(19) Using the recursion method with $x[n] = c$, we obtain

$$y[1] = (1+i)\,y[0] - c$$
$$y[2] = (1+i)\,y[1] - c$$
$$= (1+i)^2\,y[0] - [(1+i)+1]\,c$$
$$y[3] = (1+i)^3\,y[0] - [(1+i)^2 + (1+i) + 1]\,c$$
$$\vdots$$
$$y[n] = (1+i)^n\,y[0] - [(1+i)^{n-1} + \dots + 1]\,c.$$

Using the sum of a geometric series, we know that

$$(1+i)^{n-1} + \dots + (1+i) + 1 = \frac{1-(1+i)^n}{-i}$$

The loan is repaid when

$$y[N] = 0.$$

So we require

$$y[N] = 0 = (1+i)^N\,y[0] - \frac{c\,[1-(1+i)^N]}{-i}$$

$$\Rightarrow c\,[(1+i)^N - 1] = i(1+i)^N\,y[0]$$

$$\Rightarrow c = \frac{i(1+i)^N\,y[0]}{(1+i)^N - 1}.$$

$$= \frac{i\,y[0]}{1-(1+i)^{-N}}$$

(20) (c) $y[n+2] + 0.5\,y[n+1] + 0.25\,y[n]$

$$= x[n+1] - x[n]$$

for $n \geq -2$

Find $h[n]$, the unit pulse response for $n = 0, 1, 2, 3$. Assume $y[n] = 0$ for $n < 0$

<u>Solution</u> : $h[n] = y[n]$ when $x[n] = \delta[n]$

So to obtain $h[0]$ we use

$n = -2$:

$0.25\,h[-2] + 0.5\,h[-1] + h[0] = \delta[-1] - \delta[-2]$

$\Rightarrow 0 + 0 + h[0] = 0 + 0$

$\Rightarrow h[0] = 0.$

Use $n = -1$

$0.25\,h[-1] + 0.5\,h[0] + h[1] = \delta[0] - \delta[-1]$

$\Rightarrow 0 + 0 + h[1] = 1$

Use $n = 0$:

$h[2] + 0.5\,h[1] + 0.25\,h[0] = \delta[1] - \delta[0]$

$\Rightarrow h[2] + 0.5 + 0 = 0 - 1$

$\Rightarrow h[2] = -1.5$

Use $n = 1$:

$h[3] + 0.5\,h[2] + 0.25\,h[1] = \delta[2] - \delta[1]$

$\Rightarrow h[3] + (0.5)(-1.5) + 0.25 = 0$

$\Rightarrow h[3] = 3/4 - 1/4$

$= \frac{1}{2}.$

So

$(h[0], h[1], h[2], h[3])$

$= (0, 1, -1.5, \frac{1}{2}).$

$$h[n] = \left[(0.5)^n - (0.25)^n\right] u[n]$$

(21) To show that $h[n]$ is the impulse response of the system we need to show that

$$h[n+2] - 0.75\,h[n+1] + 0.125\,h[n] = 0.25\,\delta[n+1]$$

So let $n \geq 0$: Then

$$\begin{aligned}
LHS &= \left[(0.5)^{n+2} - (0.25)^{n+2}\right] u[n+2] \\
&\quad - 0.75\left[(0.5)^{n+1} - (0.25)^{n+1}\right] u[n+1] \\
&\quad + 0.125\left[(0.5)^n - (0.25)^n\right] u[n] \\
&= \left(\tfrac{1}{4}\right)(0.5)^n - \left(\tfrac{1}{16}\right)(0.25)^n \\
&\quad - 0.75\left[\left(\tfrac{1}{2}\right)(0.5)^n - \left(\tfrac{1}{4}\right)(0.25)^n\right] \\
&\quad + 0.125\left[(0.5)^n - (0.25)^n\right] \\
&= \left[\left(\tfrac{1}{4}\right) - \left(\tfrac{3}{8}\right) + \left(\tfrac{1}{8}\right)\right](0.5)^n \\
&\quad + \left[\left(-\tfrac{1}{16}\right) + \left(\tfrac{3}{16}\right) - \tfrac{1}{8}\right](0.25)^n \\
&= 0
\end{aligned}$$

Let $n = -1$:

$$\begin{aligned}
LHS &= (0.5)^1 - (0.25)^1 \\
&\quad - 0.75\left[(0.5)^0 - (0.25)^0\right] \\
&= \cancel{\left(\tfrac{1}{4}\right)(0.5)^n - \left(\tfrac{1}{16}\right)(0.25)^n} \\
&\quad \cancel{- 0.75\left[\left(\tfrac{1}{2}\right)(0.5)^n - \left(\tfrac{1}{4}\right)(0.25)^n\right]} \\
&= \tfrac{1}{4}
\end{aligned}$$

Let $n = -2$: $LHS = (0.5)^0 - (0.25)^0 = 0$

and for $n \leq -3$ $LHS = 0$.

Hence

$$LHS = \begin{cases} \tfrac{1}{4}, & \text{if } n = -1 \\ 0, & \text{otherwise} \end{cases} = \tfrac{1}{4}\delta(n+1)$$

as required.

(22) For the system

$$y[n] + a_1 y[n-1] = b_0 x[n]$$

the response from the unit pulse input

$$x[n] = \delta[n]$$

is given by $h[n] = 0.3(0.7)^n u[n]$

So $h[n] + a_1 h[n-1] = b_0 \delta[n]$, for $n \geq 0$.

At $n=0$: $h[0] + a_1 h[-1] = b_0 \delta[0]$

$\Rightarrow (0.3)(0.7)^0 u[0] = b_0$

$\Rightarrow \underline{0.3 = b_0}$.

At $n=1$: $h[1] + a_1 h[0] = b_0 \delta[1]$

$\Rightarrow (0.3)(0.7) + 0.3a_1 = (0.3)(0)$

$\Rightarrow \underline{a_1 = -0.7}$.

So the system is

$$y[n] - 0.7\, y[n-1] = 0.3\, x[n]$$

(b) $x[n] = \delta[n] + \delta[n-1] + \delta[n-2]$

So $y[n] = (x * h)[n]$

$= h[n] + h[n-1] + h[n-2]$

as the system is LTI

$= 0.3\left[(0.7)^n u[n] + (0.7)^{n-1} u[n-1] + (0.7)^{n-2} u[n-2]\right]$

$= 0.3\,\delta[n] + 0.51\,\delta[n-1] + 1.341\,(0.7)^n u[n-2]$

(25) The input is

$$x[n] = (n+1)\left(u[n] - u[n-2]\right)$$
$$= \begin{cases} 1, & n = 0 \\ 2, & n = 1 \\ 0, & \text{otherwise} \end{cases}$$
$$= \delta[n] + 2\,\delta[n-1].$$

We know that $(\delta[n], h[n])$ is a trajectory of the system

$$y[n+2] - 0.75\,y[n+1] + 0.125\,y[n] = 0.25\,x[n+1]$$

This system is linear and time-invariant, so the output for this input is

$$y[n] = h[n] + 2h[n-1]$$
$$= \left(\left(\tfrac{1}{2}\right)^n - \left(\tfrac{1}{4}\right)^n\right)u[n]$$
$$+ 2\left(\left(\tfrac{1}{2}\right)^{n-1} - \left(\tfrac{1}{4}\right)^{n-1}\right)u[n-1].$$

If $n=0$: $y[0] = \left(\tfrac{1}{2}\right)^0 - \left(\tfrac{1}{4}\right)^0$
$= 0$

If $n \geq 1$: $y[n] = \left(\tfrac{1}{2}\right)^n - \left(\tfrac{1}{4}\right)^n$
$+ 2\left[\left(\tfrac{1}{2}\right)^{n-1} - \left(\tfrac{1}{4}\right)^{n-1}\right]$

$$= \left(\tfrac{1}{2}\right)^n + 2\left(\tfrac{1}{2}\right)^{-1}\left(\tfrac{1}{2}\right)^n$$
$$- \left(\tfrac{1}{4}\right)^n - 2\left(\tfrac{1}{4}\right)^{-1}\left(\tfrac{1}{4}\right)^n$$
$$= 5\left(\tfrac{1}{2}\right)^n - 9\left(\tfrac{1}{4}\right)^n$$

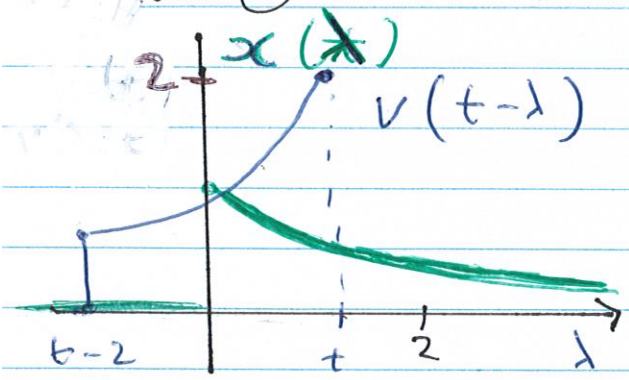
So $y[n] = \left[5\left(\tfrac{1}{2}\right)^n - 9\left(\tfrac{1}{4}\right)^n\right]u[n-1]$

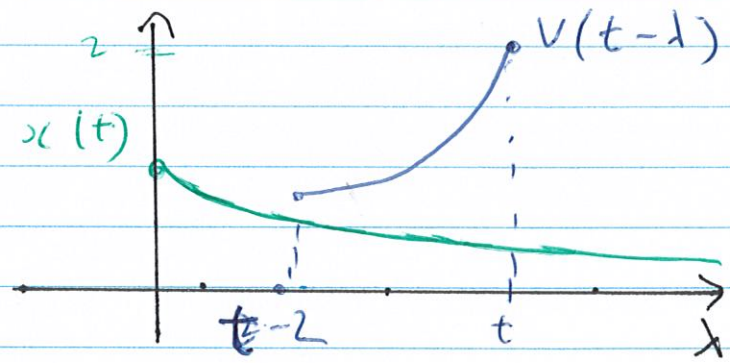
(27) (c) $x(t) = \begin{cases} e^{-t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$

$v(t) = \begin{cases} 2e^{-2t}, & 0 \leq t \leq 2 \\ 0, & \text{elsewhere} \end{cases}$

So the support of x is $[0, \infty)$
and the support of v is $[0, 2]$.
So $x * v$ will have support $[0, \infty)$.
We shall "delay and flip" signal v.

Case (1): $0 \leq t \leq 2$. Case (2): $t \geq 2$





So $(x*v)(t) = \int_0^{\infty} x(\lambda) v(t-\lambda) d\lambda$

$= \int_0^t e^{-\lambda} 2e^{-2(t-\lambda)} d\lambda$

$= \int_0^t 2e^{-2t+\lambda} d\lambda$

$= \left[2e^{-2t+\lambda}\right]_0^t$

$= 2e^{-t} - 2e^{-2t}$

$= 2e^{-t}(1 - e^{-t})$

So $(x*v)(t)$

$= \int_0^{\infty} x(\lambda) v(t-\lambda) d\lambda$

$= \int_{t-2}^t e^{-\lambda} 2e^{-2(t-\lambda)} d\lambda$

$= \left[2e^{-2t+\lambda}\right]_{t-2}^t$

$= 2e^{-t} - 2e^{-2t+t-2}$

$= 2e^{-t}(1 - e^{-2})$

So
$(x*v)(t) = \begin{cases} 2e^{-t}(1-e^{-t}), & 0 \leq t \leq 2 \\ 2e^{-t}(1-e^{-2}), & t \geq 2. \\ 0, & \text{elsewhere} \end{cases}$

28 (a) Given $x(t) = \begin{cases} t+1, & 0 \le t \le 1 \\ 2-t, & 1 < t \le 2 \\ 0, & \text{elsewhere} \end{cases}$

and $h(t) = \delta(t-2) + 2\delta(t-1)$

Then

$$(x * h)(t) = \int_{-\infty}^{\infty} x(\lambda)\, h(\lambda - t)\, d\lambda$$

$$= \int_{-\infty}^{\infty} x(\lambda)\left[\delta(\lambda-(t-2)) + 2\delta(\lambda-(t-1))\right] d\lambda$$

$$= x(t-2) + 2x(t-1)$$

by the Sifting Theorem

Then $x(t-2) = \begin{cases} (t-2)+1, & 0 \le t-2 \le 1 \\ 2-(t-2), & 1 < t-2 \le 2 \end{cases}$

$$= \begin{cases} t-1, & 2 \le t \le 3 \\ 4-t, & 3 < t \le 4 \end{cases}$$

And

$x(t-1) = \begin{cases} (t-1)+1, & 0 \le t-1 \le 1 \\ 2-(t-1), & 1 < t-1 \le 2 \end{cases}$

$$= \begin{cases} t, & 1 \le t \le 2 \\ 3-t, & 2 < t \le 3 \end{cases}$$

So

$$(x*h)(t) = \begin{cases} 2t, & 1 \le t < 2 \\ (t-1)+2(t), & t = 2 \\ (t-1)+2(3-t), & 2 < t \le 3 \\ 4-t, & 3 \le t \le 4 \end{cases}$$

$$= \begin{cases} 2t, & 1 \le t < 2 \\ 5, & t = 2 \\ 5-t, & 2 < t \le 3 \\ 4-t, & 3 < t \le 4 \end{cases}$$

28 (b) If $\delta(t) = x(t+2) + 2x(t+1)$

then

$$(x * h)(t) = x(t+2) + 2x(t+1)$$

Then $x(t+2) = \begin{cases} t+3, & 0 \le t+2 \le 1 \\ 2-(t+2), & 1 < t+2 \le 2 \\ 0, & \text{elsewhere} \end{cases}$

$= \begin{cases} t+3, & -2 \le t \le -1 \\ -t, & -1 < t \le 0 \\ 0, & \text{elsewhere} \end{cases}$

Also $x(t+1) = \begin{cases} t+2, & -1 \le t \le 0 \\ 1-t, & 0 < t \le 1 \\ 0, & \text{elsewhere} \end{cases}$

So

$(x * h)(t) = \begin{cases} t+3, & -2 \le t \le -1 \\ 4, & t = -1 \\ t+4, & -1 < t \le 0 \\ 2(1-t), & 0 < t \le 1 \\ 0, & \text{elsewhere.} \end{cases}$

(30) Let $h(t) = (e^{-2t} - e^{-t})\,u(t)$

$x(t) = u(t)$

Then the step response is, for $t \geq 0$

$$y(t) = (h * x)(t)$$

$$= \int_0^t h(t-\lambda)\,u(\lambda)\,d\lambda$$

$$= \int_0^t h(t-\lambda)\,d\lambda$$

$$= \int_0^t e^{-2(t-\lambda)} - e^{-(t-\lambda)}\,d\lambda$$

$$= \left[\frac{1}{2}e^{-2(t-\lambda)} - e^{-(t-\lambda)}\right]_0^t$$

$$= \left(\frac{1}{2} - 1\right) - \left(\frac{1}{2}e^{-2t} - e^{-t}\right)$$

$$= e^{-t} - \frac{1}{2}e^{-2t} - \frac{1}{2}$$

32 Let $h(t)$ be the impulse response of the system. As y is the output from an input x, we know that

$$y(t) = (x * h)(t).$$

$$= \int_{-\infty}^{\infty} x(\lambda)\, h(t-\lambda)\, d\lambda$$

$$= \int_{-\infty}^{\infty} x(t-\lambda)\, h(\lambda)\, d\lambda$$

So

$$\dot{y}(t) = \frac{\partial}{\partial t} \int_{-\infty}^{\infty} x(t-\lambda)\, h(\lambda)\, d\lambda$$

$$= \int_{-\infty}^{\infty} \frac{\partial}{\partial t}\left[x(t-\lambda)\, h(\lambda)\right] d\lambda$$

$$= \int_{-\infty}^{\infty} \dot{x}(t-\lambda)\, h(\lambda)\, d\lambda$$

$$= (\dot{x} * h)(t)$$

So $\dot{y}$ is the output from input $\dot{x}$, and hence $(\dot{x}, \dot{y})$ is a trajectory.

(33) For the continuous-time system

$$I \frac{d^2\theta}{dt^2} + k_d \frac{d\theta}{dt} = \tau(t), \quad t \geq 0$$

To discretize the system with sampling period $T > 0$, we use

$$\left.\frac{d\theta}{dt}\right|_{t=nT} \simeq \frac{\theta(nT+T) - \theta(nT)}{T}$$

and

$$\left.\frac{d^2\theta}{dT^2}\right|_{t=nT} \simeq \frac{\theta(nT+2T) - 2\theta(nT+T) + \theta(nT)}{T^2}$$

Then

$$\frac{I}{T^2}\left(\theta(nT+2T) - 2\theta(nT+T) + \theta(nT)\right) + \frac{k_d}{T}\left(\theta(nT+T) - \theta(nT)\right) = \tau(nT)$$

Let $\theta[n] := \theta(nT)$, we obtain the difference equation

$$I\,\theta[n+2] + (Tk_d - 2I)\,\theta[n+1] + (I - Tk_d)\,\theta[n] = T^2\,\tau[n]$$