IEOR 4102, HMWK 1, Professor Sigman

1. An asset price starts off initially at price \$3.00 at the end of a day (day 0), and at the end of each consecutive day, independent of the past, the price goes up by one dollar (with probability $p = 0.7$) or down by one dollar (with probability $q = 0.3$).

   (a) What is the probability that the stock will reach \$11.00 before going down to 0?
   SOLUTION: Gambler's Ruin Problem: With $q/p = 3/7$; $P_3(11) = \frac{1-(q/p)^3}{1-(q/p)^{11}} = 0.921$.

   (b) What is the probability that the stock will reach \$10.00 before going down to a low of \$2.00?
   SOLUTION: Starting initially at 3 and going "up by 7" (to hit 10) before "down by 1" (to hit 2) is equivalent to starting a random walk initially at $R_0 = 0$, choose $a = 7$, $b = 1$, and use the formula for the probability of hitting a before hitting $-b$:

   $$p(a) = \frac{1-(q/p)^b}{1-(q/p)^{a+b}} = \frac{1-(3/7)^1}{1-(3/7)^8} = 0.572.$$

   (c) What is the probability that the stock will (as time goes on) become infinitely valuable without ever hitting 0?
   SOLUTION: Gambler's Ruin Problem, where we have $i = 3$ and want $P_i(\infty) = 1-(q/p)^i = 1-(3/7)^3 = 0.921$. (Because we have positive drift, this probability is positive.)

   (d) (*Continuation:*) Answer (a)– (c) in the case when the two probabilities 0.7 and 0.3 are reversed.
   SOLUTION:
   $q/p = 7/3$ now, and we have negative drift now.
   (a) 0.001
   (b) 0.002 (0.00152)
   (c) Since we now have negative drift, $P_i(\infty) = 0$.

   (e) (*Continuation:*) Answer (a)– (c) in the case when $p = q = 0.5$.
   SOLUTION: Now, $P_i(N) = i/N$ for the Gambler's ruin problem, and for the random walk with $R_0 = 0$, we have $P(a) = b/(a+b)$. This yields
   (a) 3/11
   (b) 1/8
   (c) Even when $p = 1/2$, we have $P_i(\infty) = 0$.

2. An insurance risk business has a reserve of money (in units of millions of dollars). Initially, it has 3 units. Every day, it earns 1 unit (interest), but also (each day) there is a chance of a claim against the business, independent of past days, of size 2 units with probability 0.35 (with probability 0.65 no such claim comes in). (A claim removes the 2 units from the reserve.)

   (a) What is the probability that the risk business will get ruined (run out of money)?

SOLUTION: Recall that the reserve process is exactly a simple random walk with $p = 0.65$ starting from $i = 3$. ($q/p = (.35/.65 = 7/13)$ Thus we want $1 - P_i(\infty)$, where $P_i(\infty) = \lim_{N\to\infty} P_i(N) = 1 - (q/p)^i$. Thus we want $(q/p)^i = (7/13)^3 = 0.156$.

(b) What is the smallest value i (units) the business would need to have started with to ensure that the probability of ruin is *less* than 1/2?

SOLUTION: Noting that $(7/13)^1 = 0.538 > 1/2$ and $(7/13)^2 = 0.29 < 1/2$, we see that the answer is $i = 2$.

3. A jumping bean moves on the integers according to a simple random walk taking one step per unit time. $R_n =$ the position at time $n \geq 0$. Assume that $p = 0.35$; the probability that a step takes the bean forward (to the right), and $q = 1 - p = 0.65$ is the probability that a step takes the bean backward (to the left). It starts off initially at position $R_0 = 5$.

(a) Does this random walk have positive drift or negative drift?

SOLUTION:

Negative drift (by definition) since $p < 1/2$; $P(\lim_{n\to\infty} R_n = -\infty) = 1$.

(b) What is the probability that the bean will go down to 0 before ever reaching \$6?

SOLUTION: With $q/p = 13/7$; $1 - P_5(6) = 0.473$, where

$$P_5(6) = \frac{1 - (q/p)^5}{1 - (q/p)^6} = 0.527.$$

(c) What is the probability that the bean will go below (<) 0 before ever reaching \$6?

SOLUTION:

Going below 0 means that it hits -1 since this is a simple random walk, only taking ± 1 size steps. So we want the probability that such a random walk, goes down by $b = 6$ before going up by $a = 1$.

So we want $1 - p(a)$, where (with $a = 1$, $b = 6$),

$$p(a) = \frac{1 - (\frac{q}{p})^b}{1 - (\frac{q}{p})^{a+b}}. \tag{1}$$

Calculation gives answer 0.468.

(d) What is the probability that the bean will *never* reach as high as 6.00?

SOLUTION: We want the probability that the random walk, starting at 5 never goes up by 1 to 6. This is equivalent to the random walk, starting initially at the origin, $R_0 = 0$, never reaching as high as 1, meaning that it never goes above the origin. Letting $M = \max_{n\geq 0} R_n$, (with $R_0 = 0$) we know that because the random walk has negative drift (e.g., $p < 1/2$),M has a geometric distribution, $P(M \geq a) = (p/q)^a$, $a \geq 0$. ($p/q = 7/13$.) Thus we want $P(M \leq 0) = P(M = 0) = 1 - (p/q) = 4/13$.

4. As a more realistic model for asset pricing, suppose that the price of an asset moves (day by day) as

$$X_n = 8 \times 2^{R_n}, \ n \geq 0,$$

where $R_0 = 0$, and $R_n = \sum_{k=1}^{n} \Delta_k, \ k \geq 1$, is a simple symmetric random walk; $P(\Delta = 1) = 1/2 = P(\Delta = -1)$.

   (a) What is the probability that the asset price reaches a high of 32 before a low of 1/2?
   SOLUTION: By taking logarithms in base 2, this is equivalent to the probability that $3 + R_n$ hits 5 before -1, or that R_n (starting at 0) hits $a = 2$ before $-b = -4$: (Recall that $p = 1/2$)

$$p(a) = \frac{b}{a+b} = 2/3. \tag{2}$$

   (b) What is the probability that the asset price will ever reach as high as 2^{500}?
   SOLUTION: We want the probability that $3 + R_n$ ever hits 500, that is, the probability that R_n ever hits 497. Recall that for the simple symmetric random walk R_n, $P(M = \infty) = 1$ (and $P(m = -\infty) = 1$ meaning that it will hit any integer, however large (or however small), with certainty if we wait long enough. Thus the answer is 1.

5. *Simulating simple random walks:* For each of $p = .4$ and $p = .55$ and $p = 0.5$: Simulate (using MATLAB or PYTHON) the simple random walks starting from $R_0 = 0$ out to $n = 1000$ steps to:

   (a) Compute R_n/n to see if it is close to $E(\Delta) = 2p - 1$, as it should be by the Strong Law Of Large Numbers.
   SOLUTION: For $p = 0.4$, $E(\Delta) = -0.2$ and we got $R_n/n = -0.1900$; for $p = 0.55$, $E(\Delta) = 0.1$ and we got $R_n/n = 0.0940$; for $p = 0.5$, $E(\Delta) = 0$ and we got $R_n/n = 0.0100$. So the simulation results are fairly close to $E(\Delta)$ in all three cases.
   Sample simulation codes in MATLAB as follows:

```
clear all
n = 1000; % number of steps
p = 0.5; % can be set to be 0.4 and 0.55
R = 0; % initialize R_0 = 0
for i = 1:n
   U = rand;
   if(U < p)
       R = R + 1;
   else
       R = R - 1;
   end
end
R/n
```

   (b) Estimate the probability that the random walk will ever reach as high as $a = 100$ by time $n = 1000$. If we define

$$M_n = \max_{0 \leq k \leq n} R_k,$$

the maximum of the random walk during the first n time units (steps), then we want to estimate $P(M_n \geq 100)$, for $n = 1000$.

Here is the pseudo-code: Recall that a Δ can be simulated via generating a U (uniform over $(0, 1)$) and setting $\Delta = 1$ if $U \leq p$; $\Delta = -1$ if $U > p$.

1 With $R_0 = 0$, start simulating the random walk sequentially via $R_{k+1} = R_k + \Delta_{k+1}$.

2 If for some $k \leq n = 1000$, it holds that $R_k = 100$, then stop and output $I = 1$.

3 If $R_k < 100,\ 0 \leq k \leq n = 1000$, then stop and output $I = 0$

4 Repeat (1–3) above (independently) $m = 5,000$ times to obtain 5000 independent copies of I, denoted by $I_1, \ldots, I_{5000}$.

5 Use estimate
$$P(M_n \geq 100) \approx \frac{1}{m}\sum_{i=1}^{m} I_i.$$

SOLUTION: For $p = 0.4$, we got $\frac{1}{m}\sum_{i=1}^{m} I_i = 0$; for $p = 0.55$, we got $\frac{1}{m}\sum_{i=1}^{m} I_i = 0.5596$; for $p = 0.5$, we got $\frac{1}{m}\sum_{i=1}^{m} I_i = 0.0018$.

Sample simulation codes in MATLAB as follows:

```
clear all
p = 0.5;
m = 5000; % sample size
n = 1000; % number of steps
I = zeros(m,1); % store results
for i = 1:m
    i
    R = 0; % initialize R_0 = 0
    for j = 1:n
        U = rand;
        if(U < p)
            R = R + 1;
        else
            R = R - 1;
        end
        if(R == 100)
            I(i) = 1;
            break;
        end
        I(i) = 0;
    end
end
mean(I)
```