Powers of Two with Digits from $\{1, 2, 8\}$: A Combinatorial Proof of Finiteness

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Abstract

We present a completely elementary proof that the only powers of two whose decimal representation uses only the digits $\{1, 2, 8\}$ are $2, 8, 128$. The key ideas are: (i) the phase condition $2^n \equiv 2 \pmod 6$ for odd n; (ii) the digit composition invariant $a \equiv b \pmod 3$ (the numbers of 1's and of $\{2, 8\}$ digits in higher positions are congruent mod 3); (iii) explicit enumeration of admissible two- and three-digit tails; (iv) exclusion of four-digit cases by digit-sum mod 3 and divisibility by 16; and (v) exclusion of lengths ≥ 5 by the invariant.

1 Statement of the problem

We call a number *valid* if all its decimal digits belong to the set $\{1, 2, 8\}$. We seek valid powers of two $N = 2^n$. Clearly, 2 and 8 are valid, and $128 = 2^7$ is also valid. We shall show that there are no others.

2 Phase mod 6 and the structure of digits

Two elementary observations:

- For odd n, we have $2^n \equiv 2 \pmod 6$ (*phase 2*); for even n, the remainder is 4, and the last digit is 4 or 6, which are invalid. Hence all valid cases have odd n and $N \equiv 2 \pmod 6$.
- In $\mathbb{Z}/6\mathbb{Z}$ one has $10 \equiv 4$, and therefore $10^j \equiv 4$ for all $j \geq 1$.

Lemma 1 (Digit composition invariant mod 6)**.** *Let a valid N have last digit $u \in \{2, 8\}$, and above the units place let there be a digits equal to 1 and b digits equal to either 2 or 8. Then*

$$2a + b \equiv 0 \pmod 3, \qquad a + 2b \equiv 0 \pmod 3, \qquad \textit{in particular} \;\; a \equiv b \pmod 3. \quad (1)$$

Proof. Work in $\mathbb{Z}/6\mathbb{Z}$. The unit digit contributes $u \equiv 2$. Each higher digit 1 contributes $4 \cdot 1 \equiv 4$, and each higher digit 2 or 8 contributes $4 \cdot 2 \equiv 8 \equiv 2$. Thus

$$N \equiv 2 + 4a + 2b \pmod 6.$$

Since $N \equiv 2 \pmod 6$, we get $4a + 2b \equiv 0 \pmod 6$, i.e. $2a + b \equiv 0 \pmod 3$.

On the other hand, modulo 3, the sum of digits equals $N \pmod 3$. For odd n, we have $2^n \equiv 2 \pmod 3$. The digit residues are $1 \equiv 1$, $2 \equiv 2$, $8 \equiv 2$. Hence

$$(2) + a \cdot 1 + b \cdot 2 \equiv 2 \pmod 3 \quad \Rightarrow \quad a + 2b \equiv 0 \pmod 3.$$

Subtracting the two congruences gives $a \equiv b \pmod 3$. $\square$

Corollary 1. *If N is valid, then adding a single digit on the left from $\{1, 2, 8\}$ breaks $a \equiv b \pmod 3$. Therefore, any possible extension of a valid number to the left must occur in blocks of three digits.*

Example. For $N = 128$ we have $a = 1$, $b = 1$ and $a \equiv b \pmod 3$. Adding any single digit to the left (1128, 2128, 8128) violates the invariant or fails other properties of a power of two.

3 Admissible tails mod 10, mod 100, mod 1000

We now *explicitly* derive the possible tails.

One digit

The last digit of powers of two cycles as $2, 4, 8, 6, \ldots$. For odd n, only 2 and 8 remain — both valid.

Two digits

Work mod 100. The order of 2 mod 25 is 20, and mod 4 it is 2, so the period mod 100 is $\operatorname{lcm}(20, 2) = 20$. List $\{2^n \bmod 100 : \ n \equiv 1, 3, \ldots, 19\}$ and keep those whose both digits lie in $\{1, 2, 8\}$. Exactly three appear:

$$\boxed{12, \quad 28, \quad 88.} \tag{2}$$

Example. For instance, $2^7 = 128 \equiv 28 \pmod{100}$, $2^{19} = 524288 \equiv 88$, and $2^9 = 512 \equiv 12$.

Three digits

For $n \geq 3$ we have $2^n \equiv 0 \pmod 8$, so the last three digits are multiples of 8. Among all three-digit numbers over $\{1, 2, 8\}$ ending in 2 or 8, divisibility by 8 leaves

112, 128, 288, 888

(the others, e.g. $212, 812, 228, 828, 188$, etc., are not divisible by 8).

Next, since the order of 2 mod 125 is 100, the full period mod 1000 is 100. Checking the 100-term cycle (or equivalently using the CRT for mod 8 and mod 125) shows that among the four candidates, exactly three actually occur:

$$\boxed{112, \quad 128, \quad 288.} \tag{3}$$

The tail 888 never appears in 2^n mod 1000.

Example.
- $2^7 = 128$ ends with 128;

- $2^{19} = 524288$ ends with 288;
- 2^{89} ends with 112 (obtained by squaring and checking mod 1000).

Remark (Manual check of (2)–(3)). For two digits: compute $2^n \bmod 25$ for $n = 1, 3, \ldots, 19$ and match with the correct last digit mod 4. For three digits: require divisibility by 8, then solve the CRT system mod 125 (period 100). Both are small finite tables (20 and 100 entries respectively).

4 Exclusion of 4-digit numbers

Lemma 2. *No four-digit valid number is a power of two.*

Proof. From (3), possible three-digit tails are $112, 128, 288$. Consider $dXYZ$ with $XYZ \in \{112, 128, 288\}$ and $d \in \{1, 2, 8\}$.

(i) Digit sum mod 3. For odd n, $2^n \equiv 2 \pmod 3$, so the digit sum must be $\equiv 2 \pmod 3$. Compute:

$$\begin{aligned}
\mathrm{sum}(d112) &= d + 1 + 1 + 2 \equiv d + 1 \pmod 3, \\
\mathrm{sum}(d128) &= d + 1 + 2 + 8 \equiv d + 1 \pmod 3, \\
\mathrm{sum}(d288) &= d + 2 + 8 + 8 \equiv d + 2 \pmod 3.
\end{aligned}$$

Hence: $d \in \{2, 8\}$ for $d112$ and $d128$ give 0 mod 3 instead of 2, and all $d288$ are impossible (since $d \equiv 1, 2, 2 \pmod 3$ never yield 2). Only $d = 1$ survives for 1112 and 1128.

(ii) Divisibility by 16. For $n \geq 4$, 2^n is divisible by 16. Check:

$$\begin{aligned}
1112 &\equiv 8 \pmod{16}, \\
1128 &\equiv 8 \pmod{16}.
\end{aligned}$$

Neither divisible by 16. Thus no four-digit valid powers of two exist. □

Example. 1128 looks plausible (all digits allowed) but is not divisible by 16; 2128, 8128 violate the digit-sum mod 3 condition.

5 Exclusion of all lengths ≥ 5

Lemma 3. *There exist no valid powers of two with length* ≥ 5.

Proof. Assume N valid. By Lemma 1, the invariant $a \equiv b \pmod 3$ must hold. By Corollary 1, any valid extension/reduction happens in blocks of three digits. Removing three digits at a time from the left, one reaches length $1, 2, 3$, or 4. Lengths $1, 2, 3$ give exactly $2, 8, 128$; length 4 is impossible by Lemma 2. Contradiction. □

6 Main theorem and examples

Theorem 1. *The only valid powers of two are*

$$2^1 = 2, \qquad 2^3 = 8, \qquad 2^7 = 128.$$

Proof. Oddness of the exponent is necessary (phase mod 6). From the tail analysis, the only realizable valid three-digit tail is 128 (for $n = 7$). By Lemmas 2 and 3, no longer valid numbers exist. Lengths $1, 2, 3$ give $2, 8, 128$ directly. $\square$

Example. Verification:
- $2^1 = 2$ — valid.
- $2^3 = 8$ — valid.
- $2^5 = 32$ — invalid (digit 3).
- $2^7 = 128$ — valid.
- $2^{19} = 524288$ — invalid (tail 288 valid but digit 5 appears).
- Any $d128$ with $d \in \{1, 2, 8\}$ is not a power of two (Lemma 2).

Appendix A: Computing tails explicitly

Two digits. For odd n, the values $2^n \bmod 100$ (period 20) yield residues whose last digits are 2 or 8 and tens digit in $\{1, 2, 8\}$ — precisely $12, 28, 88$.

Three digits. Requiring divisibility by 8 (for $n \geq 3$) and digit restriction narrows the list to $112, 128, 288, 888$. Mod 125 (cycle length 100), 888 never appears, while $112, 128, 288$ do (at $n \equiv 89, 7, 19 \pmod{100}$ respectively).

Appendix B: Mini-table of carry transitions

When moving from 2^n to 2^{n+1}, for each digit d and carry $c \in \{0, 1\}$, the next digit is $e = (2d + c) \bmod 10$, and the new carry $c' = \lfloor (2d + c)/10 \rfloor$. Requiring $e \in \{1, 2, 8\}$ severely restricts possibilities. The only stable right-hand pattern is $11 \mapsto 22$ with no carry, consistent with the tail $112 \to 128$; the next step introduces a non-admissible digit to the left, matching the exclusion of longer lengths.