Homogeneous Linear Systems

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Although we are interested in solving $A\mathbf{x} = \mathbf{b}$ for any A and any $\mathbf{b}$, the particular case $\mathbf{b} = \mathbf{0}$ is particularly important. In this case, the linear system is said to be *homogeneous*.

Homogeneous and Non-Homogeneous Linear Systems

A system of linear equations all of whose constant terms are 0 is said to be **homogeneous**. Phrased in terms of matrix-vector multiplication: A linear system described by the matrix equation

$$A\mathbf{x} = \mathbf{0}$$

is called a **homogeneous** linear system. Otherwise, the system is said to be **non-homogeneous** or **inhomogeneous**.

For example, the system

$$\begin{aligned} x + y + z &= 5 \\ x - y + z &= 7 \end{aligned}$$

is non-homogeneous, whereas the system

$$\begin{aligned} x + y + z &= 0 \\ x - y + z &= 0 \end{aligned}$$

is homogeneous. Both of these systems have the coefficient matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

but the constant vector of the first system is $\begin{bmatrix} 5 \\ 7 \end{bmatrix}$ whereas the constant vector of the second system is the zero vector, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Any inhomogeneous system has a corresponding homogeneous system – just make the constants on the right equal to 0, as in the example. How do the solutions of the two systems relate? This is the question we will address in this section.

Before continuing, note one very important difference between $A\mathbf{x} = \mathbf{b}$ for a non-zero vector $\mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$ – every homogeneous linear system is consistent because the zero vector $\mathbf{x} = \mathbf{0}$ is a solution to $A\mathbf{x} = \mathbf{0}$.

If $A\mathbf{x} = \mathbf{b}$ has no solution, then there's nothing else to say about it. So let's suppose that A and $\mathbf{b}$ are a matrix and vector such that $A\mathbf{x} = \mathbf{b}$ has at least one solution. We know that this means that the number of solutions is either 0 or infinite. The solutions to $A\mathbf{x} = \mathbf{b}$ turn out to be closely related to the solutions to the corresponding homogeneous equation $A\mathbf{x} = \mathbf{0}$.

In this section, we will compare the solutions to $A\mathbf{x} = \mathbf{b}$ with the solutions to the corresponding homogeneous equation $A\mathbf{x} = \mathbf{0}$. Before getting to the theoretical result, let's see an example.

Example 1. We will compare the solutions to the two systems of equations

$$\begin{aligned} x + y + z &= 0 \\ 2x - y + z &= 0 \\ 3x + \ \ 2z &= 0 \end{aligned}$$

and

$$\begin{aligned} x + y + z &= 3 \\ 2x - y + z &= 2 \\ 3x + \ \ 2z &= 5 \end{aligned}$$

Both systems have the same coefficient matrix. We solve the first one:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & -1 & 1 & 0 \\ 3 & 0 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & -3 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2/3 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We see that x and y are leading variables and z is free. If we let $z = t$, we get $x = -2t/3$, $y = -t/3$, $z = t$, which in vector form is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} -2/3 \\ -1/3 \\ 1 \end{bmatrix}.$$

Now we solve the nonhomogeneous equation. Notice that the only difference is that the last column of the augmented matrix changes. We do the same row reduction steps.

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -1 & 1 & 2 \\ 3 & 0 & 2 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & -1 & -4 \\ 0 & -3 & -1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & -1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1/3 & 4/3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2/3 & 5/3 \\ 0 & 1 & 1/3 & 4/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The vector form of the solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5/3 \\ 4/3 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2/3 \\ -1/3 \\ 1 \end{bmatrix}.$$

Look carefully at the general nonhomogeneous solution – it's just the homogeneous solution, plus the solution $\langle 5/3, 4/3, 0 \rangle$. The homogeneous solution is a line in $\mathbf{R}^3$ spanned by $\langle -2/3, -1/3, 1 \rangle$. If you graph the nonhomogeneous solution, you get a line that is parallel to the graph of the homogeneous solution; the nonhomogeneous solution is the homogeneous solution shifted by the vector $\langle 5/3, 4/3, 0 \rangle$. $\square$

The same thing happens in general: *provided that the nonhomogeneous equation has a solution*, it is always a shift of the homogeneous solution by a particular solution to the nonhomogeneous equation.

> **Homogeneous and Non-Homogeneous Solutions**
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> Suppose that $\mathbf{x}_p$ is one solution to $A\mathbf{x} = \mathbf{b}$ (meaning that $A\mathbf{x}_p = \mathbf{b}$). The general solution to $A\mathbf{x} = \mathbf{b}$ is the set of vectors of the form $\mathbf{x}_p + \mathbf{x}_h$ where $\mathbf{x}_h$ is a solution to $A\mathbf{x} = \mathbf{0}$.
>
> In other words, once one solution, $\mathbf{x}_p$, to the equation $A\mathbf{x} = \mathbf{b}$ is known, any other solution is obtained by adding solutions to $A\mathbf{x} = \mathbf{0}$.

Proof. First, observe that if $\mathbf{x}_h$ is a homogeneous solution, then

$$A(\mathbf{x}_p + \mathbf{x}_h) = A\mathbf{x}_p + A\mathbf{x}_h = \mathbf{b} + \mathbf{0} = \mathbf{b},$$

so $\mathbf{x}_p + \mathbf{x}_h$ is also a solution to $A\mathbf{x} = \mathbf{b}$. Conversely, let $\mathbf{x}$ be any solution to $A\mathbf{x} = \mathbf{b}$. We consider the vector $\mathbf{x} - \mathbf{x}_p$ and observe that $A(\mathbf{x} - \mathbf{x}_p) = A\mathbf{x} - A\mathbf{x}_p = \mathbf{0} - \mathbf{0} = \mathbf{0}$. This shows that $\mathbf{x} - \mathbf{x}_p$ is a homogeneous solution. If we write $\mathbf{x}_h$ for the homogeneous solution $\mathbf{x} - \mathbf{x}_p$, then $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$. $\square$

In other words, the solution to $A\mathbf{x} = \mathbf{b}$ (provided it is not the empty set) is obtained by taking the solution to $A\mathbf{x} = \mathbf{0}$ and "shifting" or translating that set by a solution to $A\mathbf{x} = \mathbf{b}$. It doesn't matter which solution to $A\mathbf{x} = \mathbf{b}$ we pick; we get the same set.

Notice that for an $m \times n$ matrix A, if $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then there is no way that $A\mathbf{x} = \mathbf{b}$ can have infinitely many solutions: once you have one solution to $A\mathbf{x} = \mathbf{b}$, the rest come from the homogeneous solutions. We've seen this before: the reduced row echelon form of the matrix A has no free columns if and only if $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, if and only if for every $\mathbf{b} \in \mathbf{R}^m$, $A\mathbf{x} = \mathbf{b}$ has either 0 or 1 solution.

Exercise 1. Construct 3×2 matrices A and B such that $A\mathbf{x} = \mathbf{0}$ has only the trivial solution and $B\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

Exercise 2. For the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \end{bmatrix}$$

Solving the *homogeneous system* $A\mathbf{x} = \mathbf{0}$ is important. We will see that the solution to the homogeneneous system is closely related to the solution to the *inhomogeneous system* $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} \neq \mathbf{0}$. We will practice the steps to solve $A\mathbf{x} = \mathbf{0}$ in parametric vector form. Use variables x, y, and z.

(a) Step 1: Write out the two linear equations corresponding to the matrix-vector equation $A\mathbf{x} = \mathbf{0}$.

(b) Step 2: Identify the *pivot variable(s)* and the *free variable(s)*.

(c) Step 3: Set each of the free variables equal to a parameter (s or t is a typical choice of letter). Then solve for each of the pivot variables in terms of the parameters.

(d) Step 4: Write out the solution in parametric vector form.

(e) Step 5: you've already solved the system but in this case you can get an idea of what the solution looks like. Sketch the solution set.

Exercise 3. Suppose that A is the matrix

$$A = \begin{bmatrix} 1 & 2 & 8 \\ 5 & 6 & 28 \\ 4 & 2 & 14 \end{bmatrix}$$

(a) Compute RREF(A).

(b) Solve the homogeneous system $A\mathbf{x} = \mathbf{0}$. Write your answer in parametric vector form.

(c) Let $\mathbf{b}$ be the sum of the columns of A. Find the general solution to $A\mathbf{x} = \mathbf{b}$.

(d) Wario was working on this problem but wrote down the wrong matrix in his notebook. He accidentally changed the 28 to 38. Other than that, he did everything else right. What were his answers to a., b., and c.? You can do all of this from the beginning, but try to answer it without doing computations. Hint: Think about how many ways there are to express $\begin{bmatrix} 5 \\ 6 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

Exercise 4. For all parts of this question, A is a 2×3 matrix and its echelon form has 2 pivots. **Choose the words** that correctly complete the following sentences.

(a) For all $\mathbf{b}$ in $\mathbf{R}^2$, the equation $A\mathbf{x} = \mathbf{b}$ (always / sometimes but not always) has (a unique solution / infinitely many solutions / no solution).

(b) The span of the columns of A is (a point in / a line in / all of) $\mathbf{R}^k$. What is the number k?

(c) The set of solutions to $A\mathbf{x} = \mathbf{0}$ is a (point / line / plane) in $\mathbf{R}^d$. What is the number d?

Exercise 5. For all parts of this question, A is a 3×3 matrix and its echelon form has 3 pivots. **Choose the words** that correctly complete the following sentences.

(a) For all $\mathbf{b}$ in $\mathbf{R}^3$, the equation $A\mathbf{x} = \mathbf{b}$ (always / sometimes but not always) has (a unique solution / infinitely many solutions / no solution).

(b) The span of the columns of A is (a point in / a line in / all of) $\mathbf{R}^k$. What is the number k?

(c) The set of solutions to $A\mathbf{x} = \mathbf{0}$ is a (point / line / plane) in $\mathbf{R}^d$. What is the number d?