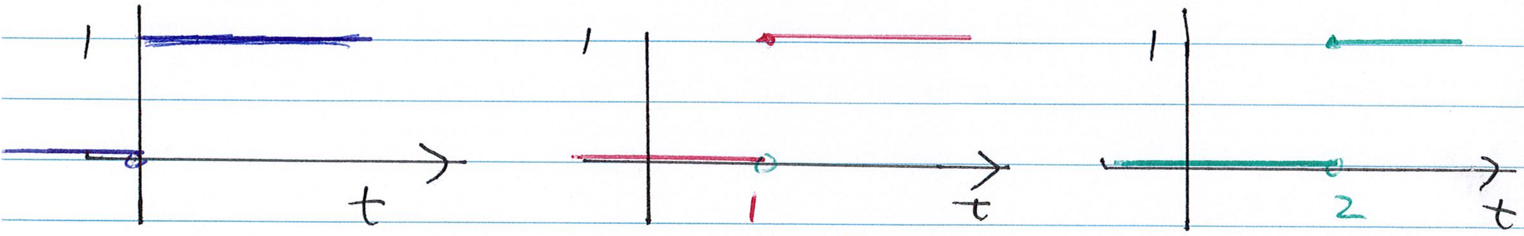
(1)

(b) Sketch

$$x(t) = (t+1)u(t-1) - tu(t) - u(t-2)$$

Solution: Firstly sketch $u(t)$, $u(t-1)$, $u(t-2)$



$u(t)$ $u(t-1)$ $u(t-2)$

Then

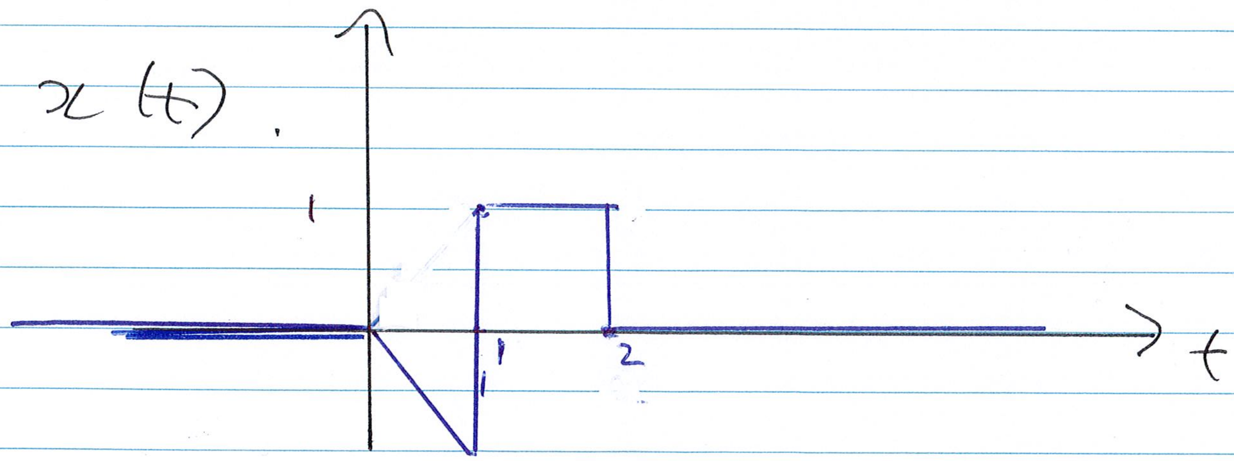
$$x(t) = t[u(t-1) - u(t)] + u(t-1) - u(t-2)$$

Note that $u(t-1) - u(t) = \begin{cases} 0 & \text{if } t \geq 1 \\ -1 & \text{if } 0 \leq t < 1 \\ 0 & \text{if } t < 0 \end{cases}$

and

$$u(t-1) - u(t-2) = \begin{cases} 0, & \text{if } t \geq 2 \\ 1, & \text{if } 1 \leq t < 2 \\ 0, & \text{if } t < 1 \end{cases}$$

So sketch $x(t)$



2 Let $x(t)$ and $v(t)$ be signals related such that

$$x(t)\,u(t-c) = v(t-c)\,u(t-c)$$

Since $u(t-c) = \begin{cases} 1 & , \text{ if } t \geq c \\ 0 & , \text{ otherwise} \end{cases}$

Hence for $t \geq c$, we know that

$$\boxed{x(t) = v(\overset{\alpha}{\underbrace{t-c}})}$$

For $\boxed{t < c}$, we do not know how $x(t)$ and $v(t)$ are related.

To find $v(t)$ we introduce

$$\alpha = t - c, \Rightarrow t = \alpha + c$$

So

$$v(\alpha) = x(\alpha + c), \alpha \geq 0$$

OR $\boxed{v(t) = x(t+c)}$, $t \geq 0$.

i) If $c = 2$ and $x(t) = t^2 - t + 1$

then

$$\begin{aligned} v_1(t) &= x(t+2) \\ &= (t+2)^2 - (t+2) + 1 \\ &= t^2 + 4t + 4 - t - 2 + 1 \\ &= t^2 + 3t + 3 \end{aligned}$$

Not unique. Could have anything that is different for $t < 0$.

eg

$$v_2(t) = (t^2 + 3t + 3)\,u(\cancel{t} + \tfrac{1}{2})$$

(3) Sketch

(d) $x[n] = \delta[n+1] - \delta[n] + u[n+1] - u[n-2]$

Note that $\delta[n+1] = \begin{cases} 1 & \text{if } n = -1 \\ 0 & \text{if } n \neq -1 \end{cases}$

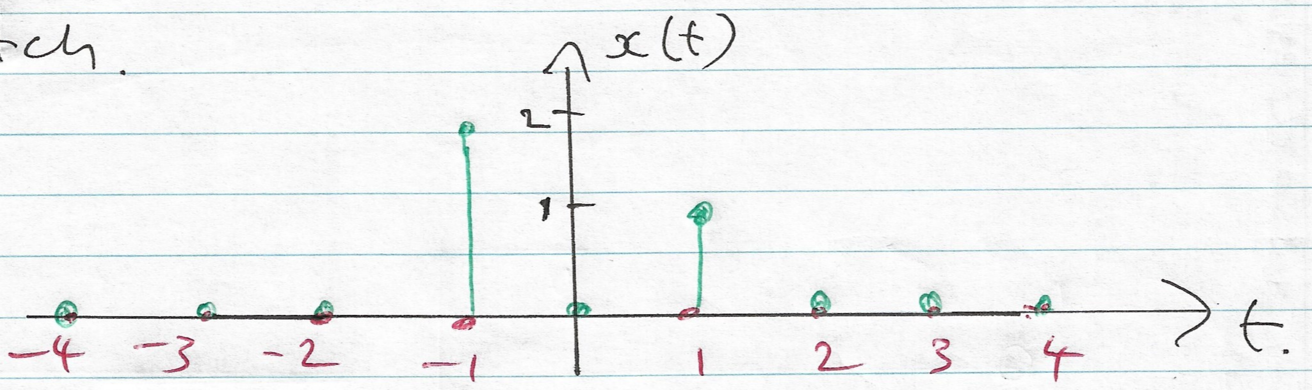
$$\delta[n] = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}$$

$$u[n+1] = \begin{cases} 1 & \text{if } n \geq -1 \\ 0 & \text{if } n < -1. \end{cases}$$

$$u[n-2] = \begin{cases} 1 & \text{if } n \geq +2 \\ 0 & n < +2. \end{cases}$$

So
$$\begin{aligned} x[-2] &= 0 \\ x[-1] &= 1 + 1 = 2 \\ x[0] &= -1 + 1 = 0 \\ x[1] &= 1 = 1 \\ x[2] &= 1 - 1 = 0 \end{aligned}$$
etc

Sketch.



(5) (a) $x[n] = \cos[n]$
$= \cos[\Omega n + \phi]$
where $\Omega = 1$ and $\phi = 0$.
Then $\frac{\Omega}{2\pi}$ is irrational
$\Rightarrow$ x is is not periodic.

(b) $x[n] = \cos[n\sqrt{5}]$
$= \cos[\Omega n + \phi]$
where $\Omega = \sqrt{5}$, $\phi = 0$
Then $\frac{\Omega}{2\pi} = \frac{\sqrt{5}}{2\pi}$ is irrational
$\Rightarrow$ x is not periodic

(c) $x[n] = \cos[n\sqrt{5}] + \cos[n\sqrt{2}]$
Since the signal $\cos[n\sqrt{5}]$ is not periodic, this signal is also not periodic

(d) $x[n] = \cos[\pi n] + \cos[\frac{3\pi}{2} n]$
Let $x_1[n] = \cos[\pi n]$
$= \cos[\Omega_1 n + \phi_1]$
where $\Omega_1 = \pi$, $\phi = 0$.
So $\frac{\Omega_1}{2\pi} = \frac{\pi}{2\pi} = \frac{1}{2}$ $\Rightarrow$ x_1 is periodic with $T_1 = 2$
Let $x_2[n] = \cos[\frac{3\pi}{2} n]$
Let $\Omega_2 = 3\pi/2$
Then $\frac{\Omega_2}{2\pi} = \frac{3\pi/2}{2\pi} = \frac{3}{4}$ $\Rightarrow$ x_2 is periodic with $T_2 = 4$
So x is periodic with period
$T = \mathrm{lcm}(T_1, T_2) = \mathrm{lcm}(2, 4)$
$= 4$.

$T = 4$

$\downarrow$

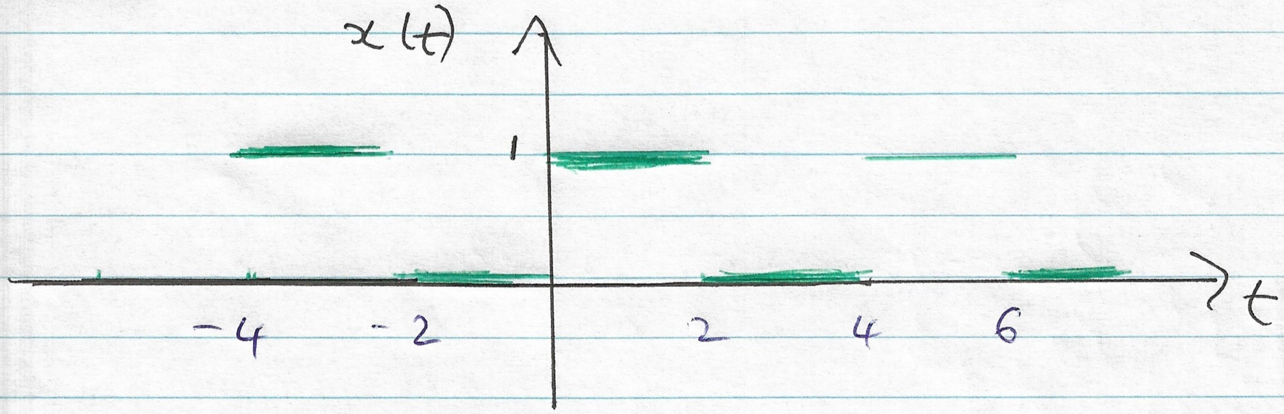
(7) Sketch

$$x(t) = \sum_{l=-\infty}^{\infty} f(t-4l) \quad \text{for all } t \in \mathbb{R}$$

where $f(t) = \begin{cases} 1 & ; \ 0 < t \leq 2 \\ 0 & ; \ \text{otherwise} \end{cases}$

Solution : Since $x(t)$ has period $T=4$.

we repeat $f(t)$ ~~at~~ intervals of length 4.



8 (b) $y(t) = \sin(t)\, x(t)$.

(a) This is causal because $y(t_0)$ only depends on values of $x(t)$ for $t \le t_0$,

(b) This is memoryless as $y(t_0)$ only depends on $x(t_0)$, for any t_0

(c) This is linear. Consider two inputs $x_1(t)$ and $x_2(t)$ and any two real numbers α and β.
Then we define
y_1 = output from x_1
y_2 = output from x_2
y_3 = output from $\alpha x_1 + \beta x_2$.

Then
$$y_1(t) = \sin(t)\, x_1(t)$$
$$y_2(t) = \sin(t)\, x_2(t)$$
and
$$y_3(t) = \sin(t)\left[\alpha x_1(t) + \beta x_2(t)\right]$$
$$= \alpha \sin(t)\, x_1(t) + \beta \sin(t) x_2(t)$$
$$= \alpha\, y_1(t) + \beta\, y_2(t).$$

(iv) This is time-varying.
Consider the signals $x_1(t) = t^2$
and $x_2(t) = x_1(t-1) = (t-1)^2$
Then let y_1 = output from x_1
y_2 = output from x_2
Then
$$y_1(t) = \sin(t)\, t^2$$
$$y_2(t) = \sin(t)\, (t-1)^2$$
$$\ne \sin(t-1)\,(t-1)^2 = y_1(t-1)$$
Hence NOT time-invariant.

(8) (d) (i) $y(t) = \int_0^t \lambda x(\lambda)\, d\lambda$

(i) The system is causal because to find the output y at time t_1, we need to know $x(t)$ for $0 \le t \le t_1$ ie only past and present values of x.

(ii) The system has memory because $y(t_1)$ depends on $x(t)$ for $t < t_1$

(iii) The system is linear: Let (x_1, y_1) and (x_2, y_2) be trajectories and let $a, b \in \mathbb{R}$. Then

$$y_1(t) = \int_0^t \lambda x_1(\lambda)\, d\lambda$$

$$y_2(t) = \int_0^t \lambda x_2(\lambda)\, d\lambda$$

Let $x_3 = a x_1 + b x_2$

$y_3 =$ output from x_3

Then $y_3(t) = \int_0^t \lambda (a x_1(\lambda) + b x_2(\lambda))\, d\lambda$

$$= a \int_0^t \lambda x_1(\lambda)\, d\lambda + b \int_0^t \lambda x_2(\lambda)\, d\lambda$$

$$= a\, y_1(t) + b\, y_2(t)$$

as required.

Hence (x_3, y_3) is also a trajectory and the system is Linear

8 (d) (iv) The system is time-varying. We show this with a counterexample to time-invariance.

Let $x(t) = t$

and let $y(t) = $ output from $x(t)$

$$= \int_0^t \lambda \, x(\lambda) \, d\lambda$$

$$= \int_0^t \lambda^2 \, d\lambda$$

$$= \frac{1}{3} t^3$$

Next let $x_1(t) = x(t-1)$

$$= t - 1.$$

Then $y_1(t) = $ output from $x_1(t)$

$$= \int_0^t \lambda \, x_1(\lambda) \, d\lambda$$

$$= \int_0^t \lambda (\lambda - 1) \, d\lambda$$

$$= \int_0^t \lambda^2 - \lambda \, d\lambda$$

$$= \left[\tfrac{1}{3} \lambda^3 - \tfrac{1}{2} \lambda^2 \right]_0^t$$

$$= \frac{1}{6} t^2 (2t - 3).$$

AND $y(t-1) = \frac{1}{3}(t-1)^3$

$$= \tfrac{1}{3}(t^3 - 3t^2 - 3t + 1)$$

$$\neq y_1(t).$$

So the system is <u>not time-invariant</u>

because $(x(t-1), y(t-1))$ is <u>NOT</u> a trajectory.

(10) (b) $y[n] = 1 + v[n-2]$

This system is nonlinear. We will show this with counterexamples.

The Additivity Property says that if (v_1, y_1) and (v_2, y_2) are input-output pairs then $(v_1 + v_2, y_1 + y_2)$ is also an input-output pair.

Consider $v_1[n] = n^2$

$v_2[n] = n^3$

Also let $v_3[n] = v_1[n] + v_2[n]$.

Then

$= n^2 + n^3$

$$y_1[n] = 1 + v_1[n-2]$$

$$= 1 + (n-2)^2 \leftarrow$$

$$y_2[n] = 1 + (n-2)^3 \leftarrow$$

Let $y_3[n]$ be the output from $v_3[n]$. Then

$$y_3[n] = 1 + v_3[n-2]$$

$$= 1 + (n-2)^2 + (n-2)^3$$

Clearly

$$y_3[n] \neq y_1[n] + y_2[n]$$

So the Additivity Property fails and the system is nonlinear.

(10) (b) We can also consider the Homogeneity Property, which says that if (v, y) is an input-output pair and $\alpha \in \mathbb{R}$ is any real scalar, then

$$(\alpha v, \alpha y) \text{ is an input-output pair.}$$

So let $v[n] = n^2$

and let $\alpha = 3$

Then let $y[n]$ = output from $v[n]$

$$= 1 + (n-2)^2$$

Also let $y_1[n]$ = output from $\alpha v[n]$

$$= 1 + 3(n-2)^2$$

Then $y_1[n] \neq \alpha y[n]$

$$= 3 + 3(n-2)^2$$

So the Homogeneity Property Fails

$\Rightarrow$ System is non linear.

NOTE (i) System is nonlinear if either Additivity or Homogeneity Property Fail.

(ii) The choice of $v[n] = n^2$ and $\alpha = 3$ is not unique. Many other inputs and scalars could have ~~been~~ used to show Homogeneity Fails.

(12) Given signals x_1 and x_2 with periods T_1 and T_2, with

$$\frac{T_1}{T_2} = \frac{n}{m}, \quad (n, m) \text{ are coprime}$$

Show that the signal $x_1 + x_2$ has period $T_0 = mT_1 = nT_2$.

Proof: We know that $x_1(t) = x_1(t + T_1)$

$$x_2(t) = x_2(t + T_2).$$

Also know that

$$mT_1 = nT_2.$$

So then $(x_1 + x_2)(t + mT_1)$

$$= x_1(t + mT_1) + x_2(t + mT_1)$$

$$= x_1(t) + x_2(t + nT_2)$$

$$= x_1(t) + x_2(t)$$

$$= (x_1 + x_2)(t)$$

So $x_1 + x_2$ has period $T_0 = mT_1 = nT_2$.

(13) Let $x[n]$ be a 2-periodic discrete signal, and let

$$y[n] = x[n] + x[n-1]$$

Show that y is a constant signal

Proof: A discrete signal is constant if it is 1-periodic, i.e.

$$y[n+1] = y[n].$$

So we consider

$$y[n+1] = x[n+1] + x[n-1+1]$$

$$= x[n+1] + x[n]$$

$$= y[n]$$

So $y[n]$ is a constant signal.

(14) $x_1(t) = \cos(10\pi t)$

(a) So $w_s = 10\pi$, $f_s = 5$ Hz, $T_s = \frac{1}{5}$ s.

(b) Since $f_s = 6$ Hz, we have $T_s = \frac{1}{6}$

$f_2 = 6 - 5$
$= 1$ Hz $\Rightarrow T_2 = 1$

and $x_2(t) = \cos(2\pi t)$.

Then $y_1[n] = x_1(nT_s)$
$= \cos\left(\frac{10\pi n}{6}\right)$
$= \cos\left(\frac{5\pi n}{3}\right) \longleftarrow$

and $y_2[n] = x_2(nT_s)$
$= \cos\left(\frac{2\pi n}{6}\right)$
$= \cos\left(\frac{\pi n}{3}\right). \longleftarrow$

Hence
$= \cos\left(2\pi n - \frac{\pi n}{3}\right)$
$= \cos\left(\frac{5\pi n}{3}\right)$
$= y_1[n]$.

NOTE: We used $\cos(\theta) = \cos(2\pi - \theta)$.

(c) Shannon-Nyquist Theorem says that to avoid aliasing we need to sample at twice the frequency of the signal x_1, ie. at half its period.
So we need $T_s = \frac{1}{10}$ S.

(1b) (b) If $g[n] = n$ then the system is

$$y[n] = x[n]\,(n + (n-1))$$
$$= (2n-1)\,x[n].$$

This is not time-invariant. To see this we introduce

$$x[n] = n$$

Then the output from this x is

$$y[n] = n(2n-1)$$
$$= 2n^2 - n.$$

Now consider $T = 1$. and let

$$x_1[n] = x[n-1]$$
$$= n-1$$

Let $y_1[n]$ be the output from $x_1[n]$.

Then

$$y_1[n] = (2n-1)\,x_1[n]$$
$$= (2n-1)(n-1)$$
$$= 2n^2 - 3n + 1.$$

And

$$y[n-1] = 2(n-1)^2 - (n-1)$$
$$= 2n^2 - 5n + 3$$

Since

$$y_1[n] \neq y[n-1]$$

the system is not time invariant.

(17) (a) Assume a system is time-invariant, with trajectories $(x(t), y(t))$
Let x be an input signal that is periodic, and let y be its output.
Then y is also periodic, with the same period as x.

Proof: We know that $x(t)$ has period $T > 0$
so

$$x(t) = x(t+T) \quad \text{for all } t. \quad — (1)$$

We know that $(x(t), y(t))$ is a trajectory and the system is time-invariant, so

$$(x(t+T), y(t+T)) \quad — (2)$$

is a trajectory

Substituting (1) into (2) gives

$$(x(t), y(t+T)) \quad — (3)$$

is a trajectory

But since each input produces a unique output, we conclude that outputs $y(t)$ and $y(t+T)$ must be the same, i.e.

$$y(t) = y(t+T)$$

so y is also periodic with period T.