Simple Parameter Estimation Technique for Three-Parameter Generalized Extreme Value Distribution

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Abstract: This note simplifies the widely used parameter estimator methods, namely, method of moments (MOM) and probability weighted moment (PWM) for the three-parameter generalized extreme value (GEV) distribution for improved parameter estimation. The important shape parameter (κ) of the GEV distribution used in annual maximum series and partial duration series models is expressed as a function of skewness (γ). Validity of the approximate PWM estimates of shape (κ) and, in turn, scale (α) and location (ξ) parameters is checked using the ratio of variance and bias with simulated data, whereas the MOM-approximated estimates are tested using standard error of quantile estimates. The resulting expressions exhibit better accuracy when applied to field data of five watersheds.

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Introduction

The three-parameter generalized extreme value (GEV) distribution was recommended for flood frequency analysis in the Flood Studies Report (NERC 1975). Subsequently the index-flood procedure made use of this distribution for regional flood frequency analysis. There are three parameters that describe the GEV distribution and they are: scale (α), shape (κ), and location (ξ). These parameters are estimated by method of moment (MOM), maximum likelihood method (MLM), method of textiles (NERC 1975), and probability weighted moments (PWM) or equivalent L moments (Hosking et al. 1985). As the true parameters of any probability distribution function are always unknown, researchers have tried to find the best estimation methods based on the criteria of their corresponding bias and variance of the quantile estimations. Hosking et al. (1985) showed that the probability weighted moments quantile estimators for the GEV distribution are better than the MLM for small sample sizes and Madsen et al. (1997) showed that MOM quantile estimators perform well when the sample sizes are modest.

Hosking et al. (1985) and Stedinger et al. (1993) proposed the MOM- and PWM-based procedures, respectively, to express the GEV distribution parameters as a function of sample statistics like mean (μ), standard deviation (σ), and skewness (γ). The first procedure involves iteration, whereas the latter uses two relations. As these do not give an exact analytical solution, attempts have been made in the past to refine these methods and propose alternate procedures. For instance, Farquharson et al. (1987) have used the at-site coefficient of variation (C_v) and κ relationship to reliably estimate the value of regional κ for a homogeneous region. Similarly, Donaldson (1996) has given approximation for the GEV shape parameter in terms of the skewness and L skewness of the distribution. The results from such studies have not only simplified the equations for solving these parameters, but have also assisted in initializing the parameters in certain type of hydrological simulations. For example, in the generation of annual maximum series (AMS) and partial duration series (PDS) data, assigning initial parameter values is a prerequisite for the simulation procedure. This approach was used in an algorithm by Madsen et al. (1997) and Rosbjerg et al. (1992) to test the performance evaluation of AMS and PDS models. Similarly, Madsen et al. (1997) used the $C_v-\kappa$ relationship and expressed the goodness-of-fit of PDS models in terms of κ.

Keeping in mind the popularity of the GEV distribution in hydrological analysis and the need for simpler equations for computing the parameters using different estimation methods, this study was taken up with the following objectives: (1) Propose a simple, noniterative, and approximate expression for evaluating the GEV parameters using MOM and PWM estimation methods that meet the accuracy required in AMS and PDS model analysis; (2) test the validity of the proposed procedure; and (3) demonstrate the workability of the proposed approximate relations to example field data. In summary, to avoid iteration in the former approach and to improve the accuracy of the latter are the objectives of this study.

GEV Distribution in AMS Model

The cumulative distribution function for GEV distribution is given by Jenkinson (1969) as

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$$F(q)=\begin{cases}\exp\left[-\left(1-\kappa\dfrac{q-\xi}{\alpha}\right)^{1/\kappa}\right] & \text{for } \kappa\neq 0 \quad (1a)\\ \exp\left[-\exp\left(-\dfrac{q-\xi}{\alpha}\right)\right] & \text{for } \kappa=0 \quad (1b)\end{cases}$$

where q=flood exceedance value; the range of the variable q depends on the sign of κ. When κ is negative the variable q can take values in the range of $(\xi+\alpha/\kappa)>q<\infty$, making it suitable for flood frequency analysis; when κ is positive the variable q becomes upper bounded; and for $\kappa=0$, the distribution reduces to a two-parameter Extreme Value type I (EV1) or Gumbel distribution.

The MOM estimators of the GEV parameters are given by Stedinger et al. (1993) as

$$\hat{\xi}=\hat{\mu}+\hat{\alpha}^{*}/\hat{\kappa}[\Gamma(1+\hat{\kappa})-1] \qquad (2a)$$

$$\hat{\alpha}^{*}=\text{sign}(\hat{\kappa})\hat{\sigma}\hat{\kappa}/\{[\Gamma 1+2\hat{\kappa}]-(\Gamma 1+2\hat{\kappa})^{2}\}^{1/2} \qquad (2b)$$

$$\hat{\gamma}=\text{sign}(\hat{\kappa})(\{-\Gamma(1+3\hat{\kappa})+3\Gamma(1+\hat{\kappa})\Gamma(1+2\hat{\kappa}) - 2[\Gamma(1+\hat{\kappa})]^{3}\})/(\{\Gamma(1+2\hat{\kappa})-\Gamma(1+\hat{\kappa})^{2}\})^{3/2} \qquad (2c)$$

where sign is plus or minus depending on the κ value, and the $\hat{\gamma},\hat{\kappa}$ represents the corresponding population estimate. The PWM estimators are given by Hosking et al. (1985) as

$$\xi=\lambda_1-\alpha[1-\Gamma(1+\kappa)]/\kappa \qquad (3a)$$

$$\hat{\alpha}^{*}=\frac{\hat{\lambda}_2}{\Gamma(1+\hat{\kappa})(1-2^{-\hat{\kappa}})}\hat{\kappa} \qquad (3b)$$

$$\hat{\kappa}=7.859c+2.9554c^{2} \qquad (3c)$$

$$c=2/(\tau_3+3)-\ln(2)/\ln(3) \qquad (3d)$$

where the L-moment estimators $\hat{\lambda}_1$, $\hat{\lambda}_2$, $\hat{\lambda}_3$, and τ_3 are obtained by unbiased estimators of the first three PWMs (Hosking 1990) as follows:

$$\lambda_1=\beta_0;\quad \lambda_2=2\beta_1-\beta_0;\quad \lambda_3=6\beta_2-6\beta_1+\beta_0; \quad \lambda_4=20\beta_3-30\beta_2+12\beta_1-\beta_0 \qquad (4)$$

and β_0, β_1, β_2, and β_3 in the previous equations are PWMs given by $\beta_r=E\{X[F(x)]^r\}$, where $F(x)$=cumulative distribution function of variate x. For $r=0$, β_0=mean, and ratio L skewness $(\tau_3)=\lambda_3/\lambda_2$. As the maximum likelihood estimators are determined numerically using a modified Newton–Raphson algorithm (Hosking 1985), approximate equations for parameters are difficult.

Table 1. Estimated κ Using MOM Estimation Method

Skewness (γ)	κ using Eq. (2c)[a]	κ [Eq. (6)]	Error =Column 2−Column 3
−0.63111	0.5	0.500750	−7.50E−04
−0.06874	0.3	0.300070	−6.97E−05
0.637637	0.1	0.099675	3.25E−04
1.811925	−0.09	−0.08535	−4.64E−03
1.910339	−0.1	−0.09470	−5.30E−03
13.48355	−0.3	−0.29664	−3.36E−03

[a]Stedinger et al. 1993.

Approximate Equations for GEV Parameters

Method of Moments

The method of moments given by Eq. (2) contains multiple gamma function of κ, whose sign is not known. Therefore the present practice of evaluating κ involves an iterative procedure with an initial guess of γ. To avoid this, a simple expression is derived using the following procedure:

1. For random κ values (in a practical range), corresponding values of γ are computed from Eq. (2c).
2. The generated sets of (κ,γ) can be used to fit them in a simple function as
$$\kappa=\phi(\gamma,a_i) \qquad (5)$$
where ϕ=anbitrary function and a_i=parameters of Eq. (5).
3. The parameters of the function ϕ are determined using Marquardt (1962) algorithm to give Eq. (5) as a simple relationship between κ and γ.

So, for a known skewness of the sample data, shape parameter κ can be computed using Eq. (5). The previous procedure is used to yield simple relationships for the other two GEV parameters α and ξ.

Relation between γ and κ

To derive the $\gamma-\kappa$ relationship, 1,000 values of κ were generated using a random number generation scheme. On the basis of the reports of Rosbjerg et al. (1992), Madsen et al. (1997), and Lu and Stedinger (1992), the range of κ was fixed between −0.5 and 0.5. The corresponding γ values [Eq. (2c)] were computed, and the generated (γ,κ) values were used to derive the following approximate relations for κ parameter (MOM):

$$\kappa=\begin{cases}0.0087\gamma^{3}+0.0582\gamma^{2}-0.32\gamma+0.2778 & \text{for } -0.7\leqslant\gamma\leqslant 1.15 \quad (6a)\\ -0.31158\{1-\exp[-0.4556(\gamma-0.97134)]\}+1.13828 & \text{for } 1.15\leqslant\gamma \quad (6b)\end{cases}$$

Eq. (6a) is a simple third-degree polynomial, and Eq. (6b) is an exponential equation. It may be noted here that the form of Eq. (5) changes with the nature of the variable curve considered. The coefficient of determination (R^2) for both Eqs. (6a) and (6b) was approximately 1. Table 1 reports the results of κ values derived using Eqs. (6a) and (6b) that are more accurate than 4.9×10^{-4} for $-0.7\leqslant\gamma\leqslant 1.0$, and 1.1×10^{-2} for $\gamma\geqslant 1.0$. A similar inference holds for the results (Table 1) of Eq. (5) (see Stedinger et al. 1993).

Relation between α and κ

On rearranging Eq. (2b), the ratio (α/σ) can be written as a function of κ. Using the same procedure as outlined earlier,

J. Hydrol. Eng., 2007, 12(6): 682-689

Table 2. Estimated α/σ and $(\zeta-\mu)/\sigma$ Using MOM Estimation Method

	α/σ			$[\zeta-\mu]/\sigma$		
κ	Eq. (2*b*)[a]	Eq. (7)	Error=Column 2−Column 3	Eq. (2*a*)	Eq. (8)	Error=Column 2−Column 3
0.5	1.079328	1.078113	1.22E−03	−0.2456	−0.2448	−7.88E−04
0.3	1.010945	1.011313	−3.67E−04	−0.34551	−0.3456	9.58E−05
0.1	0.873689	0.873176	5.13E−04	−0.42504	−0.4250	3.16E−05
−0.09	0.681815	0.682118	−3.03E−04	−0.45979	−0.4620	2.24E−03
−0.1	0.670215	0.670562	−3.47E−04	−0.45996	−0.4617	1.83E−03
−0.3	0.410839	0.410329	5.09E−04	−0.40818	−0.4098	1.68E−03

[a]Stedinger et al. (1993).

an approximate relation between α and κ was obtained as follows:

$$\frac{\alpha}{\sigma} = -0.1429\kappa^3 - 0.7631\kappa^2 + 1.0145\kappa + 0.7795,$$
$$-0.5 \leqslant \kappa \leqslant 0.5 \qquad (7)$$

The results of (α/σ) values using Eq. (7) (Table 2) have an accuracy better than 1.22×10^{-3} for $-0.5\leqslant\kappa\leqslant0.3$, and similar comparison holds for Eq. (2*b*) (Stedinger et al. 1993).

Relation between ξ and κ

Eq. (2*a*) is first used to evaluate $[(\xi-\mu)/\sigma]$ values for different κ values. Then the computed set of $[(\xi-\mu)/\sigma,\kappa]$ is used to derive an approximate relationship between the two as follows:

$$\frac{\xi-\mu}{\sigma} = \begin{cases} 0.514075(\kappa)^{1.33199} - 0.44901, & 0.01 \leqslant \kappa \leqslant 0.5 \qquad (8a) \\ 19.357\kappa^4 + 13.749\kappa^3 + 4.484\kappa^2 + 0.5212\kappa - 0.4427, & -0.5 \leqslant \kappa \leqslant 0.01 \qquad (8b) \end{cases}$$

The results of $(\xi\text{-}\mu)/\sigma$ values using Eqs. (8) and (3) (Stedinger et al. 1993) are given in Table 2 and these have an accuracy better than 4×10^{-4} for $0.01\leqslant\kappa\leqslant0.5$, and 1.5×10^{-2} for $-0.5\leqslant\kappa\leqslant0.01$.

The GEV parameters given by Eqs. (6)–(8) yield results with a good accuracy, and avoid the iterative procedure involved in the original Eq. (2) of Stedinger et al. (1993). The reliability of these equations is checked using a goodness-of-fit test as discussed in a subsequent section of this note.

Probability Weighted Moments

Relation between τ_3 and κ

The relationship between *L* skewness (τ_3) and κ for GEV distribution using PWM (Hosking et al. 1985) is expressed as follows:

$$\tau_3 = 2(1-3^{-\kappa})/(1-2^{-\kappa}) - 3 \qquad (9)$$

As no explicit solution is available for Eq. (9), Hosking et al. (1985) proposed an alternate procedure [Eq. (3*c*)] that involves solving of two equations. One is used for calculating a variable coefficient (*c*) and the other utilizes *c* to calculate κ For simplification of Eq. (9), the steps are given as follows:

1. For random values of κ, compute τ_3 from Eq. (9) and
2. Generate large sets of (τ,τ_3) in a practical range to fit a relationship of the form $\kappa=\phi(\tau_3,a_i)$ to the previous data set.

In the present case also, 1,000 data (κ,τ_3) were used for the range of $0.25\leqslant\kappa\leqslant-95$, which covers a wide range of κ value for GEV distribution (Lu and Stedinger 1992; Rosbjerg et al. 1992; Madsen et al. 1997; Hosking and Wallis 1997). The Marquardt algorithm (1962) was used to optimize the parameters of function $[\kappa=\phi(\tau_3,a_i)]$ to derive for $\kappa-\tau_3$ relation in the following form:

$$\kappa = \begin{cases} 0.488138(\tau_3)^{1.70839} - 1.7631(\tau_3)^{0.981824} + 0.285706, & 0.01 \leqslant \tau_3 \leqslant 0.5 \qquad (10a) \\ 0.483706(\tau_3)^{1.679096} - 1.73786(\tau_3)^{1.008948} + 0.255108, & 0.5 \leqslant \tau_3 \leqslant 0.95 \qquad (10b) \end{cases}$$

It may be noted here that the range $0.25\leqslant\kappa\leqslant-0.95$ in Eq. (10) corresponds to $0.01\leqslant\tau_3\leqslant0.95$ of Eq. (9). The use of Eq. (10) is simple compared to Eq. (3*c*) and the estimated κ has accuracy better than 5.46×10^{-4}, which is an improvement over Eq. (3*c*) (Hosking and Wallis 1997) with an accuracy of 9×10^{-4} for $-0.5\leqslant\tau_3\leqslant0.5$ Results of both the methods are compared withthe actual value [given by Eq. (9)] for some selected κ in Table 3. It can be observed from the results that Eq. (10) estimates the value of κ with a better accuracy than does Eq. (3*c*) (Hosking et al. 1985), and the present method uses a single relationship to compute κ, instead of two used in the latter case.

J. Hydrol. Eng., 2007, 12(6): 682-689

Table 3. Estimated κ Using PWM Estimation Method

τ_3	κ Actual [Eq. (9)]	Eq. (3*b*)[a]	Eq. (10)	Error=actual-estimated Eq. (3*b*)	Eq. (10)
0.00951	0.266771	0.267643	0.26763	−8.72E−04	−8.59E−04
0.17	−0.00012	−0.00012	−0.00018	0.00E+00	6.32E−05
0.20016	−0.0465	−0.04674	−0.0464	2.44E−04	−9.36E−05
0.30057	−0.19362	−0.19443	−0.19332	8.07E−04	−2.98E−04
0.49977	−0.45716	−0.45687	−0.45736	−2.82E−04	2.07E−04
0.54902	−0.51739	−0.51623	−0.51717	−1.16E−03	−2.18E−04
0.60776	−0.58698	−0.58443	−0.58677	−2.55E−03	−2.09E−04
0.70466	−0.69689	−0.6912	−0.69692	−5.69E−03	3.62E−05
0.95904	−0.96049	−0.94158	−0.96004	−1.89E−02	−4.55E−04

[a]Hosking et al. (1985).

Relation between α and κ

Eq. (7) is rearranged to express (α/λ_2) as a function of κ, and using the procedure outlined above the following approximate equation is derived:

$$\frac{\alpha}{\lambda_2} = 1.023602813(\tau_3)^{1.8850974} - 2.95087636(\tau_3)^{1.195591244} + 1.759614982, \quad -0.5 \leqslant \kappa \leqslant 0.3 \text{ or } -0.0095 \leqslant \tau_3 \leqslant 0.5 \tag{11a}$$

The computed value of (α/λ_2) using Eq. (11*a*) and that given by Eq. (3*b*) (Hosking et al., 1985) for different values of κ are given in Table 4. The following equation can be used for $\kappa > 0.3$

$$\frac{\alpha}{\lambda_2} = 1.5954866(\tau_3)^{1.5816175} - 3.886135(\tau_3)^{0.89522} + 2.310643, \quad \kappa > 0.3 \text{ or } 0.5 \geqslant \tau_3 \geqslant 0.95 \tag{11b}$$

Eq. (11*a*) has an accuracy of 9×10^{-4} and Eq. (11*b*) has an accuracy better than 2.24×10^{-4}.

Relation between ξ and τ_3

Eq. (2*a*) and the simulation procedure is used to derive the following expression for ξ:

$$\frac{\xi - \lambda_1}{\lambda_2} = -0.0937(\tau_3)^4 - 0.2198(\tau_3)^3 + 1.407(\tau_3)^2 - 1.4825(\tau_3) - 0.6205, \quad 0.01 \leqslant \tau_3 \leqslant 0.95 \tag{12}$$

As observed from Eq. (12), the sample *L* moments (λ_1 and λ_2) and *L* skewness (τ_3) can be used to compute the ξ-parameter of GEV distribution with an accuracy of more than 3.3×10^{-4} (Table 4).

The results of Eqs. (10)–(12) show that the estimates of the parameters have a better accuracy and are simple to use. The validity of these results is further checked using a suitable goodness-of-fit test as given in the following section.

Goodness-of-Fit

To check the reliability of the proposed equations for estimation of the GEV distribution parameters, goodness-of-fit test is carried out on the basis of the variance and bias of the quintile estimators. The *T*-year event based on the AMS is defined as the 1-1/*T* quantile in annual maximum distribution, and is denoted by $q(T)$.

Method of Moments

The standard error of quantile estimate (S_E^2) for GEV (MOM) distribution is given as follows (Kite 1977):

$$S_E^2 = \frac{\mu_2}{N}\left\{1 + K_T g_1 + K_T^2/4[g_2 - 1] + \frac{\partial K_T}{\partial g_1}[2g_2 - 3g_1^2 - 6] + K_T[g_3 - 6g_1g_2/4 - 10g_1/4]\right\} + \frac{\mu_2}{N}\left\{\frac{\partial K_T^2}{\partial g_1}[g_4 - 3g_3g_1 - 6g_2 + 9g_1^2g_2/4 + 35g_1^2/4 + 9]\right\}$$

where K_T=frequency factor, and g_i are given by

$$g_1 = \mu_3/\mu_2^{3/2}, \quad g_2 = \mu_4/\mu_2^2, \quad g_3 = \mu_5/\mu_2^{5/2}, \quad g_4 = \mu_6/\mu_2^3 \tag{13}$$

Here, μ_i=*i*th central moment. To describe the degree of error for the approximate method [Eqs. (6)–(8)], the ratio of S_E^2 is used and it is given by

$$R_{SE} = [S_E^2 \text{ for } \kappa \text{ using Eq. } (2c)]/[S_E^2 \text{ for } \kappa \text{ using Eq. } (6)] \tag{14}$$

A value of R_{SE} greater than 1 indicates that the κ estimate by the proposed method yields less error in quantile estimate than that of κ estimate from Eq. (2*c*). Fig. 1 presents the R_{SE} for selected κ in the range from −0.3 to 0.5 for *T*=100, 500, and 1,000 years. It is observed that R_{SE} is near one for $0.5\leqslant\kappa\leqslant0.2$ It deviates from 1.0 for $\kappa\leqslant0.02$. However, for $\kappa=0.02\pm0.005$, the ratio is

Table 4. Estimated α/λ_1 $[-0.5\leqslant\tau_3\leqslant0.30]$ and Estimated $(\xi-\lambda_1)/\lambda_2$ $[0.01\leqslant\tau_3\leqslant0.95]$ Using PWM Estimation Method

τ_3	κ	Error in α/λ_2 =actual−estimated Eq. (3*a*)[a]	Eq. (1)	Error in $[\xi-\lambda_1]/\lambda_2$ =actual−estimated
0.00951	0.266771	−8.28E−04	1.25E−03	−3.68E−04
0.09967	0.112444	−6.47E−04	−1.03E−03	3.39E−04
0.17	−0.00012	0.00E+00	1.38E−03	1.94E−04
0.20016	−0.0465	3.39E−04	1.72E−03	8.30E−05
0.30057	−0.19362	1.23E−03	−1.64E−03	−1.86E−04
0.53919	−0.50551	−1.53E−03	3.23E−03	4.24E−05

[a]Hosking et al. (1985).

J. Hydrol. Eng., 2007, 12(6): 682-689

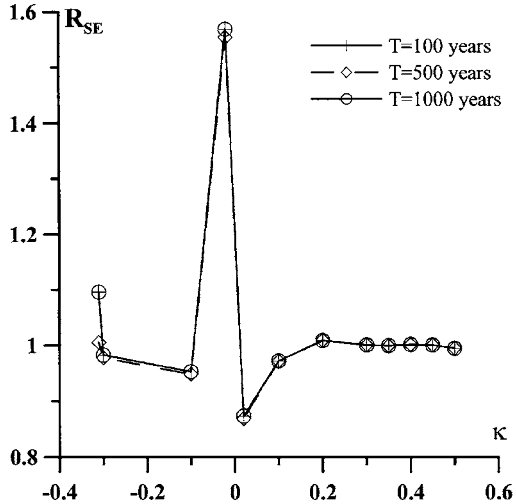

Fig. 1. Ratio of standard error of quantile estimates for different κ values (MOM)

more than 1 implying that the standard error of quantile estimate using the approximate equation is less. Further, the R_{SE} values increase with increase in T for negative value of κ. For $\kappa=-0.31$, $R_{\mathrm{SE}}=1.0049$, 0.0053, and 1.1193 for $T=100$, 500, and 1,000 years, respectively. Fig. 2 shows $R_{\mathrm{SE}}-\kappa$ variations for different T.

The previous analysis was based on simulated values. To further validate the results five Indian watersheds were selected for application of the present as well as the available approaches. The characteristics of the watershed data are shown in Table 5. The parameter estimates of the GEV distribution using the field data along with the corresponding computed values of κ, α, and ξ are shown in Tables 6 and 7. It is evident from the results that the computed parameter values using the simplified, noniterative, and less complex proposed relations exhibit in computations the lesser error than do the available ones, which are not only iterative but also complex in nature. Thus, the former seems to be more apt to field applications than the latter.

Probability Weighted Moments

To check the validity of the proposed equations for parameter estimation using PWM [Eqs. (10)–(12)], the goodness-of-fit based on sampling variance of GEV quantile estimators is compared employing different κ value. The asymptotic variance of the GEV (PWM) estimators for κ, denoted as $\mathrm{Var}[q_F(\kappa)]$ given by Lu and Stedinger (1992) as

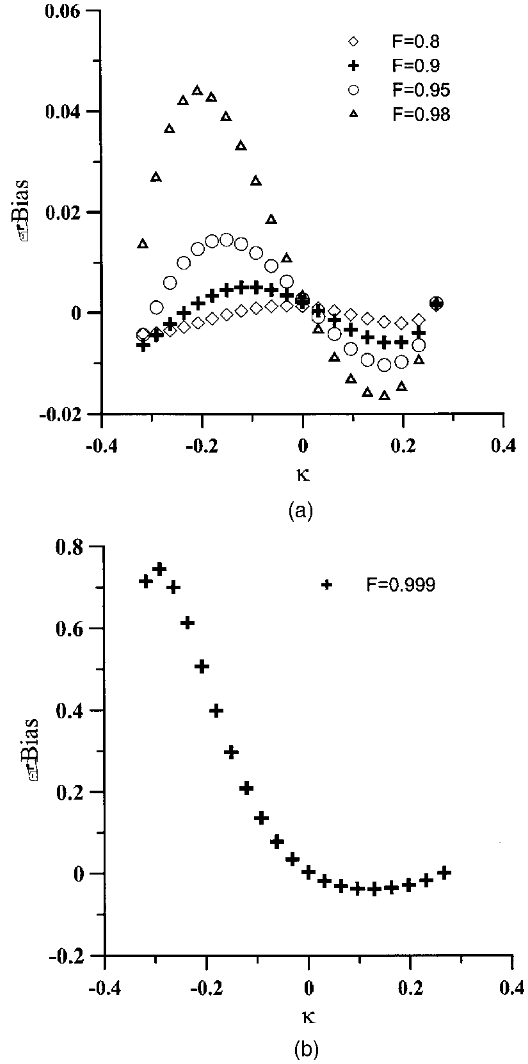

Fig. 2. (a) Bias in $q(T)$ for different κ values (PWM) at (b) F=0.99; note: ΔBias=Difference in bias

$$\mathrm{Var}[q_F(\kappa)] = \alpha^2[c_1 + c_2 y + c_3 y^2]/n \tag{15}$$

where q_F $(\kappa)=F$th quantile estimate for a given κ, $F=1-1/T$; n=number of years of record for analysis, c_i=variables depending upon κ, and y is given as

$$y = \begin{cases} \{1-[\ln(F)^{\kappa}]\}/\kappa & \text{if } \kappa \neq 0 \\ -\ln[-\ln(F)] & \text{if } \kappa = 0 \end{cases}$$

Good approximations for c_1, c_2, and c_3 are

Table 5. Characteristics of Study Area

Catchment name	Region	Length of data (years)	Sample statistics μ ($\mathrm{m^3/s}$)	σ ($\mathrm{m^3/s}$)	Coefficient of variation
476/1(3B)	Lower Narmada-Tapi catchment	14	294.07	186.77	0.635
8 (3A)	Mahi-Sabarmati catchment	25	74.00	72.31	0.977
105 (3B)	Narmada-Tapi catchment	28	223.82	245.67	1.097
5 (3A)	Mahi-Sabarmati catchment	18	352.72	416.39	1.180
294 (3C)	Upper Narmada-Tapi catchment	30	919.6	561.88	0.611

J. Hydrol. Eng., 2007, 12(6): 682-689

Table 6. Estimated κ for Study Area

	Sample statistics			κ estimate using MOM		κ estimate using PWM	
Catchment	Skew	λ_2	τ_3	Stedinger et al. (1993)	Eq. (6)	Hosking et al. (1985)	Eq. (10)
476/1(3B)	0.0576	0.37643	0.02229	0.2589	0.2595	0.2450	0.2443
8 (3A)	1.8284	0.47995	0.44987	−0.0916	−0.0869	−0.3946	−0.3943
105 (3B)	3.3964	0.46381	0.45905	−0.1946	−0.1945	−0.4062	−0.4060
5 (3A)	1.6875	0.60064	0.46854	−0.0674	−0.0729	−0.4062	−0.4181
294 (3C)	0.6347	0.34867	0.14711	0.1008	0.1003	0.0360	0.0356

$$c_1 = 1.1128 - 0.2384\kappa + 0.0908\kappa^2 + 0.1084\kappa^3 \quad \text{for } -0.3 \leqslant \kappa \leqslant 0.3$$

$$c_2 = 0.4580 - 3.0561\kappa + 1.1104\kappa^2 - 0.4071\kappa^3 \quad \text{for } \kappa \geqslant 0$$

$$c_3 = 0.8046 - 2.8890\kappa + 8.7874\kappa^2 - 10.375\kappa^3 \quad \text{for } \kappa \geqslant 0$$

$$c_2 = 0.4580 - 7.512\kappa + 5.0832\kappa^2 - 11.623\kappa^3 + 2.250 \ln(1+2\kappa) \quad \text{for } \kappa < 0$$

$$c_3 = 0.8046 - 2.6215\kappa + 6.8989\kappa^2 + 0.003\kappa^3 - 0.10 \ln(1+3\kappa) \quad \text{for } \kappa < 0 \qquad (16)$$

To provide a measure for comparing the performance of one method with another, the ratio of variance, which is independent of n, is described as follows:

$$R_{A/H} = \mathrm{Var}[q_F(\kappa)] \text{ obtained using actual } \kappa \text{ [Eq.(9)]}/\mathrm{Var}[q_F(\kappa)] \text{ obtained using } \kappa \text{ estimated with Hosking's Eq.}(3c)$$

$$R_{A/P} = \mathrm{Var}[q_F(\kappa)] \text{ obtained using actual } \kappa \text{ [Eq. (9)]}/\mathrm{Var}[q_F(\kappa)] \text{ obtained using } \kappa \text{ estimated with approximate equations [Eq. (10)]} \qquad (17)$$

The steps followed for evaluating $R_{A/P}$ or $R_{A/H}$ are as follows:

1. Generate a random κ value in the required range, and corresponding τ_3 is estimated using Eq. (9),
2. τ_3 deduced in Step (1) is substituted in Eq. (3*c*) (Hosking et al. 1985), and Eq. (10) (approximate equations) to get the respective κ value. The κ obtained at Step (1) is referred here as the actual.
3. κ estimated at Step (2) for both the methods are used in Eqs. (3*b*), and (11), to get corresponding α/λ_2 and subsequently $\mathrm{Var}[x_p(\kappa)]$. As the sample L moment λ_2 is same for both the methods, it cancels out in $R_{A/H}$ or $R_{A/P}$ calculation [Eq. (17)].

$R_{A/H}$ greater than 1 means that the variance of quantile q_F is more when actual κ is used in the GEV distribution, implying a better Fth quantile estimation from Eq. (3*c*). The resulting $R_{A/H}$ or $R_{A/P}$ for F=0.8, 9, 95, 0.98, and 0.999 are given in Table 8. Clearly, for all these quantiles, the estimated $\mathrm{Var}[q_p(\kappa)]$ is less using approximate equations ($R_{A/H}$ is greater than one) for positive κ, and the Hosking et al. (1985) method yields comparatively low values of $\mathrm{Var}[q_F(\kappa)]$ for $-0.015 \leqslant \kappa \leqslant -0.24$.

When the shape parameter differs from the actual value (κ_R), the bias of a quantile estimator increases and it leads to misinterpretations. The $\mathrm{Bias}[q_F(\kappa)]$ due to incorrectly specified shape parameter is given as (Lu and Stedinger 1992) as

$$\mathrm{Bias}[q_F(\kappa_R)|\kappa] = \frac{\alpha}{\kappa}\left([-\ln(F)]^{\kappa} - \Gamma(1+\kappa) + \frac{(1-2^{-\kappa})\Gamma(1+\kappa)}{(1-2^{-\kappa_R})\Gamma(1+\kappa_R)}\{\Gamma(1+\kappa_R) - [-\ln(F)]^{\kappa_R}\}\right) \qquad (18)$$

Table 9 present the $\mathrm{Bias}[q_F(\kappa)]$ computed using κ of Eqs. (3*c*) and (10) for F=0.85, 0.9, 0.95, and 0.99. The difference is seen to be quite insignificant for most of the cases. However, for higher quantile (F=0.99), the difference in $\mathrm{Bias}[q_F(\kappa)]$ increases with Eq. (10) giving less bias than Eq. (3*c*) for shape parameter less than zero. For quantitative evaluation, the difference in bias is obtained as

$$\text{Difference} = \text{Bias}\{\text{with } \kappa \text{ estimate [Eq. } (3c)]\} \; \text{Bias}\{\text{with } \kappa \text{ estimate [Eq.(10)]}\} \qquad (19)$$

Here, a positive difference implies less bias of Eq. (10). Fig. 2 shows that the difference is more for κ less than zero, and it is less for positive values of κ. It increases with increase in quantile (F). However, the estimated bias is nearly equal for both the methods for positive values of κ for F=0.999.

Table 7. Estimated α and ξ for Study Area

	α-parameter estimate (MOM)		α-parameter estimate (PWM)		ξ-parameter estimate (MOM)		ξ-parameter estimate (PWM)	
Catchment	Stedinger et al. (1993)	Eq. (7)	Hosking et al. (1985)	Eq. (11)	Stedinger et al. (1993)	Eq. (8)	Hosking et al. (1985)	Eq. (12)
476/1(3B)	184.6	184.7	191.3	191.4	226.1	226.1	191.4	191.4
8 (3A)	49.2	49.6	30.2	30.2	40.7	40.6	37.5	37.5
105 (3B)	136.1	136.2	86.2	86.4	113.2	113.8	116.9	116.9
5 (3A)	290.4	292.1	171.9	172.1	161.5	160.5	171.9	172.2
294 (3C)	491.3	490.8	477.7	477.4	680.9	680.9	660.3	660.3

J. Hydrol. Eng., 2007, 12(6): 682-689

Table 8. Ratio of Var$[x_p(\kappa)]$ for Different κ Using PWM Estimation Method

κ	Ratio of Var$[x_p(\kappa)]$ for two methods for different quantiles.							
	$F=0.8$		$F=0.9$		$F=0.95$		$F=0.999$	
	$R_{A/H}$	$R_{A/P}$	$R_{A/H}$	$R_{A/P}$	$R_{A/H}$	$R_{A/P}$	$R_{A/H}$	$R_{A/P}$
0.26677	0.99805	0.99572	0.99734	0.99502	0.99683	0.99453	0.99497	0.99270
0.14577	0.99866	1.00421	0.99806	1.00469	0.99760	1.00507	0.99565	1.00667
0.11244	0.99889	1.00303	0.99838	1.00349	0.99798	1.00385	0.99625	1.00541
0.09598	0.99903	1.00233	0.99856	1.00276	0.99821	1.00309	0.99663	1.00456
−0.0157	1.00017	0.99785	1.00026	0.99785	1.00033	0.99785	1.00066	0.99786
−0.1066	1.00101	0.99664	1.00162	0.99635	1.00212	0.99611	1.00467	0.99490
−0.2077	1.00125	0.99867	1.00222	0.99833	1.00306	0.99803	1.00766	0.99642
−0.3036	1.00108	1.00224	1.00213	1.00206	1.00303	1.00191	1.00798	1.00106

The parameter estimates of the GEV distribution using the field data along with the corresponding computed values of κ, α, and ξ using PWM estimation method are shown in Tables 6 and 7. It is evident from the results that the proposed equations computes the parameters same as the existing methods for the field data and with more simplicity.

Results and Discussion

Based on the existing PWM and MOM methods, the present analysis focused on the development of a simple and direct (noniterative) parameter estimation procedure for GEV distribution. Approximate equations (6)–(8) (MOM) and (10)–(12) (PWM) show simple parametric relations. The sample statistic (γ, λ_1, λ_2, and τ_3) was presented in the form of algebraic functions. In practice, certain estimation method is evaluated on the basis of its performance in quantile estimation, which is judged using certain error criteria. For this study, S_E^2 was used to compare the estimator method [Eqs. (6)–(8)] and the existing method for MOM. Similarly, Var$[q_F(\kappa)]$ and Bias$[q_p(\kappa)]$ were used for PWM to compare Eqs. (10)–(12) and the existing method. The ratio of S_E^2, and Var$[q_F(\kappa)]$ and difference in Bias$[q_F(\kappa)]$ were computed for different quantiles to check the validity of the proposed approximate equations.

For a given κ, the value of skewness can be computed using Eq. (2) (Stedinger et al. 1993) and their corresponding values are referred to as the actual γ and κ in the present analysis. However, the reverse process of computing κ parameter that is usually practiced needs iteration, often yielding erroneous κ computations. Table 1 compares computed κ values using proposed equation [Eq. (6)] with the actual κ, and the difference between the two is used as a criterion. The difference lies in the range from -3.36×10^{-3} to 1.12×10^{-3} which is satisfactory.

The validity of parameters is further checked using the basis of ratio of standard error of quantile estimate (S_E^2) denoted as R_{SE} defined as follows:

$$R_{\mathrm{SE}}=\frac{\lfloor S_E^2\rfloor_{\text{Eq. }(2c)}}{[S_E^2]_{\text{Eqs. (7) and (8)}}} \tag{20}$$

The ratio greater than one implies a better evaluation of κ by Eqs. (7) and (8) than that from Eq. (2c). The results of R_{SE} for κ in the range from −0.3 to 0.5 are shown in Fig. 1. As the standard

Table 9. Bias $[x_p(\kappa)]$ for Different κ (PWM) Using Eqs. (3) and (10)

κ	Bias $[x_p(\kappa)]$ for different quantiles							
	$F=0.8$		$F=0.9$		$F=0.95$		$F=0.99$	
	Eq. (3)	Eq. (10)	Eq. (3)	Eq. (10)	Eq. (3)	Eq. (10)	Eq. (3)	Eq. (10)
0.266771	1.171595	1.170214	1.498908	1.497167	1.619706	1.61785	1.235262	1.233975
0.214067	1.216411	1.218398	1.653206	1.658451	1.873488	1.882059	1.691129	1.714063
0.162639	1.255148	1.257026	1.816137	1.822049	2.163922	2.17435	2.398051	2.432736
0.112444	1.287102	1.287921	1.987299	1.991476	2.495411	2.503819	3.502301	3.540014
0.095979	1.296122	1.296546	2.046076	2.049418	2.615805	2.623007	3.994973	4.031533
0.06344	1.311551	1.31124	2.16603	2.167543	2.872452	2.876718	5.231741	5.262004
−0.0157	1.333675	1.332321	2.47783	2.475024	3.613746	3.609315	10.53776	10.52003
−0.06172	1.334618	1.333361	2.670809	2.66632	4.132661	4.123373	16.19191	16.11367
−0.09182	1.329708	1.328811	2.800688	2.795661	4.511432	1.499559	21.5837	21.4478
−0.10669	1.325539	1.324901	2.865717	2.860619	4.710854	4.697968	24.91594	24.7455
−0.16508	1.296785	1.297574	3.124165	3.120148	5.577582	5.563077	44.13458	43.78796
−0.20773	1.262111	1.26417	3.312862	3.311002	6.300013	6.287342	67.4121	66.90455
−0.24944	1.215767	1.219003	3.493071	3.494214	7.08298	7.074943	102.3243	101.6636
−0.29024	1.157574	1.161625	3.660561	3.665017	7.922603	7.921578	154.1491	153.4046
−0.30364	1.135484	1.139681	3.712746	3.718246	8.214188	8.215851	176.4009	175.6598

J. Hydrol. Eng., 2007, 12(6): 682-689

error of quantile estimate is a function of return period of flood i.e., T, the variation of R_{SE} is also reported with respect to T. The results show no trend in R_{SE} with respect to T. However, S_E^2 increases with increase in T for positive values of κ for $\kappa=0.5$, $S_E^2=77.17$, 83.02, and 84.45 for $T=100$, 500, and 1,000 years, respectively; and for $\kappa=0.02$, $S_E^2=-529.6$, -762.6, and -909.9. The same is not true for negative κ; there is an increase in κ with increase in T. The results strongly indicate that the computed κ using MOM estimation method with the proposed equation is closer to true value when $0.2<\kappa<0.5$. The ratio (R_{SE}) is greater than one for most of the cases (Fig. 1) implying less error in quantile estimates. For $\kappa=0.02$, the error is relatively high and for $\kappa=-0.02$, it is minimum. The nature of $\kappa-R_{SE}$ graph (Fig. 1) indicates an equal error ($R_{SE}\approx 1$) in the ranges of $0.2<\kappa<0.5$ and $-0.01<\kappa<-0.03$.

For PWM estimation of κ, the ratio of variance of quantile estimate ($R_{A/P}$) was seen to be more than one (implying less error) for $0.046<\kappa<0.267$; $R_{A/P}\approx 0.998$ for $-0.015<\kappa-0.2357$; and $R_{A/P}>1$ for $-0.249<\kappa<-0.316$. Similarly, the bias of quantile [Eq. (18)] was evaluated for different quantiles and shape parameters (Fig. 2). For higher quantiles, viz, $p=0.95$, 098, and 0.999, the Bias$[x_F(\kappa)]$ was less for present method [Eq. (10)] than for Eq. (3) for $0.0157<\kappa<-0.31694$. The difference of bias becomes more prominent for $F=0.99$. Fig. 2 shows the κ-difference in the bias plot. The difference is more (greater than zero) for $0.0157<\kappa<-0.31694$.

Conclusion

This study has proposed to derive simple equations for estimation of GEV distribution parameters, by avoiding the existing iterative procedure and may be advantageous to use in certain hydrologic simulation procedures. This was carried out for method of moments and probability weighted moments. All the three GEV-distribution parameters are expressible as a simple function of the sample skewness, quite useful in AMS or PDS simulation. The proposed relations exhibit good accuracy when applied to simulated and field data of five watersheds.

Notation

The following symbols are used in this technical note:

Bias$[q_F(\kappa)]$ = bias in quantile estimate when κ is used;
$F(q)$ = Fth quantile;
F = probability of nonexceedance;
q = discharge (m^3/s, Ft^3/s);
$q(T)$ = T-year return year flood;
R_{SE} = ratio of standard error of quantile estimate (S_E^2);
S_E^2 = standard error of quantile estimate;
Var$[q_F(\kappa)]$ = variance in quantile estimate when κ is used;
$\beta_0, \beta_1, \beta_2, \beta_3$ = PWMs;
γ = sample skewness;
κ, α, ξ = shape, scale, and location parameters of GEV distribution;
$\lambda_1, \lambda_2, \lambda_3$ = L moments; and
τ_3 = L skewness.

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J. Hydrol. Eng., 2007, 12(6): 682-689