Probability Weighted Moments: Definition and Relation to Parameters of Several Distributions Expressable in Inverse Form

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Probability Weighted Moments: Definition and Relation to Parameters of Several Distributions Expressable in Inverse Form

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Distributions whose inverse forms are explicitly defined, such as Tukey's lambda, may present problems in deriving their parameters by more conventional means. Probability weighted moments are introduced and shown to be potentially useful in expressing the parameters of these distributions.

INTRODUCTION

The generalized form of *Tukey's* [1960] lambda distribution may be expressed as

$$x = m + aF^b - C(1 - F)^d \qquad (1)$$

where $F \equiv F(x) = P(X \leq x)$. To approximate the expected value of the range for a sample of size n from a normal population, *Joiner and Rosenblatt* [1971] considered the special symmetric case where $m = 0$ and $a^{-1} = b = c^{-1} = d = 0.135$. Of a similar mathematical structure is the Thomas' Wakeby distribution,

$$x = m + a[1 - (1 - F)^b] - c[1 - (1 - F)^{-d}] \qquad (2)$$

which *Houghton* [1977, 1978a, b] introduced to flood frequency analysis, observing that the distribution can explain the condition of separation. (The condition of separation was noted by *Matalas et al.* [1975] to be a characteristic of regional flood sequences which distinguishes them from simulated sequences derived from distributions commonly used in hydrology.) *Landwehr et al.* [1978] further discussed properties of the Wakeby distribution and qualified Houghton's observation by noting that if its parameters are positive in sign, then the condition of separation can be explained when $b > 1$, but not when $0 < b < 1$.

The Wakeby distribution is potentially useful to flood frequency analysis in particular and to flow frequency analysis in general for several reasons. First, it offers a simple explanation of the condition of separation. Second, it is characterized by five parameters suggesting better capability of fitting data than that of distributions characterized by fewer parameters. Moreover, because the left tail of the distribution is more strongly influenced by b and the right tail by d, the distribution can accommodate various types of flows ranging from low flows to floods. Third, the utility ascribed to the generalized lambda distribution by *Ramberg* [1975] applies equally well to Wakeby: in Monte Carlo experiments the distributions may be used (1) to approximate other distributions of interest, (2) to represent the unknown underlying distribution of data, and (3) to facilitate robustness and sensitivity studies. Certain analy-

ses, such as those concerned with plotting positions and properties of order statistics, can be handled mathematically more easily with distributions, like Wakeby, whose inverse forms are analytically defined.

Only the inverse forms $x = x(F)$ of the Wakeby and generalized lambda distributions are analytically defined. Although their moments can be expressed as functions of their parameters, the inverse relations cannot be readily derived, and thus moment estimates of their parameter values are not feasible. Moreover, maximum likelihood estimates of the parameter values are not easily obtained. To realize more fully the potential of these and other such distributions, techniques for estimating the parameter values and an understanding of the sampling properties of the estimates are in order. For the Wakeby distribution, *Houghton* [1977] suggested an estimating technique, referred to as the incomplete mean algorithm. The algorithm involves iterating on assumed values of b and d until what is defined as a 'best fit' is achieved.

In the following paragraphs, probability weighted moments are introduced, and their potential usefulness in deriving explicit expressions for the parameters of distributions whose inverse forms $x = x(F)$ can be explicitly defined is considered. The relations between the parameters and the probability weighted moments for the generalized lambda, in the case where $a = c$ and $b = d$ (symmetric lambda), and the Wakeby distributions are presented, as well as for some other distributions for which both $F = F(x)$ and $x = x(F)$ are explicitly defined.

PROBABILITY WEIGHTED MOMENTS

A distribution function $F \equiv F(x) = P(X \leq x)$ may be characterized by probability weighted moments, which are defined as

$$M_{l,j,k} \equiv E[X^l F^j (1 - F)^k] = \int_0^1 [x(F)]^l F^j (1 - F)^k dF \qquad (3)$$

where l, j, and k are real numbers. If $j = k = 0$ and l is a nonnegative integer, then $M_{l,0,0}$ represents the conventional moment about the origin of order l. If $M_{l,0,0}$ exists and X is a continuous function of F, then $M_{l,j,k}$ exists for all nonnegative real numbers j and k. If j and k are nonnegative integers, then

$$M_{l,0,k} = \sum_{j=0}^{k} \binom{k}{j} (-1)^j M_{l,j,0}$$

$$M_{l,j,0} = \sum_{k=0}^{j} \binom{j}{k} (-1)^k M_{l,0,k} \tag{4}$$

where if $M_{l,0,k}$ exists and X is a continuous function of F, $M_{l,j,0}$ exists. In general, even if $M_{l,j,k}$ exists, it may be difficult to derive its analytical form, particularly if the inverse $x = x(F)$ of the distribution $F = F(x)$ cannot be analytically defined.

In the special case where $l, j,$ and k are nonnegative integers, $M_{l,j,k}$ is proportional to $E[X_{j+1,k+j+1}^l]$, the lth moment about the origin of the $(j + 1)$th order statistic for a sample of size $k + j + 1$. More specifically,

$$E[X_{j+1,k+j+1}^l] = M_{l,j,k}/B[j + 1, k + 1] \tag{5}$$

where $B[\cdot,\cdot]$ denotes the beta function. If $j = 0$, $(k + 1)M_{l,0,k}$ represents the lth moment about the origin of the first-order statistic for a sample of size $k + 1$ and if $k = 0$, $(j + 1)M_{l,j,0}$ represents the lth moment about the origin of the $(j + 1)$th order statistic for a sample of size $j + 1$.

Note that the expected value of the range of X in a sample of size $n = k + 1 = j + 1$ is

$$E[X_{n,n} - X_{1,n}] = n(M_{1,n-1,0} - M_{1,0,n-1}) \tag{6}$$

Among distributions for which only the inverse form $x = x(F)$ is explicitly defined are the generalized lambda (equation (1)) and Wakeby (equation (2)). There are many distributions which may be explicitly defined as both $F = F(x)$ and $x = x(F)$, among them being the kappa, recently introduced by Mielke [1973] in analyzing precipitation data, and the more familiar Weibull, Gumbel, and logistic. In Table 1 the analytical forms of $x = x(F)$ are presented, as well as those of $F = F(x)$, where possible, for these specific distributions. For each, the conventional moments $M_{l,0,0}$ and the probability weighted moments $M_{1,j,0}$ or $M_{1,0,k}$, whichever has the simpler analytical structure, are given in Table 2.

From Table 2 it is noted that the relations between the probability weighted moments and the distributions' parameters are of simpler analytical structure than those between the conventional moments and the parameters; the simpler structure is due in part to x being taken to only the first power. This suggests that it might be possible to derive the parameters as functions of the probability weighted moments, particularly in those cases where the parameters cannot be derived as functions of the conventional moments. For example, in Table 2, except for the Gumbel and logistic distributions, the parameters cannot be defined explicitly as functions of the conventional moments. Given the values of the conventional moments, iterative techniques may be used to derive one or more, as the case may be, of the parameter values, allowing the remaining parameter values to be determined directly. On the other hand, the relations between the parameters and the probability weighted moments can be explicitly defined for each of the distributions, with the exception of the kappa and generalized lambda. If $b = d$ and $a = c$, the generalized lambda distribution is symmetric, in which case the parameters m, a, and b can be explicitly expressed as functions of the probability weighted moments. The specific relations are given below, illustrating potential usefulness of probability weighted moments.

PARAMETER EXPRESSIONS

In this section the following convention is adopted:

$$M_{(k)} \equiv M_{1,0,k} \tag{7}$$

Table 3 presents expressions for the parameters of the Weibull, Gumbel, symmetric lambda, and logistic distributions as functions of probability weighted moments. The Gumbel and logistic parameters may also be given as functions of their conventional moments as shown in the table.

The Wakeby relationships are more complex and are displayed in a separate table. Although $M_{l,j,k}$ for $j \neq 0$ and $k \neq 0$ may be expressed in terms of the parameters, the resulting equations would be more complicated. And in the case of the Wakeby distribution the resulting equations would involve beta functions whereby explicit expressions for the parameters cannot be obtained. Thus solutions are given in terms of the $M_{(k)}$.

TABLE 1. Distribution Functions

Distribution	x	F	Range of x	Reference[*]
Weibull	$m + a[-\ln(1 - F)]^{1/b}$	$1 - \exp\left[-\left(\dfrac{x-m}{a}\right)^b\right]$	m to ∞	1
Gumbel	$m - a \ln[-\ln F]$	$\exp\left\{-\exp\left[-\left(\dfrac{x-m}{a}\right)\right]\right\}$	$-\infty$ to ∞	2
Generalized lambda	$m + aF^b - c(1 - F)^d$	not explicitly defined	$m + a(0)^b - c$[†] to $m + a - c(0)^d$[†]	3
Logistic	$m + a[\ln(F) - \ln(1 - F)]$	$\left\{1 + \exp\left[-\left(\dfrac{x-m}{a}\right)\right]\right\}^{-1}$	$-\infty$ to ∞	4
Wakeby	$m + a[1 - (1 - F)^b]$ $- c[1 - (1 - F)^{-d}]$	not explicitly defined	m to $m + a[1 - (0)^b]$ $- c[1 - (0)^{-d}]$[†]	5
Kappa	$m + a[bF^b/(1 - F^b)]^{1/bc}$	$\left\{\left(\dfrac{x-m}{a}\right)^{bc} \Big/ \left[b + \left(\dfrac{x-m}{a}\right)^{bc}\right]\right\}^{1/b}$	m to ∞	6

[*]Ranges of parameter values can be found in the work of Johnson and Kotz [1970] for 1, 2, and 4; Ramberg and Schmeiser [1974] for 3; Landwehr et al. [1978] for 5; and Mielke [1973] for 6.
[†]The value of this term is either 0 or ∞, depending upon whether the parameter in the exponent is positive or negative ($0^0 = 1$).

TABLE 2. Moment Expressions

Distribution	Conventional Moments $M_{l,0,0}$ (integer $l \geq 0$)	Probability Weighted Moments $M_{1,j,k}$ (real j, $k \geq 0$)
Weibull	$\sum_{s=0}^{l} \binom{l}{s} m^{l-s} a^s \Gamma[(b+s)/b]^*$	$M_{1,0,k} = \dfrac{m}{1+k} + \dfrac{a\Gamma(1+1/b)}{(1+k)^{1+1/b}}$
Gumbel	$d^l[e^{\theta m}\Gamma(1-a\theta)]/d\theta^l\|_{\theta=0}\dagger$	$M_{1,j,0} = \dfrac{m}{1+j} + \dfrac{a\{\ln(1+j)+\epsilon\}}{1+j}\ddagger$
Generalized lambda	$\sum_{s=0}^{l} \binom{l}{s} m^{l-s} \left\{\sum_{t=0}^{s} \binom{s}{t} a^{s-t}(-c)^t B[1+b(s-t), 1+dt]\right\}\S$	$M_{1,0,k} = \dfrac{m}{1+k} + aB[1+b, 1+k] - \dfrac{c}{1+k+d}$
Logistic	$d^l[e^{\theta m}\Gamma(1-a\theta)\Gamma(1+a\theta)]/d\theta^l\|_{\theta=0}$	$M_{1,0,k} = \dfrac{m}{1+k} - \dfrac{a}{1+k}\sum_{s=0}^{k}\dfrac{1}{s}\|\|$
Wakeby	$\sum_{s=0}^{l} \binom{l}{s}(m+a-c)^{l-s}\left\{\sum_{t=0}^{s}\binom{s}{t}(-a)^{s-t}c^t \right.$ $\left. \cdot [b(s-t)-dt+1]^{-1}\right\}$	$M_{1,0,k} = \dfrac{m}{1+k} + \dfrac{a-c}{1+k} - \dfrac{a}{1+k+b}$ $+ \dfrac{c}{1+k-d}$
Kappa	$\sum_{s=0}^{l}\binom{l}{s}m^{l-s}a^s b^{-1+s/bc}B[(s+c)/bc, 1-s/bc]$	$M_{1,j,0} = \dfrac{m}{1+j} + ab^{-1+1/bc}B[(1+j+1/c)/b, 1-1/bc]$

$*\Gamma[\cdot]$ denotes the gamma function.
$\dagger$Here, $d^l[\cdot]/d\theta^l\|_{\theta=0}$ denotes the lth derivative of [ ] evaluated at $\theta = 0$.
$\ddagger$Here, ϵ is Euler's number equal to $0.5772\ldots$
$\S B[\cdot,\cdot]$ denotes the beta function.
$\|\|$ Integer k. Summation is 0 when $k = 0$.

Neither the parameters of the generalized lambda distribution nor those of the kappa distribution easily lend themselves to expression as functions of either conventional or probability weighted moments.

Weibull Parameters

The parameters of the Weibull distribution can be expressed explicitly as functions of probability weighted moments. Two distinct cases must be considered: first, when m, the lower bound on the range of x, is known and without loss of generality can be taken to be zero and, second, when m is unknown and must be estimated. As was noted previously, it is not possible to express Weibull parameters explicitly as functions of the conventional moments.

Gumbel Parameters

For the Gumbel distribution it is easier to obtain probability weighted moments of the form $M_{1,j,0}$ rather than $M_{(k)}$. However, as can be seen from (4),

$$M_{1,1,0} = M_{(0)} - M_{(1)} \tag{8}$$

and for consistency the parameters in Table 3 are expressed as functions of $M_{(k)}$.

It is noted that the parameters of the Gumbel distributions can be expressed in terms of both conventional and probability weighted moments. Thus alternative estimators for the parameters derived from finite length samples may be obtained by using probability weighted moments as well as the methods of conventional moments and maximum likelihood.

Symmetric Lambda Parameters

Tukey [1962] defined the symmetric lambda distribution as an implicit function of X:

$$x = F^\lambda - (1 - F)^\lambda \tag{9}$$

A more general but still symmetric form for this distribution is

$$x = m + a[F^b - (1 - F)^b] \tag{10}$$

Equation (1) gives the most general and not necessarily symmetric form of the distribution as stated by *Joiner and Rosenblatt* [1971].

Explicit expressions for the parameters as functions of attributes of the distribution are not readily obtained by using either the maximum likelihood or the method of moments technique. However, it is possible to express the parameters of the general symmetric form ((10) or (1) with $c = a$ and $d = b$) explicitly as functions of the probability weighted moments.

Given the expression for $M_{(k)}$ from Table 2, it can be shown for integer $k \geq 2$ that

$$\frac{J_k}{R} = \left[\frac{\Gamma(b+k+1)}{\Gamma(b+1)} - k!(b+1)\right]\frac{\Gamma(b+3)}{\Gamma(b+3+k)b} \tag{11}$$

where

$$J_k = M_{(k)} - M_{(k+1)} - M_{(0)}/(k+1)(k+2) \tag{12}$$

and

$$R = M_{(1)} - M_{(0)}/2 \tag{13}$$

In particular, given the expression in (11) for $k = 2$, the parameters of the generalized symmetric lambda distribution can be expressed as functions of probability weighted moments, as shown in Table 3.

It is noted that two values for the parameter b are obtained; additional criteria would need to be applied to determine which value is appropriate for the particular distribution function.

Logistic Parameters

The logistic distribution is a special case of the symmetric lambda distribution, occurring in the limit as $a \to \infty$ and $b \to 0$

TABLE 3. Parameter Expressions

Distribution	Parameter	Using Probability Weighted Moments $M_{(k)}$	Using Conventional Moments $M_{l,0,0}$
Weibull	$m = 0$	0	cannot be expressed explicitly
	a	$M_{(0)}/\Gamma\{\ln[M_{(0)}/M_{(1)}]/\ln(2)\}*$	
	b	$\ln(2)/\ln[M_{(0)}/2M_{(1)}]$	
	$m \neq 0$	$4\{M_{(3)}M_{(0)} - [M_{(1)}]^2\}/\{4M_{(3)} + M_{(0)} - 4M_{(1)}\}$	
	a	$\{M_{(0)} - m\}/\Gamma\left[\ln\left(\dfrac{M_{(0)} - 2M_{(1)}}{M_{(1)} - 2M_{(3)}}\right)\Big/ \ln(2)\right]$	
	b	$\ln(2)/\ln\left(\dfrac{M_{(0)} - 2M_{(1)}}{2(M_{(1)} - 2M_{(3)})}\right)$	
Gumbel	m	$M_{(0)} - \epsilon a$†	$M_{1,0,0} - \epsilon a$
	a	$[M_{(0)} - 2M_{(1)}]/\ln(2)$	$\{6[M_{2,0,0} - (M_{1,0,0})^2]\}^{1/2}/\pi$
Symmetric lambda	m	$M_{(0)}$	$M_{1,0,0}$
	a	$-R(b + 1)(b + 2)/b$	cannot be expressed explicitly
	b	$[R - 7J_2 \pm (J_2^2 - 10J_2R + R^2)^{1/2}]/2J_2$	cannot be expressed explicitly
	J_2	$M_{(2)} - M_{(3)} - M_{(0)}/12$	
	R	$M_{(1)} - M_{(0)}/2$	
Logistic	m	$M_{(0)}$	$M_{1,0,0}$
	a	$M_{(0)} - 2M_{(1)}$	$\{3[M_{2,0,0} - (M_{1,0,0})^2]\}^{1/2}/\pi$

*$\Gamma[\cdot]$ denotes the gamma function.
†ϵ is Euler's number, equal to 0.5772. . . .

if and only if ab approaches a constant. As is shown in Table 3, the parameters of the logistic distribution may be expressed as functions of both conventional and probability weighted moments. Again, these two sets of expressions form the basis for constructing alternative parameter estimators from finite length records.

Wakeby Parameters

The parameters of the Wakeby distribution are not readily obtained in terms of the conventional moments. The expressions for probability weighted moments are more complex than those for the parameters of the previously discussed distributions. Several additional definitions are adopted in this section. Define

$$\{k\} \equiv (k + 1)(k + 1 + b)(k + 1 - d)M_{(k)} \qquad (14)$$

so that

$$\{k\} = m(k + 1 + b)(k + 1 - d) + ab(k + 1 - d) + cd(k + 1 + b) \qquad (15)$$

Also, define

$$k_i \equiv k + i \qquad (16)$$

Then, for the real $k \geq 0$, it can be shown that if $m = 0$, then

$$-\{k_2\} + 2\{k_1\} - \{k_0\} = 0 \qquad (17)$$

but regardless of the value of m,

$$\{k_3\} - 3\{k_2\} + 3\{k_1\} - \{k_0\} = 0 \qquad (18)$$

Equations (17) and (18) hold for any arbitrary $k \geq 0$. Thus d can be expressed as a function of b, and b can be derived explicitly as a function of the probability weighted moments $M_{(K)}$ and $M_{(I)}$, where $I > K > 0$ and $i = 0, 1, 2, 3$. When the parameter m is not zero, it can be expressed as a function of only the forementioned probability weighted moments plus b and d. Finally, a and c can be obtained in terms of b, d, m, $M_{(K)}$, and $M_{(I)}$. These relationships are shown in Table 4.

For example, let $K = 0$ and $I = 1$. Table 5 contains the equations for this specific solution of Wakeby parameters in terms of the probability weighted moments. Note that because

TABLE 4. Explicit Solution of Wakeby Parameters

Parameter	Expression
b	$\dfrac{(N_3C_1 - N_1C_3) \pm [(N_1C_3 - N_3C_1)^2 - 4(N_1C_2 - N_2C_1)(N_2C_3 - N_3C_2)]^{1/2}}{2(N_2C_3 - N_3C_2)}$
d	$(N_1 + bN_2)/(N_2 + bN_3)$
m	$[\{K + 3\} - \{K + 2\} - \{K + 1\} + \{K\}]/4$
a	$\dfrac{(K + 1 + b)(K + 2 + b)}{b(b + d)}\left[\dfrac{\{K + 1\}}{(K + 2 + b)} - \dfrac{\{K\}}{(K + 1 + b)} - m\right]$
c	$\dfrac{(K + 1 - d)(K + 2 - d)}{d(b + d)}\left[-\dfrac{\{K + 1\}}{(K + 2 - d)} + \dfrac{\{K\}}{(K + 1 - d)} + m\right]$
N_{4-j}	$A(j, K)$
C_{4-j}	$A(j, I)$ for $I > K$
$\forall m: A(j, k)$	$(k + 4)^j M_{(k+3)} - 3(k + 3)^j M_{(k+2)} + 3(k + 2)^j M_{(k+1)} - (k + 1)^j M_k$
$m = 0: A(j, k)$	$-(k + 3)^j M_{(k+2)} + 2(k + 2)^j M_{(k+1)} - (k + 1)^j M_k$
$\{k\}$	$(k + 1)(k + 1 + b)(k + 1 - d)M_{(k)}$
K, I	Specific but arbitrarily chosen values of k such that $0 \le K < I$

TABLE 5. Specific Solution of Wakeby Parameters

Parameter	Expression
b	$\dfrac{(N_3C_1 - N_1C_3) \pm [(N_1C_3 - N_3C_1)^2 - 4(N_1C_2 - N_2C_1)(N_2C_3 - N_3C_2)]^{1/2}}{2(N_2C_3 - N_3C_2)}$
d	$(N_1 + bN_2)/(N_2 + bN_3)$
$m \ne 0$	$[\{3\} - \{2\} - \{1\} + \{0\}]/4$
a	$\dfrac{(b + 1)(b + 2)}{b(b + d)}\left[\dfrac{\{1\}}{2 + b} - \dfrac{\{0\}}{1 + b} - m\right]$
c	$\dfrac{(1 - d)(2 - d)}{d(b + d)}\left[\dfrac{-\{1\}}{2 + d} + \dfrac{\{0\}}{1 - d} + m\right]$
If $m = 0$	
N_{4-j}	$-(3)^j M_{(2)} + (2)^{1+j} M_{(1)} - M_{(0)}$
C_{4-j}	$-(4)^j M_{(3)} + 2(3)^j M_{(2)} - (2)^j M_{(1)}$
$\forall m$	
N_{4-j}	$(4)^j M_{(3)} - (3)^{1+j} M_{(2)} + 3(2)^j M_{(1)} - M_{(0)}$
C_{4-j}	$(5)^j M_{(4)} - 3(4)^j M_{(3)} + (3)^{1+j} M_{(2)} - (2)^j M_{(1)}$
$\{k\}$	$(k + 1)(k + 1 + b)(k + 1 - d)M_{(k)}$ $k = 0, 1, 2, 3, 4$

$M_{(0)} = E[X]$, when K is chosen equal to 0, the mean is assumed to exist and it is required that $b > -1$ and $d < 1$ [see *Landwehr et al.*, 1978].

From Tables 4 and 5 it appears that there are two possible solutions for the value of b, depending upon whether the positive or negative value of the square root is used in the computation. This would indicate that the method does not yield a unique solution. However, there is a symmetry in the Wakeby definition: if the signs of a, b, c, and d are reversed, the definitional form is still maintained. This is reflected by the two possible solutions for b. Denote the value of b computed by using the positive square root as $b(+)$ and that computed by using the negative square root as $b(-)$. It can be shown that the corresponding values for d are $-b(-)$ and $-b(+)$, respectively. Similarly, $c(+) = -a(-)$ and $c(-) = -a(+)$. Thus the solution yielded is indeed unique, and with no loss of generality the value of b can be chosen as the larger of $b(+)$ and $b(-)$.

SUMMARY

Distributions that can be expressed in inverse form, particularly those that can only be so expressed, may present problems in deriving explicit expressions for their parameters as functions of conventional moments. Of the latter type are Thomas' Wakeby and the generalized form of Tukey's lambda. The kappa and the more familiar Weibull, Gumbel, and logistic distributions can also be written in inverse form.

Such distributions may be characterized by probability weighted moments defined as $E[X^l F^j(1 - F)^k]$, where the random variable X is distributed as $F \equiv F(x) = P(X \leq x)$ and where l, j, and k are real numbers. If j and k are nonnegative integers, then the probability weighted moment of order (l, j, k) is proportional to the lth moment about the origin of the $(j + 1)$th order statistic for a sample of size $n = k + j + 1$.

For the specific distributions noted above, the relations between the probability weighted moments and the parameters are of a simpler analytical structure than those between the conventional moments and the parameters. The simpler analytical structure suggests that it may be possible to derive the relations between the parameters and the probability weighted moments even though it may not be possible to derive the relations between the parameters and the conventional moments. This is exemplified by the Wakeby, symmetric lambda, and Weibull distributions. For the Gumbel and logistic distributions their parameters may be explicitly expressed as functions of both probability weighted moments and conventional moments. In the case of the kappa distributions, neither type of moments appears to lend itself to functional expression of the parameters.

The fact that the parameters of the Wakeby and symmetric lambda distributions can be expressed as functions of probability weighted moments but not as functions of conventional moments illustrates the potential usefulness of probability weighted moments. Moreover, for distributions whose parameters can be expressed as functions of both kinds of moments, probability weighted moments provide a basis for an alternative method for parameter estimation to the methods of conventional moments and maximum likelihood.

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