Homework 1 for Stochastic Processes (Spring 2021)

Due: March 8th, 2021

Problem 1. (Geometric Distribution) For any $p \in [0,1]$, a p-coin is the one who gives HEAD with probability p and TAIL with probability $1-p$. Recall that we say a random variable X follows the *geometric distribution* with parameter p if X is the number of independent p-coins tossed before the first HEAD comes out. This is in fact an informal definition. What is the probability space $(\Omega, \mathcal{F}, P)$ here? Write down the formal definition of X (as a function from Ω to $\mathbb{R}$) and compute $\mathbf{Pr}\left[X = k\right]$ for every $k \geq 0$.

Problem 2. (Monte Carlo Method) Consider the use of Monte Carlo method to estimate the volume of a n-dimensional ball. Let $B_n = \{\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n : \forall i \in [n], x_i \in [-1,1]\}$ be the box such that you can draw points from B_n uniformly at random. Let $\mathrm{Ball}(\mathbf{0}, 1)$ be the n-dimensional ball of radius 1 centered at $\mathbf{0}$ and let V denote its volume. Suppose you draw T points in total and the number of points in $\mathrm{Ball}(\mathbf{0}, 1)$ is k. Then an unbiased estimator of V is $\hat{V} = \frac{k}{T} \cdot 2^n$.

1. Prove that $\mathbf{E}\left[\hat{V}\right] = V$.
2. If we want to guarantee that with probability at least 99%, it holds that

$$0.99 \leq \frac{\hat{V}}{V} \leq 1.01,$$

   how large should we choose T?[1]

Problem 3. (Conditional Expectation)

1. Let X be a random variable and $f(\cdot) : \mathbb{R} \to \mathbb{R}$ be a function. We usually use $f(X)$ to denote the random variable such that $\omega \in \Omega \mapsto f(X(\omega)) \in \mathbb{R}$. Prove that $f(X)$ is always $\sigma(X)$-measurable.
2. Let Y, Y' be two random variables such that $\sigma(Y) = \sigma(Y')$. Prove that $\mathbf{E}\left[X \mid Y\right] = \mathbf{E}\left[X \mid Y'\right]$.
3. This fact you just proved should convince you that the conditional expectation $\mathbf{E}\left[X \mid Y\right]$ only depends on the σ-algebra $\sigma(Y)$ (but not the value of Y). Let Ω be the set of outcomes and $X : \Omega \to \mathbb{R}$ be a random variable. Let $\mathcal{F}$ be a σ-algebra on Ω. Can you define the notation $\mathbf{E}\left[X \mid \mathcal{F}\right]$?
4. (bonus) Let $\mathcal{F}_1, \mathcal{F}_2$ be two σ-algebras such that $\mathcal{F}_1 \subseteq \mathcal{F}_2$ and $X : \Omega \to \mathbb{R}$ be a random variable. Prove that $\mathbf{E}\left[\mathbf{E}\left[X \mid \mathcal{F}_1\right] \mid \mathcal{F}_2\right] = \mathbf{E}\left[\mathbf{E}\left[X \mid \mathcal{F}_2\right] \mid \mathcal{F}_1\right] = \mathbf{E}\left[X \mid \mathcal{F}_1\right]$.

[1] Hint: You can look for keywords like "Chernoff bounds" on internet.