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MACRO (ECON 210 C)
Problem set # 1
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① take the costly state verification model we developed in class. Imagine two contracts:

$$P = \begin{cases} D & \text{if } Y \geq D \\ Y & \text{if } Y < D \end{cases} \qquad P' = \begin{cases} D' & \text{if } Y > D' \\ f(Y) < Y & \text{if } Y \leq D' \end{cases}$$

Show that the first contract is preferable to the second contract

(Notes)

Costly State verification Model:

- Entrepreneur has a project that requires 1 unit of resources and has wealth $w<1$, must obtain $1-w$ of outside financing
- outside investor bears some risk
- Socially efficient to undetake project if $\gamma > 1+r$ (γ: expected output)
- entrepreneur undertakes project if

  Y = actual output

  $$E[Y-P] > (1+r)w \qquad Y \sim U(0, 2\gamma)$$

  E[output - payment to investor] > (outside option)*(their wealth)

- Investor: projects with $\gamma \geq 1+r$ gets financing, others dont.
  - ↳ expected payment: $(1+r)\underbrace{(1-w)}$, w is entrepreneurs wealth
    - ↳ so $(1-w)$ is amount outside investor will invest.
- Equity contract: outside investor gets fraction $\frac{(1+r)(1-w)}{\gamma}$ of output.

$$E\left[\underbrace{\frac{(1+r)(1-w)}{\gamma}}_{\text{constant}} Y\right] = (1+r)(1-w)$$

$$= \frac{(1+r)(1-w)}{\gamma} \cdot E[Y] = (1+r)(1-w)$$

↳ expected output $= \gamma$

$$= \frac{(1+r)(1-w)}{\gamma} \cdot \gamma = (1+r)(1-w)$$

$$= (1+r)(1-w) = (1+r)(1-w)$$

- Entrepreneur's expected income is

  $$\gamma - (1+r)(1-w)$$

  $$= \gamma - (1+r) + (1+r)w > (1+r)w$$ how much entrepreneur is investing + return

  if $\gamma \geq (1+r)$

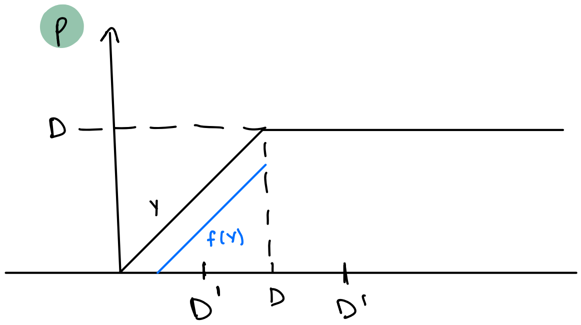
- Outside investor's expected payment: $(1+r)(1-w)$ + expected cost of verifying
- Optimal contract: Simple debt contract
  - if $Y \geq D$, investor gets D, doesnt verify
  - if $Y < D$, investor verifies output and gets Y

Answer starts here

$P = \begin{cases} D & \text{if } Y \geq D \\ Y & \text{if } Y < D \end{cases}$ —

$P' = \begin{cases} D' & \text{if } Y > D' \\ f(Y) < Y & \text{if } Y \leq D' \end{cases}$ → $f(y) < y$ if default then investor is making less

D



$$E[P] = D \cdot P(y \geq D) + E(y \mid y < D) \cdot P(y < D)$$

$$= \left[D \cdot \frac{2\gamma - D}{2\gamma}\right] + \left[\frac{D}{2} \cdot 1 - \frac{2\gamma - D}{2\gamma}\right]$$

$$= \left[D \cdot \frac{2\gamma}{2\gamma} - \frac{D}{2\gamma}\right] + \left[\frac{D}{2} \cdot 1 - \frac{2\gamma}{2\gamma} + \frac{D}{2\gamma}\right]$$

$$= \left[D \cdot \left(1 - \frac{D}{2\gamma}\right)\right] + \left[\frac{D}{2} \cdot 1 - 1 + \frac{D}{2\gamma}\right]$$

$$= \left[D \cdot \left(1 - \frac{D}{2\gamma}\right)\right] + \left[\frac{D}{2} \cdot \frac{D}{2\gamma}\right]$$

$$= D - \frac{D^2}{2\gamma} + \frac{D^2}{4\gamma}$$

$$= D - \frac{2D^2}{4\gamma} + \frac{D^2}{4\gamma}$$

$$= D - \frac{D^2}{4\gamma}$$

$$E(P) = D - D^2/4\gamma$$

$$E[P'] = D' \cdot P(y \geq D') + E(f(y) \mid y < D') \cdot P(y < D')$$

$< D'/2$

$$= \left(D' - \frac{D'^2}{2\gamma}\right) + E\left[f(y) \mid y < D'\right] P(y < D')$$

$$R(D) = E(P) - C \cdot P(y < D)$$

$$= D - \frac{D^2}{4\gamma} - C \frac{D}{2\gamma}$$

$$R(D') = D' - \frac{(D')^2}{2\gamma} + E[f(y) \mid y < D') \frac{D'}{2\gamma} - C \frac{D'}{2\gamma}$$

$$< D' - \left(\frac{D'}{2\gamma}\right)^2 + \frac{(D')^2}{2\gamma} - \frac{C D'}{2\gamma}$$

$$4D\gamma - D^2 - 2CD < 4D'\gamma - (D')^2 - 2CD'$$

$$4\gamma - D - 2C < \frac{D'}{D}(4\gamma - 2C) - \frac{(D')^2}{D}$$

$$(4\gamma - 2C) - (4\gamma - 2C)\frac{D'}{D} < \frac{D^2 - (D')^2}{D}$$

$(4\gamma-2C)(D-D') < D^2-(D')^2$

$(4\gamma-2C)(D-D') < (D+D')(D-D')$

$\rightarrow D \neq D'$

$(4\gamma-2C) < (D+D') \leq 2D$

$2(2\gamma-C) < 2D \Rightarrow 2\gamma-C < D$

which implies $D' > D$

$\Rightarrow D' > D \Rightarrow P(y<D') > P(y<D)$ so D' has higher expected cost

$\Rightarrow$ Contract P' is less preferable :)

② Take the costly state verification model in class

$$P = \begin{cases} D & \text{if } Y > D \\ Y & \text{if } Y \leq D \end{cases} \qquad P' = sY$$

Answer:

- Under the contract: $P' = sY$, the lender will always have to verify, so they'll pay verification cost with probability 1 : $V_{P'} = 1C$ in contract p its $V_P = \frac{D}{2\gamma} \cdot C$ and since $D < 2\gamma - C$, $D < 2\gamma$, $\frac{D}{2\gamma} < 1$ which implies $V_P < V_{P'}$

③ Start with market clearing of capital in Moll (2014) and using $\omega(z)$ and definition of $\Omega(z)$, show they imply

$$1 = \lambda(1 - \Omega(\underline{z}))$$

explain what its capturing

market clearing condition of capital : $\int k_t(a,z)\, dG_t(a,z) = \int a\, dG_t(a,z)$

$$\int k_t(a,z)\, dG(a,z) = K(t)$$

$$\int_a \int_z k_t(a,z)\, dG(a,z) = K(t)$$

$$\int_a \int_0^{\underline{z}} 0\, dG_t(a,z) + \int_a \int_{\underline{z}}^{\infty} \lambda a\, dG(a,z) = K(t)$$

$$\int_a \int_{\underline{z}}^{\infty} \lambda a\, dG_t(a,z) = K(t)$$

$$\lambda \int_{\underline{z}}^{\infty} \int_a a\, dG_t(a,z) = K(t)$$

$$\lambda \int_{\underline{z}}^{\infty} \left(\int_a a\, g_t(a,z) \right) dz = K(t)$$

$$\lambda \int_{\underline{z}}^{\infty} \left(\frac{1}{K(t)} \int_a a\, g_t(a,z)\, da \right) dz = 1 \qquad \omega(z,a) = \frac{1}{K(t)} \int_a a\, g_t(a,z)\, da$$

$$\lambda \int_{\underline{z}}^{\infty} \omega(z,a)\, dz = 1$$

$$\Omega(\underline{z}) = \int_0^{\underline{z}} w(z,a)\, dz \quad = \quad 1 - \Omega(\underline{z}) = \int_{\underline{z}}^{\infty} \omega(z,a)\, dz$$

$$\lambda(1 - \Omega(\underline{z})) = 1$$

↑ frictions $\lambda \to 1$ binding financial frictions

↓ frictions $\lambda \to \infty$: any firm can borrow any amount

$(1 - \Omega(\underline{z}))$: share of total wealth held by active entrepreneurs

$1 = \lambda(1 - \Omega(\underline{z}))$ any entrepreneur is an active entrepreneur.

④ last equation is market clearing condition in the capital market. It pins down $\underline{z}$ given wealth shares $\omega(z)$ that you can assume exogeneous, and the parameter λ. The eq doesnt depend on the return on savings r. is th r a free parameter in the model?

- r is not a free parameter since the marginal entrepreneur's productivity depends on r: $\underline{z}\pi = r + \delta$

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$$k_{t+1} = \eta\left[k\bar{\omega} - \int_0^{\underline{\omega}} \pi_1 r\, p(\omega)\, \partial\omega - \int_{\underline{\omega}}^{\bar{\omega}} k(1-g(\omega))\, \partial\omega\right]$$

$$\hat{q}k - r x(\bar{\omega}) = 0$$
$$\hat{q}k - r x(\underline{\omega}) - \hat{q}\pi_1 r = 0$$

$$\downarrow$$

$x(\omega)$: inc in $\omega \Rightarrow x'(\omega) > 0$

$$(x^{-1})'(y) > 0$$

$$p(\omega) = \frac{r(x(\omega) - s^e) - \hat{q}k_1}{\pi_2\hat{q}(k_2 - k_1) - \pi_1\hat{q}k_1} = \frac{r/\hat{q}\,(x(\omega) - s^e) - k_1}{\pi_2(k_2 - k_1) - \pi_1 k_1}$$

$$g(\omega) = s^e/s(\omega)^* = \frac{s^a}{x(\omega) - (\hat{q}/r)k_1}$$

$$\bar{\omega} = x^{-1}\left(\frac{\hat{q}k}{r}\right) \Rightarrow \partial/\partial\hat{q}\ \bar{\omega} > 0\ ,\quad \underline{\omega} = x^{-1}\left(\frac{\hat{q}}{r}(k - \pi_1 r)\right) \Rightarrow \frac{\partial}{\partial\hat{q}}\underline{\omega} > 0$$

$$\partial/\partial\hat{q}\ p(\omega) = \frac{(s^e - x(\omega))r}{\hat{q}^2(\pi_2(k_2 - k_1) - \pi_1 k_1)} < 0 \qquad \frac{d}{\partial\hat{q}} g(\omega) > 0$$

$$\partial/\partial\hat{q}\ \eta k\bar{\omega} > 0$$

$$\partial/\partial\hat{q}\ \eta\pi_1 r\int_0^{\underline{\omega}} p(\omega)\,\partial\omega = \left[p(\underline{\omega})\cdot\left(\frac{d}{\partial\hat{q}}\underline{\omega}\right) + \int_0^{\underline{\omega}} \partial/\partial\hat{q}\ p(\omega)\,\partial\omega\right]\eta\pi_1 \cdot r < 0$$

$$\partial/\partial\hat{q}\ \eta k\int_{\underline{\omega}}^{\bar{\omega}} g(\omega)\,\partial\omega = \eta k\left[\underbrace{g(\bar{\omega})}_{>0}\underbrace{\left(\frac{d}{\partial\hat{q}}\bar{\omega}\right)}_{>0} - \underbrace{g(\underline{\omega})}_{>0}\underbrace{\left(\frac{d}{\partial\hat{q}}\underline{\omega}\right)}_{>0} + \int_{\underline{\omega}}^{\bar{\omega}} \underbrace{\partial/\partial\hat{q}\ g(\omega)}_{>0}\,\partial\omega\right] > 0$$

$$\partial/\partial\hat{q}\ k_{t+1} = \eta\left[(+) - (-) + (+)\right] > 0$$

6. Solve the utility maximization problem in Bernanke (1989). Derive the expression for p in slide 11.

$$\max \ \pi_1 (p c^a + (1-p) c_1) + \pi_2 c_2$$

$$\text{s.t.} \quad \pi_1(\hat{q} k_1 - (1-p)c_1 - p(c^a + \gamma \hat{q})) + \pi_2(\hat{q} k_2 - c_2) \geq r(x - s^e)$$
$$c_2 \geq p(0) + (1-p)(c_1 + (k_2 - k_1)\hat{q})$$
$$c_1 \geq 0 \quad c^a \geq 0 \quad 0 \leq p \leq 1$$

$$\mathcal{L} = \pi_1 (p c^a + (1-p) c_1) + \pi_2 c_2 + \lambda[-r(x-s^e) + \pi_1(\hat{q} k_1 - (1-p) c_1 - p(c^a \gamma \hat{q})) +$$
$$+ \pi_2[\hat{q} k_2 - c_2)] + \mu[c_2 - (1-p)(c_1 + (k_2 - k_1)\hat{q})] + \theta[c_1] + \psi[c^a] + \alpha[p] + \beta[1-p]$$

FOC:

$$c^a: \ \pi_1^p - \lambda \pi_1 p + \psi = 0$$
$$c_1: \ \pi_1(1-p) - \lambda \pi_1 (1-p) - \mu(1-p) + \theta = 0$$
$$c_2: \ \pi_2 - \lambda \pi_2 + \mu = 0$$

C.S.

$$\lambda[\quad] = 0 \qquad \lambda \geq 0$$
$$\mu[\quad] = 0 \qquad \mu \geq 0$$
$$\theta(c_1) = 0 \qquad \theta \geq 0$$
$$\psi(c^a) = 0 \qquad \psi \geq 0$$
$$\alpha(p) = 0 \qquad \alpha \geq 0$$
$$\beta(1-p) = 0 \qquad \beta \geq 0$$

$$\lambda = 1 + \mu/\pi_2 = 1 + \psi/\pi_1 p = 1 - \mu/\pi_1 + \theta/\pi_1(1-p) \Rightarrow$$ if $\theta = 0$: $\mu/\pi_2 = -\mu/\pi_1 \Rightarrow \pi_2 = -\pi$ which is a contradiction, so $\theta > 0$ and $c_1 = 0$

if $\psi = 0$ then $\mu = 0$

Since $\mu/\pi_2 = \psi/\pi_1 p \Rightarrow \psi/\mu = \frac{\pi_1}{\pi_2} p$

$$\pi_1(\hat{q} k_1 - (1-p)c_1 - p(c^a + \gamma \hat{q})) + \pi_2(\hat{q} k_2 - c_2) = r(x - s^e)$$
$$c_2 = p(0) + (1-p)(c_1 + (k_2 - k_1)\hat{q}) \quad c_1 = 0, \ c^a = 0 \quad 0 \leq p \leq 1$$

$$c_2 = \frac{\pi_1 \hat{q} k_1 + \pi_2 \hat{q} k_2 - r(x - s^e)}{\pi_2} - \frac{\pi_1 \hat{q} \gamma}{\pi_2} p$$

$$p\left(\frac{(k_2 - k_1)\hat{q} \pi_2 - \pi_1 \hat{q} \gamma}{\pi_2}\right) = \frac{r(x - s^e) - (\pi_2 + \pi_1) k_1 \hat{q}}{\pi_2}$$

$$\boxed{p = \frac{r(x - s^e) + k_1 \hat{q}}{(k_2 - k_1)\hat{q} \pi_2 - \pi_1 \hat{q} \gamma}}$$