in \mathbf{R}$, and define the function $f\colon \mathbf{R} \to \mathbf{R}$ by $f(x) = mx + b$. The function f is called a linear function because its image is a line. But is f a linear map? We first show that f

is a linear map when $b = 0$. In this case, $f(x) = mx$. We have

$$\begin{aligned} f(cx_1 + dx_2) &= m(cx_1 + dx_2) \\ &= mcx_1 + mdx_2 \\ &= c(mx_1) + d(mx_2) \\ &= cf(x_1) + df(x_2). \end{aligned}$$

This shows that the function f defined by $f(x) = mx$ is a linear map. When $b \neq 0$, however, $f(0) = b \neq 0$, so f is not a linear map.

Exercise 10. Show that the map T defined by $T(x_1, x_2) = (4x_1 - 2x_2, 3|x_2|)$ is not linear.

Exercise 11. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the map defined by

$$T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ 5x_1x_2 \end{bmatrix}.$$

Decide whether T is linear. Justify your answer in the following way.

- If you believe T is linear, prove it using the definition of linear map.
- If you believe T is not linear, provide a **specific counterexample** (i.e., using specific numbers) to one of the conditions in the **definition** of linear map.

Exercise 12. Let T be a linear map from $\mathbf{R}^2$ to $\mathbf{R}^2$. If $T \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ and $T \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, find $T \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

Exercise 13. True or False: The map $T(x, y) = (x, 1)$ is a linear map.

3 Properties of Linear Maps

Let's look at where linear maps send lines.

The line segment from $\mathbf{0}$ to a vector $\mathbf{u}$ is the set of points of the form $t\mathbf{u}$, where $0 \le t \le 1$. A linear map T maps this segment into the segment between $\mathbf{0}$ and $T(\mathbf{u})$. To see this, let $\mathbf{x} = t\mathbf{u}$ for some $0 \le t \le 1$. Since T is linear, $T(t\mathbf{u}) = tT(\mathbf{u})$, which is a point on the line segment between $\mathbf{0}$ and $T(\mathbf{u})$.

The line through vectors $\mathbf{p}$ and $\mathbf{q}$ in $\mathbf{R}^n$ may be written in the parametric form

$$\mathbf{x} = (1 - t)\mathbf{p} + t\mathbf{q}.$$

The line segment from $\mathbf{p}$ to $\mathbf{q}$ is the set of points of the form $(1 - t)\mathbf{p} + t\mathbf{q}$ for $0 \le t \le 1$. We have

$$T(\mathbf{x}) = (1 - t)T(\mathbf{p}) + tT(\mathbf{q}).$$

T maps this line segment onto a line segment if $T(\mathbf{p}) \neq T(\mathbf{q})$ and onto a single point if $T(\mathbf{p}) = T(\mathbf{q})$.

Given $\mathbf{v} \neq \mathbf{0}$ and $\mathbf{p} \in \mathbf{R}^n$, the line through $\mathbf{p}$ in the direction of $\mathbf{v}$ has the parametric equation $\mathbf{x} = \mathbf{p} + t\mathbf{v}$. Where does a linear map $T\colon \mathbf{R}^n \to \mathbf{R}^n$ map this line? We have

$$T(\mathbf{p} + t\mathbf{v}) = T(\mathbf{p}) + tT(\mathbf{v}).$$

This is a line through $T(\mathbf{p})$ in the direction of $T(\mathbf{v})$ if $T(\mathbf{v}) \neq \mathbf{0}$. If $T(\mathbf{v}) = \mathbf{0}$, T maps the line onto a single point (a *degenerate line*).

Where do linear maps map planes? Let $\mathbf{u}$ and $\mathbf{v}$ be linearly independent vectors in $\mathbf{R}^3$, and let P be the plane through $\mathbf{u}, \mathbf{v}$, and $\mathbf{0}$. The parametric equation of P is $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$ (with s, t in $\mathbf{R}$). A linear map $T\colon \mathbf{R}^3 \to \mathbf{R}^3$ maps a typical vector $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$ to

$$T(s\mathbf{u} + t\mathbf{v}) = sT(\mathbf{u}) + tT(\mathbf{v}).$$

If $T(\mathbf{u}) = T(\mathbf{v}) = \mathbf{0}$ then T maps P onto just the origin in $\mathbf{R}^3$. If either $T(\mathbf{u})$ or $T(\mathbf{v})$ is nonzero and they are scalar multiples of each other, then T maps P onto a line through $\mathbf{0}$. If $T(\mathbf{u})$ and $T(\mathbf{v})$ are linearly independent, then T maps P onto a plane through $\mathbf{0}$.

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbf{R}^n$. It can be shown that the set P of all points in the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$ has the form $a\mathbf{u} + b\mathbf{v}$, for $0 \le a \le 1$, $0 \le b \le 1$. Let $T\colon \mathbf{R}^n \to \mathbf{R}^n$ be a linear map. Then

$$T(a\mathbf{u} + b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v}).$$

Therefore the image of a point in P under the map T lies in the parallelogram determined by $T(\mathbf{u})$ and $T(\mathbf{v})$.

Exercise 14. Find where the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

maps the line $\mathbf{r}(t) = \langle 1, 0, 0\rangle + t\langle 2, -1, 3\rangle$.

Exercise 15. Let ℓ be the set of all points (x, y, z) such that

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

for some real t, and let P be the plane with equation

$$x + y + z = 1.$$

(a) Prove that the matrix

$$\mathbf{A} = \begin{pmatrix} -2 & -2 & 2 \\ 2 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}$$

maps all points on line ℓ to points on plane P.

(b) Prove that the matrix

$$\mathbf{B} = \begin{pmatrix} 1 & 1 & -1 \\ 3 & 3 & -1 \\ 5 & 5 & -1 \end{pmatrix}$$

maps all points on plane P to points on line ℓ.

Exercise 16. Show that the matrix

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

takes the unit circle to the ellipse with equation

$$\frac{x^2}{4} + \frac{y^2}{9} = 1.$$

Exercise 17. The matrix

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

takes the parabola given by $y = x^2$ to another parabola. Find an equation for this parabola.

4 The Matrix of a Linear Map

Whenever a linear map T arises geometrically or is described in words, we usually want a "formula" for $T(\mathbf{x})$. The discussion that follows shows that every linear map from $\mathbf{R}^n$ to $\mathbf{R}^m$ is actually a matrix map $\mathbf{x} \mapsto A\mathbf{x}$ and that important properties of T are related to familiar properties of A. The key to finding A is to observe that T is completely determined by what it does to the columns of the $n \times n$ identity matrix I_n. In this section, you will

- Calculate the standard matrix of a linear mapping from $\mathbf{R}^n$ to $\mathbf{R}^m$.

The **standard basis vectors** $\mathbf{e}_1, ..., \mathbf{e}_n$ for $\mathbf{R}^n$ are defined as follows: $\mathbf{e}_i$ is the vector whose ith component is 1 and all the other components are 0.

Notice that for any m by n matrix A, $A\mathbf{e}_j$ is the jth column of A.

Every linear map is determined by where it sends the standard basis vectors. For every vector $\mathbf{v} = \langle a_1, ..., a_n \rangle$ can be expressed as

$$\mathbf{v} = a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n,$$

and by the linearity of T,

$$T(\mathbf{v}) = a_1 T(\mathbf{e}_1) + \cdots + a_n T(\mathbf{e}_n).$$

Therefore, once the vectors $T(\mathbf{e}_1)$,...,$T(\mathbf{e}_n)$ are given, one can determine $T(\mathbf{v})$.

To every linear map $T\colon \mathbf{R}^n \to \mathbf{R}^m$, let A be the matrix whose columns are $T(\mathbf{e}_1), ..., T(\mathbf{e}_n)$ in that order. This is called the **standard matrix** associated with T. Since $A\mathbf{e}_j$ is the jth column of A, we see that

$$A\mathbf{e}_1 = T(\mathbf{e}_1),\ A(\mathbf{e}_2) = T(\mathbf{e}_2), ..., A(\mathbf{e}_n) = T(\mathbf{e}_n).$$

Since T and the function $\mathbf{x} \mapsto A\mathbf{x}$ are linear maps that agree on $\mathbf{e}_1, ..., \mathbf{e}_n$, they must be the same linear map. Therefore,

$$T(\mathbf{x}) = A\mathbf{x}$$

for all $x \in \mathbf{R}^n$.

Standard Matrix of a Linear Map from $\mathbf{R}^n$ into $\mathbf{R}^m$

If T is a linear map from $\mathbf{R}^n$ into $\mathbf{R}^m$, the matrix whose columns are $T(\mathbf{e}_1)$,...,$T(\mathbf{e}_n)$, in that order, is called the **standard matrix** of T. It is an m by n matrix, A. It has the property that

$$T(\mathbf{x}) = A\mathbf{x}.$$

Exercise 18. Assume that T is a linear transformation. Find the standard matrix of T.

(a) $T\colon \mathbf{R}^2 \to \mathbf{R}^4$, $T(\mathbf{e}_1) = (3, 1, 3, 1)$ and $T(\mathbf{e}_2) = (-5, 2, 0, 0)$.

(b) $T\colon \mathbf{R}^3 \to \mathbf{R}^2$, $T(\mathbf{e}_1) = (1, 3), T(\mathbf{e}_2) = (4, -7)$, and $T(\mathbf{e}_3) = (-5, 4)$.

(c) $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ rotates points (about the origin) through $3\pi/2$ radians (counterclockwise).

(d) $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ first reflects points through the horizontal x_1-axis and then reflects points through the line $x_2 = x_1$.

Theorem 1. *Every linear map $T\colon \boldsymbol{R}^n \to \boldsymbol{R}^m$ has the form*

$$T(x_1, ..., x_n) = (a_{11}x_1 + \cdots + a_{1n}x_n, ..., a_{m1}x_1 + \cdots + a_{mn}x_n)$$

for some mn scalars $a_{11}, ..., a_{mn}$.

Any function that does not have this form is not a linear map. Keep in mind that none of the components have any "constant" part! This takes some getting used to, since the function $f(x) = mx + b$ is a linear function . The function $f\colon \mathbf{R} \to \mathbf{R}$ defined by $f(x) = mx + b$ is not a linear map unless $b = 0$: we need $f(0) = 0$. However, $f(x) = mx$ *is* a linear map.

Exercise 19. Find a matrix A such that

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_1 - 2x_3 \\ 4x_1 \\ x_1 - x_2 + x_3 \end{bmatrix}$$

for all values of the variables.

Exercise 20. Show that T is a linear transformation by finding a matrix that implements the mapping:

$$T(x_1, x_2, x_3, x_4) = (0, x_1 + x_2, x_2 + x_3, x_3 + x_4).$$

Exercise 21. Let $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ be a linear transformation such that $T(x_1, x_2) = (x_1 + x_2, 4x_1 + 5x_2)$. Find $\mathbf{x}$ such that $T(\mathbf{x}) = (3, 8)$.

Exercise 22. (a) Show that $T(x, y) = (x^2 + xy, y)$ is not a linear mapping from $\mathbf{R}^2$ to $\mathbf{R}^2$.

(b) Gregg was thinking about this problem. He wrote $T(x, y)$ like this:

$$\begin{bmatrix} x & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^2 + xy \\ y \end{bmatrix}$$

Since this is a matrix times (x, y), Gregg thought that T is a linear mapping. Explain what the mistake is in his reasoning.

Exercise 23. Find a 2 by 2 matrix that reflects points across the line $y = 2x$.

Exercise 24. Two vertices of an equilateral triangle are $(0, 0)$ and $(10, 5)$. What are the possibilities for the third vertex? Hint: Use a rotation matrix.

Exercise 25. Suppose that T is a linear map that sends $\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 5 \\ 4 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Explain why T must also send $\begin{bmatrix} -9 \\ 19 \\ 20 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Exercise 26. Suppose that T is a linear map that sends $\begin{bmatrix} 8 \\ -2 \\ 4 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Explain why T must also send $\begin{bmatrix} -4 \\ 1 \\ -2 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Exercise 27. Give an example of a linear map $T : \mathbf{R}^2 \to \mathbf{R}^2$ that sends both $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Are there any other possible linear maps? Try to prove it.

5 1-1, Onto, Invertible

In this section, you will:

- Identify whether a linear transformation is one-to-one and/or onto and whether it has an inverse

As we have seen in the previous sections, matrices give rise to linear mappings from $\mathbf{R}^n$ to $\mathbf{R}^m$, and conversely, every linear mapping from $\mathbf{R}^n$ to $\mathbf{R}^m$ is determined by a matrix. This means that we can think of linear mappings from $\mathbf{R}^n$ to $\mathbf{R}^m$ in terms of a matrix, and vice versa. In this section, we'll see how properties of special linear mappings can be seen in terms of the corresponding matrix. This will provide a better understanding of what properties of the pivot positions of a matrix tell us about the behavior of the matrix.

A function $f\colon X \to Y$ is said to be **one-to-one** (or **injective**) provided that, if $x_1, x_2 \in X$ and $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$. In other words, two distinct elements in the domain must be mapped to two distinct elements in the codomain. This condition is equivalent to: For every element $y \in Y$, there is *at most one* element $x \in X$ for which $f(x) = y$; there is at most one (i.e., 0 or 1) element $x \in X$ that gets mapped to y.

Example 9. Consider the function $f\colon \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x(x-1)(x-2)$. The function f is not one-to-one because there is more than one value of x such that $f(x) = 0$, namely, 0, 1, and 2.

Although this might be overkill, let's clarify what this says in the case of a mapping $T\colon \mathbf{R}^n \to \mathbf{R}^m$. The mapping T is said to be **one-to-one** if each $\mathbf{b}$ in $\mathbf{R}^m$ is the image of at most one $\mathbf{x}$ in $\mathbf{R}^n$. Equivalently, T is one-to-one if, for each $\mathbf{b}$ in $\mathbf{R}^m$, the equation $T(\mathbf{x}) = \mathbf{b}$ has either a unique solution or none at all. "Is T one-to-one?" is a uniqueness question. The mapping T is *not* one-to-one when some $\mathbf{b}$ in $\mathbf{R}^m$ is the image of more than one vector in $\mathbf{R}^n$. If there is no such $\mathbf{b}$, then T is one-to-one.

One-to-One Linear Maps

Let $T\colon \mathbf{R}^n \to \mathbf{R}^m$ be a linear map. Then T is one-to-one precisely when the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution ($\mathbf{x} = \mathbf{0}$).

Proof. Since T is linear, $T(\mathbf{0}) = \mathbf{0}$. If T is one-to-one, then the equation $T(\mathbf{x}) = \mathbf{0}$ has at most one solution and hence only the trivial solution.

Conversely, suppose that $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution. We claim that T is one-to-one. Suppose that there are two (not necessarily distinct) vectors $\mathbf{u}$ and $\mathbf{v}$ such $T(\mathbf{u}) = T(\mathbf{v})$. We will show that $\mathbf{u} = \mathbf{v}$. Subtracting both sides of the equation $T(\mathbf{u}) = T(\mathbf{v})$ by $T(\mathbf{v})$ gives

$$T(\mathbf{u}) - T(\mathbf{v}) = \mathbf{0}.$$

By the linearity of T, $T(\mathbf{u}) - T(\mathbf{v}) = T(\mathbf{u} - \mathbf{v})$, so

$$T(\mathbf{u} - \mathbf{v}) = \mathbf{0}.$$

Since the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution, $\mathbf{u} - \mathbf{v} = \mathbf{0}$. In other words, $\mathbf{u} = \mathbf{v}$. This shows that T is one-to-one. $\square$

So, when T is a linear map, we can tell if T is one-to-one or not just by examining which vectors are mapped to the zero vector. If the only vector that is mapped to the zero vector is the zero vector, then T is one-to-one.

Notice that for general functions, you cannot tell if a function f from $\mathbf{R}$ to $\mathbf{R}$ is one-to-one or not just by looking at the solutions to $f(x) = 0$. For example, the function $f(x) = e^x$ is one-to-one but there are no values of x for which $f(x) = 0$. The function $g(x) = -2 + \sin x$ also has no roots but $g(x)$ is certainly not one-to-one. The function $h(x) = x(x-1)(x-2) - 24$ crosses the x-axis once (at $x = 4$), but h is not one-to-one, since $h(0) = h(1)$ (they are both -24). This isn't surprising. For general functions, knowing that a function crosses the x-axis once doesn't tell you about how the function behaves elsewhere.

Since $T(\mathbf{x}) = A\mathbf{x}$, we see that T is 1-1 if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. We have seen this before! This is when the columns of A are linearly independent. Recall that this is when the echelon form of A has a pivot in every column. (Let's review the reason: If there is a free column, then $A\mathbf{x} = \mathbf{0}$ will have an infinite number of solutions, and a non-trivial solution gives a way to express the zero vector as a non-trivial linear combination of the columns of A. If there are no free columns, then there are no free variables, so $A\mathbf{x} = \mathbf{0}$ has only the solution $\mathbf{x} = \mathbf{0}$.)

> **Pivot Criterion for One-to-One Linear Maps**
>
> Let $T\colon \mathbf{R}^n \to \mathbf{R}^m$ be a linear map and let A be the standard matrix of T. Then T is onto precisely when an echelon form of A has a pivot in every *column*.

A function $f\colon X \to Y$ is said to be **onto** (or **surjective**) if every element of y is the image of some element of x, i.e., for every $y \in Y$, there is *at least one* (i.e., 1 or more) element $x \in X$ such that $f(x) = y$. In other words, every element of Y is an output of the function.

Suppose that $T\colon \mathbf{R}^n \to \mathbf{R}^m$ is a linear transformation. How can you tell if it is onto? Let A be the standard matrix of T. Then A is an $m \times n$ matrix. T being onto means: For every $\mathbf{y} \in \mathbf{R}^m$, there is at least one $\mathbf{x} \in \mathbf{R}^n$ such that $T(\mathbf{x}) = \mathbf{y}$.

What does that mean in terms of the matrix A? Since $T(\mathbf{x}) = A\mathbf{x}$, it means: For every $\mathbf{y} \in \mathbf{R}^m$, there is at least one $\mathbf{x} \in \mathbf{R}^n$ such that $A\mathbf{x} = \mathbf{y}$. In other words, no matter what $\mathbf{y} \in \mathbf{R}^m$ is, the equation $A\mathbf{x} = \mathbf{y}$ always has a solution. In order for this to happen, the echelon form of A needs a pivot in every row. Do you remember what happens if $REF(A)$ does not have a pivot in the last row? It would have a row of zeros. And then what goes wrong? You would be able to come up with a vector $\mathbf{b}$ such that, when the augmented matrix is row reduced, the vector of constants $\mathbf{b}$ will turn into a vector whose last component is not 0, and so $A\mathbf{x} = \mathbf{b}$ would have no solution.

Moreover, if $REF(A)$ does have a pivot in every row, then $A\mathbf{x} = \mathbf{b}$ will have a solution for every $\mathbf{b} \in \mathbf{R}^m$. This leads us to the following criterion:

> **Pivot Criterion for Onto Linear Maps**
>
> Let $T\colon \mathbf{R}^n \to \mathbf{R}^m$ be a linear map and let A be the standard matrix of T. Then T is onto precisely when an echelon form of A has a pivot in every *row*.

Keep in mind that this is not the *meaning* of "onto" – the word "onto" can be used to describe functions that are not linear maps. The pivot criterion is a way to tell if a linear map is onto, not what it means to be onto.

Example 10. Let T be the linear transformation whose standard matrix is

$$A = \begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -2 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Does T map $\mathbf{R}^4$ onto $\mathbf{R}^3$? Is T a one-to-one mapping?

Since A happens to be in echelon form, we can see at once that A has a pivot position in each row. So T maps $\mathbf{R}^4$ onto $\mathbf{R}^3$. However, there is no pivot in column 3. So T is not one-to-one.

If a function $f\colon X \to Y$ is both one-to-one and onto, it is said to be **bijective** or **invertible**: For every element $y \in Y$, there is *exactly one* element $x \in X$ such that $f(x) = y$. In this case, we can define the **inverse** of f: $f^{-1}\colon Y \to X$ is the function that sends y back to the (unique) element $x \in X$ such that $f(x) = y$. Therefore, $f^{-1} \circ f = \mathrm{id}_X$.

Invertible Linear Maps

A linear transformation $T\colon \mathbf{R}^n \to \mathbf{R}^m$ is one-to-one and onto (i.e. it is invertible) precisely when its standard matrix A has a pivot in every row and column.

In particular, A must be a square matrix. Naturally, we call A an **invertible matrix** if the corresponding linear map T_A is an invertible function. We will return to invertible matrices later.

Invertible Matrices

A matrix A is **invertible** if the linear map T_A is invertible. These matrices are the square matrices that have a pivot in every row and column.

Example 11. Let T be the linear map whose standard matrix is

$$A = \begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Since A happens to be in echelon form, we can see that A has a pivot position in each row. Therefore, T maps $\mathbf{R}^4$ (its domain) *onto* $\mathbf{R}^3$. However, since the equation $A\mathbf{x} = \mathbf{b}$ has a free variable, T is not one-to-one.

Actually, we can see that

If $m < n$, then a linear map $T\colon \mathbf{R}^n \to \mathbf{R}^m$ is never one-to-one.

if $m > n$, then a linear map $T\colon \mathbf{R}^n \to \mathbf{R}^m$ cannot be onto because there are more rows than columns, and so its standard matrix cannot have more pivots than there are rows.

If there is a linear transformation $T\colon \mathbf{R}^n \to \mathbf{R}^m$ that maps $\mathbf{R}^n$ onto $\mathbf{R}^m$, how are m and n related? If $T\colon \mathbf{R}^n \to \mathbf{R}^m$ maps $\mathbf{R}^n$ *onto* $\mathbf{R}^m$, then it must be that $m \leq n$.

If there is a linear transformation $T\colon \mathbf{R}^n \to \mathbf{R}^m$ that is one-to-one, what can you say about m and n? If T is one-to-one, it must be that $n \leq m$.

NOTE: Let's be clear exactly what we are saying. *Nobody is saying* that *if $n \leq m$, then T* must be one-to-one. For example, you could make $n = 2$ and $m = 3$. This 3×2 matrix

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

does *not* have a pivot in the second column, so T is *not* one-to-one. What we are saying is that in order for T to even have a chance to be one-to-one, the matrix A cannot have more columns than rows, because then it would be impossible for there to be a pivot in every column.

Also, nobody is saying that if $m \leq n$, then T is onto. For example, this 2×3 matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 8 & 12 \end{bmatrix}$$

does not have a pivot in row 2, so T is not onto. What we are saying is that in order for T to even have a chance to be onto, the matrix A cannot have more rows than columns, because then it would be impossible for there to be a pivot in every row.

For example, if A is a 7×5 matrix and $T(\mathbf{x}) = A\mathbf{x}$, then T is not onto. If A has 5 pivots, then T is 1-1; otherwise, if A has fewer than 5 pivots, then T is not 1-1.

Exercise 28. Suppose that T is a linear map from $\mathbf{R}^3$ to $\mathbf{R}^6$ and that T is one-to-one. If A is the standard matrix of T, give ALL possibilities for the RREF (reduced row echelon form) of A.

Exercise 29. Give an example of a non-zero matrix that takes the plane spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Give an explicit description of the image of the transformation defined by this matrix.

Exercise 30. Suppose that A is a matrix, $\mathbf{b}$ is a vector, and the general solution to $A\mathbf{x} = \mathbf{b}$ is

$$\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

(a) By choosing a certain value of t, show that $A \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} = \mathbf{b}$.

(b) By choosing a certain value of t, show that $A \left(\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right) = \mathbf{b}$.

(c) Show, using parts (a) and (b), that $A \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \mathbf{0}$.

(d) Explain why part (c) shows that for all t, $A \left(t \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right) = \mathbf{0}$.

6 Why Linear maps?

You might ask: Every linear map from $\mathbf{R}^n$ to $\mathbf{R}^m$ is a matrix map. So what's the point of learning about linear maps?

Once you learn about vector spaces, you will learn about linear maps from one vector space to another. This concept of a linear map, not necessarily from $\mathbf{R}^n$ to $\mathbf{R}^m$, but from one vector space V to another vector space W, goes beyond matrices. Linear maps, also known as linear mappings, from one vector space to another, arise in many situations. For example, the set of twice differentiable functions on the open interval $(0, 1)$ is a vector space. Differentiation is a linear map from this vector space to the set of differentiable functions on $(0, 1)$, which is another vector space.