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① Productivity Shocks in Three-Equation Model

log-linearized NK model

$\hat{y}_t = -\sigma\left(\hat{i}_t - E_t\{\hat{\pi}_{t+1}\}\right] + E_t\{\hat{y}_{t+1}\}$ Euler Equation ↘ Dynamic (Investment-Saving) Curve

$\hat{\pi}_t = \kappa(\hat{y}_t - \hat{y}_t^{flex}) + \beta E_t\{\hat{\pi}_{t+1}\}$ Pricing, MC equation → NK phillips curve

$\hat{i}_t = \phi_\pi \hat{\pi}_t + v_t$ central bank Monetary Rule.

where

$$\hat{y}_t^{flex} = \frac{1+\varphi}{\gamma+\varphi}\hat{a}_t \qquad v_t = 0 \qquad \hat{a}_t = \rho_a \hat{a}_{t-1} + \varepsilon_t$$

a) using method of undetermined coefficients, solve for $\hat{y}_t$ and $\hat{\pi}_t$ as a function of $\hat{a}_t$.

method of undetermined coefficient : used to find particular solutions to linear nonhomogeneous differential equation with constant coefficients.

$$\hat{y}_t = \eta_{ya}\hat{a}_t$$

$$\hat{\pi}_t = \eta_{\pi a}\hat{a}_t \quad \rightarrow \quad \hat{\pi}_{t+1} = \eta_{\pi a}\hat{a}_{t+1}$$

$$\hat{i}_t = \eta_{ia}\hat{a}_t$$

① $\underset{v_t=0}{\hat{i}_t = \phi_\pi \hat{\pi}_t + v_t} \Rightarrow \eta_{ia}\hat{a}_t = \phi_\pi \eta_{\pi a}\hat{a}_t + 0 \Rightarrow \boxed{\eta_{ia} = \phi_\pi \eta_{\pi a}}$

② $\hat{y}_t = -\sigma(\hat{i}_t - E_t\{\hat{\pi}_{t+1}\}) + E_t\{\hat{y}_{t+1}\} \Rightarrow \eta_{ya}\hat{a}_t = -\sigma\left(\eta_{ia}\hat{a}_t - \underbrace{E_t\{\eta_{\pi a}\hat{a}_{t+1}\}}_{\substack{= \eta_{\pi a}E_t[\hat{a}_{t+1}] \\ = \eta_{\pi a}\rho_a\hat{a}_t}}\right] + \underbrace{E_t\{\eta_{ya}\hat{a}_{t+1}\}}_{\substack{\eta_{ya}E_t[\hat{a}_{t+1}] \\ \eta_{ya}\rho_a\hat{a}_t}}$

$$\eta_{ya}\cancel{\hat{a}_t} = -\sigma\left(\eta_{ia}\cancel{\hat{a}_t} - \eta_{\pi a}\rho_a\cancel{\hat{a}_t}\right) + \eta_{ya}\rho_a\cancel{\hat{a}_t}$$

$$\eta_{ya} = -\sigma(\eta_{ia} - \eta_{\pi a}\rho_a) + \eta_{ya}\rho_a$$

$$\eta_{ya} - \eta_{ya}\rho_a = -\sigma(\eta_{ia} - \eta_{\pi a}\rho_a)$$

$$\eta_{ya} = \frac{-\sigma(\eta_{ia} - \eta_{\pi a}\rho_a)}{(1-\rho_a)} \Rightarrow \frac{-\sigma(\phi_t\eta_{\pi a} - \eta_{\pi a}\rho_a)}{1-\rho_a} \Rightarrow -\sigma\eta_{\pi a}\frac{(\phi_t - \rho_a)}{(1-\rho_a)}$$

$$\boxed{\eta_{ya} = -\sigma\eta_{\pi a}\frac{(\phi_t - \rho_a)}{(1-\rho_a)}}$$

(3) $\hat{\pi}_t = \kappa(\hat{y}_t - \hat{y}_t^{flex}) + \beta E_t\{\hat{\pi}_{t+1}\}$

$$\eta_{\pi a}\hat{a}_t = \kappa\left(\eta_{ya}\hat{a}_t - \frac{1+\varphi}{\gamma+\varphi}\hat{a}_t\right) + \beta \underbrace{E_t\{\eta_{\pi a}\hat{a}_{t+1}\}}$$

$$= \eta_{\pi a} E[\hat{a}_{t+1}]$$
$$= \eta_{\pi a} \rho_a \hat{a}_t$$

$$\eta_{\pi a}\cancel{\hat{a}_t} = \kappa\left(\eta_{ya}\cancel{\hat{a}_t} - \frac{1+\varphi}{\gamma+\varphi}\cancel{\hat{a}_t}\right) + \beta\,\eta_{\pi a}\rho_a\cancel{\hat{a}_t}$$

$$\eta_{\pi a} = \kappa\left(\eta_{ya} - \frac{1+\varphi}{\gamma+\varphi}\right) + \beta\,\eta_{\pi a}\,\rho_a$$

$$\eta_{\pi a} - \beta\,\eta_{\pi a}\rho_a = \kappa\left(\eta_{ya} - \frac{1+\varphi}{\gamma+\varphi}\right)$$

$$\boxed{\eta_{\pi a} = \frac{\kappa\left(\eta_{ya} - \frac{1+\varphi}{\gamma+\varphi}\right)}{(1-\beta\rho_a)}}$$

- Plug in $\eta_{\pi a}$ into η_{ya} to get η_{ya} as function of $\hat{a}_t$

$$\eta_{ya} = -\sigma\,\eta_{\pi a}\left(\frac{\phi_t - \rho_a}{1-\rho_a}\right)$$

$$= -\sigma\left[\frac{\kappa\left(\eta_{ya} - \frac{1+\varphi}{\gamma+\varphi}\right)}{1-\beta\rho_a}\right]\left(\frac{\phi_t-\rho_a}{1-\rho_a}\right)$$

$$= -\sigma\left(\frac{\kappa\eta_{ya} - \kappa\left(\frac{1+\varphi}{\gamma+\varphi}\right)}{1-\beta\rho_a}\right)\left(\frac{\phi_t-\rho_a}{1-\rho_a}\right) \cdot (1-\beta\rho_a)(1-\rho_a)$$

$$= -\sigma\left(\kappa\eta_{ya} - \kappa\left(\frac{1+\varphi}{\gamma+\varphi}\right)\right)\left(\frac{\phi_t-\rho_a}{(1-\rho_a)(1-\beta\rho_a)}\right)$$

$$= -\sigma\kappa\eta_{ya} + \sigma\kappa\left(\frac{1+\varphi}{\gamma+\varphi}\right)\left(\frac{\phi_t-\rho_a}{(1-\rho_a)(1-\beta\rho_a)}\right)$$

$$= -\sigma\kappa\eta_{ya}\left(\frac{\phi_t-\rho_a}{(1-\rho_a)(1-\beta\rho_a)}\right) + \sigma\kappa\left(\frac{1+\varphi}{\gamma+\varphi}\right)\left(\frac{\phi_t-\rho_a}{(1-\rho_a)(1-\beta\rho_a)}\right)$$

$$\eta_{ya} + \sigma\kappa\eta_{ya}\left(\frac{\phi_t-\rho_a}{(1-\rho_a)(1-\beta\rho_a)}\right) = \sigma\kappa\left(\frac{1+\varphi}{\gamma+\varphi}\right)\left(\frac{\phi_t-\rho_a}{(1-\rho_a)(1-\beta\rho_a)}\right)$$

$$\eta_{ya} = \frac{\sigma\kappa\left(\frac{1+\varphi}{\gamma+\varphi}\right)\left(\frac{\phi_t-\rho_a}{(1-\rho_a)(1-\beta\rho_a)}\right)}{\left(1+\sigma\kappa\eta_{ya}\left(\frac{\phi_t-\rho_a}{(1-\rho_a)(1-\beta\rho_a)}\right)\right)} = \frac{\sigma\kappa\left(\frac{1+\varphi}{\gamma+\varphi}\right)(\phi_t-\rho_a)}{(1-\beta\rho_a)(1-\rho_a)+\sigma\kappa(\phi_t-\rho_a)} = \eta_{ya}$$

$$\hat{y}_t = \eta_{ya}\hat{a}_t = \frac{\sigma\kappa\left(\frac{1+\varphi}{\gamma+\varphi}\right)(\phi_t - \rho_a)}{(1-\beta\rho_a)(1-\rho_a) + \sigma\kappa(\phi_t - \rho_a)}\hat{a}_t$$

- plug in η_{ya} into $\eta_{\pi a}$ to get $\eta_{\pi a}$ as function of $\hat{a}_t$

$$\eta_{\pi a} = \frac{\kappa\left(\eta_{ya} - \frac{1+\varphi}{\gamma+\varphi}\right)}{(1-\beta\rho_a)} = \frac{\kappa\eta_{ya} - \kappa\left(\frac{1+\varphi}{\gamma+\varphi}\right)}{(1-\beta\rho_a)}$$

$$\eta_{\pi a} = \frac{-\left(\frac{1+\varphi}{\gamma+\varphi}\right)\kappa(1-\rho_a)}{(1-\rho_a)(1-\beta\rho_a) + \sigma(\phi_\pi - \rho_a)\kappa}$$

$$\hat{\pi}_t = \eta_{\pi a}\hat{a}_t = \frac{-\left(\frac{1+\varphi}{\gamma+\varphi}\right)\kappa(1-\rho_a)}{(1-\rho_a)(1-\beta\rho_a) + \sigma(\phi_\pi - \rho_a)\kappa} \cdot \hat{a}_t$$

b) plot the impulse response function for list to a one unit shock to $\hat{a}_t$

- see python file

c.) Intuitively explain results

shock to productivity increases output, decreases labor, increases optimal y, output gap temporarily decreases. Interest rates and real rates decrease and banks respond through MPR (taylor rule).

d.) see code.

Impulse Response functions

- used to analyze dynamic effects of a shock to one of the variables in the system on the current and future values of all the variables in the model.

VAR Models

- captures linear interdependencies among multiple time series.

$$Y_t = A_1 Y_{t-1} + A_2 Y_{t-2} + \cdots + A_p Y_{t-p} + u_t$$

Y_t : vector of n-endogenous variables

A_i : matrices of coefficients

u_t : vector of error terms

IRF

- describes the reaction of the system to a shock in one of the error terms u_t.

Steps

1. estimate VAR model → SVAR
2. identify structural shock : sometimes shocks need to be orthogonalized (made independent of one another) — by Cholesky decomposition
3. simulate impulse response : calculate response of each var to the shock at each future time period.

Math $\quad \mathrm{IRF}_j(h) = \dfrac{\partial Y_{t+h}}{\partial u_{j,t}}$: a shock at time t to the j^{th} variable

IRF at horizon h

$\hat{y}_t$, output

$\hat{\pi}_t$, inflation

$\hat{y}_t^{flex}$, output with flexible prices

$\hat{y}_t - \hat{y}_t^{flex}$ gaps in output

$\hat{i}_t$ nominal interest rate

$\mathbb{E}_t \hat{r}_{t+1}$ fisher : relationship between inflation and both real and nominal interest rate

n_t labor $\quad E_t\{R_{t+1}\} \equiv E_t\{Q_t P_t / P_{t+1}\}$

a_t productivity

Dynamic Investment Savings Curve

- represents the relationship between output gap and the real interest rate in an economy over time.
- aggregate demand depends negatively on the real interest rate and positively on future expected output.
- provides insight into how economic activity is influenced by interest rates and expectations.

$$\hat{y}_t = -\sigma\left(\hat{i}_t - E_t\{\pi_{t+1}\}\right) + E_t\{\hat{y}_{t+1}\}$$

$\hat{y}_t$	output level
$-\sigma$	intertemporal rate of substitution
$\hat{i}_t$	nominal interest rate
$E_t\{\pi_{t+1}\}$	expected inflation for next period
$E_t\{\hat{y}_{t+1}\}$	expected output in next period

New Keynesian Phillips Curve

- relationship that describes how current inflation is determined by expected future inflation and the level of economic activity, typically measured by the output gap.

$$\hat{\pi}_t = K(\hat{y}_t - \hat{y}_t^{flex}) + \beta E_t\{\hat{\pi}_{t+1}\}$$

$\hat{\pi}_t$	inflation rate at time t
K	parameter reflecting sensitivity of inflation to output gap.
$(\hat{y}_t - \hat{y}_t^{flex})$	output gap at time t
β	discount factor (time preference)
$E_t\{\hat{\pi}_{t+1}\}$	expected inflation rate for next period

Expected Future Inflation : firms set prices based on what they expect future inflation to be, making expectations crucial

Output gap : when the economy is producing above its potential (positive gap), there is upward pressure on prices (inflation). Conversely, negative output gap exerts downward pressure on prices.

Monetary Policy Rule

- central banks set nominal interest rate according to an interest Rate rule. (taylor Rule).
- systematic way for central banks to set nominal interest rate based on economic conditions (particularly inflation and output gap). Helps stabilize the economy by influencing aggregate demand through interest rate adjustment.

$$\hat{i}_t = \phi_\pi \hat{\pi}_t + v_t$$

$\hat{i}_t$ nominal interest rate
ϕ_π response coefficient to the output gap.
$\hat{\pi}_t$ inflation rate at time t
v_t monetary policy shock

NKPC: describes how inflation evolves based on expectations and economic slack output gap
MPR: describes how the central bank sets the interest rate in response to deviations of inflation from the target and output gap

NKPC + MPR + DIS describes how the output gap is influenced by real interest rates and expectations.

Parameters β σ κ ρ_a ϕ_π γ

σ

β

κ

ϕ_π

ϕ_y

② Non linear NK Model in Jupyter

a.) real reset price equation for firm:

$$p_t^* \equiv \frac{P_t^*}{P_t} = (1+\mu) E_t \left\{ \sum_{s=0}^{\infty} \frac{\theta^s \Lambda_{t,t+s} Y_{t+s} \left(\frac{P_{t+s}}{P_t}\right)^{\varepsilon-1}}{\sum_{k=0}^{\infty} \theta^k \Lambda_{t,t+k} Y_{t+k} \left(\frac{P_{t+k}}{P_t}\right)^{\varepsilon-1}} \frac{W_{t+s}/P_t}{A_{t+s}} \right\}$$

Why is this expression not recursive?

we have the left side is optimum reset price and RHS is some expectation of non optimal price reset so this isnt recursive in that the RHS doesnt have an expression for the continuation of "optimal" price reset

basically we need $\hat{p}_{t+1}^*$ on RHS.

NOW, WTS: $B_t = E_t \left(\frac{F_{1t}}{F_{2t}} \right)$ where F_{1t} and F_{2t} are recursive.

$\Lambda_{t,t,k} = \Lambda_{t,t+1} \Lambda_{t+1,t+k}$

how I discount today versus k = how I discount today versus tomorrow x how I discount tomorrow versus k.

b.) show F_{2t} can be written $Y_t + \theta \Pi_{t+1}^{\varepsilon-1} \Lambda_{t,t+1} F_{2,t+1}$

$$F_{2t} \equiv \sum_{k=0}^{\infty} \theta^k \Lambda_{t,t+k} Y_{t+k} \left(\frac{P_{t+k}}{P_t} \right)^{\varepsilon-1}$$

$$= \underset{\theta^0=1}{\theta^0} \underset{\Lambda_{t,t}=1}{\Lambda_{t,t+0}} \underset{Y_t}{Y_t} \left(\underset{\frac{P_t}{P_t}=1}{\frac{P_{t+0}}{P_t}} \right)^{\varepsilon-1} + \sum_{k=1}^{\infty} \theta^k \Lambda_{t,t+k} Y_{t+k} \left(\frac{P_{t+k}}{P_t} \right)^{\varepsilon-1}$$

$$= Y_t + \sum \theta^k \Lambda_{t,t+k} Y_{t+k} \left(\frac{P_{t+k}}{P_t} \right)^{\varepsilon-1}$$

$$= Y_t + \sum_{k=1}^{\infty} \theta^k \Lambda_{t,t+1} \Lambda_{t+1,t+k} Y_{t+k} \left(\frac{P_{t+k}}{P_t} \right)^{\varepsilon-1}$$

$$= Y_t + \sum_{k=0}^{\infty} \theta^{k+1} \Lambda_{t,t+1} \Lambda_{t+1,t+k+1} Y_{t+k+1} \left(\frac{P_{t+k+1}}{P_t} \right)^{\varepsilon-1}$$

$$= Y_t + \theta^1 \Lambda_{t,t+1} \sum_{k=0}^{\infty} \theta^k \Lambda_{t+1,t+k+1} Y_{t+k+1} \left(\frac{P_{t+k+1}}{P_t} \right)^{\varepsilon-1}$$

$$= Y_t + \theta' \Lambda_{t,t+1} \sum_{k=0}^{\infty} \theta^k \Lambda_{t+1,t+k+1} Y_{t+k+1} \left(\frac{P_{t+k+1}}{P_t} \cdot \frac{P_{t+1}}{P_{t+1}} \right)^{\varepsilon-1}$$

$$= \frac{P_{t+k+1}}{P_{t+1}} \quad \frac{P_{t+1}}{P_t}$$

$$= Y_t + \theta \Lambda_{t,t+1} \left(\frac{P_{t+1}}{P_t}\right)^{\varepsilon-1} \sum_{k=0}^{\infty} \theta^k \Lambda_{t+1,t+k+1} Y_{t+k+1} \left(\frac{P_{t+k+1}}{P_{t+1}}\right)^{\varepsilon-1}$$

$$= Y_t + \theta \Lambda_{t,t+1} \Pi_{t+1}^{\varepsilon-1} F_{2,t+1}$$

c.) Show $F_{1t} \equiv (1+\mu) \sum_{s=0}^{\infty} \theta^s \Lambda_{t,t+s} Y_{t+s} (P_{t+s}/P_t)^{\varepsilon-1} \dfrac{W_{t+s}/P_t}{A_{t+s}}$

$$\equiv (1+\mu)\left[\theta^0 \Lambda_{t,t+0} Y_{t+0} \left(\frac{P_{t+0}}{P_t}\right)^{\varepsilon-1} \frac{W_{t+0}/P_t}{A_{t+0}} + \sum_{s=1}^{\infty} (*)\right.$$

$$\equiv (1+\mu)\left[Y_t \frac{W_t/P_t}{A_t} + \underbrace{\sum_{s=1}^{\infty} \theta^s \Lambda_{t,t+s} Y_{t+s} \left(\frac{P_{t+s}}{P_t}\right)^{\varepsilon-1} \frac{W_{t+s}/P_t}{A_{t+s}}}_{\sum_{s=0}^{\infty} \theta^{s+1} \underbrace{\Lambda_{t,t+s+1}}_{\Lambda_{t,t+1} \cdot \Lambda_{t+1,t+s+1}} Y_{t+s+1} \left(\frac{P_{t+s+1}}{P_{t+1}}\right)^{\varepsilon-1} \frac{W_{t+s+1}/P_t}{A_{t+s+1}}}\right]$$

$$\equiv (1+\mu)\left[Y_t \frac{W_t/P_t}{A_t} \Lambda_{t,t+1}\, \theta \sum_{s=1}^{\infty} \theta^s \Lambda_{t+1,t+s+1} \left(\frac{P_{t+s+1}}{P_{t+1}} \cdot \frac{P_{t+1}}{P_t}\right)^{\varepsilon-1} \frac{W_{t+s+1}/P_t}{A_{t+s+1}}\right.$$

$$\equiv (1+\mu)\left[Y_t \frac{W_t/P_t}{A_t} \Lambda_{t,t+1} \left(\frac{P_{t+1}}{P_t}\right)^{\varepsilon} F_{1,t+1}\right]$$

$$\equiv (1+\mu) \quad Y_t \frac{W_t/P_t}{A_t} \Lambda_{t,t+K} \Pi_{t+1}^{\varepsilon} F_{1,t+1}$$

d.) Show gross inflation can be written as $1 = \theta \Pi_t^{\varepsilon-1} + (1-\theta) P_t^{*\,1-\varepsilon}$

$$P_t = \left[\theta P_{t-1}^{1-\varepsilon} + (1-\theta) P_t^{*\,1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}$$

$$P_t^{1-\varepsilon} = \theta P_{t-1}^{1-\varepsilon} + (1-\theta) P_t^{*\,1-\varepsilon}$$

$$1 = \theta \left(\frac{P_{t-1}}{P_t}\right)^{1-\varepsilon} + (1-\theta)\left(\frac{P_t^*}{P_t}\right)^{1-\varepsilon}$$

$$= \theta \left(\frac{P_t}{P_{t-1}}\right)^{\varepsilon-1} + (1-\theta)\left(\frac{P_t^*}{P_t}\right)^{1-\varepsilon}$$

$$1 = \theta \Pi_t^{\varepsilon-1} + (1-\theta) B_t^{1-\varepsilon}$$

e.) explain how $p_t^* > 1$, then $\Pi_t > 1$

if $p_t^* > 1$: optimal price > price firm is charging

when $p^* > p$ inflation will be > 1

f/g see code

h how does IRF depend on value of θ

higher θ means more sticky prices.
output gap is larger when θ is higher.
θ is higher since prices cant move
in response to change in productivity.

i what would you see from same shock in RBC
with no capital?

In RBC model, prices arent sticky so output and
consumption adjust so there will be no output gap.