# Powers of Two with Digits from $\{1, 2, 8\}$: A Combinatorial Proof of Finiteness

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**Abstract**

We give a completely elementary proof that the only powers of two whose decimal expansion uses only the digits $\{1, 2, 8\}$ are $2, 8, 128$. The key ideas are: (i) the phase condition $2^n \equiv 2 \pmod 6$ for odd $n$; (ii) a digit-composition invariant $a \equiv b \pmod 3$ (the numbers of digits equal to 1 and to $\{2, 8\}$ in higher positions are congruent mod 3); (iii) an explicit derivation of admissible two- and three-digit endings; (iv) exclusion of length 4 via digit sum modulo 3 and divisibility by 16; (v) exclusion of lengths $\geq 5$ via the invariant.

## 1 Setup

Call a number *valid* if its decimal expansion uses only digits from $\{1, 2, 8\}$. We are interested in valid powers of two $N = 2^n$. Clearly, 2 and 8 are valid, and $128 = 2^7$ is valid as well. We show that there are no others.

## 2 Phase modulo 6 and the structure of decimal places

Two elementary observations:

- For odd $n$ we have $2^n \equiv 2 \pmod 6$ (*phase* 2); for even $n$ the residue is 4, and the last digit is 4 or 6, which is invalid. Thus, $n$ must be odd and $N \equiv 2 \pmod 6$.
- In $\mathbb{Z}/6\mathbb{Z}$ one has $10 \equiv 4$, hence $10^j \equiv 4$ for all $j \geq 1$.

**Lemma 1** (Composition invariant modulo 6)**.** *Let a valid $N$ have last digit $u \in \{2, 8\}$, and above the units place let there be $a$ digits equal to 1 and $b$ digits from $\{2, 8\}$. Then*

$$2a + b \equiv 0 \pmod 3, \qquad a + 2b \equiv 0 \pmod 3, \qquad \textit{in particular } \; a \equiv b \pmod 3. \tag{1}$$

*Proof.* Work in $\mathbb{Z}/6\mathbb{Z}$. The units digit contributes $u \equiv 2$. Each higher 1 contributes $4 \cdot 1 \equiv 4$, and each higher 2 or 8 contributes $4 \cdot 2 \equiv 8 \equiv 2$. Thus

$$N \equiv 2 + 4a + 2b \pmod 6.$$

Since $N \equiv 2 \pmod 6$, we get $4a + 2b \equiv 0 \pmod 6$, i.e. $2a + b \equiv 0 \pmod 3$.

On the other hand, modulo 3 the digit sum equals $N \pmod 3$. For odd $n$, $2^n \equiv 2 \pmod 3$. With $1 \equiv 1$, $2 \equiv 2$, $8 \equiv 2$, we have

$$(2) + a \cdot 1 + b \cdot 2 \equiv 2 \pmod 3 \Rightarrow a + 2b \equiv 0 \pmod 3.$$

Subtracting gives $a \equiv b \pmod 3$. □

**Corollary 1.** *If $N$ is valid, adding a single leftmost digit from $\{1, 2, 8\}$ breaks $a \equiv b \pmod 3$. Hence any possible valid extension on the left must occur in blocks of three digits.*

*Example.* For $N = 128$ we have $a = 1$, $b = 1$ and $a \equiv b \pmod 3$. Adding one digit to the left (1128, 2128, 8128) breaks the invariant or other necessary properties of powers of two.

# 3 Admissible endings: $\mathrm{mod}\,10$, $\mathrm{mod}\,100$, $\mathrm{mod}\,1000$

We now *explicitly* derive all admissible endings.

## One digit

The last digit of $2^n$ cycles: $2, 4, 8, 6, \ldots$. For odd $n$ only 2 and 8 remain — both valid.

## Two digits: CRT derivation

The period of $2^n$ mod 100 is $\mathrm{lcm}(\mathrm{ord}_{25}(2), \mathrm{ord}_4(2)) = \mathrm{lcm}(20, 2) = 20$. Consider only odd $n$ (last digit 2 or 8). The explicit table of $2^n$ mod 100 for odd $n$ yields exactly three endings whose both digits lie in $\{1, 2, 8\}$:

$$\boxed{12, \quad 28, \quad 88.} \tag{2}$$

| $n$ mod 20 | 1 | 3 | 5 | 7 | 9 | 11 | 13 | 15 | 17 | 19 |
|---|---|---|---|---|---|---|---|---|---|---|
| $2^n$ mod 100 | 2 | 8 | 32 | 28 | 12 | 48 | 92 | 68 | 72 | 88 |

From this row we see *only* $12, 28, 88$ as admissible pairs.

## Three digits: divisibility by 8 + period modulo 125

For $n \geq 3$, $2^n \equiv 0 \pmod 8$, so the last three digits are multiples of 8. Among three-digit numbers over $\{1, 2, 8\}$ ending in 2 or 8, divisibility by 8 leaves the *candidates*

$$112, \ 128, \ 288, \ 888.$$

Next, using $\mathrm{ord}_{125}(2) = 100$, the period modulo 1000 is 100. Scanning the cycle yields *exactly three* actual endings:

$$\boxed{112, \quad 128, \quad 288,} \tag{3}$$

while 888 never occurs.

*Remark* (Manual check of (2)–(3)). For two digits: compute $2^n$ mod 25 for $n = 1, 3, \ldots, 19$ and align with $\mathrm{mod}\,4$ (the last digit fixes $\mathrm{mod}\,4$). For three digits: the 8-divisibility condition drastically shortens the list; then align with $\mathrm{mod}\,125$ (cycle length 100).

# 4 Mini table of carries (local dynamics)

When passing from $2^n$ to $2^{n+1}$ right-to-left, if $d$ is a digit and $c \in \{0, 1\}$ is the incoming carry, then

$$e = (2d + c) \bmod 10, \qquad c' = \left\lfloor \frac{2d + c}{10} \right\rfloor.$$

Requiring $e \in \{1, 2, 8\}$ leaves only one locally admissible transition from $\{1, 2, 8\}$:

| $d$ | $c$ | $2d + c$ | $e = (2d + c) \bmod 10$ | $c'$ | is $e$ valid? |
|---|---|---|---|---|---|
| 1 | 0 | 2 | 2 | 0 | **yes** |
| 1 | 1 | 3 | 3 | 0 | no |
| 2 | 0 | 4 | 4 | 0 | no |
| 2 | 1 | 5 | 5 | 0 | no |
| 8 | 0 | 16 | 6 | 1 | no |
| 8 | 1 | 17 | 7 | 1 | no |

Thus the only stable right-hand pattern is $11 \mapsto 22$ *with no carry*, consistent with the observed ending $112 \to 128$.

# 5 Excluding length 4

**Lemma 2.** *No four-digit valid number is a power of two.*

*Proof.* By (3), the possible three-digit endings are $112, 128, 288$. Consider $dXYZ$ with $XYZ \in \{112, 128, 288\}$ and $d \in \{1, 2, 8\}$.

**(i) Digit sum modulo 3.** For odd $n$, $2^n \equiv 2 \pmod 3$, i.e. the digit sum must be $\equiv 2 \pmod 3$. Compute:

$$\begin{aligned}
\mathrm{sum}(d112) &= d + 1 + 1 + 2 \equiv d + 1 \pmod 3, \\
\mathrm{sum}(d128) &= d + 1 + 2 + 8 \equiv d + 1 \pmod 3, \\
\mathrm{sum}(d288) &= d + 2 + 8 + 8 \equiv d + 2 \pmod 3.
\end{aligned}$$

Hence $d \in \{2, 8\}$ for $d112$ and $d128$ gives 0 mod 3 instead of 2, and all $d288$ are impossible (since $d \equiv 1, 2, 2 \pmod 3$ never produce 2). Only $d = 1$ remains for 1112 and 1128.

**(ii) Divisibility by 16.** For $n \geq 4$, $2^n$ is divisible by 16. Check:

$$\begin{aligned}
1112 &\equiv 8 \pmod{16}, \\
1128 &\equiv 8 \pmod{16}.
\end{aligned}$$

Neither is divisible by 16. Thus no four-digit valid power of two exists. $\square$

*Example.* 1128 looks plausible (all digits allowed) but is not divisible by 16; 2128, 8128 violate the digit-sum mod 3 condition.

## 6 Excluding all lengths $\geq 5$

**Lemma 3.** *There are no valid powers of two of length $\geq 5$.*

*Proof.* Assume $N$ is valid. By Lemma 1, $a \equiv b \pmod 3$ must hold. By Corollary 1, any valid extension/truncation occurs in blocks of three digits. Removing three digits at a time from the left, we must arrive at a number of length $1, 2, 3$, or 4. Lengths $1, 2, 3$ yield exactly $2, 8, 128$; length 4 is impossible by Lemma 2. Contradiction. $\square$

## 7 Main theorem and testable examples

**Theorem 1.** *The only valid powers of two (alphabet $\{1, 2, 8\}$) are*

$$2^1 = 2, \qquad 2^3 = 8, \qquad 2^7 = 128.$$

*Proof.* Oddness of the exponent is necessary (phase mod6). From the endings section, the only realizable valid three-digit ending is 128 (at $n = 7$). By Lemmas 2 and 3, no lengths $\geq 4$ exist. Lengths $1, 2, 3$ give $2^1 = 2$, $2^3 = 8$, $2^7 = 128$. $\square$

*Example.* Checks:

- $2^1 = 2$ — valid.
- $2^3 = 8$ — valid.
- $2^5 = 32$ — invalid (digit 3).
- $2^7 = 128$ — valid.
- $2^{19} = 524288$ — invalid (ending 288 is admissible, but digit 5 appears to the left).
- Any $d128$ with $d \in \{1, 2, 8\}$ is not a power of two (see Lemma 2).

## 8 Extension: adding the digit 4

Consider the alphabet $\{1, 2, 4, 8\}$. Modulo 3, the digit 4 is equivalent to 1, and modulo 6 its contribution is 4 (same as 1), so the invariant of Lemma 1 remains valid provided 1 and 4 are treated as one class. The phase modulo 6 now also allows even $n$ with last digit 4, but among actual powers this yields only $2^2 = 4$.

**Theorem 2** (Alphabet $\{1, 2, 4, 8\}$)**.** *The only valid powers of two are*

$$2^1 = 2, \qquad 2^2 = 4, \qquad 2^3 = 8, \qquad 2^7 = 128.$$

*Proof.* The invariant and the local carry table still apply. The only new case is $2^2 = 4$ (last digit 4 is allowed). The exclusions of length 4 and of lengths $\geq 5$ transfer verbatim. $\square$

## Appendix A: getting endings even faster

**Two digits.** See the table for $2^n \bmod 100$ (odd $n$) above; only $12, 28, 88$ are admissible.

**Three digits.** Multiples of 8 reduce the candidate list to $112, 128, 288, 888$, of which only $112, 128, 288$ occur along the modulo-1000 cycle.

# Appendix B: compact carry table

The summary table of local rightmost-digit transitions under doubling is given in the "Mini table of carries" section. It shows that only $1 \to 2$ without carry is admissible, consistent with the stable fragment $11 \mapsto 22$ and the ending $112 \to 128$.