Error Theories & Data Processing (English)

误差理论与数据处理（英文）

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Quick Review of last lesson

Detection of abnormal errors 粗大误差的判别:

(1) 3σ method $|v_d| = |x_d - \bar{x}| (? >) 3\sigma$

(2) t method (Romanowsky Method)

$x_1, x_2, \cdots, x_n$ remove x_j

$$\bar{x}' = \frac{1}{n-1}\sum_{i=1, i\neq j}^{n} x_i \qquad \sigma' = \sqrt{\frac{\sum_{i=1, i\neq j}^{n} v_i'^2}{n-2}} \qquad \left|x_j - \bar{x}'\right| (? >) K\sigma'$$

(3) Grubbs method

$$x_1, x_2, \cdots, x_n \quad \Rightarrow \quad x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)}$$

$$let\ g_{(n)} = \frac{x_{(n)} - \bar{x}}{\sigma}\ and\ g_{(1)} = \frac{\bar{x} - x_{(1)}}{\sigma} \qquad g_{(i)} = \frac{x_{(i)} - \bar{x}}{\sigma}\ (? \ge)\ g_0(n, \alpha)$$

Chapter3: Synthesis and assignment of error
第三章：误差的合成与分配

Section1: Error of function 函数误差

In chapter2, we studied the errors of direct measurements, but some objects are not suitable to be measured directly, so we need to measure them indirectly via certain functions. An indirect measurement is a function of direct measurements, so its errors are also functions of errors of those direct measurements. This is basically the study of error transfer problem, also called <u>error synthesis误差合成</u>.

System error of a function
函数的系统误差

Mathematic model of indirect measurement
间接测量的数学模型:

$$y = f(x_1, x_2, \ldots, x_n)$$

$x_1, x_2, \ldots, x_n$ are the direct measurements required for computing indirect measurement y.

The total differential equation（全微分方程） of function y is given as follows:

$$dy = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \cdots + \frac{\partial f}{\partial x_n} dx_n$$

Synthesis of system error of function

With approximation, system error of the function Δy is given by:

$$\Delta y = \frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2 + \ldots + \frac{\partial f}{\partial x_n}\Delta x_n$$

$\partial f / \partial x_i \; (i = 1, 2, \cdots, n)$ are the error transfer coefficients误差传递系数 at the measurement point $x_1, x_2, \ldots, x_n$

For linear function $y = a_1 x_1 + a_2 x_2 + \ldots + a_n x_n$

System error of the function $\Delta y = a_1 \Delta x_1 + a_2 \Delta x_2 + \ldots + a_n \Delta x_n$

Furthermore $if \;\; a_i = 1 \quad \Rightarrow \quad \Delta y = \Delta x_1 + \Delta x_2 + \ldots + \Delta x_n$

If the function is the sum of each direct measurements, the system error of function is also the sum of system errors of those direct measurements.

Synthesis of system error of function

$x_1, x_2, \ldots, x_n$ have system errors $\Delta x_1, \Delta x_2, \cdots, \Delta x_n$

$x_{10}, x_{20}, \cdots, x_{n0}$ have no system errors. where $x_{i0} = x_i - \Delta x_i$

Obviously $y = f(x_1, x_2, \ldots, x_n)$ has system error,

and $y_0 = f(x_{10}, x_{20}, \cdots x_{n0})$ has no system error.

$$\Delta y = y - y_0 = f(x_1, x_2, \cdots x_n) - f(x_{10}, x_{20}, \cdots x_{n0})$$

$$\approx \frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2 + \cdots + \frac{\partial f}{\partial x_n}\Delta x_n$$

Therefore the system error of a function can be obtained in 2 ways: 1) remove system errors from all independent variables and then solve the function without system error, the difference between the 2 function values is the theoretical system error; 2) synthesize the system error of the function based on those system errors of independent variables and their error transfer coefficients.

Synthesis of system error of function

Trigonometric functions三角函数:

$$\sin\varphi = f\left(x_1, x_2, \ldots, x_n\right) \qquad \Delta\sin\varphi = \frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2 + \cdots + \frac{\partial f}{\partial x_n}\Delta x_n$$

$$d\sin\varphi = \cos\varphi \cdot d\varphi \;\Rightarrow\; d\varphi = \frac{d\sin\varphi}{\cos\varphi} \;\Rightarrow\; \Delta\varphi = \frac{\Delta\sin\varphi}{\cos\varphi}$$

$$\Delta\varphi = \frac{1}{\cos\varphi}\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\Delta x_i$$

$$\cos\varphi = f\left(x_1, x_2, \ldots, x_n\right) \qquad \Rightarrow \quad \Delta\varphi = \frac{1}{-\sin\varphi}\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\Delta x_i$$

$$\tan\varphi = f(x_1, x_2, \cdots, x_n) \qquad \Rightarrow \quad \Delta\varphi = \cos^2\varphi\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\Delta x_i$$

$$\cot\varphi = f(x_1, x_2, \cdots, x_n) \qquad \Rightarrow \quad \Delta\varphi = -\sin^2\varphi\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\Delta x_i$$

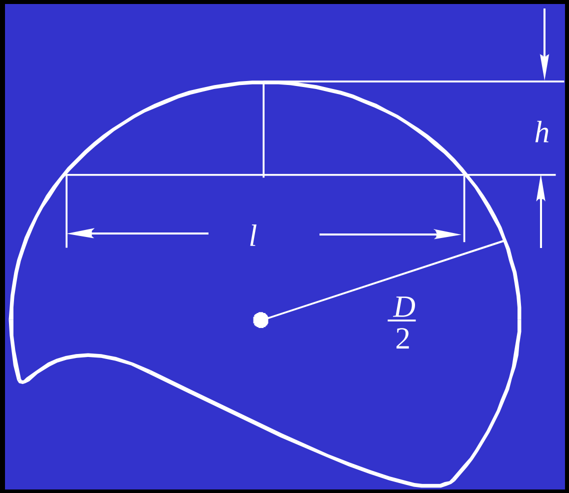
Example 3-1

Indirect measurement of the diameter according to the height and length of an arc:

Known: Height of the arc $h = 50mm$， length of the arc $l = 500mm$.

System error of the height $\Delta h = -0.1mm$

System error of the length $\Delta l = 1mm$



Please find out the system error of diameter and the corrected final result.

$$D = \frac{l^2}{4h} + h$$

At $h = 50\text{mm}$ $l = 500\text{mm}$

the diameter is calculated as: $D = \frac{l^2}{4h} + h = 1300mm$

Example 3-1 (continued)

Error transfer coefficients:

$$\frac{\partial f}{\partial h} = -\left(\frac{l^2}{4h^2} - 1\right) = -\left(\frac{500^2}{4\times 50^2} - 1\right) = -24$$

$$\frac{\partial f}{\partial l} = \frac{l}{2h} = \frac{500}{2\times 50} = 5$$

System error of D $\quad \Delta D = \frac{\partial f}{\partial l}\Delta l + \frac{\partial f}{\partial h}\Delta h = 7.4\mathrm{mm}$

Result after adjustment:

$$D_0 = D - \Delta D = 1300 - 7.4 = 1292.6mm$$

Example 3-1 (comparison method)

Remove system errors for independent variables, and we have:

$$l_0 = l - \Delta l = 500 - 1 = 499$$

$$h_0 = h - \Delta h = 50 - (-0.1) = 50.1$$

Diameter with no system error is given as:

$$D_0 = \frac{l_0^{\ 2}}{4h_0} + h_0 = 1292.6mm$$

The system error of the diameter can be solved as follows:

$$\Delta D = D - D_0 = 1300 - 1292.6 = 7.4mm$$

Computation of random error of function
函数的随机误差计算

Mathematic model:

$$y = f(x_1, x_2, ..., x_n)$$

Assume there are only random errors, we have

$$y + \delta y = f(x_1 + \delta x_1, x_2 + \delta x_2, \cdots, x_n + \delta x_n)$$

Take Taylor expansion and 1st order approximation

$$y + \delta y = f(x_1, x_2, ..., x_n) + \frac{\partial f}{\partial x_1}\delta x_1 + \frac{\partial f}{\partial x_2}\delta x_2 + \cdots + \frac{\partial f}{\partial x_n}\delta x_n$$

$$\delta y = \frac{\partial f}{\partial x_1}\delta x_1 + \frac{\partial f}{\partial x_2}\delta x_2 + \cdots + \frac{\partial f}{\partial x_n}\delta x_n$$

Variance and standard deviation of function
函数的方差和标准差

Take variance on both sides, we have:

$$\sigma_y{}^2 = \left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_{x1}{}^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_{x2}{}^2 + \cdots + \left(\frac{\partial f}{\partial x_n}\right)^2 \sigma_{xn}{}^2 + 2\sum_{1 \le i < j}^{n} \left(\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} D_{ij}\right)$$

or $$\sigma_y{}^2 = \left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_{x1}{}^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_{x2}{}^2 + \cdots + \left(\frac{\partial f}{\partial x_n}\right)^2 \sigma_{xn}{}^2 + 2\sum_{1 \le i < j}^{n} \left(\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} \rho_{ij}\sigma_{xi}\sigma_{xj}\right)$$

- σ_{xi} is the standard deviation of i^{th} direct measurement.
- ρ_{ij} is the correlation coefficient of i^{th} and j^{th} measurement.
- $D_{ij} = \rho_{ij}\sigma_{xi}\sigma_{xj}$ is the covariance of i^{th} and j^{th} measurement
- $\frac{\partial f}{\partial x_i}$ is the error transfer coefficient of the i^{th} measurement at the measurement point $(x_1, x_2, \ldots, x_n)$

Variance of function of independent variables 独立变量函数的方差

If the random errors of measurements are independent, we have: $D_{ij} = \rho_{ij} = 0$

$$\sigma_y{}^2 = \left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_{x1}{}^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_{x2}{}^2 + \cdots + \left(\frac{\partial f}{\partial x_n}\right)^2 \sigma_{xn}{}^2$$

or $$\sigma_y = \sqrt{\left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_{x1}{}^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_{x2}{}^2 + \cdots + \left(\frac{\partial f}{\partial x_n}\right)^2 \sigma_{xn}{}^2}$$

let $\dfrac{\partial f}{\partial x_i} = a_i$ then $$\boxed{\sigma_y = \sqrt{a_1{}^2 \sigma_{x1}{}^2 + a_2{}^2 \sigma_{x2}{}^2 + \cdots + a_n{}^2 \sigma_{xn}{}^2}}$$

If all random errors of measurements are Normal, we can replace the standard deviation with extreme errors and obtain the expression for extreme error of the function as follows:

$$\boxed{\delta_{\lim y} = \pm\sqrt{a_1^2 \delta_{\lim x1}^2 + a_2^2 \delta_{\lim x2}^2 + \cdots + a_n^2 \delta_{\lim xn}^2}}$$

In most cases, $a_i=1$, we can simplify the above equations to much simpler forms.

Standard deviation of trigonometric function
三角函数的标准差

$$\sin\varphi = f(x_1, x_2, \cdots, x_n) \qquad \sigma_\varphi = \frac{1}{\cos\varphi}\sqrt{\left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_{x1}^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_{x2}^2 + \cdots + \left(\frac{\partial f}{\partial x_n}\right)^2 \sigma_{xn}^2}$$

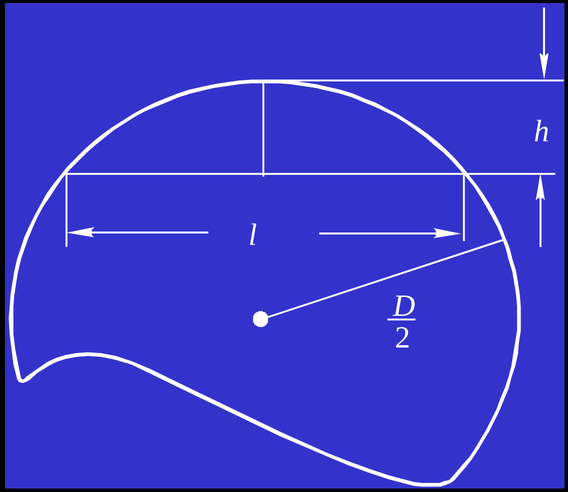
$$\cos\varphi = f(x_1, x_2, \cdots, x_n) \qquad \sigma_\varphi = \frac{1}{\sin\varphi}\sqrt{\left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_{x1}^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_{x2}^2 + \cdots + \left(\frac{\partial f}{\partial x_n}\right)^2 \sigma_{xn}^2}$$

$$\tan\varphi = f(x_1, x_2, \cdots, x_n) \qquad \sigma_\varphi = \cos^2\varphi\sqrt{\left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_{x1}^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_{x2}^2 + \cdots + \left(\frac{\partial f}{\partial x_n}\right)^2 \sigma_{xn}^2}$$

$$\cot\varphi = f(x_1, x_2, \cdots, x_n) \qquad \sigma_\varphi = \sin^2\varphi\sqrt{\left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_{x1}^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_{x2}^2 + \cdots + \left(\frac{\partial f}{\partial x_n}\right)^2 \sigma_{xn}^2}$$

Example3-2

Indirect measurement of the diameter by the height and length of an arc:



Known: Height of the arc $h = 50mm$, length of the arc $l = 500mm$.

System error of the height $\Delta h = -0.1mm$

System error of the length $\Delta l = 1\text{mm}$

$\sigma_h = 0.005\text{mm}$ $\sigma_l = 0.01\text{mm}$

Please find out the standard deviation of diameter and the corrected final result.

$$\sigma_D = \sqrt{\left(\frac{\partial f}{\partial l}\right)^2 \sigma_l^2 + \left(\frac{\partial f}{\partial h}\right)^2 \sigma_h^2} = \sqrt{5^2 \times 0.01^2 + 24^2 \times 0.005^2} = 0.13mm$$

Final result after correction is:

$$D = D_0 - \Delta D = 1292.6\text{mm} \qquad \sigma_D = 0.13\text{mm}$$

The correlation of error 误差的相关性

The correlations among the random errors directly affect the error synthesis of the function.

$$\sigma_y{}^2 = \left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_{x1}{}^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_{x2}{}^2 + \cdots + \left(\frac{\partial f}{\partial x_n}\right)^2 \sigma_{xn}{}^2 + 2\sum_{1\le i<j}^{n}\left(\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}\rho_{ij}\sigma_{xi}\sigma_{xj}\right)$$

When $\rho_{ij} = 0$ $\quad \sigma_y = \sqrt{a_1{}^2\sigma_{x1}{}^2 + a_2{}^2\sigma_{x2}{}^2 + \cdots + a_n{}^2\sigma_{xn}{}^2}$

When $\rho_{ij} = +1$ $\quad \sigma_y = \left|a_1\sigma_{x1} + a_2\sigma_{x2} + \cdots + a_n\sigma_{xn}\right|$

Assessment of correlation coefficient 相关系数的评定

The correlation coefficients ρ_{ij} reflex the level of dependence between the errors of x_i and x_j , and they have the following features:

$$-1 \le \rho_{ij} \le +1 \quad for\ all\ i \ne j$$

$$\rho_{ii} = +1$$

How to assess the correlation coefficient?

(1) Direct judgement method

If 2 errors are impossible to have any correlation or only have weak correlation, we decide $\rho=0$.

If 2 errors show strong positive or negative linear relationship, we may decide $\rho=1$ or $\rho=-1$ respectively.

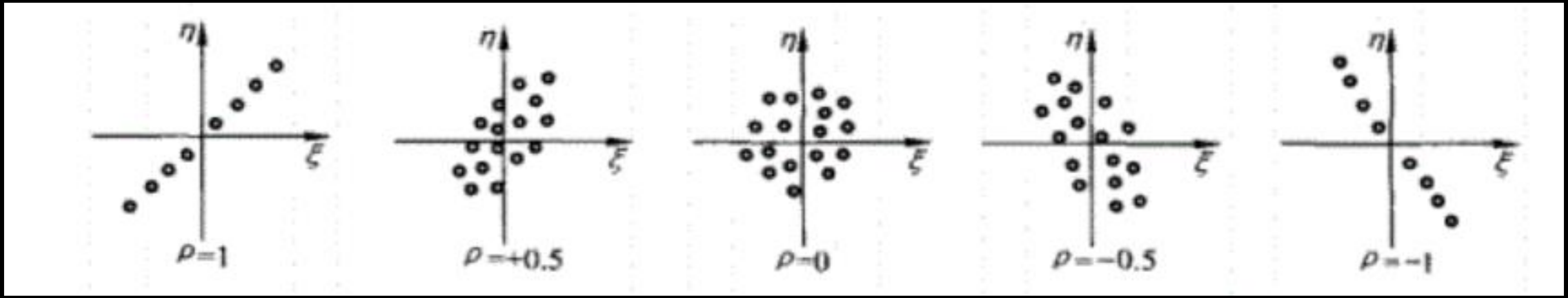
Assessment of correlation coefficient

(2) Experiment observation and simple calculation methods

a) Observation method

plot corresponding values of 2 errors at multiple measurements (ξ_i, η_i)

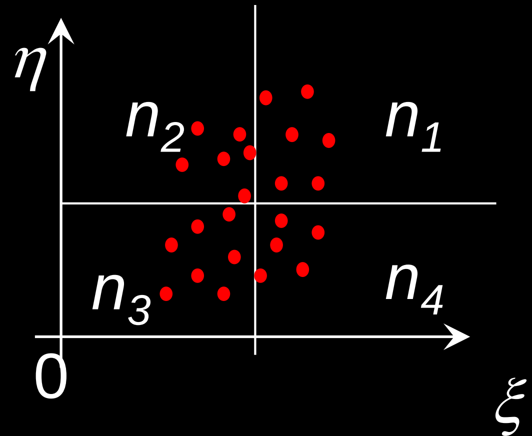
compare them with figure3-3 of the textbook to determine the value of correlation coefficient ρ



Determination of correlation coefficient

b) Simple calculation method

Plot corresponding values (ξ_i, η_i) of multiple measurements on a plane coordinates. Draw a vertical line to evenly split the points into left and right sides; draw a horizontal line to evenly split the points into upper and lower sides.



$$\rho \approx -\cos\left[\frac{n_1 + n_3}{\sum n}\pi\right] \qquad \sum n = n_1 + n_2 + n_3 + n_4$$

c) Direct computation method

$$\rho = \frac{\sum(\xi_i - \bar{\xi})\sum(\eta_i - \bar{\eta})}{\sqrt{\sum(\xi_i - \bar{\xi})^2 \sum(\eta_i - \bar{\eta})^2}}$$

Homeworks

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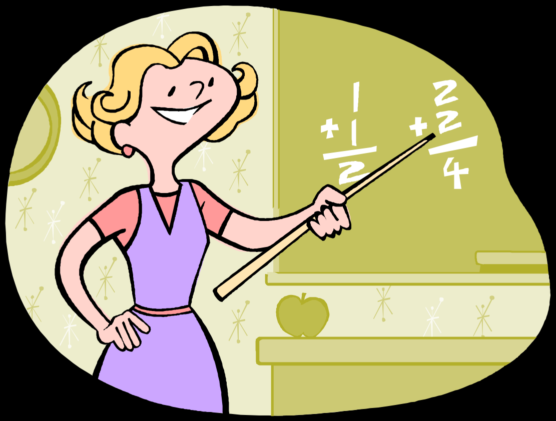
Questions 3-1, 3-3, 3-4, 3-6, 3-8



Section2: Synthesis of random error
第二节：随机误差的合成

The synthesis of random errors of an experiment can be achieved via the synthesis of standard deviation or the synthesis of extreme error.

(1) Synthesis of standard deviation (q random errors)

$$\sigma = \sqrt{\sum_{i=1}^{q}(a_i\sigma_i)^2 + 2\sum_{1\le i<j}^{q}\rho_{ij}a_i a_j \sigma_i \sigma_j}$$

$\sigma_1, \sigma_2, \cdots, \sigma_q$ are standard deviations of q random errors

$a_1, a_2, \cdots, a_q$ are transfer coefficients of q random errors $a_i = \partial f / \partial x_i$

If $\rho_{ij} = 0$

$$\boxed{\sigma = \sqrt{\sum_{i=1}^{q}(a_i\sigma_i)^2}}$$

Section2: Synthesis of random error

(2) Synthesis of extreme error (q random errors)

$$\delta = \pm t\sigma = \pm t\sqrt{\sum_{i=1}^{q}(a_i\sigma_i)^2 + 2\sum_{1\le i<j}^{q}\rho_{ij}a_ia_j\sigma_i\sigma_j}$$

If the confidence coefficients are different, i.e. $\delta_i = \pm t_i\sigma_i$

Then t and σ must be determined separately, which is annoying.

$$\delta = \pm t\sqrt{\sum_{i=1}^{q}(a_i\frac{\delta_i}{t_i})^2 + 2\sum_{1\le i<j}^{q}\rho_{ij}a_ia_j\frac{\delta_i}{t_i}\frac{\delta_j}{t_j}}$$

If $\rho_{ij} = 0 \ for\ all\ i \neq j$

then $$\boxed{\delta = \pm t\sqrt{\sum_{i=1}^{q}(a_i\frac{\delta_i}{t_i})^2}}$$

Section2: Synthesis of random error

$$\delta = \pm t\sqrt{\sum_{i=1}^{q}(a_i \frac{\delta_i}{t_i})^2 + 2\sum_{1\le i<j}^{q}\rho_{ij}a_i a_j \frac{\delta_i}{t_i}\frac{\delta_j}{t_j}}$$

If all extreme errors use the same confidence coefficient t,

i.e. $\delta_i = \pm t\sigma_i \ \ for\ all\ i$ and $\delta = \pm t\sigma$

then $$\delta = \pm\sqrt{\sum_{i=1}^{q}(a_i\delta_i)^2 + 2\sum_{1\le i<j}^{q}\rho_{ij}a_i a_j\delta_i\delta_j}$$

If $\rho_{ij} = 0 \ \ for\ all\ i \ne j$

then $$\boxed{\delta = \pm\sqrt{\sum_{i=1}^{q}(a_i\delta_i)^2}}$$

Section3: Synthesis of system error
第三节：系统误差的合成

In engineering practices, system errors of an experiment can be classified into known system errors and unknown system errors.

(1) Synthesis of known system error
已定系统误差的合成

Definition: System error whose magnitude and direction are already known.

$$\Delta = \sum_{i}^{r} a_i \Delta_i$$

- Δ is the synthesized system error of the final result
- Δ_i is the i^{th} system error, a_i is its transfer coefficient.
- Known system errors can be removed during measuring process or from the final result after the synthesis.

(2) Synthesis of unknown system error

未定系统误差的合成

Definition: Unknown system errors are errors whose magnitude and direction are unknown or unnecessary to be known, so we are only able to or only required to estimate their ranges of extremity, e.g. $\pm e_i$.

Properties of unknown system error:

- When the measuring condition does not change, unknown system error is a constant for repeated measurements and cannot be cancelled out among those measurements.
- When the measuring condition changes, unknown system error has randomness within its range, and it has certain distribution, just like random error.

Notations for unknown system error:

The extreme unknown system error is noted as "e".

The standard deviation of unknown system error is "u".

Synthesis of unknown system error

Because unknown system errors have certain randomness and distributions, the joint effect of multiple unknown system errors is certain cancellation of the errors just like the random errors. As a result, the synthesis of unknown system errors can be treated just like random errors, with great convenience.

Like random errors, unknown system errors can be synthesized in terms of standard deviation or in terms of extreme error.

a) Synthesis in terms of standard deviation:

$$u = \sqrt{\sum_{i=1}^{s}(a_i u_i)^2 + 2\sum_{1 \le i < j}^{s} \rho_{ij} a_i a_j u_i u_j}$$

(the number of unknown system errors is "s")

When $\rho_{ij} = 0$

$$u = \sqrt{\sum_{i=1}^{s}(a_i u_i)^2}$$

Synthesis of unknown system error

b) Synthesis in terms of extreme error:

The extreme errors of individual unknown system errors are given as: $e_i = \pm t_i u_i \qquad i = 1,\ 2,\ \cdots,\ s$

The overall extreme error of unknown system error is:

$$e = \pm tu$$

$$e = \pm t\sqrt{\sum_{i=1}^{s}(a_i u_i)^2 + 2\sum_{1 \le i < j}^{s}\rho_{ij} a_i a_j u_i u_j}$$

$$e = \pm t\sqrt{\sum_{i=1}^{s}\left(\frac{a_i e_i}{t_i}\right)^2 + 2\sum_{1 \le i < j}^{s}\rho_{ij} a_i a_j \frac{e_i}{t_i}\frac{e_j}{t_j}}$$

$$if\ \ \rho_{ij} = 0\ and\ t_i = t, \quad then\ \ e = \pm\sqrt{\sum_{i=1}^{s}(a_i e_i)^2}$$

Section4: Synthesis of system and random errors

第四节：系统误差与随机误差的合成

When multiple system errors and random errors of different properties are present in the same experiment, we generally use extreme error to express overall error of the final result, however standard deviation is also occasionally used.

(1) Synthesis in terms of extreme error

Assume there are r individual known system errors, s individual unknown system errors and q individual random errors during measuring process, and they are noted as follows:

$$\Delta_1, \quad \Delta_2, \quad \cdots, \quad \Delta_r$$

$$e_1, \quad e_2, \quad \cdots, \quad e_s$$

$$\delta_1, \quad \delta_2, \quad \cdots, \quad \delta_q$$

$$\Delta_{overall} = \sum_{i=1}^{r} a_i \Delta_i \pm t \sqrt{\sum_{i=1}^{s} \left(\frac{b_i e_i}{t_i} \right)^2 + \sum_{i=1}^{q} \left(\frac{c_i \delta_i}{t_i} \right)^2 + R}$$

a_i, b_i and c_i are the error transfer coefficients for known system errors, unknown system errors and random errors respectively, and R is the influence of the covariance of errors.

Synthesis of system and random errors

If all corresponding distributions are <u>Normal</u>, if all errors are <u>independent</u>, we have:

$$\Delta_{overall} = \sum_{i=1}^{r} \Delta_i \pm \sqrt{\sum_{i=1}^{s} \left(b_i e_i\right)^2 + \sum_{i=1}^{q} \left(c_i \delta_i\right)^2}$$

When known system errors are removed, the final extreme error is composed of extreme errors of unknown system errors and random errors.

$$\boxed{\Delta_{overall} = \pm\sqrt{\sum_{i=1}^{s} \left(b_i e_i\right)^2 + \sum_{i=1}^{q} \left(c_i \delta_i\right)^2}}$$

We can see that for synthesis of multiple unknown system errors and random errors, those errors are treated the same way.

Synthesis of system and random errors

Please note: previous error synthesis equations are only valid for a single experiment.

When the experiment is repeated for multiple times, the true random errors can have certain cancellation among those repeated experiments. However, system errors, known or unknown, can not be reduced by multiple experiments. The error synthesis equation of multiple experiments is given as follows:

$$\Delta_{overall} = \pm\sqrt{\sum_{i=1}^{s}\left(b_i e_i\right)^2 + \frac{1}{n}\sum_{i=1}^{q}\left(c_i \delta_i\right)^2}$$

n is the number of repeated experiments

Synthesis of system and random errors

(2) Synthesis in terms of standard deviation

When synthesizing in terms of standard deviation, only unknown system errors and random errors are considered.

Assume there are s individual unknown system errors and q individual random errors during measuring process as follows:

$$e_1,\quad e_2,\quad \cdots,\quad e_s$$

$$\delta_1,\quad \delta_2,\quad \cdots,\quad \delta_q$$

$$\sigma_{overall} = \sqrt{\sum_{i=1}^{s}(b_i u_i)^2 + \sum_{i=1}^{q}(c_i \sigma_i)^2 + R}$$

b_i and c_i are the error transfer coefficients for unknown system errors and random errors respectively, and R is the influence of the covariance of errors.

Synthesis of system and random errors

If all errors are <u>independent</u>, we have:

$$\sigma_{overall} = \sqrt{\sum_{i=1}^{s}\left(b_i u_i\right)^2 + \sum_{i=1}^{q}\left(c_i \sigma_i\right)^2}$$

If the experiment is repeated for n times, we have:

$$\boxed{\sigma_{overall} = \sqrt{\sum_{i=1}^{s}\left(b_i u_i\right)^2 + \frac{1}{n}\sum_{i=1}^{q}\left(c_i \sigma_i\right)^2}}$$

Example 3-3

On universal tool microscope, using imaging method, the length of an object is measured twice as $l_1 = 50.026mm; \quad l_2 = 50.025mm$

the height of the object H=80mm, please calculate final result and extreme error.

The average of 2 measurements is:

$$\bar{l} = \frac{1}{2}(l_1 + l_2) = \frac{1}{2}(50.026 + 50.025) = 50.0255mm$$

According to the graduation error table of the optical ruler we can know that at 50mm the correction C=-0.0008mm, which should be treated as a known system error, and the length can be corrected as:

$$L = \bar{l} + C = 50.0255 - 0.0008 = 50.0247mm$$

For measurement using image method on a universal tool microscope, the errors are given or analyzed as follows:

(1) Random errors, caused by reading and aiming errors

(a) reading extreme error $\delta_1 = \pm 0.8\mu m$

(b) aiming extreme error $\delta_2 = \pm 1\mu m$

Example 3-3 (continued)

(2) Unknown system errors. Their extreme errors are listed as follows:

(a) Abbe error:

$$e_1 = \pm\frac{HL}{4000} = \pm\frac{80\times 50}{4000}\mu m = \pm 1.0\mu m$$

(b) graduation error of the optical ruler without correction:

$$e_2 = \pm(1+\frac{1}{200})\mu m = \pm(1+\frac{50}{200})\mu m = \pm 1.25\mu m$$

(a) temperature error:

$$e_3 = \pm\frac{7L}{700}\mu m = \pm\frac{7\times 50}{700}\mu m = \pm 0.35\mu m$$

(b) correction error of the optical ruler if correction is applied:

$$e_4 = \pm 0.5\mu m$$

These 4 errors cannot cancel out and will not reduce as the number of measurement increases, so they are system error. However we only know the ranges of those errors instead of some known values, therefore they should be considered unknown system errors.

Example 3-3 (continued)

All errors in the measurement are summarized in the following table:

#	Causes of errors	Extreme Error (μm)		Remark
		Random Errors	Unknown System Errors	
1	Abbe error	—	± 1	
2	Graduation error of optical ruler	—	± 1.25	void if correction is applied
3	Temperature error	—	± 0.35	
4	Reading error	± 0.8	—	
5	Aiming error	± 1	—	
6	Correction error of optical ruler	—	± 0.5	void if no correction is applied

Example 3-3 (continued)

There are 2 situations to discuss:

Situation #1: correction is not available for the graduation error

The overall extreme error is synthesized without error #6 as follows:

$$\delta_{overall} = \pm\sqrt{\frac{1}{2}\sum_{i=1}^{2}\delta_i^2 + \sum_{i=1}^{3}e_i^{\,2}} = \pm\sqrt{\frac{1}{2}(0.8^2 + 1^2) + (1^2 + 1.25^2 + 0.35^2)}\ \mu m$$

$$= \pm 1.87\,\mu m \approx \pm 1.9\,\mu m$$

The final result can be expressed as: $L = (50.0255 \pm 0.0019)mm$

Situation #2: correction is available for the graduation error

The overall extreme error is synthesized without error #2 as follows:

$$\delta_{overall} = \pm\sqrt{\frac{1}{2}\sum_{i=1}^{2}\delta_i^2 + \sum_{i=1}^{3}e_i^{\,2}} = \pm\sqrt{\frac{1}{2}(0.8^2 + 1^2) + (1^2 + 0.35^2 + 0.5^2)}\ \mu m$$

$$= \pm 1.48\,\mu m \approx \pm 1.5\,\mu m$$

The final result can be expressed as: $L = (50.0247 \pm 0.0015)mm$

Example 3-4

Use TC328B scale and 3 standard weights to measure the mass of a stainless steel ball, a single measurement shows M=14.0040g, please calculate the standard deviation of the measurement.

According to TC328B scale and its measuring method, the major errors are analyzed as follows:

(1) Random error caused by the disturbance of the indication

$$\sigma_1 = 0.05mg$$

(2) Unknown system errors

a) Standard weight error (standard deviation u_1). There are total 3 standard weights, 1 weight of 10g and 2 weights of 2g, and their standard deviations are: $u_{11} = 0.4mg; u_{12} = 0.2mg$

$$u_1 = \sqrt{u_{11}^2 + 2u_{12}^2} = \sqrt{0.4^2 + 2\times 0.2^2}mg \approx 0.5mg$$

b) Indication error of the scale (standard deviation u_2).

$$u_2 \approx 0.03mg$$

Example 3-4 (continued)

The above mentioned random error and unknown system errors are independent, and their error transfer coefficients are 1s. Therefore, the synthesized overall standard deviation can be calculated as follows:

$$\sigma = \sqrt{\sigma_1^2 + u_1^2 + u_2^2} = \sqrt{0.05^2 + 0.5^2 + 0.03^2}\,mg \approx 0.5mg$$

The final result can be expressed as: (3σ)

$$M = (14.0040 \pm 0.0015)g$$

Section5 Assignment of errors 误差的分配

Given allowable overall error, sometime we need to reasonably assign each individual errors. Because known system error can be removed via correction, only random errors and unknown system errors need to be considered. When assigning errors, the unknown system errors are treated just like random errors. Assume all errors are random errors and independent to each other, then we have:

$$\sigma_y = \sqrt{{a_1}^2{\sigma_{x1}}^2 + {a_2}^2{\sigma_{x2}}^2 + \cdots + {a_n}^2{\sigma_{xn}}^2} \qquad \sigma_y = \sqrt{D_1^2 + D_2^2 + \cdots + D_n^2}$$

$$D_i = \frac{\partial f}{\partial x_i}\sigma_i = a_i\sigma_i$$

are called partial errors of the function, or <u>local errors</u> (局部误差).

Now the question becomes: given σ_y , how to assign D_i or corresponding σ_i satisfying

$$\sigma_y \ge \sqrt{D_1^2 + D_2^2 + \cdots + D_n^2} = \sqrt{\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2}$$

Assignment of errors 误差分配

When assigning errors, we have the following steps:

(1) Assign errors according to equal contribution rule.

按等作用原则分配误差

We make each partial error contribute equally to the overall error, i.e.

$$D_1 = D_2 = \cdots = D_n = \frac{\sigma_y}{\sqrt{n}}$$

$$D_i = \frac{\partial f}{\partial x_i}\sigma_i = a_i\sigma_i \quad \Rightarrow \quad \boxed{\sigma_i = \frac{\sigma_y}{\sqrt{n}}\frac{1}{\partial f/\partial x_i} = \frac{\sigma_y}{\sqrt{n}}\frac{1}{a_i}}$$

In terms of extreme error, we have:

$$\boxed{\delta_i = \frac{\delta}{\sqrt{n}}\frac{1}{\partial f/\partial x_i} = \frac{\delta}{\sqrt{n}}\frac{1}{a_i}}$$

- δ is the overall extreme error of the function
- δ_i is the extreme error of each individual error

Assignment of errors 误差分配

(2) Adjust errors according to their possibilities **按可能性调整误差**

Shortcomings of equal contribution rule:

1. Assigning partial errors based on equal contribution rule will have the following undesirable consequences: some of the error requirements are very easy to meet, but others are very hard to achieve, and as a result, more expensive and accurate instruments have to be used, and the experiment will need more experiments and cost more money.
2. When partial contributions are equal the corresponding error of the measurement may not be equal because the error transfer coefficients are not the same, and sometimes the differences could be very big.

From above discussion, we should make some adjustment based on the result of equal contribution assignment. For measurements whose error requirement is easy to meet, the error should be reduced, and on the contrary, for measurements whose error requirement is difficult to achieve, the error should be increased, and others keep unchanged.

Assignment of errors 误差分配

(3) Check the overall error after the adjustment 验算调整后的误差

After the errors are assigned the actual overall error should be synthesized and verified with respect to the final allowable range. If the overall error is out of the permitted range, reducible error should be further reduced; If the overall error is too small, increase the error of difficult measurement accordingly. **误差分配后，应进行总误差的合成并根据最终允许的范围验证总体误差。如果总误差超出允许的范围，能缩小的误差应再缩小；如果总误差很小，增大难测量的变量的误差。**

Please note when assigning errors with equal contribution rule: if some of the errors are fixed and unable to be changed, its contribution should be deducted from the allowable overall error, and then assign the remaining errors. **按等作用原则分配误差时需注意，当有的误差已经确定而不能改变时，应先从给定的允许总误差中去除，然后再对剩余的误差进行误差分配。**

Example 3-5

When measuring the volume of a cylinder, we can indirectly measure the diameter D and height h of the cylinder and calculate the volume according to the function: $V = \frac{\pi D^2}{4} h$

If the nominal value of diameter and height are D_0=20mm and h_0=50mm respectively, and if the relative error of the volume needs to be less than 1%, please determine the precision of diameter D and height h.

Answer:

Calculate the nominal value of the volume:

$$V_0 = \frac{\pi D_0^2}{4} h_0 = \frac{3.1416 \times 20^2}{4} \times 50 = 15708\text{mm}^3$$

Calculate the absolute error of the volume:

$$\delta_V = V_0 \times 1\% = 15708\text{mm}^3 \times 1\% = 157.08\text{mm}^3$$

Example 3-5 (continued)

(1) Assign errors according to equal contribution rule

We have the extreme error of the diameter D and height h as:

$$\delta_D = \frac{\delta_V}{\sqrt{n}} \frac{1}{\frac{\partial V}{\partial D}} = \frac{\delta_V}{\sqrt{n}} \frac{2}{\pi Dh} = 0.071\text{mm}$$

$$\delta_h = \frac{\delta_V}{\sqrt{n}} \frac{1}{\frac{\partial V}{\partial h}} = \frac{\delta_V}{\sqrt{n}} \frac{4}{\pi D^2} = 0.351\text{mm}$$

According to the specification of instruments, we can use caliper of 0.1mm division value to measure the height h=50mm, and within 50mm range the maximum error is 0.150mm. For the diameter D=20mm, we can use grade 2 micrometer, and within 20mm range, the maximum error is 0.013mm. If these 2 instruments are used, the overall extreme error of the volume is:

$$\delta_V = \pm\sqrt{\left(\frac{\partial V}{\partial D}\right)^2 \delta_D^2 + \left(\frac{\partial V}{\partial h}\right)^2 \delta_h^2} = \pm 51.36mm^3$$

$$|\delta_V| = 51.36mm^3 << 157.08mm^3$$

Example 3-5 (continue)

(2) Adjust the error assignment

The precision is a overkill, and we need to make adjustment to lower the precision. If only caliper of 0.05mm division value is used for both diameter D and height h, and within 50mm range, the extreme error is 0.08mm. Although the extreme error of measurement of diameter D is bigger than the assigned error according to equal contribution rule, it can be compensated by the redundant precision from the measurement of height h.

(3) Check the overall error after the adjustment

The overall extreme error after the adjustment is given as follows:

$$\delta_V = \sqrt{\left(\frac{\pi Dh}{2}\right)^2 \delta_D^2 + \left(\frac{\pi D^2}{4}\right)^2 \delta_h^2} = 128.15mm$$

$$\left|\delta_V\right| = 128.15mm^3 < 157.08mm^3$$

Conclusion: we need only 1 caliper of 0.05mm division value to measure the diameter and the height.

Section6 Rules to accept or ignore minor errors

第六节：微小误差取舍准则

When there are multiple errors in the measurements, some of the errors often have small contributions to the final error of the result. When the contribution of the error is small enough, it can be ignored, and such error is called minor error.

Given the standard deviation of the measurement:

$$\sigma_y = \sqrt{D_1^2 + D_2^2 + \cdots + D_{k-1}^2 + D_k^2 + D_{k+1}^2 + \cdots + D_n^2}$$

When a partial error D_k is removed:

$$\sigma_y' = \sqrt{D_1^2 + D_2^2 + \cdots + D_{k-1}^2 + D_{k+1}^2 + \cdots + D_n^2}$$

$if \ \sigma_y \approx \sigma_y'$ D_k is a minor error

Rules to accept or ignore minor errors

According to the rules of computation of significant digits, for measurement of normal precision, the error has 1 significant digit. In this circumstance, if the following condition is satisfied after a partial error is removed:

$$\sigma_y - \sigma_y' \leq (0.1 \sim 0.05)\sigma_y$$

Then we consider it has no effect on the computation of the final error. Solve the inequality we have:

$$D_k \leq (0.4 \sim 0.3)\sigma_y \quad or \quad D_k \leq \frac{1}{3}\sigma_y$$

For measurement with higher precision, 2 significant digits are used for the error, and as a result we have:

$$\sigma_y - \sigma_y' \leq (0.01 \sim 0.005)\sigma_y$$

$$D_k \leq (0.14 \sim 0.1)\sigma_y \quad or \quad D_k \leq \frac{1}{10}\sigma_y$$

Rules to accept or ignore minor errors

Conclusion:

For random error and unknown system error, the criteria for ignoring minor error is that the standard deviation of the ignored error should be less or equal to 1/3~1/10 of the overall standard deviation of the final result. 对于随机误差和未定系统误差，微小误差舍去准则是被舍去的误差必须必须小于或等于测量结果总标准差的**1/3**至**1/10**。

Rules of minor error are important for computing overall error and choosing instrument with higher precision. When computing or assigning errors, identified minor errors can be ignored for simplicity. When choosing higher precision standard instrument, its error should be between 1/10 and 3/10 of the error of the lower precision instrument which is to be examined. 微小误差取舍准则在总误差计算和选择高一级仪器上都有重要意义。计算或分配误差时，若发现微小误差可予以忽略，选择高一级精度的仪器时，其误差应为被检器具允许总误差的**1/10~3/10**。

Section7 Determine the best measuring method
第七节：最佳测量方案的确定

When the measurement result is related to multiple measuring factors, how to determine each factor in order to minimize the error of the final result.

Because known system error can be removed by correction, we only need to consider the influence of random errors and unknown system errors when discussing the best measuring method.

$$\sigma_y = \sqrt{\left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_{x1}{}^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_{x2}{}^2 + \cdots + \left(\frac{\partial f}{\partial x_n}\right)^2 \sigma_{xn}{}^2}$$

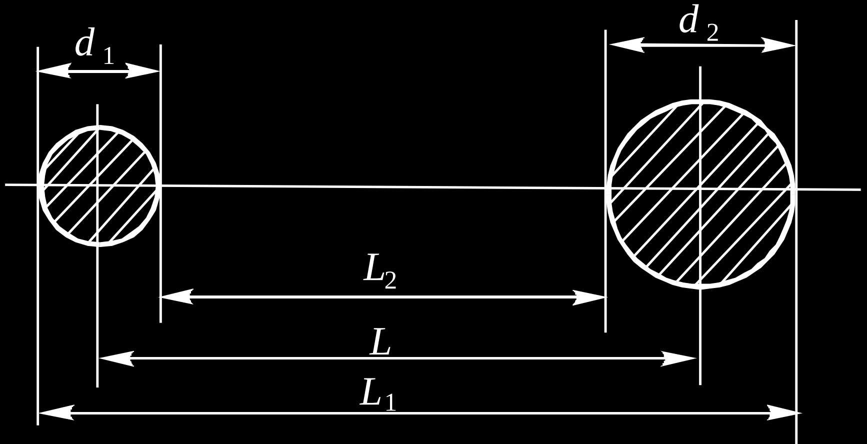
In order to minimize σ_y , we should consider some factors listed as follows:

Determine the best measuring method

1.) Select best function error equation 选择最佳函数误差公式:

For indirect measurement, if multiple functions are available, we select the function with the smallest number of direct measurements. If all functions have the same number of direct measurements, we should select the function with smaller direct errors.

To measure the distance between the centers of 2 axes, there are 3 methods:



Method 1: $L = L_1 - \frac{d_1}{2} - \frac{d_2}{2}$

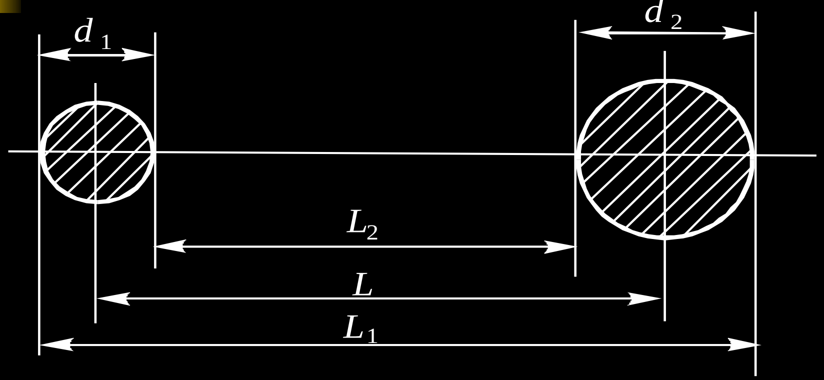
Method 2: $L = L_2 + \frac{d_1}{2} + \frac{d_2}{2}$

Method 3: $L = \frac{L_1}{2} + \frac{L_2}{2}$

$\sigma_{d1} = 0.5\mu m$ $\sigma_{d2} = 0.7\mu m$

$\sigma_{L1} = 0.8\mu m$ $\sigma_{L2} = 1.0\mu m$

Determine the best measuring method



$\sigma_{d1} = 0.5\mu m \quad \sigma_{d2} = 0.7\mu m$

$\sigma_{L1} = 0.8\mu m \quad \sigma_{L2} = 1.0\mu m$

Method 1:

$$\sigma_L = \sqrt{0.8^2 + \left(\frac{1}{2}\right)^2 0.5^2 + \left(\frac{1}{2}\right)^2 0.7^2} = 0.91\mu m$$

Method 2:

$$\sigma_L = \sqrt{1.0^2 + \left(\frac{1}{2}\right)^2 0.5^2 + \left(\frac{1}{2}\right)^2 0.7^2} = 1.09\mu m$$

Method 3:

$$\sigma_L = \sqrt{\left(\frac{1}{2}\right)^2 0.8^2 + \left(\frac{1}{2}\right)^2 1.0^2} = 0.64\mu m$$

Method 3 has the smallest error, and method 2 has the biggest error. That's because the function of method 3 is the simplest, but method 2 has more direct measurements.

Determine the best measuring method

2.) Minimize the error transfer coefficients 最小化误差传递系数:

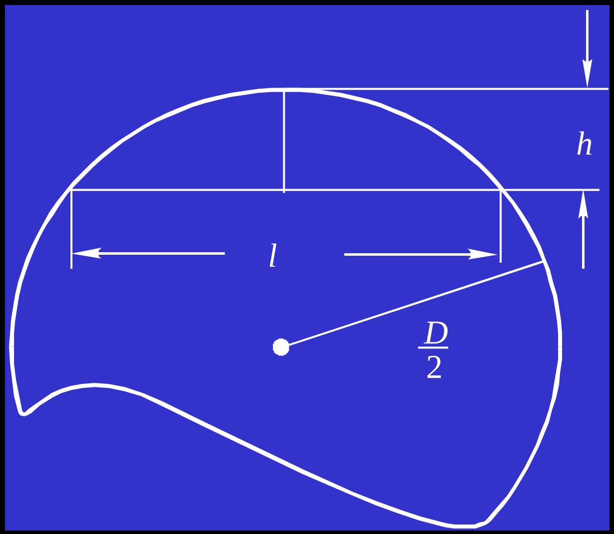
From the error equation of function, if we make the error transfer coefficients zero or minimized, the overall error of the function will be reduced accordingly.

Based on this principle, for some measurements, although we may not be able to make the error transfer coefficient zero, we can follow this direction to find the best measuring method.

Indirect measurement of the diameter by the height and span of an arc:

$$D = \frac{l^2}{4h} + h$$

Please determine the best method.



Determine the best measuring method

The error of the function $\sigma_D = \sqrt{\left(\frac{l}{2h}\right)^2 \sigma_l^2 + \left(\frac{l^2}{4h^2} - 1\right)^2 \sigma_h^2}$

In order to minimize σ_D , we can do the following:

1) Make $l/(2h) = 0$

To satisfy this we have $l = 0$ and $h = 0$, and it is meaningless.

2) Minimize $l/(2h)$

$2h$ should be maximized, i.e. the closer l is to diameter the better.

3) Make $l^2/(4h^2) - 1 = 0$

To satisfy this condition we have $l = 2h$, i.e. the diameter need to be measured directly so as to remove the influence of σ_h to the function error σ_D.

In order to minimize the function error we should measure the diameter directly, but sometime it is not feasible. From condition 2, we know h should be made as close to l/2 as possible.

Chapter 4: Uncertainty of measurement
第四章：测量不确定度

- Because of the existence of error, the true value of the object is almost impossible to be obtained 由于误差的存在，被测量的真值几乎不可能确定.
- The uncertainty of measurement describes the indetermination of the result, it is the estimate of the true value within some certain range, and it is a parameter of the result showing how dispersed the measurement is 测量的不确定度描述结果的不肯定性，是表征被测量的真值在某个量值范围的一个估计，是测量结果的一个参数，用以表示被测量值的分散性.
- A complete measurement result should include the estimate of the measurement and the parameter of dispersity 一个完整的测量结果应包含被测量值的估计与分散性参数两部分.

Result of measurement = Measurement estimate + Uncertainty

$$Y = y \pm U$$

Basic concepts of uncertainty
不确定度的基本概念

There are two ways to assess the uncertainty of measurement 测量不确定度有**2**种评定方法:

- Type A assessment: assess the uncertainty of measurement through statistic analysis of a number of observation data, e.g. for multiple measurements, calculate standard deviation σ or multiply σby a coefficient t to get extreme error;

  A类评定：通过对一系列观测值的统计分析来评定不确定度，例如，对于多次重复测量，计算标准差**σ**，或者把标准差乘以一个系数得到极限误差；

- Type B assessment: it is not based on statistic analysis but based on the probability distribution or distribution assumption determined by experience or other information which is necessary to assess the uncertainty.

  B类评定：不使用统计分析法，而是基于经验或者由其他信息决定的概率分布或分布假设来评定不确定度。

Basic concepts of uncertainty

The relationship between error and uncertainty:

Connections:

- They are all used to evaluate the precision of measurement;
- Both error and uncertainty can be presented by standard deviation, and they are caused either by random error or system error, which all affect the dispersity of measurements;
- Error is the foundation of uncertainty.

Differences:

- Error takes true value or conventional true value as the center, and uncertainty, however, uses the estimate of measured subject as the center;
- Error is normally unable to have a fixed quantity, but uncertainty can be assessed with some determined value;
- There are 3 types of error, and the classification is fuzzy and hard to differentiate; on the contrary, there are only 2 types of uncertainty, which is simpler.

Homework

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Question 3-12, 3-13, 3-14