UNIVERSITY OF MELBOURNE

ELEN30012
SIGNALS AND SYSTEMS

SEMESTER 2, 2020

LECTURE NOTES

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Contents

Subject Information for ELEN30012 Signals and Systems

Subject Description

ELEN30012 introduces the mathematical techniques that underpin the analysis and design of electrical networks, telecommunication systems, signal-processing systems and automatic control systems, among others. Topics include:

- Signals – continuously and discretely indexed signals, important signal types, frequency-domain analysis (Fourier, Laplace and Z transforms), nonlinear transformations and harmonics, sampling, and aliasing;
- Systems – viewing differential / difference equations as systems of signals, the notions of input, output and internal signals, block diagrams (series, parallel and feedback connections), properties of input-output models (causality, delay, stability, gain, shift-invariance, linearity), transient and steady state behaviour;
- Linear shift-invariant models – continuous and discrete impulse response and convolution operator models, transfer functions and frequency response, time-domain interpretation of stable and unstable poles and zeros, state-space models (construction from high-order ODEs, canonical forms, state transformations and stability), and the discretisation of models for systems of continuously indexed signals.

This material will be complemented by the frequent use of MATLAB for computation and simulation and examples from diverse areas including electrical engineering, biology and economics.

Lectures and Workshops

Signals and Systems consists of 36 lectures and 10 two-hour workshops. Students are expected to attend all lectures (or at least listen to the Lecture recording) and workshops.

Assessment

The Signals and Systems assessment will consist of:

- An end of semester exam worth 60%.
- A mid-semester test worth 10%.
- Workshops done with your workshop group partners, consisting of 15% assessment
- Three homework assignments, to be done with your workshop group partners, totaling 15%.

The grades from the continuous assessment will be posted on the CANVAS.

Intended Learning Outcomes

On completing this subject it is expected that the student will be able to:

1. Apply fundamental mathematical tools to model, analyse and design signals and systems in both time-domain and frequency-domain;
2. Recognise the broad applicability of the mathematics of signals and systems theory, particularly within electrical engineering;
3. Recognize the similarities and differences between the mathematical tools needed for dealing with continuous-time systems/signals versus their discrete-time counterparts;
4. Use MATLAB to study the behaviour of signals and systems as they arise in a variety of contexts.

Generic Skills

On completion of this subject students should have developed the following generic skills:

- Ability to apply knowledge of basic science and engineering fundamentals.
- Ability to undertake problem identification, formulation and solution.
- Ability to communicate effectively, with the engineering team and with the community at large.
- Capacity for independent critical thought, rational inquiry and self-directed learning.
- Expectation of the need to undertake lifelong learning, and the capacity to do so.

Textbooks

The recommended textbook is:
Edward W. Kamen and Bonnie S. Heck, *Fundamentals of Signals and Systems Using the Web and MATLAB*, Prentice Hall, 3rd Edition.
This text available from the University bookshop, and there are also several copies in the library. Signals and Systems is heavily based on this textbook. Many of the diagrams and practice problems used in these lecture notes have been taken from it, and are used with permission from Prentice Hall.
The recommended textbook for learning MATLAB is
Stormy Attaway, *MATLAB: A Practical Introduction to Programming and Problem Solving*, Elsevier, 5th Edition.

Calculators

Only the following calculators may be used in the mid-semester test and examination:

- Casio FX82 (any suffix)
- Casio FX100 (any suffix)

The FX100 is a little more expensive than the FX82 but it can do arithmetic with complex numbers, as well as easy conversion from rectangular to polar form, making it very useful for Signals and Systems and many other ELEN subjects.

Effective Study and Exam Preparation

Here are some guidelines for getting the most benefit from the Signals and Systems study materials.

- Begin all your study with a thorough reading of the lecture notes. The lecture notes contain many worked examples, and a good way to begin your study is to copy a problem from the lecture notes into your notebook, close the lecture notes, and do the question on your own. Then compare your answer against the solution in the lecture notes. If you get it right - great! If not, close the lecture notes and try again until you do get it right.

- At the end of each chapter, there is a large number of problems for you to practise on. Numerical answers are provided for you to check your answers. The mid-semester test and the final examination will contain questions similar to those given in this booklet, as well as those appearing in the lecture notes, the workshop exercise sheets and the assignments. It is up to you to complete as many of the problems in this booklet as possible. Obviously the more problems your are able to do successfully, the better prepared you will be for the test and examination.

- Write your own summary of the lecture material as you are doing the problem sheet questions. If you do this thoroughly, it will be about 10 A4 pages in length by end of the semester. It is essential that you write this summary yourself, it is part of the learning process!

- Do all of your work with a pen (or pencil) in a notebook dedicated to Signals and Systems. When you have done a question successfully, immediately write up a neat and clear version of your solution; this will help to reinforce the concepts in your mind.

- It is VERY IMPORTANT to work questions through to the correct answer. If you are not able to obtain the correct answer given, then you need to find out what you are doing wrong. Read the lecture notes or textbook again, discuss your work with other students, post a question to the discussion board on the LMS, or see the lecturer. Note that exam answers are marked according to the amount of progress made towards the correct answer, so if your answer goes wrong in the very beginning, you will receive little or no marks.

- It is important that you learn to write solutions that are easy for other people to read - especially the examiner! When drawing diagrams, the use of coloured pens and rulers is encouraged.

- The examination will contain some questions that are similar to those that appear on the assignments. Hence it is important that all members of your workshop group do all of the questions on each assignment. After doing the questions, arrange to meet up to discuss them and submit the best version for assessment. Do not make the mistake of using the "Divide and Conquer" method, where each student does one-third of the questions, and you just collate them together for the group assignment submission. This might seem an effective way of getting the assignment completed, but if an assignment question (or something similar to it) appears on the exam, only one member of your group will be prepared for it!

- The examination will contain at least one question that tests in detail your understanding of the exercises done in workshops. For this reason it is important that you take good notes during the workshop so you can refer to them during your exam preparation. Note: You will not be examined on the use of MATLAB.

Chapter 1

Fundamental Concepts of Signals and Systems

Signals contain information eg. sounds, images, voltage, motion, video, text. **Systems** transform signals eg. digital encoder, audio equalizer, modems, control systems, digital circuits, analog circuits. Conceptually, we may regard systems as a "black box" that converts an **input signal** into an **output signal**.

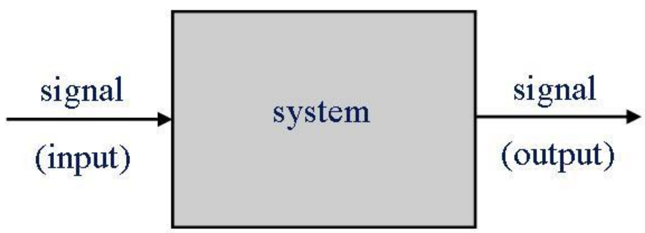


Figure 1.1: Signal and System block diagram

Examples of signals include

- Trajectory of a moving object
- Voltage signal
- Ultrasound
- Semaphore
- Scoreboard
- ECG signal

Figure 1.2: Examples of Signals

Examples of systems include

- Violin
- Mobile phone
- Robot
- Car
- Bionic ear
- Bridge
- Wine Glass
- Baby

Figure 1.3: Examples of Systems

We denote some important sets as follows:

Definition 1.0.1 (Special sets of numbers)

- $\mathbf{Z}$ *is the set of integers:* $\{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$.
- $\mathbf{Z}_0+$ *is the set of positive integers with zero:* $\{0,1,2,3,\ldots\}$.
- $\mathbf{Q}$ *is the set of rational numbers.*
- $\mathbf{R}$ *is the set of real numbers.*
- $\mathbf{C}$ *is the set of complex numbers.*
- $\mathbf{R}^n$ *is the set of real vectors of length n, e.g.* $(1,\sqrt{3},e,-.5) \in \mathbf{R}^4$.

We distinguish between two types of signals:

Definition 1.0.2 *Functions with domain* $\mathbf{R}$ *are called* **continuous-time** *signals. We write* $x : \mathbf{R} \mapsto \mathbf{R}$ *and denote the output of the signal as* $x(t)$. *Functions with domain* $\mathbf{Z}$ *(or* $\mathbf{Z}_0^+$*) are called* **discrete-time** *signals. We write* $x : \mathbf{Z} \mapsto \mathbf{R}$ *and denote the output of the signal as* $x[n]$.

Example 1.0.1 (Speech waveforms) *A speech signal is continuous-time signal; it is a function that maps real numbers (time) to real numbers (air pressure). A sampled speech signal is a discrete-time signal; it is a function that maps integers (time) to real numbers (air pressure).*

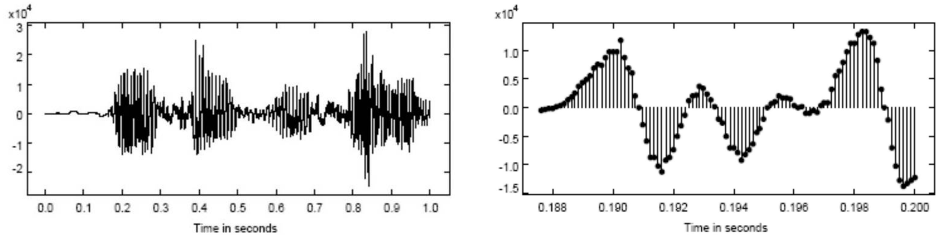


Figure 1.4: (a) Audio Signal (b) Sampled audio signal

1.1 Fundamental properties of continuous-time signals

Definition 1.1.1 (Continuous-time unit step and unit ramp signals) *The* **unit step** *function* $u : \mathbf{R} \mapsto \mathbf{R}$ *is defined as*

$$u(t) = \begin{cases} 1, & \text{if } t \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{1.1}$$

The **unit ramp** *function* $r : \mathbf{R} \mapsto \mathbf{R}$ *is defined as*

$$r(t) = \begin{cases} t, & \text{if } t \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{1.2}$$

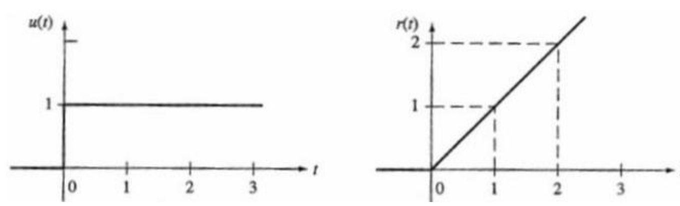


Figure 1.5: The unit step function $u(t)$ and ramp function $r(t)$

Definition 1.1.2 (Impulse signal) *The* **Dirac delta** *function* $\delta : \mathbf{R} \mapsto \mathbf{R} \cup \{\infty\}$ *is defined as the function with the properties*

1. *For all* $t \in \mathbf{R}$*, if* $t \neq 0$*, then* $\delta(t) = 0$*, and*

2. *For all* $\varepsilon > 0$

$$\int_{-\varepsilon}^{\varepsilon} \delta(\lambda)\, d\lambda = 1. \tag{1.3}$$

The δ *function is an example of a* **generalized function** *and is known as the* **impulse signal***. It is not a function in the normal sense. It may be thought of as the limit of the sequence of step functions of unit area: For each* $k \in \mathbf{Z}^+$*, define the functions* $f_k : \mathbf{R} \mapsto \mathbf{R}$ *with*

$$f_k(t) = \begin{cases} k, & \text{if } \frac{-1}{2k} \leq t \leq \frac{1}{2k} \\ 0, & \text{otherwise} \end{cases} \tag{1.4}$$

Then

$$\delta = \lim_{k \to \infty} f_k \tag{1.5}$$

The function sequence $f_k(t)$ *and* $K\delta$ *for any constant* $K > 0$ *are shown in Figure 1.6*

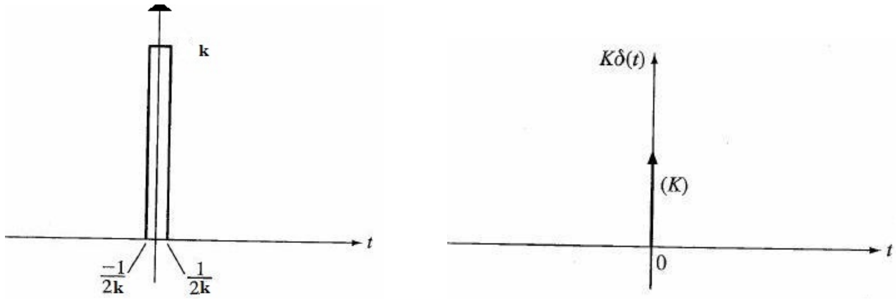


Figure 1.6: Dirac delta function $\delta(t)$.

Theorem 1.1.1 (Sifting Property of the Impulse signal) *Let $f : \mathbf{R} \mapsto \mathbf{R}$ be continuous at any point $t_0 \in \mathbf{R}$. Then for any $t \in \mathbf{R}$,*

$$\int_{-\infty}^{t} f(\lambda)\delta(\lambda - t_0)\, d\lambda = f(t_0)u(t - t_0) \tag{1.6}$$

and

$$\int_{-\infty}^{\infty} f(\lambda)\delta(\lambda - t_0)\, d\lambda = f(t_0) \tag{1.7}$$

The Sifting Theorem offers an alternative way of defining the impulse function. Together with the convolution operation that we will meet in Chapter 2, it allow us to express the outputs of any system in terms of its inputs and the impulse function.

Definition 1.1.3 (Time shift signals) *For any continuous-time signal x and for any constant $a > 0$,*

- *$x(t - a)$ is* **right time-shifted by** *a* **time units***, compared to $x(t)$.*
- *$x(t + a)$ is* **left time-shifted by** *a* **time units***, compared to $x(t)$.*

Example 1.1.1 *Sketches of $u(t-2)$, the unit step function with a right shift of two time units, and $u(t+2)$, the unit step function with a left shift of two time units are shown in Figure 1.1.1 :*

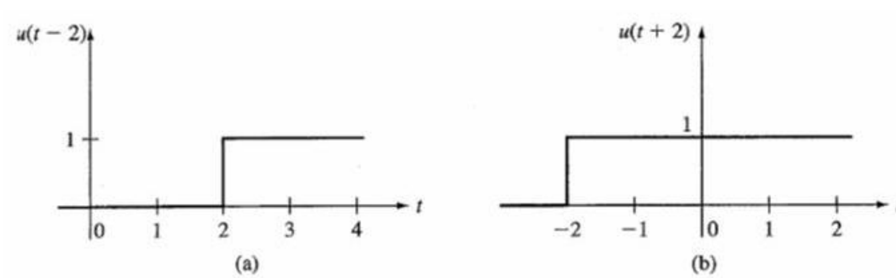


Figure 1.7: (a) Right time-shifted signal (b) Left time shifted signal

Definition 1.1.4 (Continuous signals) *Let $x : \mathbf{R} \to \mathbf{R}$ be a continuous-time signal. We say that*

- *x is* **continuous at the point** *$t_0 \in \mathbf{R}$ if*

$$x(t_0^-) = x(t_0) = x(t_0^+) \tag{1.8}$$

Here $x(t_0^-)$ and $x(t_0^+)$ denote the **left and right limits** *of x at t_0:*

$$x(t_0^-) = \lim_{h \to 0, h<0} x(t_0 + h), \qquad x(t_0^+) = \lim_{h \to 0, h>0} x(t_0 + h) \tag{1.9}$$

- *x is* **discontinuous at** t_0 *if it is not continuous at* t_0*, i.e. if Condition 1.8 is not true.*
- *x is a* **continuous signal** *if it is continuous at all points* $t \in \mathbf{R}$*.*
- *A signal x is* **piecewise continuous signal** *if it is continuous at all except finitely many points in* $\mathbf{R}$*, i.e if Condition 1.8 is true except at finitely many points.*

Example 1.1.2 *Consider $u(t-2)$, shown in Example 1.1.1(a). It is discontinuous at $t_0 = 2$, because the left limit $u(2^-) = 0$, while the right limit $u(2^+) = 1$. It is continuous at all other points $t_0 \neq 2$. Hence $u(t-2)$ is not a continuous function, however it is piecewise continuous.*

Definition 1.1.5 (Derivatives of continuous-time signals) *Let $x : \mathbf{R} \to \mathbf{R}$ be a continuous-time signal. We say that*

- *x is* **differentiable at the point** $t_0 \in \mathbf{R}$ *if its left and right derivatives both exist and are equal, i.e. if*

$$\lim_{h\to 0, h<0} \frac{x(t_0+h)-x(t_0)}{h} = \lim_{h\to 0, h>0} \frac{x(t_0+h)-x(t_0)}{h} \tag{1.10}$$

If Condition (1.10) *is satisfied, then we define the* **derivative of** x **at the point** t_0 *as*

$$\dot{x}(t_0) = \lim_{h\to 0} \frac{x(t_0+h)-x(t_0)}{h} \tag{1.11}$$

- *x is* **not differentiable at** t_0 *if Condition 1.8 is not true at* t_0*.*
- *x is a* **differentiable signal** *if it is differentiable at all points* $t \in \mathbf{R}$*.*
- *x is a* **piecewise differentiable signal** *if it is differentiable at all except finitely many points in* $\mathbf{R}$*, i.e if Condition 1.10 is true except at finitely many points.*

Example 1.1.3 *The ramp signal r defined in* (1.2) *is not differentiable at $t_0 = 0$, because the left derivative (slope) is $\dot{x}(0^-) = 0$, while the right derivative $\dot{x}(0^+) = 1$. Hence r is not a differentiable signal, however it is piecewise differentiable.*

Definition 1.1.6 (Generalised derivatives of continuous-time signals) *Let $x : \mathbf{R} \to \mathbf{R}$ be a piecewise-differentiable signal, and assume that x is not continuous at $t_0 \in \mathbf{R}$. We say that the* **generalised derivative of** x **at** t_0 *is given by*

$$\dot{x}(t_0) = [x(t_0^+) - x(t_0^-)]\delta(t-t_0) \tag{1.12}$$

and the **generalised derivative of** x *is*

$$\dot{x}(t) = \frac{dx}{dt} + [x(t_0^+) - x(t_0^-)]\delta(t-t_0) \tag{1.13}$$

where $\frac{dx}{dt}$ is the (ordinary) derivative of x at points t where condition (1.10) *holds, and $[x(t_0^-) - x(t_0^+)]\delta(t-t_0)$ applies for those points where condition* (1.10) *is not valid.*

Example 1.1.4 *The unit step signal u defined in equation 1.1 is not differentiable at $t_0 = 0$, because it is not continuous at that point: $x(0^-) = 0$ while $x(0^+) = 1$. Taking generalized derivatives, we see that*

$$\dot{u}(t) = \delta(t) \tag{1.14}$$

Hence the generalised derivative of the unit step function is the Dirac delta function. Taking antiderivatives gives

$$u(t) = \int_{-\infty}^{t} \delta(\lambda)\, d\lambda \tag{1.15}$$

Note that (1.15) *agrees with* (1.3).

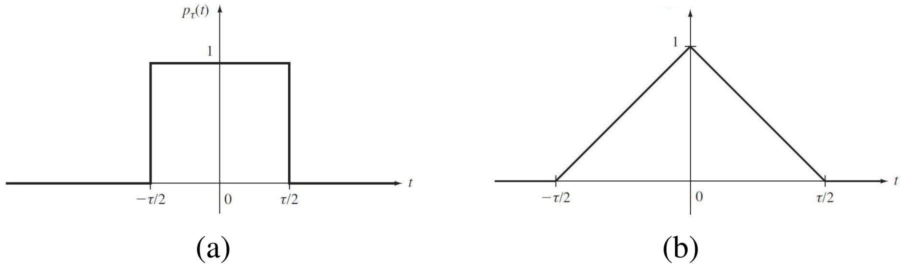


Figure 1.8: (a) Rectangular pulse signal $p_\tau(t)$ (b) Triangular pulse signal $\Lambda(t)$

Definition 1.1.7 (Finite duration signals) *A signal $x : \mathbf{R} \mapsto \mathbf{R}$ is of* **finite duration** *if there exist constants a and b such that $x(t) = 0$ for all $t < a$ and all $t > b$. If a and b are respectively the largest and smallest such numbers, then we say that $[a,b]$ is the* **support** *of the function x, and its* **duration** *d is given by $d = b - a$.*

Definition 1.1.8 (Rectangular and Triangular pulse signals) *For any $\tau > 0$, we define the* **rectangular pulse of width** *τ as*

$$p_\tau(t) = \begin{cases} 1, & \textit{if } \frac{-\tau}{2} \leq t < \frac{\tau}{2} \\ 0, & \textit{otherwise} \end{cases} \tag{1.16}$$

and we define the **triangular pulse of width** *τ as*

$$\Lambda_\tau(t) = \begin{cases} 1 - \frac{2|t|}{\tau}, & \textit{if } \frac{-\tau}{2} \leq t < \frac{\tau}{2} \\ 0, & \textit{otherwise} \end{cases} \tag{1.17}$$

These signals are shown in Figure 1.1. The rectangular pulse signal can be expressed in terms of step functions:

$$p_\tau(t) = u\left(t + \frac{\tau}{2}\right) - u\left(t - \frac{\tau}{2}\right) \tag{1.18}$$

The triangular and rectangular pulse signals are related by

$$\Lambda_\tau(t) = \left(1 - \frac{2|t|}{\tau}\right) p_\tau(t) \tag{1.19}$$

Definition 1.1.9 (Periodic signals) *A signal $x : \mathbf{R} \mapsto \mathbf{R}$ is* **periodic** *if there exists a constant $T > 0$ such that for all $t \in \mathbf{R}$,*

$$x(t) = x(t + T) \tag{1.20}$$

If T is the smallest number with this property, then T is the **fundamental period** *of x.*

Theorem 1.1.2 *Let $x_1 : \mathbf{R} \mapsto \mathbf{R}$ and $x_2 : \mathbf{R} \mapsto \mathbf{R}$ be periodic signals with fundamental periods T_1 and T_2 respectively. Then the sum of the signals x_1 and x_2 is a periodic function if and only if T_1/T_2 is a rational number, i.e. if their exist positive coprime integers m and n such that*

$$\frac{T_1}{T_2} = \frac{n}{m} \tag{1.21}$$

If m and n exist, then $x_1 + x_2$ has period $T = mT_1 = nT_2$.

Remark 1.1.1 *Here T is* **usually** *the fundamental period of $x_1 + x_2$, but there are exceptions; see Practice Problem 12.*

Example 1.1.5 (Sinusoidal signals) *For the sinusoidal signal* $x : \mathbf{R} \mapsto \mathbf{R}$

$$x(t) = A\sin(\omega t + \phi) \tag{1.22}$$

shown in Figure 1.9,

- A *is the* **amplitude**.
- ω *is* **angular frequency** *in rad/s, and* ϕ *is the* **phase** *in radians.*
- $f_0 = \omega/2\pi$ *and* $T = 1/f_0$ *are the* **fundamental frequency** *and* **fundamental period**, *respectively.*

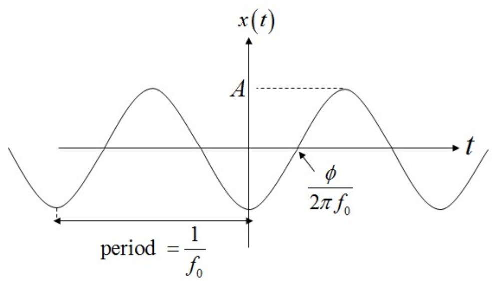


Figure 1.9: Sinusoidal signal

Definition 1.1.10 (Constructing periodic signals from finite duration signals) *Let* $x : \mathbf{R} \mapsto \mathbf{R}$ *be a finite-duration signal with support* $[a,b]$. *Let* $d \geq b - a$ *and define the signal* $y : \mathbf{R} \mapsto \mathbf{R}$ *with*

$$y(t) = \sum_{k=-\infty}^{\infty} x(t - kd) \quad \text{for all } t \in \mathbf{R} \tag{1.23}$$

Then y is called the **shift-and-add summation** *signal of x.*

Theorem 1.1.3 *The signal y defined in* (1.23) *is periodic with fundamental period* $T = d$.

Example 1.1.6 *Consider the finite duration signal x with*

$$x(t) = \begin{cases} 1, & \text{if } 0 \leq t < 1 \\ -1, & \text{if } -1 \leq t < 0 \\ 0, & \text{otherwise} \end{cases} \tag{1.24}$$

Then x has support $[-1,1]$. *Shifting x to the left and right by* $d = 2$ *time units gives*

$$x(t-2) = \begin{cases} 1, & \text{if } 2 \leq t < 3 \\ -1, & \text{if } 1 \leq t < 2 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad x(t+2) = \begin{cases} 1, & \text{if } -2 \leq t < -1 \\ -1, & \text{if } -3 \leq t < -2 \\ 0, & \text{otherwise} \end{cases} \tag{1.25}$$

If we add up all possible $2k$ *time unit shifts of x, we obtain the signal*

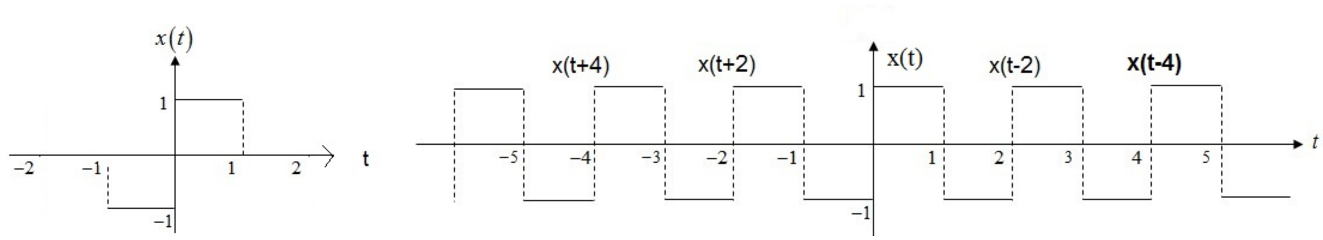
$$y(t) = \sum_{k=-\infty}^{\infty} x(t - 2k) \tag{1.26}$$

with period $T = 2$. *The graph of signal is x is shown in Figure 1.10.*

Definition 1.1.11 (Constructing finite duration signals from periodic signals) *Let* $x : \mathbf{R} \mapsto \mathbf{R}$ *be a periodic signal with period* $T > 0$. *Define the signal* $y : \mathbf{R} \mapsto \mathbf{R}$ *with*

$$y(t) = x(t)p_T(t) \tag{1.27}$$

where $p_T(t)$ *is the rectangular pulse function of width* T. *Then y has finite duration with support* $[\frac{-T}{2}, \frac{T}{2}]$. *It is called a* **rectangular window** *signal of x.*



(a) (b)

Figure 1.10: (a) $x(t)$ (b) Shift-and-add summation signal of x.

1.2 Fundamental properties of discrete-time signals

Definition 1.2.1 (Discrete-time unit step and unit ramp signals) *The* **unit step** *function* $u : \mathbf{Z} \mapsto \mathbf{R}$ *is defined as*

$$u[n] = \begin{cases} 1, & \text{if } n \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{1.28}$$

The **unit ramp** *function* $r : \mathbf{Z} \mapsto \mathbf{R}$ *is defined as*

$$r[n] = \begin{cases} n, & \text{if } n \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{1.29}$$

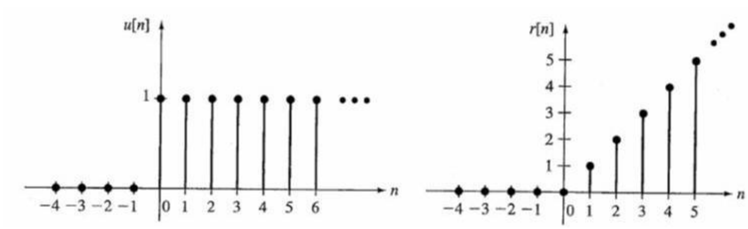


Figure 1.11: discrete unit step signal and discrete unit ramp signal

Definition 1.2.2 (Discrete-time unit pulse and rectangular pulse) *The* **unit pulse function** *function* $\delta : \mathbf{Z} \mapsto \mathbf{R}$ *is defined as*

$$\delta[n] = \begin{cases} 1, & \text{if } n = 0 \\ 0, & \text{otherwise} \end{cases} \tag{1.30}$$

It is also known as the **Kronecker delta** *function. For any positive odd integer* $L > 0$*, the discrete-time* **rectangular pulse of width** L *is*

$$p_L[n] = \begin{cases} 1, & \text{if } \frac{-(L-1)}{2} \leq n \leq \frac{L-1}{2} \\ 0, & \text{otherwise} \end{cases} \tag{1.31}$$

Their graphs are shown in Figure 1.2.2.

Theorem 1.2.1 (Sifting Property of the pulse function) *Let* $x : \mathbf{Z} \mapsto \mathbf{R}$*. Then for integer* n*,*

$$\sum_{i=-\infty}^{\infty} x[i]\delta[n-i] = x[n] \tag{1.32}$$

Definition 1.2.3 (Time shift signals) *For any discrete-time signal* x *and for any positive integer* m*,*

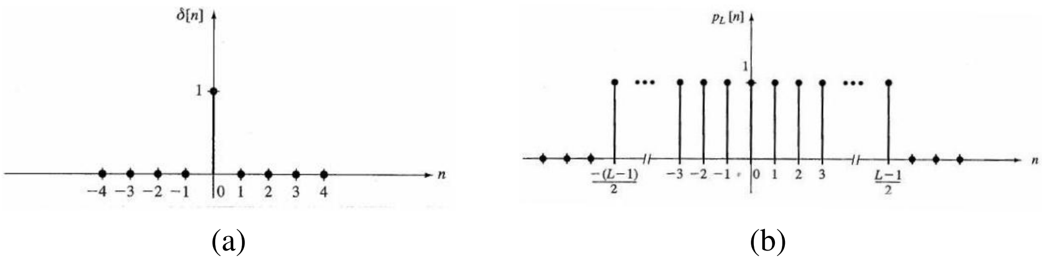


Figure 1.12: discrete (a) unit pulse and (b) rectangular pulse signals

- $x[n-m]$ *is* **right time-shifted by** m **time units**, *compared to* $x[n]$.
- $x[n+m]$ *is* **left time-shifted by** m **time units**, *compared to* $x[n]$.

Remark 1.2.1 (Continuity and differentiation of discrete-time signals) *For continuous-time signals, we defined the concepts of continuity and differentiation in Definitions 1.1.4 and 1.1.5. However, these concepts do not exist for discrete-time signals.*

Definition 1.2.4 (Periodic signals) *A signal* $x : \mathbf{Z} \mapsto \mathbf{R}$ *is* **periodic** *if there exists a positive integer L such that for all* $n \in \mathbf{Z}$,

$$x[n] = x[n+L] \tag{1.33}$$

If L is the least integer number with this property, then L is the **fundamental period** *of x.*

Theorem 1.2.2 *Let* $x_1 : \mathbf{Z} \mapsto \mathbf{R}$ *and* $x_2 : \mathbf{Z} \mapsto \mathbf{R}$ *be periodic signals with fundamental periods* L_1 *and* L_2 *respectively. Then the sum of the signals* x_1 *and* x_2 *is a periodic function with period*

$$L_0 = lcm(L_1, L_2) \tag{1.34}$$

Here $lcm(L_1, L_2)$ *denotes the* **least common multiple** *of* L_1 *and* L_2.

Remark 1.2.2 *Here* L_0 *is, in most cases, the fundamental period of* $x_1 + x_2$*, but there are exceptions.*

Theorem 1.2.3 (Discrete-time sinusoidal signals) *The signal* $x : \mathbf{Z} \mapsto \mathbf{R}$ *with*

$$x[n] = A\cos(\Omega n + \phi) \tag{1.35}$$

is periodic if and only if there exist coprime integers q and r such that

$$\frac{\Omega}{2\pi} = \frac{q}{r} \tag{1.36}$$

If q and r exist, then the period of x is

$$r = \frac{2\pi q}{\Omega} \tag{1.37}$$

Many discrete-time signals are generated by sampling a continuous-time signals. The sampling may be done by closing a switch for a very brief time interval.

Definition 1.2.5 (Uniform sampling signal) *Given a continuous-time signal x and a* **sampling interval** $T > 0$, *we define the* **sampled signal** *by*

$$x[n] = x(nT) \tag{1.38}$$

Example 1.2.1 (Sampling sinusoids) *Let* $x : \mathbf{R} \mapsto \mathbf{R}$ *be the sinusoidal signal*

$$x(t) = \sin(2\pi f_0 t) \tag{1.39}$$

with fundamental frequency f_0. *Suppose we sample* x *with period* T_s, *so the sampling frequency is* $f_s = \frac{1}{T_s}$. *Then the sampled signal is* $x_S : \mathbf{Z} \mapsto \mathbf{R}$

$$x_S[n] = x(nT_s) = \sin(2\pi f n), \quad \textit{where } f = \frac{f_0}{f_s} \tag{1.40}$$

Then x_S *is a periodic discrete-time signal if and only if* f *is a rational number.*

Example 1.2.2 (Sampling sinusoids: Aliasing) *Sampling two different continuous-time sinusoids can give the same discrete-time signal, if the sampling frequency is small enough. This is called* **aliasing** *and is shown in Figure 1.13:*

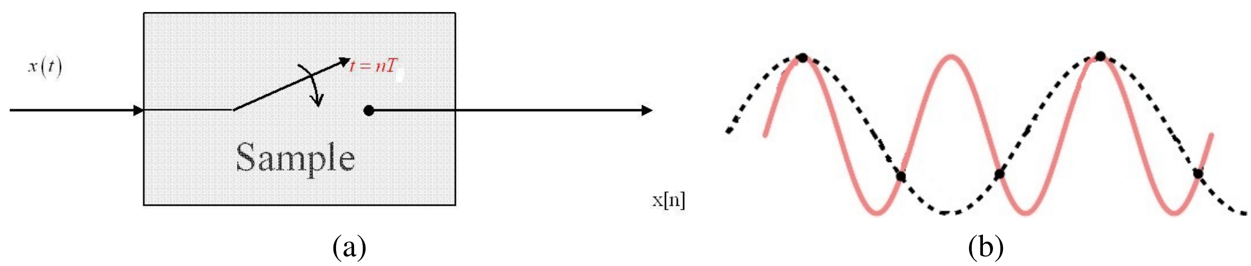


Figure 1.13: (a) Sampling (b) Aliasing

In Figure 1.13(b), the pink sinusoidal curve has been sampled at time intervals indicated by the black dots. However, the black curve is another sinusoidal signal, of smaller frequency, that also passses thorogh these black dots. Hence, using only the black time samples, we cannot know whether the continuous time signal being sampled was the pink or black curve. To avoid aliasing, we would need to increase the sampling frequency; see Problem 14.

Theorem 1.2.4 (Shannon-Nyquist Sampling Theorem) *To uniquely sample a continuous-time sinusoidal signal (and hence avoid aliasing), the sampling frequency* f_s *must be at least twice* f_0, *the fundamental frequency of the sinusoid.*

1.3 Fundamental properties of systems

Definition 1.3.1 (Systems) *Systems convert input signals into output signals:*

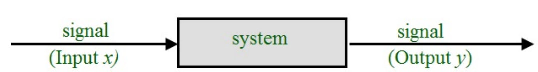
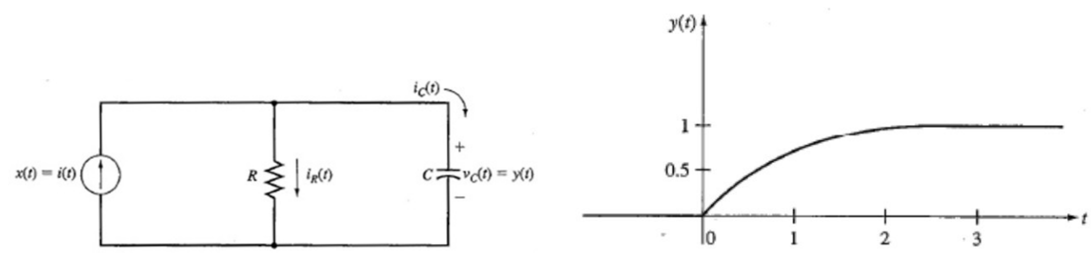


Figure 1.14: Signal and System input-output diagram

- *The* **system behaviour** *consists of all input-output pairs* (x, y), *called* **trajectories**, *that satisfy the laws of the system.*
- *A* **continuous-time system** *operates on continuous-time signals.*
- *A* **discrete-time system** *operates on discrete-time signals.*

- *Systems are usually described by a mathematical model that determines how the outputs are obtained from the inputs, e.g.*
  - *difference equations or differential equations;*
  - *convolution formula (see Chapter 2);*
  - *Laplace or z- Transfer functions (Chapter 9).*
- *If the input $x = 0$, the output y is called the* **zero-input response**. *It is also known as the* **natural response** *or the* **unforced response**. *It depends only upon the initial condition of the system.*
- *If the input $x = u$, the step signal, and the initial condition is zero, then y is called the* **step response**.
- *If the input $x = \delta$, the impulse (or unit pulse) signal, and the initial condition is zero, then y is called the* **impulse (or unit pulse) response**.
- *The input is sometimes called the* **forcing term** *or* **driving function**, *since it is usually the input that causes the system to produce an output.*

Example 1.3.1 (RC circuit) *The RC (resistive-capacitive) electric circuit in Figure 1.15(a) is an example of a continuous-time system.*



(a) (b)

Figure 1.15: (a) *RC* circuit (b) Unit step response

- *The input $x(t)$ is the current source $i(t)$, and the output $y(t)$ is the capacitor voltage $v_C(t)$.*
- *Using Kirchoff's Laws and Ohm's Law, we can derive the differential equation for the system:*

$$C\frac{dy}{dt} + \frac{1}{R}y(t) = x(t)$$

- *To solve for $y(t)$, we need to know $x(t)$ and the initial condition $y(0)$.*

To find the unit step response we must solve

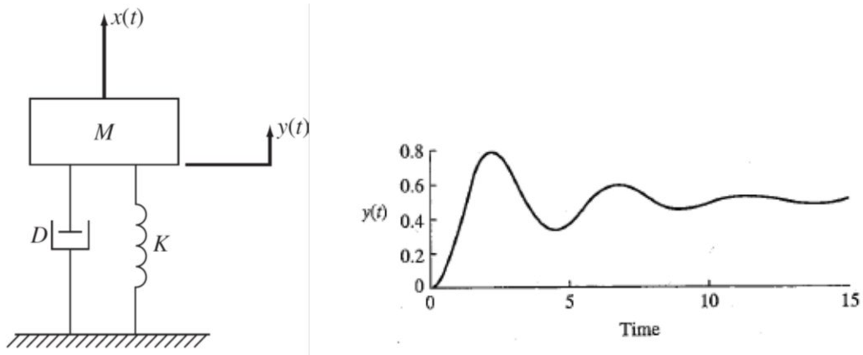
$$C\frac{dy}{dt} + \frac{1}{R}y(t) = u(t)$$

with $y(0) = 0$. The solution is found to be

$$y(t) = R[1 - e^{-t/RC}], \quad \text{for } t \geq 0$$

The step response is the voltage on the capacitor as it charges up under the dc current source. When $R = 1\ \Omega$ and $C = 1\ F$, the step response y is shown in Figure 1.15(b).

Example 1.3.2 (Mass-spring damper) *The mass-spring damper in Figure is 1.16(a) an example of a vibratory system:*



(a) (b)

Figure 1.16: (a) Mass-spring damper (b) Unit step response

- *The input $x(t)$ is the force applied to the mass, and the output $y(t)$ is the vertical displacement of the mass. The forces provided by the spring and damper are proportional to y and $\dot{y}$, respectively.*
- *Newton's Law leads to the differential equation:*

$$M\frac{d^2y}{dt^2} + D\frac{dy}{dt} + Ky(t) = x(t)$$

If we assume $M = 1$ kg, $D = 1Ns/m$, and $K = N/m$ and zero initial conditions $y(0) = \dot{y}(0) = 0$, then the unit step response is shown in Figure 1.16(b).

- *The oscillations occur due to transfers between the kinetic energy of the mass and the potential energy of the spring and damper.*
- *The oscillations eventually decay to zero as the applied force is balanced by the force of the spring.*
- *In very special cases* **resonance** *occurs, in which the oscillations grow with time, and can damage or destroy the system. We will discuss resonance in Chapter 9.*

Example 1.3.3 (Moving-average Filter) *For any positive integer N, the N***-point moving average filter** *is a discrete-time system governed by the difference equation*

$$y[n] = \frac{1}{N}\left[x[n] + x[n-1] + x[n-2] + \cdots + x[n-N+1]\right]$$

- *The filter gives the average value of the N input values $x[n]$, $x[n-1]$, $x[n-2]$, $\ldots$, $x[n-N+1]$.*
- *The filter can be used to 'smooth' data to remove noise (random variations) and reveal underlying trends.*

Definition 1.3.2 (Linear Systems) *A system is* **additive** *if for any two trajectories (x_1, y_1) and (x_2, y_2), the input-output pair $(x_1 + x_2, y_1 + y_2)$ is also a trajectory of the system.*
A system is **homogenous** *if for any trajectory (x, y) and any real number a, the input-output pair (ax, ay) is also a trajectory of the system. A system is* **linear** *if it is both additive and homogenous.*

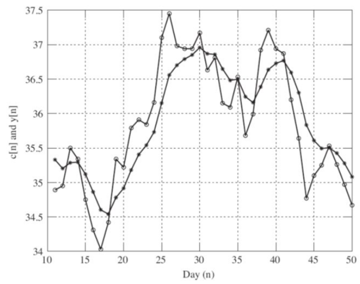


Figure 1.17: Moving Average filter

Theorem 1.3.1 (Linearity theorem) *Let (x_1, y_1) and (x_2, y_2) be trajectories of a system, and let a and b be arbitrary real numbers. Then the system is linear if and only if the input-output pair (ax_1+bx_2, ay_1+by_2) is also a trajectory of the system.*

Example 1.3.4 (Linear System) *The mass-spring damper is an example of a linear system:*

$$M\frac{d^2y}{dt^2}+D\frac{dy}{dt}+Ky(t)=x(t)$$

To see this we let (x_1, y_1) and (x_2, y_2) be trajectories of the system. Then

$$\begin{aligned} M\frac{d^2y_1}{dt^2}+D\frac{dy_1}{dt}+Ky_1(t) &= x_1(t) \\ M\frac{d^2y_2}{dt^2}+D\frac{dy_2}{dt}+Ky_2(t) &= x_2(t) \end{aligned}$$

Next we let a and b be arbitrary real numbers, and consider the signals

$$x_3(t)=ax_1+bx_2, \quad y_3=ay_1+by_2$$

Then

$$\begin{aligned} & M\frac{d^2y_3}{dt^2}+D\frac{dy_3}{dt}+Ky_3(t) \\ = \; & M\frac{d^2(ay_1+by_2)}{dt^2}+D\frac{d(ay_1+by_2)}{dt}+K(ay_1(t)+by_2(t)) \\ = \; & aM\frac{d^2y_1}{dt^2}+aD\frac{dy_1}{dt}+aKy_1(t)+bM\frac{d^2y_2}{dt^2}+bD\frac{dy_2}{dt}+bKy_2(t) \\ = \; & ax_1(t)+bx_2(t) \\ = \; & x_3(t) \end{aligned}$$

Hence (x_3, y_3) is also a trajectory of the system, and we conclude by the Linearity theorem that the mass-spring damper is a linear system.

Example 1.3.5 (Nonlinear system) *The electric circuit in Figure 1.18 contains a diode and hence is a nonlinear system.*

- *The input-output relationship can be shown to be*

$$y(t)=\begin{cases} \frac{R_2x(t)}{R_1+R_2}, & \text{if } x(t)\geq 0 \\ 0, & \text{if } x(t)<0 \end{cases}$$

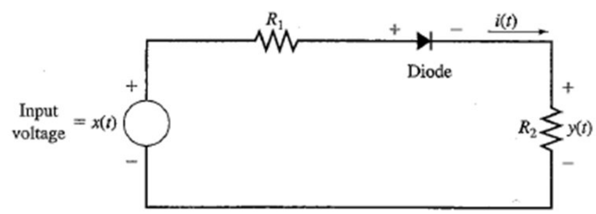


Figure 1.18: Nonlinear circuit

- *Suppose $x(t) = 1$ for all t; then $x(t) \geq 0$ and the output is $y(t) = \frac{R_2}{R_1+R_2}$. So $(1, \frac{R_2}{R_1+R_2})$ is a trajectory of the system.*

  Now consider the scalar $a = -1$ and the input $ax(t) = -1 < 0$. Hence the output is 0, and $(-1, 0)$ is a trajectory of the system. This contradicts the homogeneity requirement that $(-1, \frac{-R_2}{R_1+R_2})$ be a trajectory of the system. So the system is nonlinear.

Definition 1.3.3 (Time-invariant systems) *A system is* **time-invariant** *if, for any trajectory $(x(t), y(t))$ and any constant $T \in \mathbf{R}$, the input-output pair $(x(t-T), y(t-T))$ is also a trajectory of the system. A system is* **time-varying** *if it is not time-invariant.*

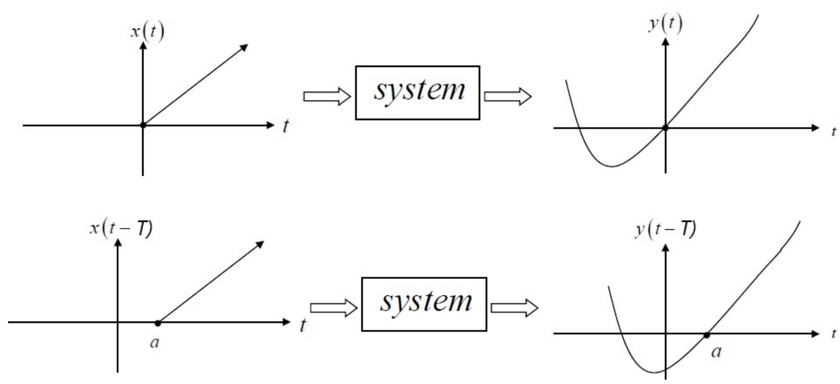


Figure 1.19: Time invariance

Example 1.3.6 (Time-invariant Systems) *The Mass-spring damper is a time-invariant system:*

$$M\frac{d^2y}{dt^2} + D\frac{dy}{dt} + Ky(t) = x(t)$$

To show this we let (x, y) be a trajectory of the system, we $T \in \mathbf{R}$ and introduce the signals

$$\begin{aligned} x_1(t) &= x(t-T) \\ y_1(t) &= y(t-T) \end{aligned}$$

We will show that (x_1, y_1) is also a trajectory of the system.

$$
\begin{aligned}
M\frac{d^2y_1(t)}{dt^2} + D\frac{dy_1(t)}{dt} + Ky_1(t) &= M\frac{d^2y(t-T)}{dt^2} + D\frac{dy(t-T)}{dt} + Ky(t-T) \\
&= M\frac{d^2y(s)}{ds^2}\left(\frac{ds}{dt}\right)^2 + D\frac{dy(s)}{ds}\left(\frac{ds}{dt}\right) + Ky(s) \\
&\quad \textit{using the change of variables} \\
&\quad\quad s = t - T;\ \textit{then } ds = dt \\
&= M\frac{d^2y(s)}{ds^2} + D\frac{dy(s)}{ds} + Ky(s) \\
&= x(s), \quad \textit{as } (x,y) \textit{ is a trajectory} \\
&= x(t-T) \\
&= x_1(t)
\end{aligned}
$$

Thus (x_1, y_1) is also a trajectory of the system, and the system is time-invariant. More examples of time-invariant systems include

- $y[n] = \frac{1}{N}\left[x[n] + x[n-1] + \cdots + x[n-N+1]\right]$ *(Moving average filter)*
- $y(t) = \frac{x^2(t) - x(t-3) + x^3(t+1)}{1 + x^4(t)}$

Example 1.3.7 (Time-varying Systems) *Some examples of time-varying systems are*

- $C\frac{dy}{dt} + \frac{1}{R(t)}y(t) = u(t)$*, where* $R(t) = 2 + \sin(t)$ *(RC circuit with variable resistor).*
- $y(t) = tx(t)$ *(Amplifier with a time-varying gain)*

Definition 1.3.4 (Causal systems) *A system is* **causal** *if, for any time t_1, the output response $y(t_1)$ does not depends on values of the input $x(t)$ for $t > t_1$.*

Example 1.3.8 (Ideal time delay system) *The ideal time delay system*

$$y(t) = x(t-1)$$

is causal because the output at time t_1 depends upon the input at time $t_1 - 1$.

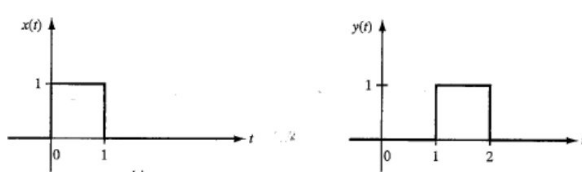


Figure 1.20: Causal system

Example 1.3.9 (Ideal predictor system) *The ideal predictor system*

$$y(t) = x(t+1)$$

is non-causal because the output at time t_1 depends upon the input at time $t_1 + 1$.

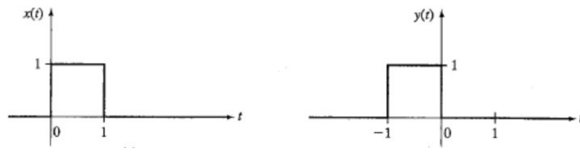


Figure 1.21: Non-causal System

Remark 1.3.1 *Non-causal systems (with time as the independent variable) are impossible to build. However, non-causal systems can occur when the independent variable(s) represent spatial coordinates rather than time, for example in image processing.*

Definition 1.3.5 (Memoryless systems) *A system is* **memoryless** *if, for any time t_1, the output response at time t_1 depends only upon the input at time t_1.*

Definition 1.3.6 (Systems with memory) *A causal system has* **memory** *if, for some time t_1, the output response at time t_1 depends upon the input at time $t < t_1$.*

Example 1.3.10

- *The (output) voltage $v_R(t)$ across a resistor with (input) current $i(t)$ is a memoryless system.*

$$v_R(t) = Ri(t) \quad \textit{(Ohm's Law)}$$

- *The capacitor voltage $v_C(t)$ from a current $i(t)$ has memory:*

$$v_C(t) = \frac{1}{C}\int_{t_0}^{t} i(s)\, ds + v_C(t_0), \quad \textit{for } t \geq t_0$$

1.4 Practice Problems

1. Sketch the following continuous-time signals for $-3 \leq t \leq 3$. Verify your answer by plotting with MATLAB.

   (a) $x(t) = u(t+1) - 2u(t-1) + u(t-3)$

   (b) $x(t) = (t+1)u(t-1) - tu(t) - u(t-2)$

2. Let $x : \mathbf{R} \to \mathbf{R}$ be a continuous time signal, and let $c \in \mathbf{R}$. Let $v : \mathbf{R} \to \mathbf{R}$ be defined such that

$$x(t)u(t-c) = v(t-c)u(t-c)$$

   (a) Assuming $x(t) = t^2 - t + 1$ and $c = 2$, write down a possible formula for $v(t)$.

   (b) Show that your answer is not unique by writing down another formula $v_1(t)$ that also satisfies this equation.

3. Sketch the following discrete-time signals for $-4 \leq n \leq 4$. Verify your answers by plotting with MATLAB.

   (a) $x[n] = \sin[\pi n/4]$

   (b) $x[n] = 2^n u[n]$

   (c) $x[n] = (n+2)u[n+2] - 2u[n] - nu[n-4]$

   (d) $x[n] = \delta[n+1] - \delta[n] + u[n+1] - u[n-2]$

4. For any positive integer N, the N-point Moving Average filter is given by the input-output relationship

$$y[n] = \frac{1}{N}\left[x[n] + x[n-1] + x[n-2] \cdots + x[n-N+1]\right]$$

   Obtain expressions for the filter's output when

   (a) $x[n] = \delta[n]$, the unit-pulse function.

   (b) $x[n] = u[n]$, the unit-step function.

   (c) $x[n] = r[n]$, the unit-ramp function.

5. Which of the following discrete-time signals are periodic? If the signal is periodic, write down its fundamental period.

   (a) $x[n] = \cos(n)$

   (b) $x[n] = \cos(n\sqrt{5})$

   (c) $x[n] = \cos(n\sqrt{5}) + \cos(n\sqrt{2})$

   (d) $x[n] = \cos(\pi n) + \cos(\frac{3\pi}{2}n)$

6. Which of the following continuous-time signals are periodic? If the signal is periodic, write down its fundamental period.

   (a) $x(t) = \cos(t)$
   (b) $x(t) = \cos(t\sqrt{5})$
   (c) $x(t) = \cos(t\sqrt{5}) + \cos(t\sqrt{2})$
   (d) $x(t) = \cos(\pi t) + \cos(\frac{3\pi}{2}t)$

7. Consider the signal $x : \mathbf{R} \to \mathbf{R}$ given by:

$$x(t) = \sum_{l=-\infty}^{\infty} f(t - 4l)$$

   where the signal f is defined for all $t \in \mathbf{R}$ as

$$f(t) = \begin{cases} 1, & 0 \leq t < 2 \\ 0, & \text{otherwise.} \end{cases}$$

   Sketch $x(t)$ for $t \in [-6, 6]$, indicating all relevant values.

8. Determine whether the following systems are (i) causal or noncausal and (ii) have memory or memoryless (iii) linear or nonlinear in (x, y) (iv) time-invariant or time-varying. In each case $x(t)$ is an arbitrary input and $y(t)$ is the output response from $x(t)$.

   (a) $y(t) = e^{x(t)}$
   (b) $y(t) = \sin(t)x(t)$
   (c) $y(t) = \int_0^t (t - \lambda)x(\lambda)\, d\lambda$ (Assume $t \geq 0$).
   (d) $y(t) = \int_0^t \lambda x(\lambda)\, d\lambda$ (Assume $t \geq 0$).

9. Which of the following continuous-time systems are linear in the input-output trajectories (v, y)? In each case, justify your answer with either a proof or a counterexample.

   (a) $\dot{y}(t) - 2y(t) = v(t)$
   (b) $y(t) = v(t - 1)$
   (c) $y(t) = \cos(v(t))$
   (d) $y(t) = v(\sin(t))$
   (e) $y(t) = \begin{cases} v(t), & \text{for } |v(t)| \leq 2 \\ 2, & \text{otherwise} \end{cases}$

10. Which of the following discrete-time systems are linear in the input-output trajectories (v, y)? Justify your answer with either a proof or a counterexample.

    (a) $y[n+1]+n^2y[n] = v[n]$

    (b) $y[n] = 1+v[n-2]$

    (c) $y[n] = v^{\frac{3}{2}}[n]$

    (d) $y[n] = \begin{cases} v[n], & \text{for } n \text{ odd} \\ v[n-2], & \text{for } n \text{ even} \end{cases}$ (here we regard $n = 0$ as an even number).

11. Consider the discrete-time signal x given by

$$x[n] = 1 - \sum_{k=3}^{\infty} \delta[n-1-k]$$

Find the values of the integers M and n_0 such that

$$x[n] = u[Mn - n_0]$$

12. Let $x_1 : \mathbf{R} \mapsto \mathbf{R}$ and $x_2 : \mathbf{R} \mapsto \mathbf{R}$ be periodic signals with fundamental periods T_1 and T_2 respectively. Assume there exist positive coprime integers m and n such that

$$\frac{T_1}{T_2} = \frac{n}{m}$$

    (a) Show that the signal $x_1 + x_2$ is a periodic function with period $T_0 = mT_1 = nT_2$.

    (b) Give an example to show that T_0 may not be the fundamental period of $x_1 + x_2$.

13. Let x be a 2-periodic discrete-time signal, i.e. $x[n] = x[n+2]$ for all $n \in \mathbf{Z}$. Let y be the discrete-time signal given by

$$y[n] = x[n] + x[n-1]$$

Show that y is a constant signal, i.e. there exists $c \in \mathbf{R}$ such that $y[n] = c$ for all $n \in \mathbf{Z}$.

14. Let $x_1 : \mathbf{R} \mapsto \mathbf{R}$ be given by $x_1(t) = \cos(10\pi t)$.

    (a) What is the fundamental frequency f_1 of x_1?

    (b) Let y_1 be the discrete-time signal obtained from sampling x_1 with sampling frequency $f_s = 6\ Hz$. Then $y_1[n] = x_1(nT_s)$, where $T_s = 1/f_s$.

    Show that aliasing of x_1 occurs at this sampling frequency by considering the sinusoidal signal $x_2 : \mathbf{R} \mapsto \mathbf{R}$ with

$$x_2(t) = \cos(2\pi f_2 t)$$

    where $f_2 = f_s - f_1$. Let y_2 be the discrete-time signal obtained from sampling x_2 with sampling frequency f_s; then $y_2[n] = x_2(nT_s)$. Show that for all $n \in \mathbf{Z}$

$$y_1[n] = y_2[n]$$

    (c) What is longest sampling period T_s that can be used to sample x_1 without aliasing?

15. A continuous-time linear system is such that the pairs of signals

$$(e^{j2t}, e^{j3t}) \quad \text{and} \quad (e^{-j2t}, e^{-j3t})$$

are both input-output trajectories of the system.

(a) Determine the output $y_1(t)$ arising from the input $x_1(t) = \cos(2t)$.

(b) Determine the output $y_2(t)$ arising from the input $x_2(t) = \cos(2(t - \frac{1}{2}))$.

(c) Is this system time-invariant? Give a reason for your answer.

Hint: Use Euler's Identities.

16. A discrete-time system is such that the input-output trajectories are given by

$$y[n] = x[n](g[n] + g[n-1])$$

Suppose that for all $n \in \mathbf{Z}$, $g[n]$ is given by

(a) $g[n] = 1$.

(b) $g[n] = n$.

(c) $g[n] = 1 + (-1)^n$.

In each case decide whether the system is time-invariant, giving reasons for your answer.

17. (a) Assume a time-invariant system has input-output trajectories(x, y). Show that if x is periodic with period T, then y is also periodic with period T.

(b) Let $x_1 : \mathbf{Z} \mapsto \mathbf{R}$ and $x_2 : \mathbf{Z} \mapsto \mathbf{R}$ be periodic signals with fundamental periods L_1 and L_2 respectively.

i. Show that the sum of the signals x_1 and x_2 is a periodic function with period

$$L_0 = lcm(L_1, L_2)$$

ii. Give an example to show that L_0 may not be the fundamental period of $x_1 + x_2$.

18. Let $N > 0$ be any integer, and let $\alpha \neq 1$ be any complex number.

(a) Use Mathematical Induction to prove that,

$$(i) \sum_{n=1}^{N} n\alpha = \frac{1}{2}N(N+1)\alpha, \quad (ii) \sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}$$

(b) Show that if $|\alpha| < 1$,

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}$$

(c) Show that for any two positive integers $k_1 < k_2$,

$$\sum_{n=k_1}^{k_2} \alpha^n = \frac{\alpha^{k_1} - \alpha^{k_2+1}}{1-\alpha}$$

Chapter 2

Time-domain Models of Systems

In this chapter we look at systems in input-output form and find ways of ways of solving the system equations to give analytic expressions for the outputs. The methods we will use are

- Recursive methods (for discrete-time systems only)
- Convolution (for both discrete-time and continuous-time systems)

2.1 Systems defined by difference equations

Definition 2.1.1 (N-th order difference equation) *Let N be a positive integer. We define the N-***th order causal linear time-invariant input/output difference equation** *to be*

$$y[n]+\sum_{i=1}^{N} a_i y[n-i]=\sum_{i=0}^{N} b_i x[n-i], \quad \text{for } n \geq 0 \tag{2.1}$$

where

- *$x[n]$ and $y[n]$ are the inputs and outputs, respectively;*
- *the coefficients a_i and b_i are real constants.*

Algorithm 2.1.1 (Recursive solution) *Given N initial conditions $y[-1],\ y[-2],\ \ldots,y[-N]$, and for a given input $x[n]$ with N initial conditions $x[-1],\ x[-2],\ \ldots,x[-N]$, the N-th order difference equation may be solved by*

1. *Set $n=0$ and obtain $y[0]$ by solving*

$$y[0]=-\sum_{i=1}^{N} a_i y[-i]+\sum_{i=0}^{N} b_i x[-i] \tag{2.2}$$

2. *Set $n=1$ and obtain $y[1]$ by solving*

$$y[1]=-\sum_{i=1}^{N} a_i y[1-i]+\sum_{i=0}^{N} b_i x[1-i] \tag{2.3}$$

3. *Repeat for $n=2,\ldots,N$. This process is called N-***th order recursion***.*

Example 2.1.1 (Bank Account) *The difference equation for the balance of a bank account is:*

$$y[n]-(1+i)y[n-1]=x[n], \quad \textit{for } n \geq 1 \tag{2.4}$$

Here i is the monthly interest rate, $y[0]$ is the initial balance of the account and $x[n]$ is the monthly deposit, commencing at month $n=1$ (and hence $x[0]=0$). Applying the recursive method yields

$$\begin{aligned}
y[1] &= (1+i)y[0]+x[1],\\
y[2] &= (1+i)y[1]+x[2],\\
&= (1+i)^2y[0]+(1+i)x[1]+x[2]\\
y[3] &= (1+i)y[2]+x[3]\\
&= (1+i)^3y[0]+(1+i)^2x[1]+(1+i)x[2]+x[3]
\end{aligned}$$

Thus for $n \geq 1$,

$$y[n]=(1+i)^n y[0]+\sum_{i=1}^{n}(1+i)^{n-i}x[i] \tag{2.5}$$

Remark 2.1.1 *The recursive method often leads to cumbersome expressions for the solution, as in equation* (2.5). *In the next section, we see how difference equations can be solved with convolution, and in Chapter 7, we will see how to solve them using the method of z-Transforms.*

Definition 2.1.2 (Unit pulse response) *For a discrete-time system defined by the difference equation* (2.1)*, the system's* **unit pulse response** *, denoted by $h[n]$, is defined to be the output of the system when the input is $x[n]=\delta[n]$, and the initial conditions are assumed to be zero:* $0=y[-1]=y[-2]=\cdots=y[-N]$.

2.2 Convolution for discrete-time systems

Convolution comes with a Health Warning:

Definition 2.2.1 (from the Oxford online dictionary)

1. *The process of becoming coiled or twisted.*
2. *A thing that is complex and difficult to follow.*
3. *A sinuous fold in the surface of the brain.*

Definition 2.2.2 (Convolution for signals) *Let x and y be discrete-time signals. The we define the* **convolution of** x **and** y *as the discrete-time signal*

$$(x\star y)[n]=\sum_{i=-\infty}^{\infty}x[i]y[n-i] \tag{2.6}$$

Definition 2.2.3 *For any discrete-time signal y and any integer i, we define the* **Delay signal of** y *as*

$$Delay_i(y[n])=y[n-i] \tag{2.7}$$

Theorem 2.2.1 (Properties of convolution)

- *Commutativity:* $x\star y=y\star x$
- *Associativity:* $(x\star y)\star z=x\star(y\star z)$
- *Distributivity:* $x\star(y+z)=x\star y+x\star z$

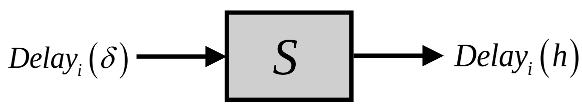
- *Homogeneity: For all* $a \in \mathbf{R}$, $\quad x \star (ay) = a(x \star y)$
- *Time-invariance: For all* $N \in \mathbf{Z}$, $Delay_N(x \star y) = x \star (Delay_N(y))$

Next we see how convolution can be used to compute the input-output relationship of a linear time-invariant system in terms of system's unit pulse response. Recall that by the Sifting Property, every discrete-time signal can be represented as

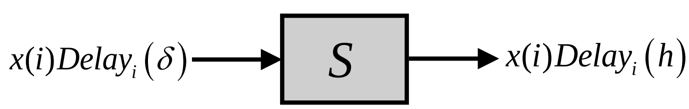
$$x[n] = \sum_{i=-\infty}^{\infty} x[i]\delta[n-i] = \sum_{i=-\infty}^{\infty} x[i]Delay_i(\delta[n]) \tag{2.8}$$

- For any linear time-invariant system S, the unit pulse response h is the response to δ:

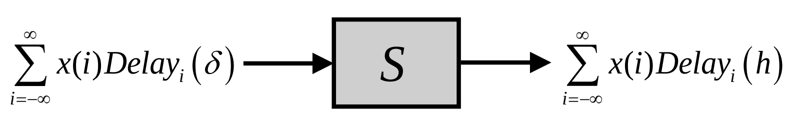
- Since S is time-invariant, the delayed input $Delay_i(\delta)$ produces output $Delay_i(h)$:



- Since S is linear, multiplying an input signal by a real value $x[i]$ will multiply the corresponding output signal by the same value:

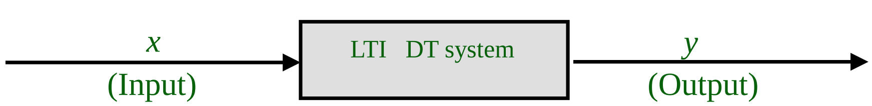


- Since S is linear, sums of inputs produce outputs that are sums of those inputs:



- Notice that

$$\sum_{i=-\infty}^{\infty} x[i]Delay_i(h[n]) = \sum_{i=-\infty}^{\infty} x[i]h[n-i] = (x \star h)[n] \tag{2.9}$$



These results naturally lead to the following theorem:

Theorem 2.2.2 *For any LTI discrete-time system with zero initial conditions, the output y from any input x can be expressed in terms of the convolution of x with the unit pulse response h:*

$$y[n] = (x \star h)[n] = \sum_{i=-\infty}^{\infty} x[i]h[n-i] \tag{2.10}$$

Theorem 2.2.3 (Convolution for finite-duration signals) *Let x and y be finite duration discrete-time signals, such that the support of x is* $[n_x, N_x]$*, and the support of y is* $[n_y, N_y]$*. Then their convolution has support* $[n_x + n_y, N_x + N_y]$*, and is given by*

$$(x \star y)[n] = \begin{cases} 0, & \text{if } n < n_x + n_y \\ \sum_{i=n_x}^{N_x} x[i]y[n-i], & \text{if } n_x + n_y \le n \le N_x + N_y \\ 0 & \text{if } n > N_x + N_y \end{cases} \tag{2.11}$$

By the commutativity of convolution, we also have

Corollary 2.2.1

$$(x \star y)[n] = \begin{cases} 0, & \text{if } n < n_x + n_y \\ \sum_{i=n_y}^{N_y} x[n-i]y[i], & \text{if } n_x + n_y \le n \le N_x + N_y \\ 0 & \text{if } n > N_x + N_y \end{cases} \tag{2.12}$$

Definition 2.2.4 (The Flip operation) *For any discrete signal* $x[n]$*, we define the signal* $Flip(x)$ *as*

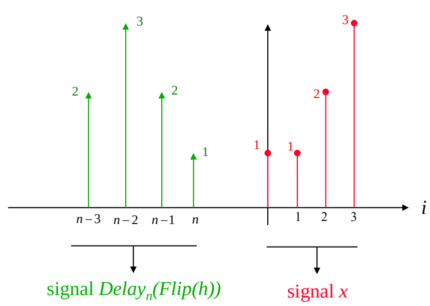
$$Flip(x[n]) = x[-n] \tag{2.13}$$

The flip operation reflects the signal in the vertical axis. This can be helpful for computations.

Example 2.2.1 (Graphical method for computing a convolution) *Let x and h be finite-duration signals with*

$$\begin{aligned} (x[0], x[1], x[2], x[3]) &= (1,1,2,3) \quad \text{with } x[n] = 0 \text{ otherwise} \\ (h[0], h[1], h[2], h[3]) &= (1,2,3,2) \quad \text{with } h[n] = 0 \text{ otherwise} \end{aligned}$$

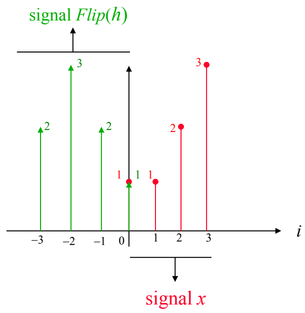
Thus $n_x = n_h = 0$*, and* $N_x = N_h = 3$*. So their convolution will be non-zero only for* $0 \le n \le 6$*.*
We will use the graphical method for computing their convolution. We begin by plotting the signals $Delay_n(Flip(h[i]))$ *and* $x[i]$ *on the same axes:*

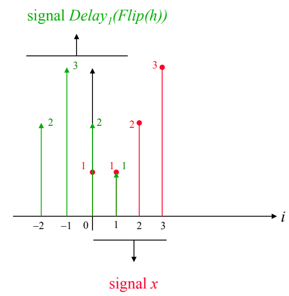


For $n < 0$ *, there is no overlap in the non-zero values of x and* $Delay_n(Flip(h))$*, so* $x[i]h[n-i] = 0$ *for all i. Hence*

$$y[n] = \sum_{i=-\infty}^{\infty} x[i]h[n-i] = 0 \quad \text{for } n < 0$$

For $n = 0$*, x and* $Delay_0(Flip(h))$ *overlap at* $i = 0$*.*
For $n = 1$*, x and* $Delay_1(Flip(h))$ *overlap at* $i = 0, 1$*.*

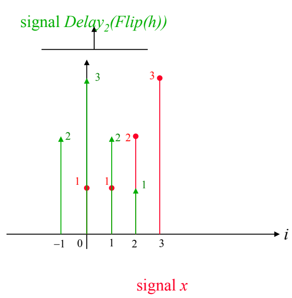
$n = 0$: 

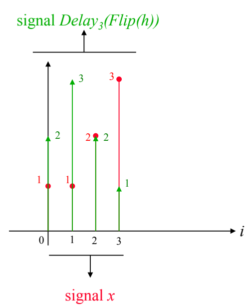
$n = 1$ 

$$y[0] = \sum_{i=-\infty}^{\infty} x[i]h[-i] = x[0]h[0] = (1)(1) = 1$$

$$y[1] = \sum_{i=-\infty}^{\infty} x[i]h[1-i] = x[0]h[1] + x[1]h[0] = (1)(2) + (1)(1) = 3$$

For $n = 2$, x and $Delay_2(Flip(h))$ overlap for $i = 0, 1, 2$.
For $n = 3$, x and $Delay_3(Flip(h))$ overlap for $i = 0, 1, 2, 3$.

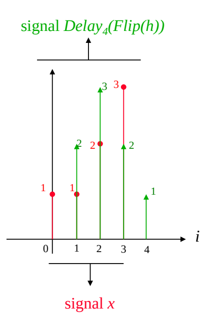
$n = 2$: 

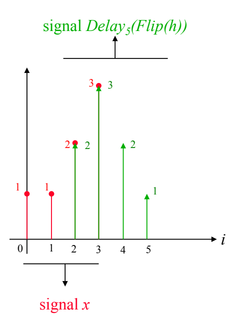
$n = 3$ 

$$y[2] = \sum_{i=-\infty}^{\infty} x[i]h[2-i] = x[0]h[2] + x[1]h[1] + x[2]h[0] = 7$$

$$y[3] = \sum_{i=-\infty}^{\infty} x[i]h[3-i] = x[0]h[3] + x[1]h[2] + x[2]h[1] + x[3]h[0] = 12$$

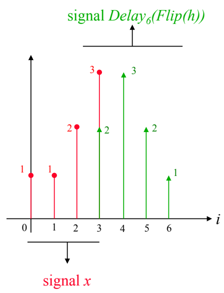
For $n = 4$, x and $Delay_4(Flip(h))$ overlap for $i = 1, 2, 3$.
For $n = 5$, x and $Delay_5(Flip(h))$ overlap for $i = 2, 3$.

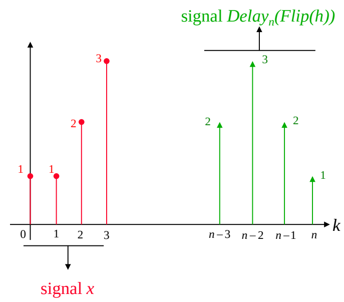
$n = 4$: 

$n = 5$: 

$$y[4] = \sum_{i=-\infty}^{\infty} x[i]h[4-i] = x[1]h[3] + x[2]h[2] + x[3]h[1] = 14$$

$$y[5] = \sum_{i=-\infty}^{\infty} x[i]h[5-i] = x[2]h[3] + x[3]h[2] = 13$$

$n = 6$: 

$n \geq 7$: 

For $n = 6$, x and $Delay_6(Flip(h))$ overlap for $i = 3$.
For $n \geq 7$, x and $Delay_n(Flip(h))$ do not overlap.

$$\begin{aligned} y[6] &= \sum_{i=-\infty}^{\infty} x[i]h[6-i] = x[3]h[3] = 6 \\ y[n] &= 0 \textit{ for } n \geq 7 \end{aligned}$$

Another method of computing convolutions is the **Array method**.

Theorem 2.2.4 *Let x and y be finite duration discrete-time signals with support $[n_x, N_x]$ and $[n_y, N_y]$ respectively. Then their* **convolution array** *is given by*

	$x[n_x]$	$x[n_x+1]$	$\dots$	$x[N_x]$
$y[n_y]$	$x[n_x]y[n_y]$	$x[n_x+1]y[n_y]$	$\dots$	$x[N_x]y[n_y]$
$y[n_y+1]$	$x[n_x]y[n_y+1]$	$x[n_x+1]y[n_y+1]$	$\dots$	$x[N_x]y[n_y+1]$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$y[N_y]$	$x[n_x]y[N_y]$	$x[n_x+1]y[N_y]$	$\dots$	$x[N_x]y[N_y]$

The values of $(x \star y)[n]$ are given by the sums of the elements on the backwards diagonals, where the diagonal beginning at $x[n_x + i]$ and finishing at $y[n_y + i]$ is summed to give $(x \star y)[n_x + n_y + i]$.

Example 2.2.2 *Let x and h be the discrete-time signals in the previous example. Their convolution array is*

		$x[0]$	$x[1]$	x[2]	$x[3]$
		1	1	2	3
$h[0]$	1	1	1	2	3
$h[1]$	2	2	2	4	6
$h[2]$	3	3	3	6	9
$h[3]$	2	2	2	4	6

Then

$$\begin{aligned} y[0] &= 1,\ y[1] = 1+2 = 3,\ y[2] = 2+2+3 = 7; \\ y[3] &= 3+4+3+2 = 12,\ y[4] = 6+6+2 = 14,\ y[5] = 9+4 = 13, \\ y[6] &= 6,\ y[n] = 0, \textit{otherwise} \end{aligned}$$

Theorem 2.2.5 *An LTI discrete-time system is* **causal** *if and only if its unit pulse response h satisfies*

$$h[n] = 0 \quad \textit{for } n < 0 \tag{2.14}$$

Theorem 2.2.6 *Let h be the unit pulse response of a causal LTI system, and let $x[n] = 0$ for $n < 0$. Then the output y, for $n \geq 0$, is given by*

$$y[n] = (h \star x)[n] \quad = \quad \sum_{i=0}^{n} x[i]h[n-i], \quad \textit{since } h[n-i] = 0 \textit{ for } i > n \tag{2.15}$$

2.3 Systems defined by differential equations

Definition 2.3.1 (N-th order differential equation) *For any positive integer N, we define N***-th order causal linear time-invariant input/output differential equation** *to be*

$$\frac{dy^N}{dt^N} + \sum_{i=0}^{N-1} a_i \frac{dy^i}{dt^i} = \sum_{i=0}^{N} b_i \frac{dx^i}{dt^i} \quad \textit{for } t \geq 0 \tag{2.16}$$

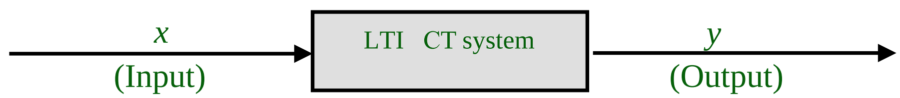
where

- *$x(t)$ and $y(t)$ are the inputs and outputs, respectively;*
- *the coefficients a_i and b_i are real constants.*

Remark 2.3.1 *For differential equations, there is no equivalent solution method to the Recursive method that we used for solving difference equations. In the next section, we see how differential equations can be solved with convolution, and in Chapter 7, we will see how to solve differential equations using the method of Laplace transforms.*

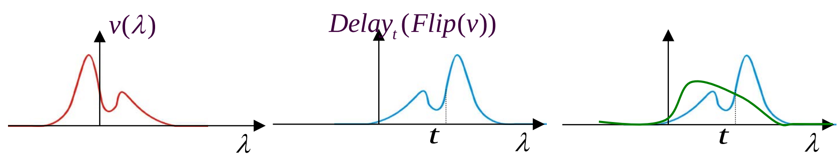
Definition 2.3.2 (Impulse response) *For a continuous-time system defined by the differential equation* (2.16)*, the system's* **impulse response***, denoted by h, is defined to be the output of the system when the input is $x(t) = \delta(t)$ (the Dirac delta function), and the initial conditions are assumed to be zero:* $0 = y(0) = \dot{y}(0) = \cdots = y^{(N-1)}(0)$.

2.4 Convolution for continuous-time systems



Definition 2.4.1 *Let x and v be continuous-time signals. The we define the* **convolution of** *x* **and** *v as the continuous-time signal*

$$(x \star v)(t) = \int_{-\infty}^{\infty} x(\lambda)v(t-\lambda)\, d\lambda \tag{2.17}$$



Theorem 2.4.1 (Properties of continuous-time convolution)

- *Commutativity:* $x \star y = y \star x$
- *Associativity:* $(x \star y) \star z = x \star (y \star z)$
- *Distributivity:* $x \star (y + z) = x \star y + x \star z$
- *Homogeneity: For all* $a \in \mathbf{R}$, $\quad x \star (ay) = a(x \star y)$
- *Time-invariance: For all* $\tau \in \mathbf{R}$, $Delay_\tau(x \star y) = x \star (Delay_\tau(y))$

Theorem 2.4.2 *For any LTI continuous-time system with zero initial conditions, the output y from any input x can be expressed in terms of the convolution of x with the impulse response h:*

$$y(t) = (x \star h)(t) = \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)\, d\lambda = \int_{-\infty}^{\infty} h(t-\lambda)x(\lambda)\, d\lambda \tag{2.18}$$

Theorem 2.4.3 *An LTI continuous-time system is* **causal** *if and only if its impulse response h satisfies*

$$h(t) = 0 \quad \text{for } t < 0 \tag{2.19}$$

For causal systems, we have a nice simplification when the input signal satisfies $x(t) = 0$ for $t < 0$.

Theorem 2.4.4 *Let h be the impulse response of a causal LTI system and let the input x satisfy* $x(t) = 0$ *for* $t < 0$. *Then the output y, for* $t \geq 0$, *is given by*

$$y(t) = (h \star x)(t) = \int_0^t h(\lambda)x(t-\lambda)\, d\lambda = \int_0^t h(t-\lambda)x(\lambda)\, d\lambda \tag{2.20}$$

Example 2.4.1 (RC circuit impulse response) *Recall the RC (resistive-capacitive) electric circuit from Example 1.3.1. It has input-output differential equation*

$$C\frac{dy}{dt} + \frac{1}{R}y(t) = x(t)$$

The impulse response h arising Dirac delta function δ *input can be evaluated as follows:*

1. *Define, for any* $\varepsilon > 0$, *the input signal*

$$x_\varepsilon(t) = \begin{cases} \frac{1}{\varepsilon}, & \text{for } t \leq \varepsilon \\ 0, & \text{otherwise} \end{cases}$$

2. *Solve the differential equation with this input* $x_\varepsilon(t)$ *to obtain the output* $y_\varepsilon(t)$.
3. *Obtain* $h(t) = \lim_{\varepsilon \to 0} y_\varepsilon(t)$.

See Workshops for the details. If $R = C = 1$, *the impulse response for the RC circuit is*

$$h(t) = \begin{cases} e^{-t}, & \text{for } t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

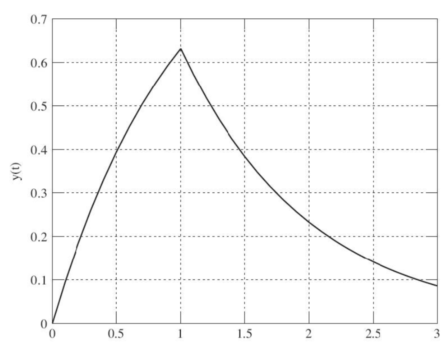
Example 2.4.2 (RC circuit pulse response) *Suppose the RC circuit in equation* (2.4.1) *has a pulse input* $x(t) = u(t) - u(t-1)$. *Assuming zero initial conditions, find the output from this input and plot its graph.*
Solution *As we have zero initial conditions, we can use the impulse response h from Equation* (2.4.1) *and Theorem 2.4.4. Noting that*

$$x(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$$

we obtain, for $t \ge 0$,

$$\begin{aligned} y(t) &= (h \star x)(t) \\ &= \int_0^t e^{-(t-\lambda)} x(\lambda)\, d\lambda \\ &= \begin{cases} \int_0^t e^{\lambda - t}\, d\lambda, & 0 \le t \le 1 \\ \int_0^1 e^{\lambda - t}\, d\lambda, & t \ge 1 \end{cases} \\ &= \begin{cases} \left[e^{\lambda - t}\right]_0^t, & 0 \le t \le 1 \\ \left[e^{\lambda - t}\right]_0^1, & t \ge 1 \end{cases} \\ &= \begin{cases} 1 - e^{-t}, & 0 \le t \le 1 \\ e^{-t}(e-1), & t \ge 1 \end{cases} \end{aligned}$$

The graph of y for $t \ge 0$ *is*



2.5 Discrete approximations to continuous-time systems

Discrete-time systems are often used as approximations for continuous-time systems. The essential link is provided by Euler:

Definition 2.5.1 (Euler approximations) *Let y be a differentiable function and let* $T > 0$. *For integer n, we define the Euler approximation to the first derivative of y as*

$$\left.\frac{dy}{dt}\right|_{t=nT} \approx \frac{y(nT+T) - y(nT)}{T} \tag{2.21}$$

and the Euler approximation to the second derivative of y as

$$\left.\frac{d^2y}{dt^2}\right|_{t=nT} \approx \frac{y(nT+2T) - 2y(nT+T) + y(nT)}{T^2} \tag{2.22}$$

T is referred to as the *step size*. These approximations become more accurate as we reduce T. Consider the first-order system defined by the differential equation

$$\frac{dy}{dt} + ay(t) = bx(t) \quad \text{for } t \geq 0 \tag{2.23}$$

For a given step size T, we introduce the discrete signals $y[n] = y(nT)$ and $x[n] = x(nT)$. Using the Euler approximation to $\frac{dy}{dt}$, we can obtain the first order difference equation

$$y[n+1] + (aT - 1)y[n] = bTx[n] \tag{2.24}$$

This difference equation can then be solved by, for example, the Recursion method to find $y[n]$, the system output y at times $t = nT$. For the system response with initial condition $y(0)$ and zero input $x(t) = 0$, solving the difference equation (2.24) gives

$$y[n] = (1 - aT)^n y(0) \tag{2.25}$$

From ODE theory (or using Laplace transforms, see Chapter 7), we know that the exact solution of the differential equation (2.23) is

$$y(t) = e^{-at} y(0)$$

Thus at time $t = nT$, the value of the exact solution is

$$\begin{aligned} y(nT) &= e^{-anT} y(0) \\ &= (e^{-aT})^n y(0) \\ &= \left(1 - aT + \frac{a^2T^2}{2} - \frac{a^3T^3}{6} + \ldots\right)^n y(0) \end{aligned}$$

from Taylor's theorem. Thus the approximated solution in (2.25) is close when

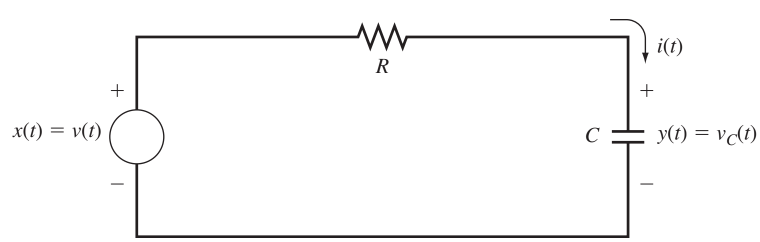
$$1 - aT \approx e^{-aT}$$

which requires $T^2 \approx 0$.

Example 2.5.1 *An RC series circuit can be described by the equation*

$$\frac{dy}{dt} + \frac{1}{RC} y(t) = \frac{1}{RC} x(t) \quad \text{for } t \geq 0$$

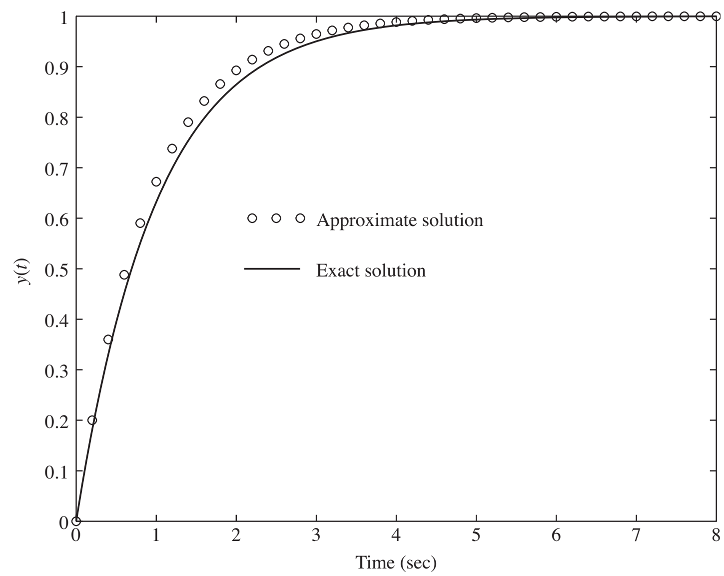
where x is the input voltage and y is the capacitor voltage.



Using Euler approximations, we obtain the difference equation

$$y[n+1] + \left(\frac{T}{RC} - 1\right) y[n] = \frac{T}{RC} x[n]$$

We assume $R = C = 1$, $x(t) = 1$, $y(0) = 0$ and plot both the exact solution and approximate the solution with $T = 0.2$:



- *The approximate solution is close to the exact solution, and this can be improved by using smaller T.*
- *Euler approximations are only one way of discretizing a continuous time system. The Runge-Kutta method is widely used and gives closer approximations.*

2.6 Practice Problems

19. A bank loan balance with monthly repayments can be described as a discrete-time system with difference equation

$$y[n]-(1+i)y[n-1]=-x[n], \quad \text{for } n\geq 1$$

where i is the monthly interest rate, $y[n]$ is the loan balance (= the amount owed to the bank) at the end of month n, and $x[n]=c$ is the constant monthly repayment, made at the end of each month, commencing at $n=1$. Then $y[0]$ is the initial amount of the loan, and assume the loan is to be repaid at the and of month N. Find an expression for c in terms of N, $y[0]$ and i.

20. Compute the unit pulse response $h[n]$ for $n=0,\ 1,\ 2,\ 3$ for the following discrete-time systems. (Note: for the unit pulse response, we assume the system has zero initial conditions , i.e. $y[n]=0$ for $n<0$.)

    (a) $y[n+1]+y[n]=2x[n]$

    (b) $y[n+2]+1.5y[n+1]+0.5y[n]=x[n]$

    (c) $y[n+2]+0.5y[n+1]+0.25y[n]=x[n+1]-x[n]$

21. Consider the discrete-time system with difference equation

$$y[n+2]-0.75y[n+1]+0.125y[n]=0.25x[n+1]$$

Show that the unit pulse response is given by

$$h[n]=((0.5)^n-(0.25)^n)u[n]$$

Hint: Let $y[n]=h[n]$ and show that $x[n]=\delta[n]$.

22. A discrete-time system has first-order difference equation

$$y[n]+a_1y[n-1]=b_0x[n]$$

    (a) Find the values of a_1 and b_0 if the unit pulse response is

$$h[n]=0.3(0.7)^n u[n]$$

    (b) Assuming zero initial conditions, find the output of the system when the input is

$$x[n]=u[n]-u[n-3]$$

    Simplify your answer as much as possible.

23. Compute the convolution of the following pairs of discrete-time signals.

    (a) $x[0]=4,\ x[1]=1,\ x[2]=-1, x[n]=0$ for all other integers n;
    $v[0]=1,\ v[1]=-2,\ v[2]=3, v[3]=-4,\ v[n]=0$ for all other integers n.

    (b) $x[n]=2^n$ for all $n\leq 3$, and $x[n]=0$ for all integers $n\geq 4$; $v[0]=2,\ \ v[1]=-3,\ \ v[2]=0,$ $v[3]=6,\ v[n]=0$ for all other integers n.

(c) $x[n] = \delta[n] - \delta[n-2]$, and $v[n] = \cos(n\pi/3)$ for all integers $n \geq 0$, and $v[n] = 0$ for all integers $n < 0$.

24. (a) Compute the convolution $(u \star u)[n]$, the convolution of the unit step function with itself.
    (b) Let $q \geq 1$ be an integer, and define a discrete-time signal x that is a truncated version of the unit step function as follows: $x[n] = u[n]$ for $n = 0, 1, 2, \ldots, q$, and $x[n] = 0$ otherwise. Find a closed-form expression for $(x \star x)[n]$
    (c) For which values of n does $(u \star u)[n] = (x \star x)[n]$?

25. Assuming zero initial conditions, find the output of the system in Question 21 when the input is

$$x[n] = (n+1)(u[n] - u[n-2])$$

Simplify your answer as much as possible.

26. Evaluate the following integrals

$$(a) \int_{-\infty}^{\infty} \delta(\lambda) e^{-j\omega\lambda}\, d\lambda \quad (b) \int_{-\infty}^{\infty} \delta(2-\lambda) e^{-2(t-\lambda)}\, d\lambda$$

27. For the continuous-time signals shown in Figure 2.1 below, compute the convolution $(x \star v)(t)$ for all $t \geq 0$, and plot the resulting signal.

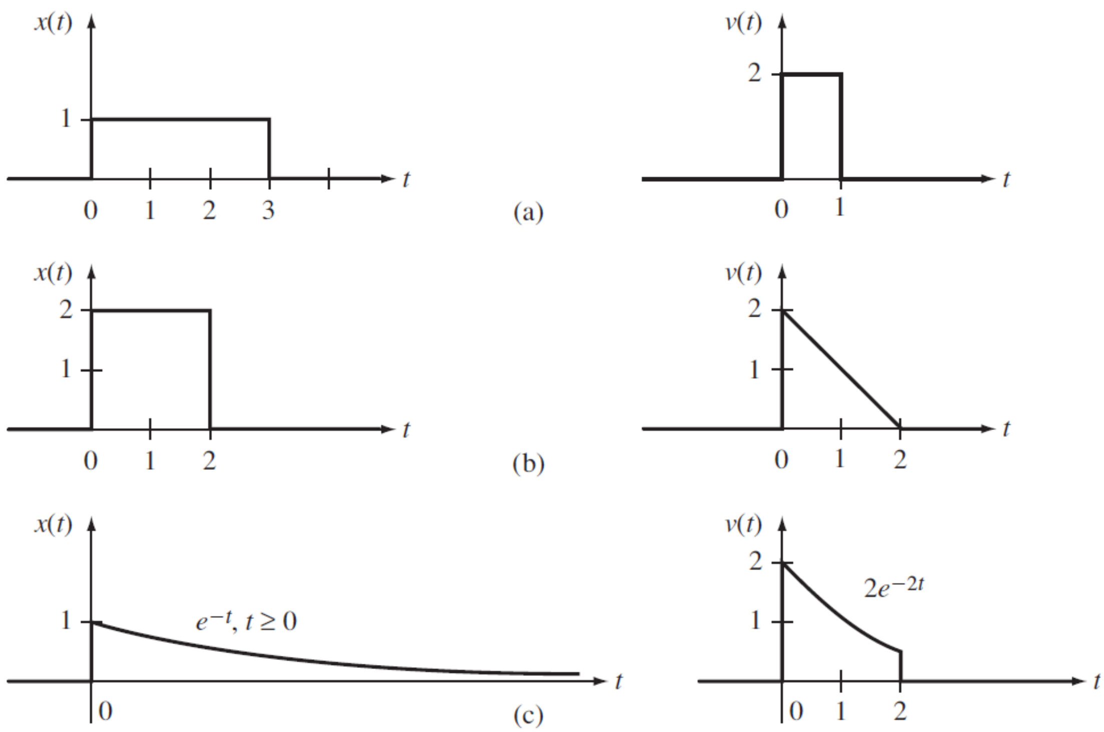


Figure 2.1: Signals for Problem 27

28. (a) Determine the convolution of the following two signals:

$$x(t) = \begin{cases} t+1, & 0 \le t \le 1 \\ 2-t, & 1 < t \le 2 \\ 0, & \text{elsewhere} \end{cases}, \quad h(t) = \delta(t-2) + 2\delta(t-1)$$

(b) Suppose instead $h(t) = \delta(t+2) + 2\delta(t+1)$. Obtain $(x \star h)(t)$ for this h.

29. A continuous-time linear time-invariant system has the input/output relationship

$$y(t) = \int_{-\infty}^{t} (t - \lambda + 2)x(\lambda)\, d\lambda$$

(a) Determine the impulse response of the system.

(b) For $1 \le t \le 2$, determine the output $y(t)$ arising from the input $x(t) = u(t) - 2u(t-1)$.

30. Consider the system with input/output differential equation

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y(t) = x(t)$$

This system has impulse response

$$h(t) = (e^{-2t} - e^{-t})u(t)$$

Assuming zero initial conditions, find the step response of this system.

31. Assume x and h are continuous-time signals such that

$$x(t) = 0, \text{for all } |t| \ge T_1, \quad h(t) = 0, \text{for all } |t| \ge T_2$$

for some positive real numbers T_1 and T_2. Show that

$$(x \star h)(t) = 0, \text{for all } |t| \ge T_3$$

where $T_3 = T_1 + T_2$.

32. Suppose $y(t)$ is the output to an LTI continuous-time system with input $x(t)$. Show that the output of this system to the input $\dot{x}(t)$ is $\dot{y}(t)$. You may use the fact that for a suitably smooth (differentiable) function $f : \mathbf{R}^2 \to \mathbf{R}$, the function $g : \mathbf{R} \to \mathbf{R}$ defined by

$$g(x) = \int_{-\infty}^{\infty} f(t,x)\, dt$$

satisfies

$$\frac{dg(x)}{dx} = \int_{-\infty}^{\infty} \frac{\partial f(t,x)}{\partial x}\, dt$$

33. A dc motor with load can be modelled by the second-order equation

$$I\frac{d^2\theta}{dt^2} + k_d\frac{d\theta}{dt} = \tau(t), \quad \text{for } t \ge 0$$

where θ is the position of the rotor shaft relative to a reference position, τ is the applied torque, I is the moment of inertia of the motor and k_d is the viscous friction coefficient.

Use Euler's method to obtain the difference equation of the discretized system with step size $T > 0$.

Chapter 3

Fourier Series Representations of Signals

In this chapter we study periodic continuous-time signals and introduce Fourier series methods for expressing them as the sum of sinusoidal functions. This leads to the notion of the frequency spectrum of a signal, which describes how a signal may be constructed in terms of its sinusoidal frequency components.

3.1 Properties of sinusoids

Music consists of sinusoidal signals (notes) played simultaneously.

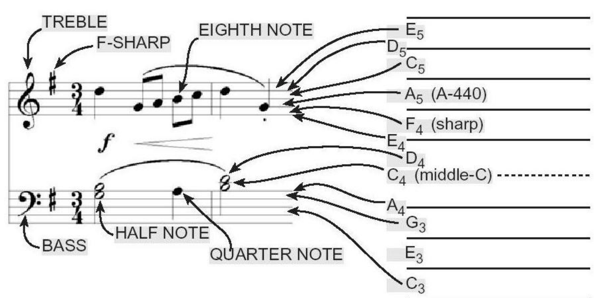


Figure 3.1: Some musical notes

Some of these notes are

- C4 (also known as middle C) is 262 Hz.
- A4 (also known A-440) is 440 Hz.
- F4 ♯ is 370 Hz.

The note A4 is the sinusoidal function

$$A4(t) = A_m \sin(880\pi t)$$

Musicologists describe three properties of a musical note:

- **Loudness**, which is the amplitude A_m.
- **Pitch**, which is the frequency $f = \frac{\omega}{2\pi} = 440\ Hz$.
- **Timbre**, which describes everything about the note that is not loudness or pitch!

When the note A4 is played on a violin and on a piano, the pitch (and perhaps loudness) are the same. Timbre describes the unique sound qualities of a particular instrument - these cannot be precisely described mathematically! So timbre is often described in subjective terms, like colour and warmth.

Example 3.1.1 (Sums of sinusoids) *Consider the signal*

$$x(t) = 0.5\cos t + \cos(4t + \pi/3) + 0.5\cos(8t + \pi/2)$$

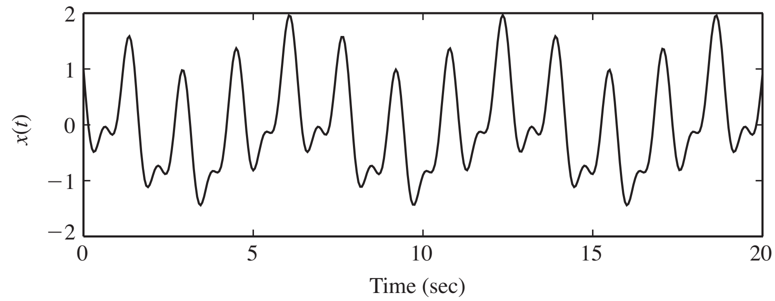
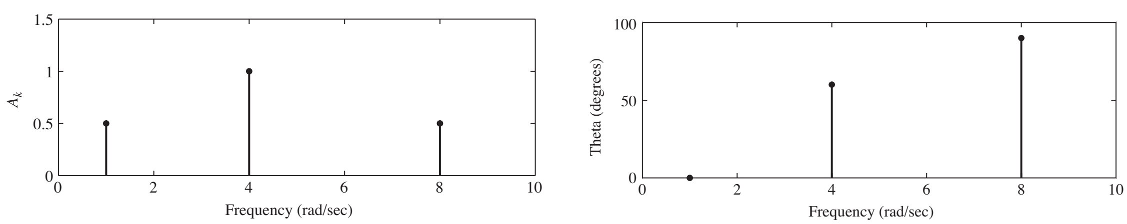


Figure 3.2: Signal $x(t)$

The three sinusoidal signals making up $x(t)$ have frequencies $\omega_1 = 1$, $\omega_2 = 4$, and $\omega_3 = 8$, with corresponding periods $T_1 = 2\pi \approx 6.3$, $T_2 = \frac{\pi}{2} \approx 1.6$ and $T_3 = \frac{\pi}{4} \approx 0.8$.

- *The amplitudes and phase angles of the three sinusoids are $A_1 = 0.5$, $A_2 = 1$ and $A_3 = 0.5$, and $\theta_1 = 0$, $\theta_2 = \pi/3 = 60°$ and $\theta_3 = \pi/2 = 90°$.*
- *If we plot these as functions of the frequencies ω_1, ω_2, and ω_3, we obtain the* **amplitude spectrum** *and* **phase spectrum** *of $x(t)$.*



(a) (b)

Figure 3.3: (a) Amplitude and (b) Phase spectrum of x

- *The amplitude spectrum shows the magnitudes of the three frequency components of $x(t)$.*
- *The frequency spectrum shows which frequencies are represented in the signal.*

3.2 Fourier series

Definition 3.2.1 (Trigonometric Fourier series) *Let x be a periodic continuous-time signal with fundamental period T, and fundamental frequency ω_0. Then the* **trigonometric Fourier series** *for x is given by*

$$x(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)) \tag{3.1}$$

The coefficients a_0, a_k and b_k are known as the **Fourier coefficients** *and may be calculated from* **Euler's formulae***:*

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt \tag{3.2}$$

$$a_k = \frac{2}{T}\int_0^T x(t)\cos(k\omega_0 t)\,dt, \quad k=1,2,\ldots \tag{3.3}$$

$$b_k = \frac{2}{T}\int_0^T x(t)\sin(k\omega_0 t)\,dt, \quad k=1,2,\ldots \tag{3.4}$$

The Fourier coefficients can be computed by integration over any interval of length T:

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt \tag{3.5}$$

$$a_k = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(k\omega_0 t)\,dt, \quad k=1,2,\ldots \tag{3.6}$$

$$b_k = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin(k\omega_0 t)\,dt, \quad k=1,2,\ldots \tag{3.7}$$

Example 3.2.1 *Find the Fourier series for the square wave defined by*

$$x(t) = \begin{cases} -1, & -\pi \le t < 0 \\ 1, & 0 \le t < \pi \end{cases}$$

with $x(t) = x(t+2\pi)$. First we sketch $x(t)$:

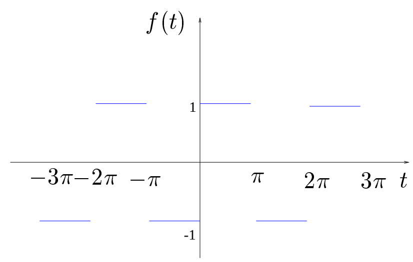


Figure 3.4: Graph of $x(t)$.

x has $T = 2\pi$, so $\omega_0 = 1$. We calculate the Fourier coefficients using Euler's formulae.

$$\begin{aligned}
a_0 &= \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{0} -1\,dt + \frac{1}{2\pi}\int_0^{\pi} 1\,dt \\
&= 0 \\
a_k &= \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(k\omega_0 t)\,dt, \quad \textit{for } k \ge 1 \\
&= \frac{1}{\pi}\int_{-\pi}^{0} -\cos(kt)\,dt + \frac{1}{\pi}\int_0^{\pi}\cos(kt)\,dt \\
&= \frac{1}{\pi}\left[\frac{-\sin(kt)}{k}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{\sin(kt)}{k}\right]_0^{\pi} \\
&= 0
\end{aligned}$$

$$\begin{aligned} b_k &= \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin(k\omega_0 t)\,dt, \quad \textit{for } k \geq 1 \\ &= \frac{1}{\pi}\int_{-\pi}^{0} -\sin(kt)\,dt + \frac{1}{\pi}\int_{0}^{\pi} \sin(kt)\,dt \\ &= \frac{1}{\pi}\left[\frac{\cos(nt)}{k}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{-\cos(nt)}{k}\right]_{0}^{\pi} \\ &= \frac{2(1-\cos(k\pi))}{k\pi} \\ &= \frac{2(1-(-1)^k)}{k\pi} \end{aligned}$$

noting that $\cos(k\pi) = (-1)^k$. *The first few Fourier coefficients are*

$$a_0 = a_k = 0,\ b_1 = \frac{4}{\pi},\ b_2 = 0,\ b_3 = \frac{4}{3\pi},\ b_4 = 0$$

Theorem 3.2.1 (Fourier series representation) *Let x be a periodic continuous-time signal with fundamental period T. Then x can be represented by its Fourier series if it satisfies the* **Dirichlet conditions***:*

1. *x is* **absolutely integrable** *over one period, i.e.*

$$\int_0^T |x(t)|\,dt < \infty \tag{3.8}$$

2. *x has only finitely many distinct maxima and minima over one period.*

3. *x has only finitely many points of discontinuity over one period.*

Remark 3.2.1 *Here are some examples of functions that do not satisfy at least one of the Dirichlet conditions. Their graphs are shown in Figure 3.2.*

1. *Let* $x_1(t) = \frac{1}{t}$ *for* $0 \leq t < T$, *and* $x_1(t) = x_1(t+T)$. *Then*

$$\begin{aligned} \int_0^T |x_1(t)|\,dt &= \int_0^T \frac{dt}{t} \\ &= \log(T) - \log(0) \\ &= \infty \end{aligned}$$

   Other examples of functions that fail the first Dirichlet condition include $x(t) = t^{-\alpha}$, *for any* $\alpha > 1$.

2. *Let* $x_2(t) = \sin\left(\frac{1}{t}\right)$ *for* $0 \leq t < T$, *and* $x_2(t) = x_2(t+T)$. *As* $\sin(t)$ *has local maxima at times* $t_k > 0$ *such that*

$$t_k = 2\pi k + \frac{\pi}{2}$$

   for any positive integer k, it follows that x_2 *has maxima at times* $0 \leq t_k < T$ *such that*

$$\frac{1}{t_k} = 2\pi k + \frac{\pi}{2} \Longleftrightarrow t_k = \frac{1}{2\pi k + \frac{\pi}{2}}$$

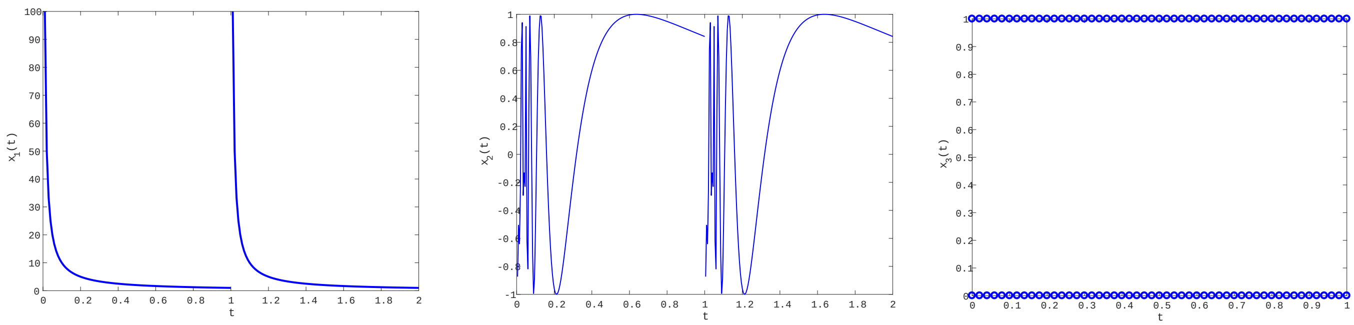
   for all positive integer k. Thus x_2 *has infinitely many distinct local maxima on the interval* $[0,T]$. *It also has infinitely many distinct local minima on the interval* $[0,T]$, *at times* $t_k = \frac{1}{2\pi k - \frac{\pi}{2}}$.

3. *Let x_3 be defined as*

$$x_3(t) = \begin{cases} 1, & \text{if } t \text{ is an irrational number} \\ 0, & \text{if } t \text{ is a rational number} \end{cases}$$

and let $x_3(t) = x_3(t+T)$. Then x_3 is discontinuous everywhere.

It is generally believed that 'virtually all' periodic signals of interest to engineers satisfy the Dirichlet conditions, and hence can be represented by Fourier series.



(a) Condition 1 (b) Condition 2 (c) Condition 3

Figure 3.5: Functions that do not satisfy the Dirichlet conditions

Definition 3.2.2 (Finite Fourier series) *Let x be a periodic continuous-time signal with fundamental period T. For any integer N, we define the* **finite Fourier series** *x_N of x as*

$$x_N(t) = a_0 + \sum_{k=1}^{N} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)\right) \tag{3.9}$$

Theorem 3.2.2 (Convergence of Fourier series I) *If x satisfies the Dirichlet conditions, then*

$$\lim_{N\to\infty} \frac{1}{T} \int_0^T |x(t) - x_N(t)|^2 \, dt = 0, \tag{3.10}$$

where x_N denotes the finite Fourier series for x.

We say that x_N **converges to x in the L_2 norm**, because the mean-squared error converges to zero.

Theorem 3.2.3 (Convergence of Fourier series II) *Let x be a periodic continuous-time signal, and assume x satisfies the Dirichlet conditions. Let x_N be the finite Fourier series of x, and let $t_0 \in \mathbf{R}$. Then*

1. *If x is continuous at t_0, then*

$$\lim_{N\to\infty} x_N(t_0) = x(t_0). \tag{3.11}$$

2. *If x is discontinuous at t_0, then*

$$\lim_{N\to\infty} x_N(t_0) = \frac{1}{2}\left(x(t_0^-) + x(t_0^+)\right), \tag{3.12}$$

where $x(t_0^-)$ and $x(t_0^+)$ are the left and right limits of x at t_0.

Remark 3.2.2 *Theorem 3.2.3 says that*

- *If x is continuous at t_0, then $x_N(t_0)$ converges pointwise to $x(t_0)$.*
- *If x is discontinuous at t_0, then $x_N(t_0)$ converges to the average of the left and right limits of $x(t_0)$.*

Definition 3.2.3 (Even and odd functions)

1. *A signal x is an* **even function** *if*

$$x(t) = x(-t) \quad \textit{for all } t \in \mathbf{R} \tag{3.13}$$

The graph of an even function is symmetric about a reflection in the y-axis. Examples of even functions are: any constant function, $\cos(\omega t)$, and t^2.

2. *A signal x is an* **odd function** *if*

$$x(t) = -x(-t) \quad \textit{for all } t \in \mathbf{R} \tag{3.14}$$

The graph of an odd function is symmetric about a reflection in the y-axis, followed by a reflection in the x-axis. Examples of odd functions are: $\sin(\omega t)$, t and t^3.

Theorem 3.2.4 (Fourier cosine series) *Suppose x is an even periodic signal with period $T = 2L$. Then x may be represented by a* **Fourier cosine series**

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) \tag{3.15}$$

$$\textit{where } a_0 = \frac{1}{L}\int_0^L x(t)\,dt \quad \textit{and} \quad a_k = \frac{2}{L}\int_0^L x(t)\cos(k\omega_0 t)\,dt \tag{3.16}$$

Theorem 3.2.5 (Fourier sine series) *Suppose x is an odd periodic signal with period $T = 2L$. Then x may be represented by a* **Fourier sine series**

$$x(t) = \sum_{k=1}^{\infty} b_k \sin(k\omega_0 t) \tag{3.17}$$

$$\textit{where} \qquad b_k = \frac{2}{L}\int_0^L x(t)\sin(k\omega_0 t)\,dt \tag{3.18}$$

3.3 Alternative Fourier Series Representations

Definition 3.3.1 (Cosine-with-phase Fourier series) *Let x be a periodic continuous-time signal with fundamental period T, and frequency ω_0. The* **cosine-with-phase Fourier series** *for x is given by*

$$x(t) = a_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k) \tag{3.19}$$

The coefficients A_k and θ_k may be obtained from the trigonometric series coefficients in Equations (3.2), (3.3), *and* (3.4) *as follows:*

$$A_k = \sqrt{a_k^2 + b_k^2} \tag{3.20}$$

$$\theta_k = \begin{cases} \tan^{-1}\left(\frac{-b_k}{a_k}\right), & k = 1, 2, \ldots, \textit{when } a_k \geq 0 \\ \pi + \tan^{-1}\left(\frac{-b_k}{a_k}\right), & k = 1, 2, \ldots, \textit{when } a_k < 0 \end{cases} \tag{3.21}$$

Definition 3.3.2 (Complex Fourier series) *Let x be a periodic continuous-time signal with fundamental period T, and frequency ω_0. The* **complex Fourier series** *for x is given by*

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \tag{3.22}$$

The **complex Fourier coefficients** *c_k may be calculated from*

$$c_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}\,dt, \quad k = 0, \pm1, \pm2, \ldots \tag{3.23}$$

Alternatively, we may integrate over any interval of length T, e.g.

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}\,dt, \quad k = 0, \pm1, \pm2, \ldots \tag{3.24}$$

Note that

$$c_k = \bar{c}_{-k}, \qquad k = 1, 2, \ldots \tag{3.25}$$

Theorem 3.3.1 (Complex Fourier series) *For any periodic continuous-time signal x with fundamental period T, and fundamental frequency ω_0, the trigonometric and complex Fourier coefficients are related by*

$$\begin{aligned} c_0 &= a_0 && (3.26)\\ c_k &= \frac{1}{2}(a_k - jb_k), \quad k = 1, 2, \ldots && (3.27)\\ c_{-k} &= \frac{1}{2}(a_k + jb_k), \quad k = 1, 2, \ldots && (3.28) \end{aligned}$$

or equivalently,

$$\begin{aligned} a_0 &= c_0 && (3.29)\\ a_k &= c_k + c_{-k}, \quad k = 1, 2, \ldots && (3.30)\\ b_k &= j(c_k - c_{-k}), \quad k = 1, 2, \ldots && (3.31) \end{aligned}$$

Remark 3.3.1 *From the cosine-with-phase Fourier series of a periodic signal xs*

$$x(t) = a_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$$

it is apparent that the component of x represented by the cosine signal $A_k\cos(k\omega_0 t + \theta_k)$ at frequency $k\omega_0$ has both an amplitude component A_k and a phase component θ_k.
Similarly, from the complex Fourier series given by

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

it is apparent that the component of x represented by the exponential signal $c_k e^{jk\omega_0 t}$ at frequency $k\omega_0$ has both an amplitude component $|c_k|$ and a phase component $\angle c_k$.

Definition 3.3.3 (Amplitude and phase spectra) *The* **amplitude spectrum** *of x is $|c_k|$, and the* **phase spectrum** *is $\angle c_k$.*

Theorem 3.3.2 (Amplitude and phase spectra)

- $|c_k| = |c_{-k}|$ *for all* $k = 1, 2, \ldots$, *so the amplitude spectrum is an even function of* k.
- $\angle c_{-k} = -\angle c_k$, *for all* $k = 1, 2, \ldots$, *so the phase spectrum is an odd function of* k.
- *The trigonometric Fourier coefficients* a_k *and* b_k *are related to the complex Fourier coefficients by:*

$$|c_k| = \frac{1}{2}\sqrt{a_k^2 + b_k^2}, \quad k = 1, 2, \ldots \tag{3.32}$$

$$\angle c_k = \begin{cases} \tan^{-1}\left(\frac{-b_k}{a_k}\right), & k = 1, 2, \ldots \text{when } a_k \geq 0 \\ \pi + \tan^{-1}\left(\frac{-b_k}{a_k}\right), & k = 1, 2, \ldots \text{when } a_k < 0 \end{cases} \tag{3.33}$$

- *The cosine-with-phase coefficients* A_k *and* θ_k *are related to the complex Fourier coefficients by:*

$$|c_k| = \frac{1}{2}A_k, \quad k = 1, 2, \ldots \tag{3.34}$$

$$\angle c_k = \theta_k, \quad k = 1, 2, \ldots \tag{3.35}$$

Example 3.3.1 (Example) *Consider the signal*

$$x(t) = \cos(t) + 0.5\cos(4t + \pi/3) + \cos(8t + \pi/2)$$

The signal is in cosine-with-phase form with coefficients

$$A_1 = 1, \theta_1 = 0; \quad A_4 = 0.5, \theta_4 = \pi/3; \quad A_8 = 1, \theta_8 = \pi/2$$

and hence the complex Fourier coefficients are

$$\begin{aligned} c_1 &= \frac{1}{2}, \quad c_4 = \frac{0.5}{2}e^{j\pi/3} = 0.25\angle 60^\circ, c_8 = \frac{1}{2}e^{j\pi/2} = 0.5\angle 90^\circ \\ c_{-1} &= \frac{1}{2}, \quad c_{-4} = \bar{c}_4 = 0.25\angle{-60^\circ}, c_{-8} = \bar{c}_8 = 0.5\angle{-90^\circ} \end{aligned}$$

We observe that $|c_k|$ *is an even function and* $\angle c_k$ *is an odd function:*

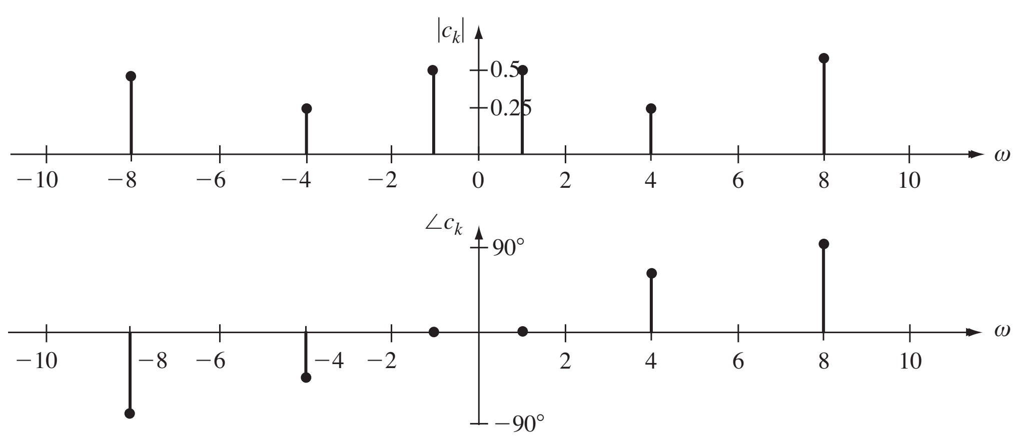


Figure 3.6: Amplitude and phase spectrum of x.

3.4 Practice Problems

34. Express the following signals in trigonometric form $a_k \cos(\omega t) + b_k \sin(\omega t)$:

    (a) $2\cos(3t) - \cos(3t - \pi/4)$

    (b) $\sin(2t - \pi/4) + 2\cos(2t - \pi/3)$

    (c) $10\cos(\pi t + \pi/3) + 8\cos(\pi t - \pi/3)$

    Hint: do NOT use Euler's formulae for the Fourier coefficients.

35. Express the signals in Question 34 in cosine-with-phase form $A_k \cos(\omega t + \theta_k)$.

36. Consider the function

$$f_1(t) = \begin{cases} 2, & 0 \le t < 2 \\ -1, & 2 \le t < 3 \end{cases} \qquad \text{and } f_1(t) = f_1(t+3).$$

    (a) Sketch the graph of the function showing at least three periods,

    (b) Explain why the function satisfies the Dirichlet conditions.

    (c) Use the definition of the trigonometric Fourier coefficients (a_k and b_k) to compute a general expression for these coefficients, and evaluate them for $k = 0, 1, 2$.

37. Repeat Question 36 for the function

$$f_2(t) = 1 - \left|\frac{t}{2}\right| \quad \text{for } -2 \le t < 2, \text{ and } f_2(t) = f_2(t+4).$$

38. Repeat Question 36 for the function

$$f_3(t) = \begin{cases} 0, & -\pi \le t < 0 \\ t, & 0 \le t < \pi \end{cases} \qquad \text{and } f_3(t) = f_3(t+2\pi)$$

.

39. In the following let f be a function with fundamental period $T > 0$.

    (a) Give an example of a function f that fails the first Dirichlet condition:

$$\int_0^T |f(t)|\, dt = \infty$$

    (b) Give an example of a function f that fails the second Dirichlet condition: it has infinitely many maxima or minima in the interval $[0, T]$.

    (c) Give an example of a function f that fails the third Dirichlet condition: it has infinitely many discontinuities in the interval $[0, T]$.

    Explain in each case why your function fails the Dirichlet condition.

40. Decide which of the following statements are true. For those that are true, give a proof. For those that are false, give an explicit counterexample.

(a) The product of two even functions is an even function.

(b) The product of two odd functions is an odd function.

(c) The product of an even function and an odd function is an even function.

41. (a) Let $a > 0$ and let x and v both be even functions. Show that

$$\int_{-a}^{a} x(t)v(t)\,dt = 2\int_{0}^{a} x(t)v(t)\,dt$$

(b) Let $a > 0$, let x be an even function and let v an odd function. Show that

$$\int_{-a}^{a} x(t)v(t)\,dt = 0$$

42. Give examples of two functions f and g that meet all of the following three conditions:

(a) f and g are periodic functions with the same fundamental period T and frequency ω_0.

(b) f and g have trigonometric Fourier series

$$\begin{aligned} f(t) &= a_0 + \sum_{k=1}^{\infty} (a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)) \\ g(t) &= \alpha_0 + \sum_{k=1}^{\infty} (\alpha_k \cos(k\omega_0 t) + \beta_k \sin(k\omega_0 t)) \end{aligned}$$

and $a_k = \alpha_k$, $b_k = \beta_k$ for all $k = 0, 1, 2, \ldots$.

(c) There exists at least one $t_0 \in [0, T]$ such that $f(t_0) \neq g(t_0)$.

Hint: consider discontinuous functions.

43. Consider again the function f_1 in Question 36.

(a) Use the definition of the complex Fourier coefficients (c_k) to compute a general expression for these coefficients, and evaluate these for $k = -2, -1, 0, 1, 2$.

(b) Confirm that your answer for part (a) is in agreement with Question 36.(c) , for $k = -2, -1, 0, 1, 2$.

44. Consider again the function f_3 in Question 38. Repeat Question 43 for the function f_3.

45. Let x be a periodic continuous-time signal, and let $\{c_k : k \in \mathbf{Z}\}$ be the coefficients of its complex exponential Fourier series.

(a) Show that if x is an even function, then for all $k \in \mathbf{Z}$, the c_k are real-valued.

(b) Show that if x is an odd function, then $c_0 = 0$, and for all non-zero $k \in \mathbf{Z}$, the c_k are imaginary-valued.

46. Suppose the function x has fundamental period $T > 0$ and fundamental frequency ω_0. Let its trigonometric Fourier series be given by

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)$$

Assume its cosine-with-phase Fourier series is given by

$$x(t) = a_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$$

and that its complex Fourier series is given by

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

(a) Show that

$$A_k = \sqrt{a_k^2 + b_k^2} \quad \text{and} \quad \theta_k = \begin{cases} \tan^{-1}\left(\frac{-b_k}{a_k}\right), & k = 1, 2, \ldots \text{when } a_k \geq 0 \\ \pi + \tan^{-1}\left(\frac{-b_k}{a_k}\right), & k = 1, 2, \ldots \text{when } a_k < 0 \end{cases}$$

(b) Show that

$$c_0 = a_0, \quad c_k = \frac{1}{2}(a_k - jb_k), \quad c_{-k} = \frac{1}{2}(a_k + jb_k), \quad k = 1, 2, \ldots$$

47. Suppose the periodic function f has fundamental frequency ω_0 and complex Fourier coefficients $\{c_k : k \in \mathbf{Z}\}$. Let $t_0 \in \mathbf{R}$ and define

$$f_1(t) = f(t - t_0), \quad f_2(t) = f(-t),$$

(a) What are the fundamental frequencies of f_1 and f_2?

(b) Let $\{\tilde{c}_k : k \in \mathbf{Z}\}$ and $\{\hat{c}_k : k \in \mathbf{Z}\}$ denote the the complex Fourier coefficients of f_1 and f_2 respectively. Express $\tilde{c}_k$ and $\hat{c}_k$ in terms of c_k.

Chapter 4

Fourier Transform for Continuous-time Signals

In this chapter we continue our study of continuous-time signals and introduce the Fourier transform. This generalizes the concept of Fourier series to include functions that are not periodic. The frequency spectrum of a non-periodic signal is defined for all real values of the frequency variable.

4.1 Definition of Fourier Transforms

Definition 4.1.1 (Fourier Transform) *For a continuous-time signal x, we define its* **Fourier Transform** *(CTFT) to be*

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt \tag{4.1}$$

- We say that x and X are Fourier Transform pairs, and write

$$x(t) \longleftrightarrow X(\omega) \tag{4.2}$$

- The variable $\omega \in \mathbf{R}$ is called the **frequency variable**.
- We use lower case letters like x and f to denote continuous-time signals, and capital letters X and F to denote their Fourier transform.
- The Fourier transform is complex-valued, and to plot the transform requires separate graphs for the amplitude $|X(\omega)|$ and phase $\angle X(\omega)$.
- The amplitude and phase spectra are generalizations of the spectra for periodic signals.

Definition 4.1.2 (Inverse Fourier Transform) *For a continuous-time signal x with Fourier Transform $X(\omega)$, the* **Inverse Fourier Transform** *of $X(\omega)$ is given by*

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\, d\omega \tag{4.3}$$

- Recalling that the complex Fourier series for a periodic signal is

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \tag{4.4}$$

  we see that $X(\omega)$ is a generalization of the Fourier complex coefficients c_k. Thus a non-periodic function $x(t)$ can also be represented in terms of its frequency spectrum.

Theorem 4.1.1 (Existence of the Fourier Transform) *A continuous-time signal x has a Fourier Transform if it satisfies the following conditions:*

1. *x is* **absolutely integrable** *over* **R**, *i.e.*

$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty \tag{4.5}$$

2. *x has only finitely many maxima, minima and points of discontinuity on any interval of finite length.*

Remark 4.1.1

- *Functions that satisfy these conditions are said to be* **well-behaved**. *Kamen and Heck argue that all signals that can be physically generated are well-behaved.*
- *The Fourier transform generalizes the concepts of amplitude and phase spectra to accommodate non-periodic signals. If $|X(\omega)|$ is large for certain values of ω, it means that those frequencies make up a large component of the signal $x(t)$.*

Example 4.1.1 *Consider the function*

$$x(t) = e^{-bt}u(t)$$

where $b > 0$ and $u(t)$ is the unit step function. Then

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt \\ &= \int_{0}^{\infty} e^{-bt}e^{-j\omega t}\, dt \\ &= \int_{0}^{\infty} e^{-(b+j\omega)t}\, dt \\ &= \left[\frac{-e^{-(b+j\omega)t}}{b+j\omega}\right]_0^{\infty} \\ &= \frac{1}{b+j\omega} \end{aligned}$$

so the amplitude and phase spectra are

$$|X(\omega)| = \frac{1}{\sqrt{b^2+\omega^2}}, \quad \angle X(\omega) = -\tan^{-1}\left(\frac{\omega}{b}\right)$$

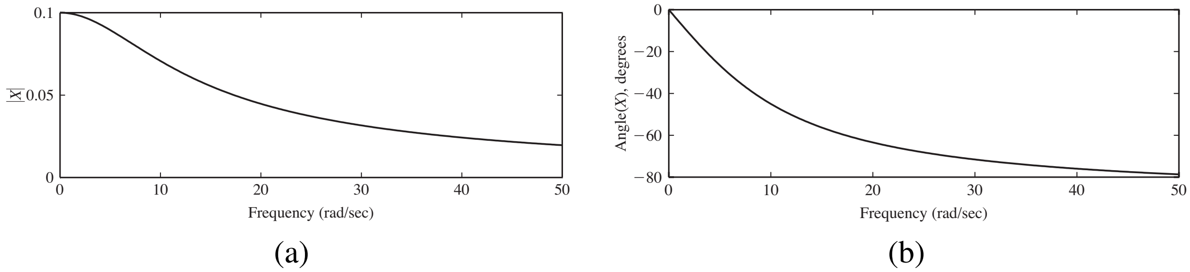


Figure 4.1: (a) Amplitude and (b) phase spectra of x

Definition 4.1.3 (Rectangular form of the Fourier Transform) *Using Euler's formulae* $e^{j\theta} = \cos\theta + j\sin(\theta)$, *we can write*

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} x(t)\cos(\omega t)\,dt - j\int_{-\infty}^{\infty} x(t)\sin(\omega t)\,dt \tag{4.6}$$

Let

$$\begin{aligned} R(\omega) &= \int_{-\infty}^{\infty} x(t)\cos(\omega t)\,dt, && (4.7)\\ I(\omega) &= -\int_{-\infty}^{\infty} x(t)\sin(\omega t)\,dt && (4.8)\end{aligned}$$

Then

$$X(\omega) = R(\omega) + jI(\omega) \tag{4.9}$$

This the **Rectangular form** *of the Fourier transform* $X(\omega)$.

Definition 4.1.4 (Polar form of the Fourier Transform) *Let*

$$|X(\omega)| = \sqrt{R^2(\omega) + I^2(\omega)} \tag{4.10}$$

$$\angle X(\omega) = \begin{cases} \tan^{-1}\left(\frac{I(\omega)}{R(\omega)}\right), & R(\omega) \geq 0 \\ \pi + \tan^{-1}\left(\frac{I(\omega)}{R(\omega)}\right), & R(\omega) < 0 \end{cases} \tag{4.11}$$

Then the **Polar form** *of the Fourier transform* $X(\omega)$ *is*

$$X(\omega) = |X(\omega)|\exp(j\angle X(\omega)) \tag{4.12}$$

Theorem 4.1.2 *For any continuous-time signal x with Fourier Transform X,*

$$\begin{aligned} |X(-\omega)| &= |X(\omega)| && (4.13)\\ \angle X(-\omega) &= -\angle X(\omega) && (4.14)\end{aligned}$$

Thus $|X(\omega)|$ *is an even function, and* $\angle X(\omega)$ *is an odd function of* ω.

Theorem 4.1.3 (Even and Odd signals)

1. *For any continuous-time even signal x with Fourier Transform X,*

$$\begin{aligned} R(\omega) &= 2\int_0^{\infty} x(t)\cos(\omega t)\,dt, && (4.15)\\ I(\omega) &= 0 && (4.16)\end{aligned}$$

$$\text{and so} \quad X(\omega) = 2\int_0^{\infty} x(t)\cos(\omega t)\,dt \tag{4.17}$$

2. *For any continuous-time odd signal x with Fourier Transform X,*

$$\begin{aligned} R(\omega) &= 0 && (4.18)\\ I(\omega) &= -2\int_0^{\infty} x(t)\sin(\omega t)\,dt && (4.19)\end{aligned}$$

$$\text{and so} \quad X(\omega) = -j2\int_0^{\infty} x(t)\sin(\omega t)\,dt \tag{4.20}$$

Example 4.1.2 (Rectangular Pulse) *Recall the rectangular pulse function of width τ:*

$$p_\tau(t) = \begin{cases} 1, & -\frac{\tau}{2} \le t < \frac{\tau}{2} \\ 0, & \textit{otherwise} \end{cases}$$

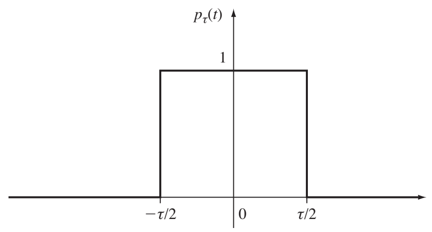


Figure 4.2: Graph of the rectangular pulse $p_\tau(t)$

p_τ is an even function and hence we can use

$$\begin{aligned} P_\tau(\omega) &= 2\int_0^\infty x(t)\cos(\omega t)\, dt \\ &= 2\int_0^{\tau/2} \cos(\omega t)\, dt \\ &= \frac{2}{\omega}\sin\left(\frac{\omega\tau}{2}\right) \end{aligned}$$

The transform of the rectangular pulse can be expressed in terms of the sinc *function, which is defined as*

$$\operatorname{sinc}(a\omega) = \frac{\sin(a\pi\omega)}{a\pi\omega}, \quad \textit{for any } a \in \mathbf{R} \tag{4.21}$$

We saw that

$$\begin{aligned} P_\tau(\omega) &= \frac{2}{\omega}\sin\left(\frac{\omega\tau}{2}\right) \\ &= \frac{2}{\omega}\left[\frac{\sin\left(\frac{\pi\omega\tau}{2\pi}\right)}{\left(\frac{\pi\omega\tau}{2\pi}\right)}\left(\frac{\pi\omega\tau}{2\pi}\right)\right] \\ &= \tau\operatorname{sinc}\left(\frac{\tau\omega}{2\pi}\right) \end{aligned}$$

using $a = \frac{\tau}{2\pi}$. Since $P_\tau(\omega)$ is real-valued, we can plot it as a function of ω:

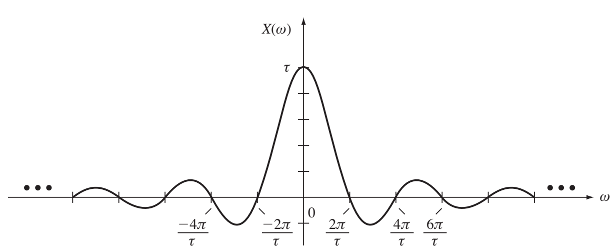


Figure 4.3: Graph of $P_\tau(\omega)$, the Fourier transform of p_τ.

- *For small values of the pulse width τ, the spectrum spreads out across a larger frequency band.*
- *For large values of τ, the spectrum becomes narrow and taller (more like a spike).*

Example 4.1.3 (Triangular Pulse) *Recall the triangular pulse function of width τ:*

$$\Lambda_\tau(t) = \begin{cases} 1 - \frac{2|t|}{\tau}, & -\frac{\tau}{2} \le t < \frac{\tau}{2} \\ 0, & \textit{otherwise} \end{cases}$$

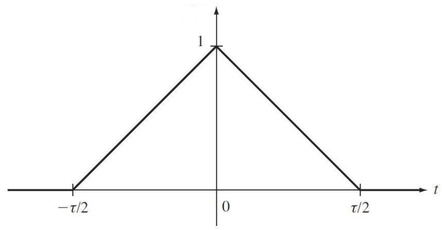


Figure 4.4: Graph of the triangular pulse $\Lambda_\tau(t)$

Then Λ_τ is an even function and hence we can use

$$\begin{aligned} L_\tau(\omega) &= 2\int_0^\infty \Lambda_\tau(t)\cos(\omega t)\,dt \\ &= 2\int_0^{\tau/2} \left(1 - \frac{2t}{\tau}\right)\cos(\omega t)\,dt \\ &= \frac{8}{\tau\omega^2}\left(\sin^2\left(\frac{\tau\omega}{4}\right)\right) \\ &= \frac{8}{\tau\omega^2}\left[\left(\frac{\sin\left(\frac{\tau\pi\omega}{4\pi}\right)}{\frac{\tau\pi\omega}{4\pi}}\right)\left(\frac{\tau\pi\omega}{4\pi}\right)\right]^2 \\ &= \frac{\tau}{2}\operatorname{sinc}^2\left(\frac{\tau\omega}{4\pi}\right) \end{aligned}$$

using $a = \frac{\tau}{4\pi}$ and $\operatorname{sinc}(a\omega) = \frac{\sin(a\pi\omega)}{a\pi\omega}$.

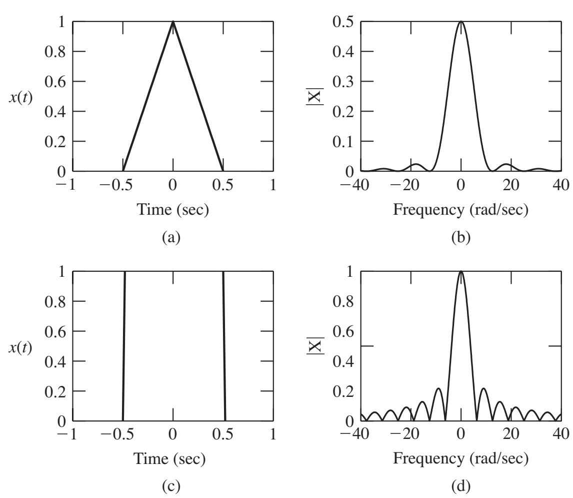


Figure 4.5: Graphs of (a) $\Lambda(t)$ (b) $|L_\tau(\omega)|$ (c) $p_\tau(t)$ (d) $|P_\tau(\omega)|$

The principal differences in the magnitude spectra of the rectangular and triangular pulse signals are

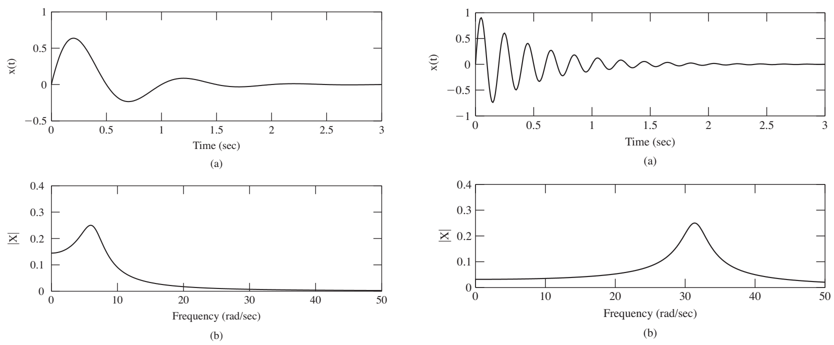
- *Rapid changes in time-domain (e.g. discontinuities) leads to more high frequency content in the spectrum. Thus the side lobes of the rectangular pulse are larger.*

- *The main lobe of the triangular pulse is wider, showing more low frequency content.*

Example 4.1.4 (Decaying sinusoid) *The amplitude spectrum of the decaying sinusoid*

$$x(t) = e^{-2t} \sin(\alpha \pi t) u(t)$$

is shown below for the cases $\alpha = 2$ *and* $\alpha = 10$.



(a) (b)

Figure 4.6: $x(t)$ and $|X(\omega)|$ for (a) $\alpha = 2$ and (b) $\alpha = 10$.

- *When* $\alpha = 2$ *we have a dominant frequency at* $\omega = 2\pi$.
- *When* $\alpha = 10$ *the time-domain signal has more rapid fluctuations, leading to more high frequency content and a dominant frequency at* $\omega = 10\pi$.

4.2 Properties of the Continuous-time Fourier Transform

The Fourier Transform has many properties that allow us to compute transforms of signals without using the definition of the transform. Some of these are

Theorem 4.2.1 (Linearity) *The Fourier Transform is linear: if* $x_1(t) \longleftrightarrow X_1(\omega)$ *and* $x_2(t) \longleftrightarrow X_2(\omega)$, *and* a *and* b *are any two scalars, then*

$$ax_1(t) + bx_2(t) \longleftrightarrow aX_1(\omega) + bX_2(\omega) \tag{4.22}$$

Theorem 4.2.2 (Time shifting) *If* $x(t) \longleftrightarrow X(\omega)$ *and* $c \in \mathbf{R}$, *then*

$$x(t-c) \longleftrightarrow X(\omega) e^{-j\omega c} \tag{4.23}$$

Theorem 4.2.3 (Time scaling) *If* $x(t) \longleftrightarrow X(\omega)$ *and* $a > 0$, *then*

$$x(at) \longleftrightarrow \frac{1}{a} X\left(\frac{\omega}{a}\right) \tag{4.24}$$

Theorem 4.2.4 (Time reversal or Flipping) *If* $x(t) \longleftrightarrow X(\omega)$ *then*

$$x(-t) \longleftrightarrow X(-\omega) \tag{4.25}$$

Corollary 4.2.1 *If* $x(t) \longleftrightarrow X(\omega)$ *and* $a \in \mathbf{R}$ *with* $a \neq 0$, *then*

$$x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right) \tag{4.26}$$

Theorem 4.2.5 (Modulation) *If* $x(t) \longleftrightarrow X(\omega)$ *and* $\omega_0 \in \mathbf{R}$, *then*

$$x(t)e^{j\omega_0 t} \quad \longleftrightarrow \quad X(\omega - \omega_0) \tag{4.27}$$

$$x(t)\cos(\omega_0 t) \quad \longleftrightarrow \quad \frac{1}{2}[X(\omega + \omega_0) + X(\omega - \omega_0)] \tag{4.28}$$

$$x(t)\sin(\omega_0 t) \quad \longleftrightarrow \quad \frac{j}{2}[X(\omega + \omega_0) - X(\omega - \omega_0)] \tag{4.29}$$

Theorem 4.2.6 (Convolution) *If* $x(t) \longleftrightarrow X(\omega)$ *and* $v(t) \longleftrightarrow V(\omega)$, *then*

$$(x \star v)(t) \longleftrightarrow X(\omega)V(\omega) \tag{4.30}$$

Theorem 4.2.7 (Duality) *If* $x(t) \longleftrightarrow X(\omega)$, *then*

$$X(t) \longleftrightarrow 2\pi x(-\omega) \tag{4.31}$$

4.3 Generalized Fourier Transform

Some important signals like $\cos(\omega t)$ and $\sin(\omega t)$ do not have Fourier transforms in the ordinary sense because they are not absolutely integrable, i.e.

$$\int_{-\infty}^{\infty} |\sin(\omega t)|\, dt = \infty \tag{4.32}$$

Nonetheless these functions have Fourier series representations - so they *should* have Fourier transforms. Next we see how to define the Fourier transform of these functions using the generalized function $\delta(t)$. Recall the Sifting Theorem

$$\int_{-\infty}^{\infty} f(\lambda)\delta(\lambda - t_0)\, d\lambda = f(t_0) \tag{4.33}$$

Hence

$$\int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}\, dt = e^0 = 1 \tag{4.34}$$

so $\delta(t)$ and 1 are a **generalized Fourier transform** pair:

$$\delta(t) \longleftrightarrow 1 \tag{4.35}$$

Hence by duality we have

$$1 \longleftrightarrow 2\pi\delta(\omega) \tag{4.36}$$

as $\delta(-\omega) = \delta(\omega)$. We recall the modulation theorem

$$x(t)e^{j\omega_0 t} \quad \longleftrightarrow \quad X(\omega - \omega_0) \tag{4.37}$$

$$x(t)\cos(\omega_0 t) \quad \longleftrightarrow \quad \frac{1}{2}[X(\omega + \omega_0) + X(\omega - \omega_0)] \tag{4.38}$$

$$x(t)\sin(\omega_0 t) \quad \longleftrightarrow \quad \frac{j}{2}[X(\omega + \omega_0) - X(\omega - \omega_0)] \tag{4.39}$$

to obtain the generalized Fourier transform pairs

Theorem 4.3.1

$$e^{j\omega_0 t} \quad \longleftrightarrow \quad 2\pi\delta(\omega - \omega_0) \tag{4.40}$$

$$\cos(\omega_0 t) \quad \longleftrightarrow \quad \pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)] \tag{4.41}$$

$$\sin(\omega_0 t) \quad \longleftrightarrow \quad j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)] \tag{4.42}$$

A frequency plot of the transform of $\cos(\omega_0 t)$ looks like

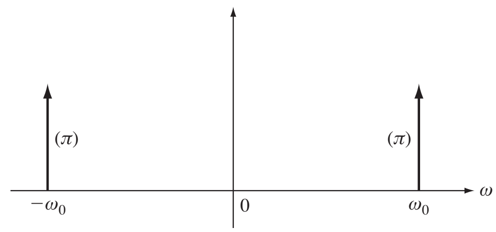


Figure 4.7: Fourier transform of $\cos(\omega_0 t)$.

Example 4.3.1 *For a periodic signal with complex Fourier series*

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

the Fourier transform is a train of impulse signals

$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi c_k \delta(\omega - k\omega_0)$$

Definition 4.3.1 (Analogue signal modulation) *Transmission of signal over a channel may be done by modulating the signal with a carrier signal. The signal $x(t)$ is modulated by the carrier signal $A\cos(\omega_c t)$*

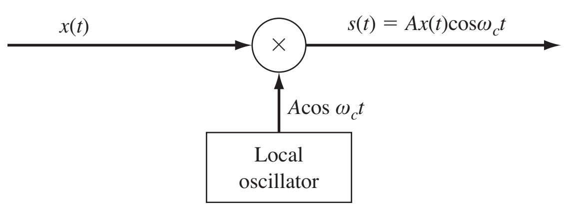


Figure 4.8: Analogue signal modulation

The transmitted signal is $s(t) = Ax(t)\cos(\omega t)$ with transform

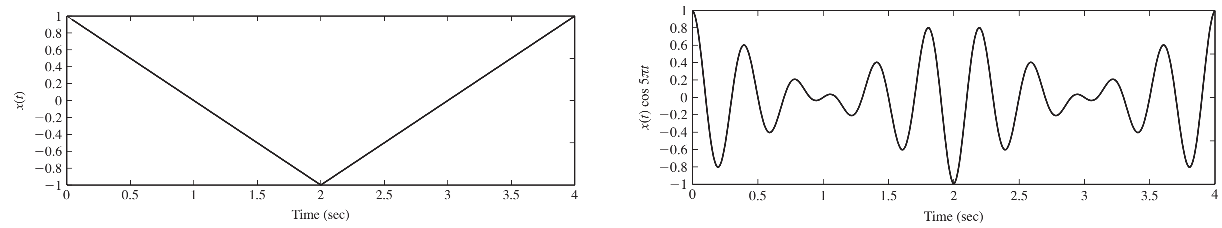
$$S(\omega) = \frac{A}{2}[X(\omega + \omega_c) + X(\omega - \omega_c)]$$

We refer to $x(t)$ as the original **baseband** *signal, and $s(t)$ is the modulated* **passband** *signal.*

Example 4.3.2 *Consider the signal*

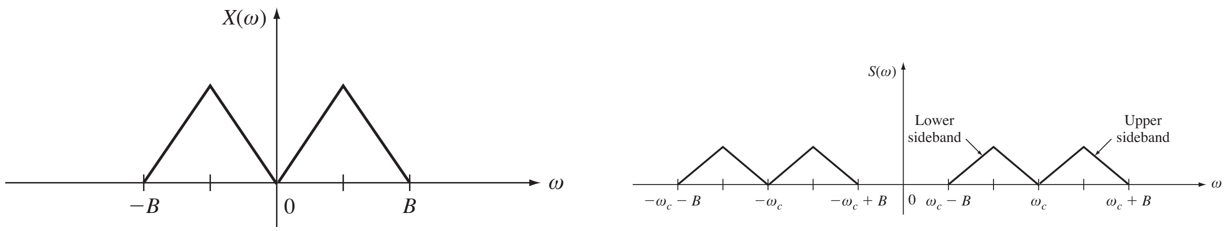
$$x(t) = \frac{1}{2}|t - 2|$$

The baseband signal and the modulated passband signal with $\omega_c = 5\pi$, are shown in Figure 4.9. Their amplitude spectra are shown in Figure 4.10. We observe that the effect of the modulation is to split the amplitude spectrum of x into two parts, each with half the magnitude, and shifted by $\pm\omega_c$ in the frequency range.



(a) (b)

Figure 4.9: (a) $x(t)$ and (b) $s(t)$, the modulation of $x(t)$.



(a) (b)

Figure 4.10: (a) $|X(\omega)|$ and (b) $|S(\omega)|$, the transforms of $x(t)$ and $s(t)$.

Usually the frequency ω_c of the carrier signal is much higher than the frequencies of the base signal, so that the spectrum of the passband signal is in a range that is suitable for transmission through cables or free space.

4.4 Practice Problems

48. Compute the Fourier transforms of the following functions.

    (a) $f_1(t) = \begin{cases} 0, & t < 0 \text{ and } t > 2 \\ -1, & 0 \le t < 1 \\ 1, & 1 \le t < 2 \end{cases}$

    (b) $f_2(t) = \begin{cases} 0, & t < -1 \text{ and } t > 2 \\ 2, & -1 \le t < 0 \\ 2(1-t), & 0 \le t < 1 \\ 2(t-1), & 1 \le t \le 2 \end{cases}$

    (c) $f_3(t) = \begin{cases} 0, & |t| > 0.5 \\ \cos(\pi t), & |t| \le 0.5 \end{cases}$

    (d) $f_4(t) = \begin{cases} 0, & |t| > 1 \\ e^{-|t|}, & |t| \le 1 \end{cases}$

    Hint: Express f_1, f_2 and f_3 in terms of rectangular and triangular pulse functions. For (d), you may use the fact that

$$\int e^{at}\cos(bt)\,dt = \frac{e^{at}(a\cos(bt) + b\sin(bt))}{a^2+b^2}$$

49. Consider the signal $f : \mathbf{R} \to \mathbf{R}$ defined by

$$f(t) = t(u(t) - u(t-\pi))$$

    (a) Sketch the graph of $f(t)$ for $-2\pi \le t \le 2\pi$.

    (b) Use the definition of Fourier transform to find the Fourier transform of f.

50. Compute the inverse Fourier transforms of the following signals

    (a) $X_1(\omega) = \cos(4\omega)$

    (b) $X_2(\omega) = \sin^2(3\omega)$

    (c) $X_3(\omega) = 2\pi\delta(\omega) + \pi\delta(\omega - 4\pi) + \pi\delta(\omega + 4\pi)$

51. Find the Fourier transform of the function $g : \mathbf{R} \to \mathbf{R}$ defined by

$$g(t) = (t-\pi)(u(t-\pi) - u(t-2\pi))$$

52. Find the Fourier transform of the function $g : \mathbf{R} \to \mathbf{R}$ defined by

$$g(t) = 2t(u(t) - u(t-\pi/2))$$

    Hint: Firstly show that $u(at) = u(t)$ for any $a \ge 0$. Then show that $g(t) = f(2t)$, where is f is given in Question 49.

53. Consider the signal $x : \mathbf{R} \to \mathbf{R}$ defined by

$$x(t) = \cos(t) + \frac{1}{2}(\cos(4t + \frac{\pi}{3}))$$

   (a) Obtain the complex exponential Fourier series for x.

   (b) Obtain the Fourier transform of x.

   (c) Express the Fourier transform in terms of the complex Fourier coefficients.

54. Match the time-domain signals in Figure 4.11 to the appropriate amplitude spectra in Figure 4.12. In each case explain why the amplitude spectrum matches the time-domain signal.

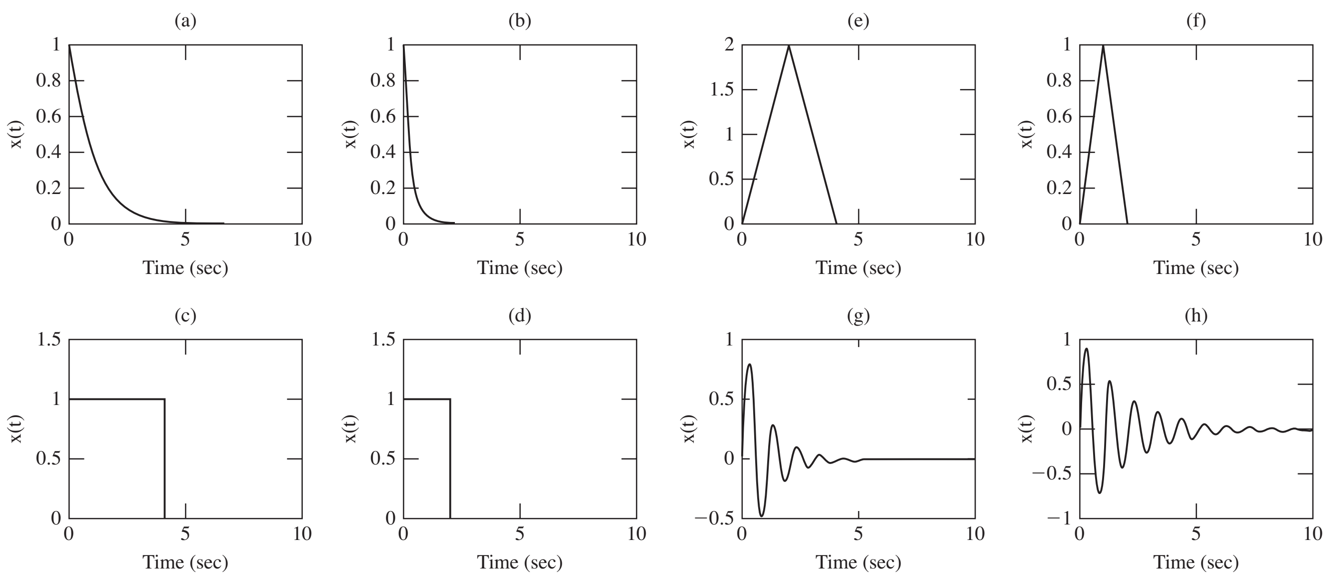


Figure 4.11: Time domain signals for Problem 54

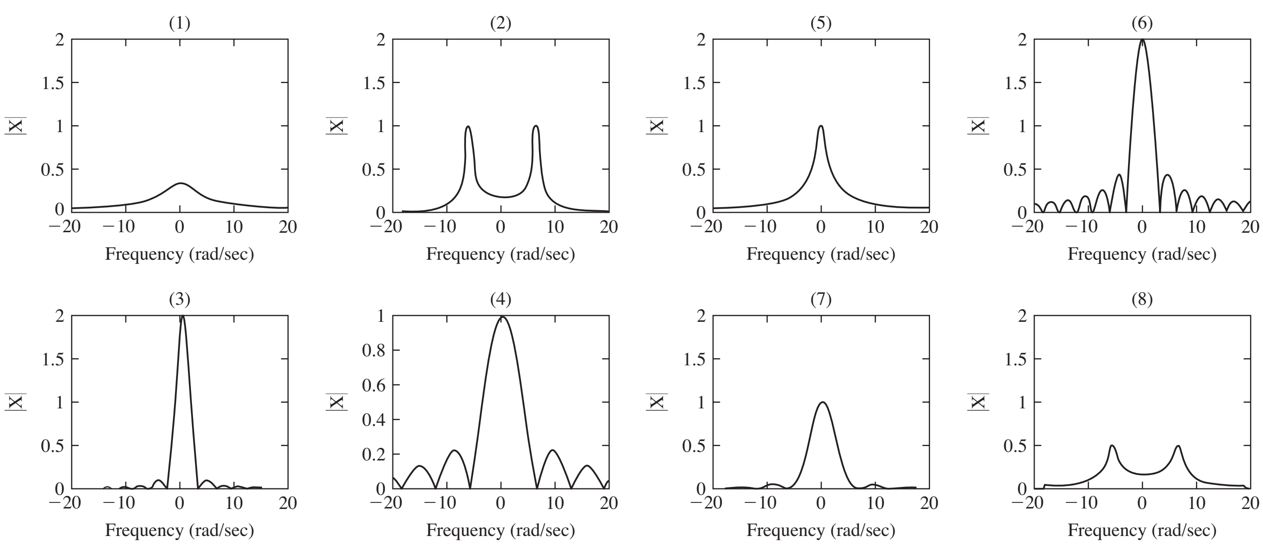


Figure 4.12: Fourier spectra for Problem 54

55. The continuous-time signal $x(t) = e^{-bt}u(t)$, with $b > 0$, has Fourier transform

$$X(\omega) = \frac{1}{j\omega + b}$$

where b is a constant. Use properties of the Fourier transform to obtain the transforms of following signals:

(a) $x_1(t) = x(5t - 4)$
(b) $x_2(t) = e^{j2t}x(t)$
(c) $x_3(t) = (x \star x)(t)$
(d) $x_4(t) = \frac{1}{jt-b}$

56. Let $x(t)$ and $X(\omega)$ be a Fourier transform pair. Using the definition of Fourier transform, prove the following Fourier transform pairs:

$$\begin{array}{lcll}
x(t-c) & \longleftrightarrow & X(\omega)e^{-j\omega c},\ c \in \mathbf{R} & \text{(Time shift)} \\
x(-t) & \longleftrightarrow & X(-\omega) & \text{(Flipping Theorem)} \\
x(at) & \longleftrightarrow & \frac{1}{a}X\left(\frac{\omega}{a}\right),\ a > 0 & \text{(Time scaling)} \\
x(t)e^{j\omega_0 t} & \longleftrightarrow & X(\omega - \omega_0),\ \omega_0 \in \mathbf{R} & \text{(Modulation)} \\
X(t) & \longleftrightarrow & 2\pi x(-\omega) & \text{(Duality)}
\end{array}$$

57. Suppose the function $x(t)$ has Fourier transform $X(\omega)$. Let

(a) $x_1(t) = x(1-t) + x(-1-t)$
(b) $x_2(t) = x(3t - 6)$

For each of these functions, express their Fourier transforms in terms of $X(\omega)$.

Chapter 5

Fourier Transform for Discrete–time Signals

In this chapter we develop the discrete-time counterpart to the study of continuous time signals given in the previous chapter. We introduce the discrete-time Fourier transform. The frequency spectrum of a non-periodic signal is defined for all real values of the frequency variable. We also consider the discrete Fourier transform which involves representing a signal in terms of finitely many frequencies.

5.1 Definition of Fourier Transforms for discrete signals

Definition 5.1.1 (Discrete-time Fourier Transform) *For a discrete-time signal x, we define its* **discrete-time Fourier Transform (DTFT)** *to be*

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} \tag{5.1}$$

- The variable $\Omega \in \mathbf{R}$ is called the **frequency variable**. We use lower case letters like x and f to denote discrete-time signals, and capital letters X and F to denote their Fourier transform.
- We may compare it with the continuous-time Fourier transform (CTFT)

  $$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt \tag{5.2}$$

  Since n is an integer, we naturally replace the integral (dt) in the CTFT with a sum in the DTFT. We also use Ω and ω to distinguish the discrete-time and continuous-time frequency variables.

Definition 5.1.2 (Inverse DTFT) *For a discrete-time signal x with DTFT $X(\Omega)$, the* **Inverse DTFT** *of $X(\Omega)$ is given by*

$$x[n] = \frac{1}{2\pi}\int_0^{2\pi} X(\Omega)e^{j\Omega n}\, d\Omega \tag{5.3}$$

We say that x and X are DTFT pairs, and write

$$x[n] \longleftrightarrow X(\Omega) \tag{5.4}$$

Theorem 5.1.1 *For any discrete-time signal x, its DTFT X is a periodic function with period 2π:*

$$X(\Omega + 2\pi) = X(\Omega), \quad \textit{for all } \Omega \in \mathbf{R} \tag{5.5}$$

Since $X(\Omega)$ and $e^{j\Omega n}$ are both periodic functions of Ω with period 2π, we can integrate over any interval of length 2π:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n}\, d\Omega \tag{5.6}$$

Theorem 5.1.2 (Existence of the DTFT) *A discrete-time signal x has a DTFT if it is* **absolutely summable** *over* **Z***, i.e.*

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty \tag{5.7}$$

- Functions that satisfy this condition are said to be have a DTFT **in the ordinary sense**.
- Any time-limited signal (a signal that has finite support) will have a DTFT in the ordinary sense.

Example 5.1.1 *The Kronecker delta function $\delta[n]$ has DTFT*

$$\delta[n] \longleftrightarrow 1$$

This follows from the Sifting Theorem for discrete signals:

$$\sum_{n=-\infty}^{\infty} \delta[n]e^{-j\Omega n} = e^0 = 1$$

Example 5.1.2 *Consider the discrete-time signal x with*

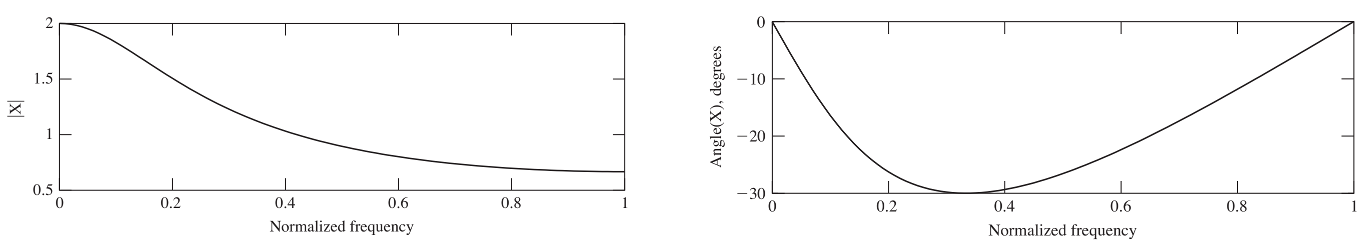
$$x[n] = a^n u[n]$$

where $|a| < 1$, with $a \neq 0$. It can be shown that

$$X(\Omega) = \frac{1}{1 - ae^{-j\Omega}}$$

so the amplitude and phase spectra are

$$|X(\Omega)| = \frac{1}{\sqrt{1 - 2a\cos(\Omega) + a^2}}, \quad \angle X(\Omega) = -\angle 1 - ae^{-j\Omega}$$



(a) (b)

Figure 5.1: (a) Amplitude and (b) Phase spectra for x, with $a = \frac{1}{2}$.

Definition 5.1.3 (Rectangular form of the Fourier Transform) *Let*

$$R(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\cos(\Omega n) \tag{5.8}$$

$$I(\Omega) = -\sum_{n=-\infty}^{\infty} x[n]\sin(\Omega n) \tag{5.9}$$

Then

$$X(\Omega) = R(\Omega) + jI(\Omega) \tag{5.10}$$

This the **Rectangular form** *of the Fourier transform* $X(\Omega)$.

Definition 5.1.4 (Polar form of the Fourier Transform) *The* **Polar form** *of the Fourier transform* $X(\Omega)$ *is*

$$X(\Omega) = |X(\Omega)|\exp(j\angle X(\Omega)) \tag{5.11}$$

$$\text{where} \quad |X(\Omega)| = \sqrt{R^2(\Omega) + I^2(\Omega)} \tag{5.12}$$

$$\angle X(\Omega) = \begin{cases} \tan^{-1}\left(\frac{I(\Omega)}{R(\Omega)}\right), & R(\Omega) \geq 0 \\ \pi + \tan^{-1}\left(\frac{I(\Omega)}{R(\Omega)}\right), & R(\Omega) < 0 \end{cases} \tag{5.13}$$

Theorem 5.1.3 *For any discrete-time signal x with Fourier Transform X,*

$$X(-\Omega) = \overline{X(\Omega)} \tag{5.14}$$

$$|X(-\Omega)| = |X(\Omega)|, \quad \text{so } |X(\Omega)| \text{ is an even function.} \tag{5.15}$$

$$\angle X(-\Omega) = -\angle X(\Omega), \quad \text{so } \angle X(\Omega) \text{ is an odd function.} \tag{5.16}$$

Theorem 5.1.4 (Even and Odd signals)

1. *For any discrete-time even signal x with DTFT* $X = R(\Omega) + jX(\Omega)$,

$$R(\Omega) = x[0] + 2\sum_{n=1}^{\infty} x[n]\cos(\Omega n) \tag{5.17}$$

$$I(\Omega) = 0n \tag{5.18}$$

$$\text{and so} \quad X(\Omega) = x[0] + 2\sum_{n=1}^{\infty} x[n]\cos(\Omega n) \tag{5.19}$$

2. *For any discrete-time odd signal x with DTFT* $X = R(\Omega) + jX(\Omega)$,

$$R(\Omega) = x[0] \tag{5.20}$$

$$I(\Omega) = -2\sum_{n=1}^{\infty} x[n]\sin(\Omega n) \tag{5.21}$$

$$\text{and so} \quad X(\Omega) = x[0] - j2\sum_{n=1}^{\infty} x[n]\sin(\Omega n) \tag{5.22}$$

Example 5.1.3 (Rectangular Pulse) *Recall that the discrete-time rectangular pulse function of width L, for any odd integer $L > 0$, is*

$$p_L[n] = \begin{cases} 1, & -(L-1)/2 \le n \le (L-1)/2 \\ 0, & \textit{otherwise} \end{cases}$$

If we introduce $q = (L-1)/2$, then it can be shown that

$$P_L(\Omega) = \frac{\sin((q+1/2)\Omega)}{\sin(\Omega/2)}$$

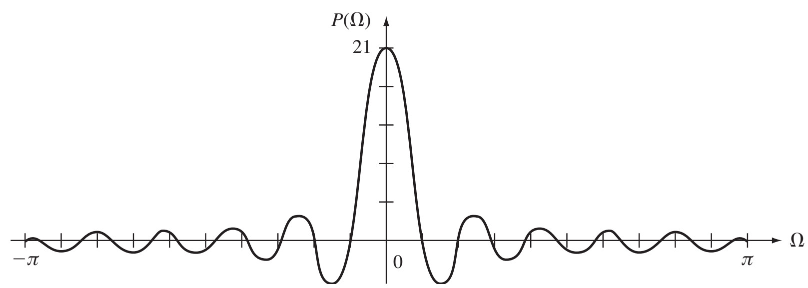


Figure 5.2: Graph of $P_L(\Omega)$.

For large values of L, the spectrum of $P_L(\Omega)$ resembles a sinc *function on the interval $-\pi \le \Omega \le \pi$.*

Example 5.1.4 (Comparison of frequency spectra) *We saw that $a^n u[n] \longleftrightarrow \frac{1}{1-ae^{-j\Omega}}$, for $|a| < 1$. We compare $a = 0.5$ with $a = -0.5$:*

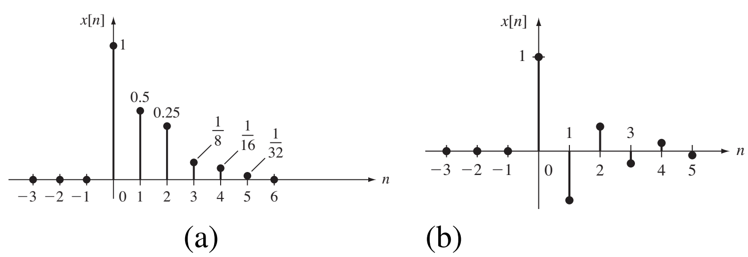


Figure 5.3: (a) $(\frac{1}{2})^n u[n]$ and (b) $(\frac{-1}{2})^n u[n]$

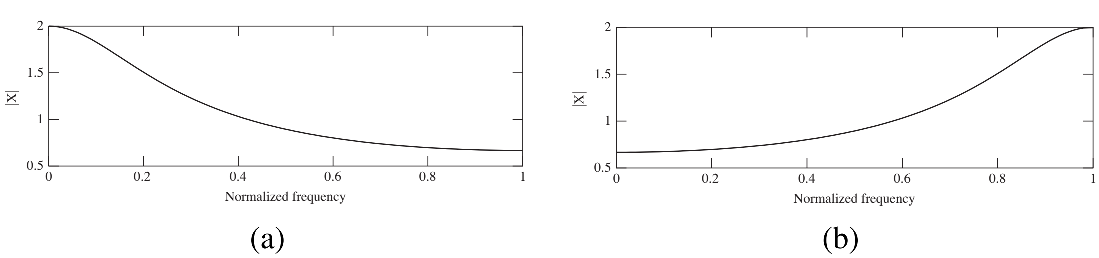


Figure 5.4: (a) Amplitude spectra of $(\frac{1}{2})^n u[n]$ and (b) $(\frac{-1}{2})^n u[n]$

Since the signal with $a = -0.5$ has greater time variations, its DTFT has more high-frequency components.

5.2 Properties of the DTFT

The DTFT has very similar properties to the CTFT that can be used to compute transforms of signals without using the definition of the transform. Some of these are

Theorem 5.2.1 (Linearity) *If $x_1[n] \longleftrightarrow X_1(\Omega)$ and $x_2[n] \longleftrightarrow X_2(\Omega)$, and a and b are any two scalars, then*

$$ax_1[n] + bx_2[n] \longleftrightarrow aX_1(\Omega) + bX_2(\Omega) \tag{5.23}$$

Theorem 5.2.2 (Time shifting) *If $x[n] \longleftrightarrow X(\Omega)$ and $q \in \mathbf{Z}$, then*

$$x[n-q] \longleftrightarrow X(\Omega)e^{-j\Omega q} \tag{5.24}$$

Theorem 5.2.3 (Time reversal or Flipping) *If $x[n] \longleftrightarrow X(\Omega)$ then*

$$x[-n] \longleftrightarrow X(-\Omega) \tag{5.25}$$

Theorem 5.2.4 (Modulation) *If $x[n] \longleftrightarrow X(\Omega)$ and $\Omega_0 \in \mathbf{R}$, then*

$$\begin{aligned}
x[n]e^{j\Omega_0 n} &\longleftrightarrow X(\Omega - \Omega_0) && (5.26)\\
x[n]\cos(\Omega_0 n) &\longleftrightarrow \frac{1}{2}[X(\Omega+\Omega_0) + X(\Omega-\Omega_0)] && (5.27)\\
x[n]\sin(\Omega_0 n) &\longleftrightarrow \frac{j}{2}[X(\Omega+\Omega_0) - X(\Omega-\Omega_0)] && (5.28)
\end{aligned}$$

Theorem 5.2.5 (Convolution) *If $x[n] \longleftrightarrow X(\Omega)$ and $v[n] \longleftrightarrow V(\Omega)$, then*

$$(x \star v)[n] \longleftrightarrow X(\Omega)V(\Omega) \tag{5.29}$$

The discrete-time signals 1, $\cos(\Omega n)$ and $\sin(\Omega n)$ do not have DTFT transforms in the ordinary sense because they are not absolutely summable , i.e.

$$\sum_{n=-\infty}^{\infty} |\sin(\Omega n)| = \infty \tag{5.30}$$

To define the **generalized DTFT** of these functions we use the Dirac delta function $\delta(t)$.

Theorem 5.2.6 *The DTFT of the constant signal $x[n] = 1$ is*

$$1 \longleftrightarrow 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k) \tag{5.31}$$

Proof: Let $X(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k)$ and take its inverse DTFT:

$$\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n}\, d\Omega\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k)e^{j\Omega n}\, d\Omega\\
&= \int_{-\pi}^{\pi} \delta(\Omega)e^{j\Omega n}\, d\Omega\\
&= e^0\\
&= 1, \quad \text{for all } n
\end{aligned}$$

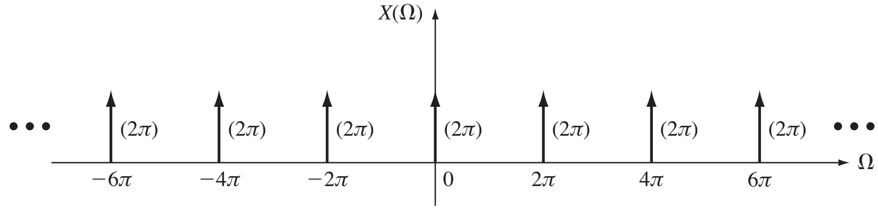


Figure 5.5: Fourier transform of the constant function 1

by the Sifting theorem. So the constant function 1 and $X(\Omega)$ are a DTFT pair. The impulse train

$$X(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k)$$

may be plotted as
Note that $X(\Omega)$ is periodic with period 2π, as expected for the DTFT. We can use equation 5.2 for $X(\Omega)$ together with the Modulation theorem 5.2.4 to obtain the generalized DTFT pairs

Theorem 5.2.7

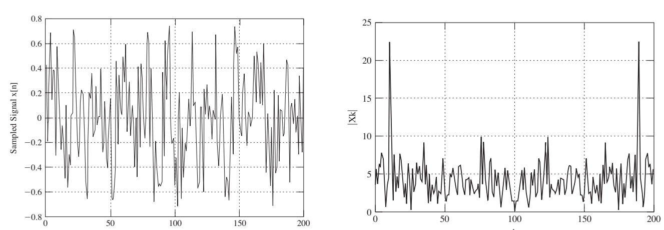
$$e^{j\Omega_0 n} \longleftrightarrow 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2\pi k) \tag{5.32}$$

$$\cos(\Omega_0 n) \longleftrightarrow \pi \sum_{k=-\infty}^{\infty} [\delta(\Omega + \Omega_0 - 2\pi k) + \delta(\Omega - \Omega_0 - 2\pi k)] \tag{5.33}$$

$$\sin(\Omega_0 n) \longleftrightarrow j\pi \sum_{k=-\infty}^{\infty} [\delta(\Omega + \Omega_0 - 2\pi k) - \delta(\Omega - \Omega_0 - 2\pi k)] \tag{5.34}$$

5.3 Discrete Fourier Transform

Up until now we have assumed that the signals we want analyse via Fourier methods were available in closed form, meaning that we have a mathematical formula for $x(t)$ or $x[n]$ that defines the signal. This is seldom the case with real-world data:



(a) (b)

Figure 5.6: (a) disctrete time-domain signal from measured data and (b) the amplitude spectrum

- On the left we see the plot of a time-domain signal obtained from 200 measurements, taken over a certain time-interval.
- The right graph shows its amplitude spectrum, revealing a dominant frequency.
- How do we obtain the frequency spectrum from the measurements?

Definition 5.3.1 (Discrete Fourier Transform) *Let x be a finite duration discrete-time signal with support $[0, L-1]$, for some integer $L > 1$. For any integer $N \geq L$, we define the N*-**point discrete Fourier transform (DFT)** *X_k of x to be*

$$X_k = \sum_{n=0}^{L-1} x[n] e^{-j2\pi kn/N}, \quad \text{where } k = 0, 1, \ldots, N-1 \tag{5.35}$$

- As x has finite duration, $x[n] = 0$ for $n < 0$, and $n \geq L$. The integer L is called the **record length**. The DFT X_k of x always exists.
- X_k is a function of the discrete variable k, and takes N complex values $X_0, \ldots, X_{N-1}$.
- The DFT is a periodic discrete function with period N.

Example 5.3.1 *Find the 4-point DFT of the signal $x[n]$ given by*

$$x[0] = 1,\ x[1] = 2,\ x[2] = 2,\ x[3] = 1, \quad x[n] = 0 \text{ otherwise}$$

Then x has support $[0, 3]$ so $L = 4$, and the 4-point DFT is

$$\begin{aligned} X_k &= \sum_{n=0}^{3} x[n] e^{-j\pi kn/2}, \\ &= x[0] + x[1]e^{-j\pi k/2} + x[2]e^{-j\pi k} + x[3]e^{-j\pi 3k/2} \end{aligned}$$

Hence

$$\begin{aligned} X_0 &= 1 + 2e^0 + 2e^0 + e^0 = 6 \\ X_1 &= 1 + 2e^{-j\pi/2} + 2e^{-j\pi} + e^{-j3\pi/2} = \sqrt{2}e^{j5\pi/4} \\ X_2 &= 1 + 2e^{-j\pi} + 2e^{-j2\pi} + e^{-j3\pi} = 0 \\ X_3 &= 1 + 2e^{-j3\pi/2} + 2e^{-j3\pi} + e^{-j9\pi/2} = \sqrt{2}e^{j3\pi/4} \end{aligned}$$

Theorem 5.3.1 (Symmetry) *If X_k is the N-point DFT of a signal x with record length $L \leq N$, then*

$$X_{N-k} = \bar{X}_k \quad \text{for all } k = 0, 1, \ldots, N-1 \tag{5.36}$$

A finite duration signal x can be recovered from X_k via the inverse DFT:

Definition 5.3.2 (Inverse Discrete Fourier Transform) *The* **inverse discrete Fourier transform** *of an N-point DFT X_k is defined to be*

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_k e^{j2\pi kn/N}, \quad \text{where } n = 0, 1, \ldots, L-1 \tag{5.37}$$

where $L \leq N$ is the record length of x.

Recall that the DTFT of a discrete signal x of record length L is

$$X(\Omega) = \sum_{n=0}^{L-1} x[n] e^{-j\Omega n} \tag{5.38}$$

and the N-point DFT of x is

$$X_k = \sum_{n=0}^{L-1} x[n] e^{-j2\pi kn/N}, \quad \text{where } k = 0, 1, \ldots, N-1 \tag{5.39}$$

Hence the connection between the DTFT and DFT is given by

Theorem 5.3.2 *Let x be a finite duration discrete-time signal with support $[0, L-1]$, let $X(\Omega)$ be the DTFT of x, and let X_k be its N-point DFT, where $N \geq L$. Then*

$$X_k = X\left(\frac{2\pi k}{N}\right) \quad \text{for all } k = 0, 1, \ldots, N-1 \tag{5.40}$$

Thus the DFT is equivalent to sampling the DTFT at the N frequency values $\Omega = 0, \frac{2\pi}{N}, \frac{4\pi}{N}, \ldots, \frac{2(N-1)\pi}{N}$.

Example 5.3.2 *Let $x[n] = p_{11}[n-5]$, where p_{11} is the rectangular discrete pulse of width $L = 11$. Then*

$$x[n] = \begin{cases} 1, & \text{if } 0 \leq n \leq 10 \\ 0, & \text{otherwise} \end{cases}$$

so x has support $[0, 10]$. We saw that before that the DTFT of the rectangular pulse p_{11} is

$$P_{11}(\Omega) = \frac{\sin(5.5\Omega)}{\sin(\Omega/2)}$$

Since $x[n] = p_{11}[n-5]$, by the time-shifting theorem, the DTFT of x is

$$X(\Omega) = P_{11}(\Omega)e^{-j5\Omega} = \frac{\sin(5.5\Omega)}{\sin(\Omega/2)}e^{-j5\Omega}$$

Hence the amplitude spectrum of the DTFT of x is $|X(\Omega)| = \frac{|\sin(5.5\Omega)|}{|\sin(\Omega/2)|}$.

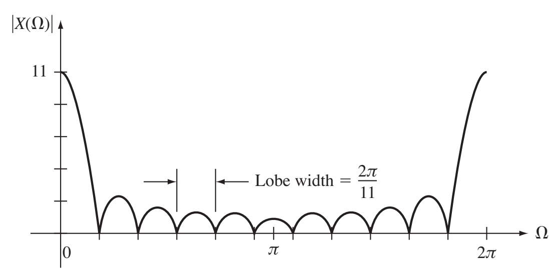
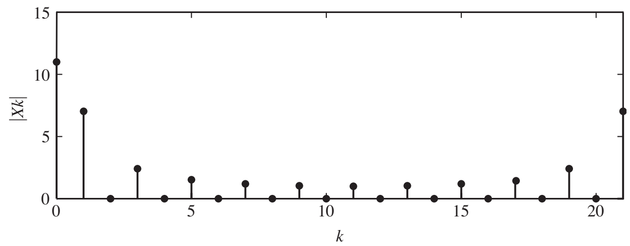


Figure 5.7: Amplitude spectrum of X

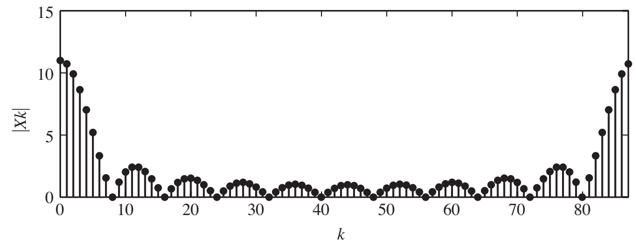
Choosing $N = L = 11$, we use Theorem 5.3.2 to obtain the 11-point DFT of x:

$$\begin{aligned} |X_k| &= \left|X\left(\frac{2\pi k}{11}\right)\right| \\ &= \left|\frac{\sin(\pi k)}{\sin(\pi k/11)}\right| \\ &= \begin{cases} 11, & \text{if } k = 0 \\ 0, & \text{if } k = 1, 2, \ldots, 10 \end{cases} \end{aligned}$$

(Here L'Hopital's Rule was used to evaluate $|X_0(0)|$.)
Unfortunately, for $N = 11$, $|X_k|$ gives a poor approximation to $|X(\Omega)|$. It samples $X(\Omega)$ at $\Omega = 0, \frac{2\pi}{11}, \ldots \frac{20\pi}{11}$. The last 10 of these occur at the zero points between the sidelobes, and hence the shape of the graph of $|X(\Omega)|$ is not apparent.
To improve the approximation, we increase the number of samples N in the DFT so the sampling frequencies $\frac{2\pi k}{N}$ are closer together. The plots below in Figure 5.8 show $|X_k|$ for $N = 22$ and $N = 88$. We observe the improvement in the approximation of the DFT to the DTFT as the sample size N increases.



(a)



(b)

Figure 5.8: DFT X_k of x with (a) $N = 22$ and (b) $N = 88$.

5.4 Practice Problems

58. Let the discrete-time signal $x[n]$ be given by

$$x[n] = \begin{cases} 2, & \text{if } n = 0,1,2,3,4 \\ -2, & \text{if } n = 5,6,7 \\ 0, & \text{otherwise} \end{cases}$$

Find $X(\Omega)$, the DTFT of x. Hint: Use the rectangular pulse of width L.

59. Let $x[n]$ be a discrete-time signal and let $X(\Omega)$ be its DTFT. Prove the following:

(a) If $x[n]$ is an even function then

$$X(\Omega) = x[0] + 2\sum_{n=1}^{\infty} x[n]\cos(\Omega n)$$

(b) If $x[n]$ is an odd function then

$$X(\Omega) = x[0] - j2\sum_{n=1}^{\infty} x[n]\sin(\Omega n)$$

60. Let $f[n] = a^n u[n]$ for some $a \in \mathbf{R}$.

(a) For which values of a does f have a DTFT?

(b) Find $F(\Omega)$, the DTFT of f, assuming it exists.

(c) Obtain an expression for the amplitude $|F(\Omega)|$ and plot its graph when $a = 0.5$.

61. Let $x[n]$ and $X(\Omega)$ be a DTFT pair. Using the definition of the DTFT to prove the following DTFT pairs:

$x[n-q]$	$\longleftrightarrow$	$X(\Omega)e^{-jq\Omega},\ q \in \mathbf{Z}$	(Time shift)
$x[-n]$	$\longleftrightarrow$	$X(-\Omega)$	(Flipping Theorem)
$x[n]e^{j\Omega_0 n}$	$\longleftrightarrow$	$X(\Omega - \Omega_0),\ \Omega_0 \in \mathbf{R}$	(Modulation)
$x[n]\cos[\Omega_0 n]$	$\longleftrightarrow$	$\frac{1}{2}[X(\Omega+\Omega_0) + X(\Omega-\Omega_0)]$	
$x[n]\sin[\Omega_0 n]$	$\longleftrightarrow$	$\frac{j}{2}[X(\Omega+\Omega_0) - X(\Omega-\Omega_0)]$	

62. The discrete-time signal $x[n]$ has DTFT

$$X(\Omega) = \frac{1}{e^{j\Omega} + b}$$

where $b \neq 0$ is an arbitrary constant. Find the DTFT of the following signals

(a) $v_1[n] = x[n-5]$

(b) $v_2[n] = x[-n]$

(c) $v_3[n] = (x \star x)[n]$

(d) $v_4[n] = x[n]e^{j2n}$

63. Find the inverse DTFT of the following frequency functions:

    (a) $X_1(\Omega) = \sin(\Omega)$

    (b) $X_2(\Omega) = \cos(\Omega)$

    (c) $X_3(\Omega) = \cos^2(\Omega)$

    (d) $X_4(\Omega) = \sin(\Omega)\cos(\Omega)$

64. The autocorrelation of the signal $x[n]$ is defined to be

$$R_x[n] = \sum_{i=-\infty}^{\infty} x[i]x[n+i]$$

    (a) Let $Flip_x[n] = x[-n]$. Show that $R_x[n] = (x \star Flip_x)[n]$.

    (b) Is $R_x[n]$ an even function, an odd function, or neither? Justify your answer.

    (c) Let $X(\Omega)$ denote the DTFT of $x[n]$. Show that

$$X(-\Omega) = \overline{X(\Omega)}$$

    (d) Hence obtain an expression for $P_x(\Omega)$, the DTFT of $R_x[n]$.

    (e) Express $P_x(0)$ in terms of $x[n]$.

65. Compute the 4-point DFT for each of the following signals, which are zero- valued except as specified:

    (a) $x[0] = x[2] = 1$.

    (b) $x[0] = x[1] = 1$, $x[2] = x[3] = -1$.

66. A signal x is known to have record length $L = 4$. Its 4-point DFT is given by

$$X_0 = 2,\ X_1 = -2 + j2,\ X_2 = -2,\ X_3 = -2 - j2$$

    Use the inverse DFT to obtain $x[n]$.

67. Prove the symmetry property of the DFT. Let x be a discrete-time signal with record length L, and let $N \geq L$ be an integer. If X_k is an N-point DFT of x, then

$$X_{N-k} = \bar{X}_k \quad \text{for all } k = 0, 1, \ldots, N-1$$

68. Let $x[n]$ be the discrete-time signal

$$x[n] \quad = \quad \begin{cases} 1, & n = 0, 1 \\ 0, & \text{otherwise} \end{cases}$$

    (a) Compute the DTFT of this signal and show that its magnitude is an even function of the frequency Ω.

(b) Let w be a periodic function of Ω with fundamental period γ. The *average power* of w is defined by

$$P(w) = \frac{1}{\gamma}\int_0^{\gamma} |w(\Omega)|^2 d\Omega.$$

Compute the average power $P(X)$ of the Fourier Transform X that you computed in (a).

(c) According to a famous result in Fourier theory ("Parseval's theorem"), the average power of the discrete time Fourier Transform can also be computed in the time domain. This is due to the following equality:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_0^{2\pi} |X(\Omega))|^2 d\Omega.$$

Verify that this equality is true for this signal x.

(d) Let $N \geq 2$ be an integer and let $(X_0, X_1, \ldots, X_{N-1})$ be the N-point DFT of the signal $x[n]$.

i. For $k = 0, 1, \ldots, N-1$ express X_k as a function of k and N.

ii. Sketch $|X_k|$ as a function of k ($k = 0, 1, \ldots, N-1$) and explain what happens as N approaches infinity.

69. Let $x[n]$ be a discrete-time signal that is zero outside of the interval $k = 0, \ldots, N_1 - 1$. Let N be an even integer with $N \geq N_1$. Let $(X_0, X_1, \ldots, X_{N-1})$ be the N-point DFT of x. Let $f[n] = x[2n]$ and $g[n] = x[2n+1]$ represent the even-indexed and odd-indexed samples of the signal x.

(a) Show that f and g are zero outside the interval $0 \leq n \leq (N/2) - 1$.

(b) Show that

$$x[n] = \begin{cases} f[\frac{n}{2}], & \text{if } n \text{ is even} \\ g[\frac{n-1}{2}], & \text{if } n \text{ is odd} \end{cases}$$

(c) Hence show that for all $k = 0, \ldots, N-1$,

$$X_k = \sum_{n=0}^{(N/2)-1} f[n] W_{N/2}^{nk} + W_N^k \sum_{n=0}^{(N/2)-1} g[n] W_{N/2}^{nk}$$

where

$$W_N = e^{-j2\pi/N}$$

Chapter 6

Fourier Analysis of Systems

In this chapter we develop the Fourier analysis of continuous-time and discrete-time LTI systems.

6.1 Frequency Response of Continuous-time Systems

Recall from Chapter 2 that for any LTI continuous-time system, the output y from any input v can be expressed in terms of the convolution of the impulse response h as:

$$y(t) = (v \star h)(t) = \int_{-\infty}^{\infty} h(\lambda)v(t-\lambda)\, d\lambda \tag{6.1}$$

In this chapter we will assume the following is true for the impulse response of any system we consider:

Assumption 6.1.1 *The impulse response is absolutely integrable:*

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty \tag{6.2}$$

Under this assumption the CTFT of h always exists

$$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\, dt \tag{6.3}$$

Recall also that the CTFT of the convolution of two signals is the product of their CTFTs:

$$(x \star v)(t) \longleftrightarrow X(\omega)V(\omega) \tag{6.4}$$

Applying this to the impulse response $y(t) = (v \star h)(t)$ we get

$$Y(\omega) = H(\omega)V(\omega) \tag{6.5}$$

where $V(\omega)$ and $Y(\omega)$ are the CTFT of the input v and output y, respectively.

Definition 6.1.1 *The equations*

$$y(t) = (v \star h)(t) \quad \textit{and} \quad Y(\omega) = H(\omega)V(\omega) \tag{6.6}$$

are called the **time-domain** *and* **frequency-domain** *representations of the system.*

We may represent this relationship visually as

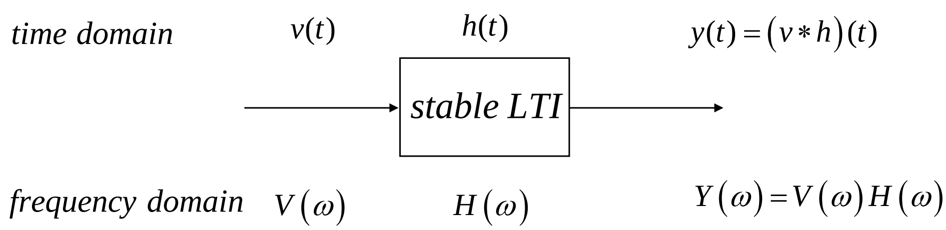


Figure 6.1: Time domain and frequency domain relationships

Theorem 6.1.1 *For any input v, the* **amplitude spectrum** *of the output y is given by*

$$|Y(\omega)| = |H(\omega)||V(\omega)| \tag{6.7}$$

and the **phase spectrum** *of the output y is given by*

$$\angle Y(\omega) = \angle H(\omega) + \angle V(\omega) \tag{6.8}$$

The CTFT $H(\omega)$ of h is called the **frequency response** of the system. It provides insight into how a system converts inputs into outputs.

Theorem 6.1.2 *Let $H(\omega)$ be the frequency response of a stable LTI system, and let $\omega_0 \in \mathbf{R}$. Then the input $v(t) = e^{j\omega_0 t}$ has output*

$$y(t) = e^{j\omega_0 t} H(\omega_0) \tag{6.9}$$

and the input $v(t) = \cos(\omega_0 t)$ has output

$$y(t) = |H(\omega_0)| \cos(\omega_0 t + \angle H(\omega_0)) \tag{6.10}$$

We may represent these relationships visually as

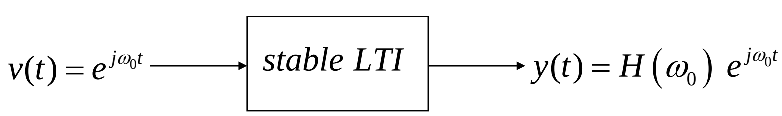


Figure 6.2: LTI system response from input $e^{j\omega_0 t}$

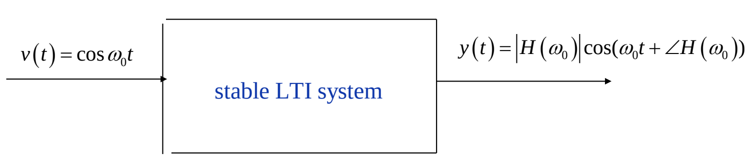


Figure 6.3: LTI system response from input $\cos(\omega_0 t)$

Example 6.1.1 (Frequency analysis of an RC circuit) *In the RC circuit shown below, the input is the source voltage v and the output y is the capacitor voltage v_C. We assume zero initial conditions, i.e. $y(0) = 0$.*
The differential equation that describes the circuit is

$$\frac{dy}{dt} + \frac{1}{RC} y(t) = \frac{1}{RC} v(t)$$

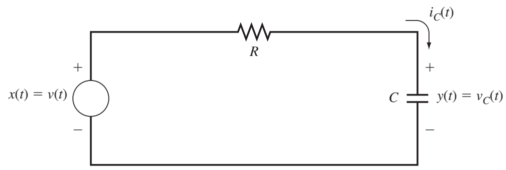


Figure 6.4: *RC* circuit

To find the frequency response $H(\omega)$, we can apply the input $v(t) = e^{j\omega t}$ with output $y(t) = H(\omega)e^{j\omega t}$. Substituting these gives

$$\begin{aligned} RC\frac{dy}{dt} + y(t) &= v(t) \\ \Rightarrow\ RC\frac{d}{dt}(H(\omega)e^{j\omega t}) + H(\omega)e^{j\omega t} &= e^{j\omega t} \\ \Rightarrow\ j\omega RCH(\omega)e^{j\omega t} + H(\omega)e^{j\omega t} &= e^{j\omega t} \\ \Rightarrow\ H(\omega)(j\omega RC + 1) &= 1 \\ \Rightarrow\ H(\omega) &= \frac{1}{1 + j\omega RC} \end{aligned}$$

Hence the magnitude response and phase response of the system are

$$|H(\omega)| = \frac{1}{\sqrt{1 + (RC\omega)^2}}, \quad \underline{/H(\omega)} = -\tan^{-1}(\omega RC)$$

Assume $RC = 1$ and the input is $v(t) = 5\cos(t)$. Then $\omega_0 = 1$, so

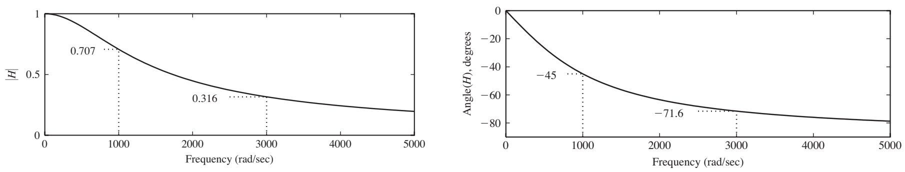
$$\begin{aligned} |H(1)| &= \frac{1}{\sqrt{1 + (RC\omega)^2}} \\ &= \frac{1}{\sqrt{2}} \\ \underline{/H(1)} &= -\tan^{-1}(\omega RC) \\ &= -\tan^{-1}(1) \\ &= -\frac{\pi}{4} \end{aligned}$$

Hence by Theorem 6.1.2,

$$\begin{aligned} y(t) &= |H(\omega_0)| 5\cos(\omega_0 t + \underline{/H(\omega_0)}) \\ &= \frac{5}{\sqrt{2}}\cos(t - \pi/4) \end{aligned}$$

If $RC = 1/1000$, the circuit's magnitude and phase responses are shown in Figure 6.5.
For an input $v(t) = \cos(\omega_0 t)$, Table 6.1 shows the magnitude and phase of the output y for different values of ω_0.
We observe that

$$\begin{aligned} |H(\omega)| \to 0 &\quad as \quad \omega \to \infty \\ \underline{/H(\omega)} \to -90^\circ &\quad as \quad \omega \to \infty \end{aligned}$$



(a) (b)

Figure 6.5: (a) Magnitude and (b) phase response for the *RC* circuit.

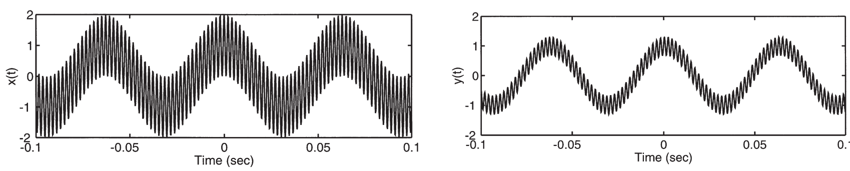
	$\omega_0 = 0$	$\omega_0 = 1000$	$\omega_0 = 3000$
$\lvert y\rvert$	1	0.707	0.316
$\angle y$	0	$-45°$	$-71.6°$

Table 6.1: Magnitude and phase of output y from input $v(t)$ with different values of ω_0.

The system is a **low pass filter** *because high frequency signals have their amplitude attenuated by the system. Figure 6.6(a) show the filter input signal*

$$x(t) = \cos(100t) + \cos(3000t)$$

and Figure 6.6(b) shows the output of the low pass filter.



(a) (b)

Figure 6.6: (a) $x(t)$ and (b) output $y(t)$ of low pass filtering $x(t)$

The low frequency component ($\omega = 100$) is approximately the same, but the high frequency component ($\omega = 3000$) is strongly attenuated.

Theorem 6.1.3 (Response to periodic inputs) *Let $H(\omega)$ be the frequency response of a stable LTI system, and let v be a periodic signal with Fourier series*

$$v(t) = a_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k) \tag{6.11}$$

Then the output y from the input v is

$$y(t) = a_0 H(0) + \sum_{k=1}^{\infty} A_k |H(k\omega_0)| \cos(k\omega_0 t + \theta_k + \angle H(k\omega_0)) \tag{6.12}$$

If we use A_k^y and θ_k^y to denote the Fourier coefficients of y, and likewise let A_k^v and θ_k^v be the coefficients for v, then

$$\begin{aligned} A_k^y &= A_k^v |H(k\omega_0)| & (6.13)\\ \theta_k^y &= \theta_k^v + \underline{/H(k\omega_0)} & (6.14) \end{aligned}$$

Corollary 6.1.1 (Response to periodic inputs) *Let $H(\omega)$ be the frequency response of a stable LTI system, and let v be a periodic signal with complex Fourier series*

$$v(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \tag{6.15}$$

Then the output y from the input v is

$$y(t) = \sum_{k=-\infty}^{\infty} c_k H(k\omega_0) e^{jk\omega_0 t} \tag{6.16}$$

If we use c_k^y and c_k^v to denote the complex Fourier coefficients of y and v, then

$$c_k^y = c_k^v H(k\omega_0) \tag{6.17}$$

Theorem 6.1.4 (Response to non-periodic inputs) *Let $H(\omega)$ be the frequency response of a stable LTI system, and let v be a non-periodic signal with CTFT $V(\omega)$. Then the output y from the input v has CTFT*

$$Y(\omega) = H(\omega)V(\omega) \tag{6.18}$$

and can be obtained by taking the inverse Fourier transform

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)V(\omega)e^{j\omega t}\, d\omega \tag{6.19}$$

In some cases the integral may be difficult to compute, so we often work with tables of Fourier transform pairs.

6.2 Filters

Example 6.2.1 (Low pass filter response) *Consider the rectangular periodic pulse train with amplitude spectrum below.*

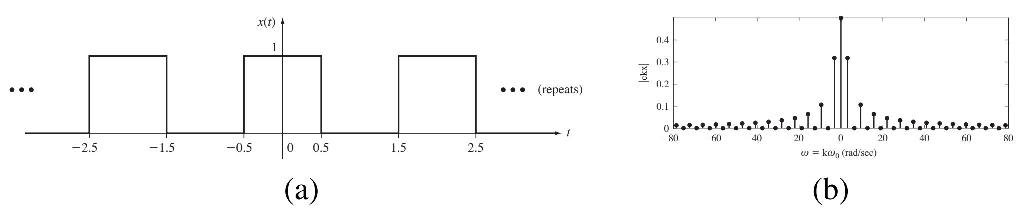


Figure 6.7: (a) rectangular pulse and (b) its amplitude spectrum

We apply this input to the RC lowpass filter circuit with parameters (i) $RC = 1$, (ii) $RC = 1/10$, and (iii) $RC = 1/100$. The responses and their corresponding ampltude spectra are shown in Figure 6.8.

- The pulse train input signal has discontinuities and hence its spectrum contains many high-frequency components. The spectrum takes discrete values as the input signal is periodic.
- A low pass *RC* circuit filters (attenuates) higher frequencies, but the circuit with $RC = 1$ does so much more sharply than the circuit with $RC = 1/100$.
- The circuit with $RC = 1/100$ passes the rectangular pulse with little distortion. Some fuzziness is apparent near the points of discontinuity due to the loss of high frequency spectrum.
- The circuit with $RC = 1$ distorts the signal substantially.
- We say that the circuit with $RC = 1/100$ has greater **bandwidth** because it passes much more of the frequency content of the input signal to the output signal.

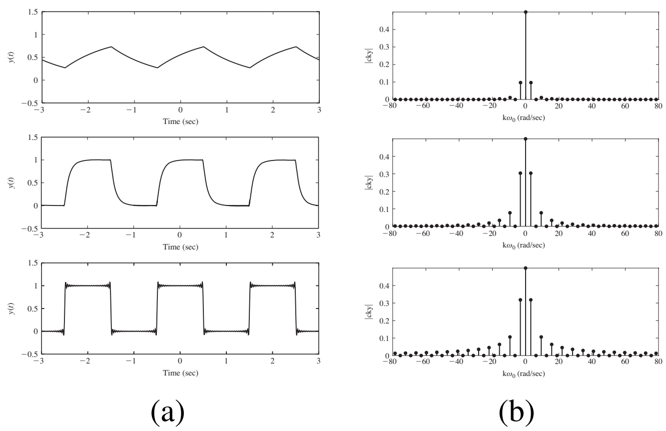


Figure 6.8: (a) outputs from the three *RC* filters and (b) their corresponding ampltude spectra

Example 6.2.2 (Low pass filter response II) *Consider the single pulse $x(t) = p_1(t)$ with amplitude spectrum below.*

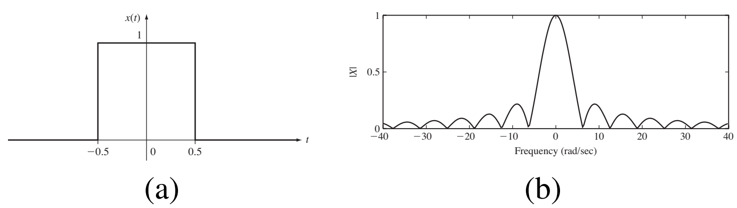


Figure 6.9: (a) $p_1(t)$ and (b) its amplitude spectrum

We apply this input to the RC low pass filter circuit with parameters (i) $RC = 1$, and (ii) $RC = 1/10$:

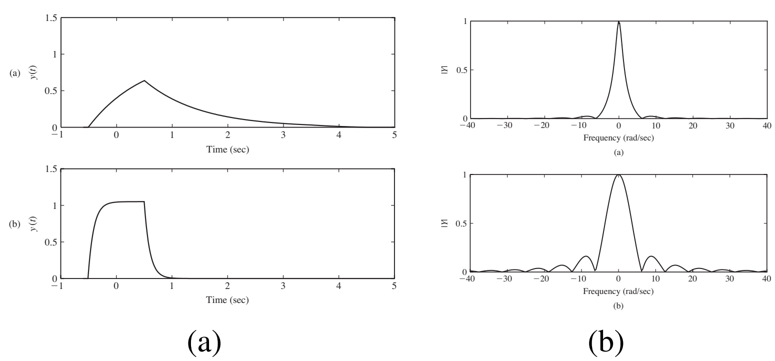


Figure 6.10: (a) outputs from the two *RC* filters and (b) their corresponding ampltude spectra

The amplitude spectrum takes continuous frequency values as the input signal is non-periodic. As expected the greater bandwidth of the circuit with $RC = 1/10$ passes more of the frequency spectrum and has less signal distortion.

Definition 6.2.1 (Ideal filters) *An* **ideal filter** *completely attenuates signals of the form $v(t) = A\cos(\omega t)$ for ω in a certain range of frequency values, while passing without attenuation signals outside of the specified range. There are four main types:*

- **Ideal low pass**: $$|H(\omega)| = \begin{cases} 1, & -B \leq \omega \leq B \\ 0, & |\omega| > B \end{cases}$$

- **Ideal high pass**: $|H(\omega)| = \begin{cases} 0, & -B \leq \omega \leq B \\ 1, & |\omega| > B \end{cases}$

- **Ideal band pass**: $|H(\omega)| = \begin{cases} 1, & B_1 \leq |\omega| \leq B_2 \\ 0, & otherwise \end{cases}$

- **Ideal band stop**: $|H(\omega)| = \begin{cases} 0, & B_1 \leq |\omega| \leq B_2 \\ 1, & otherwise \end{cases}$

where B, B_1 and B_2 are positive real numbers.

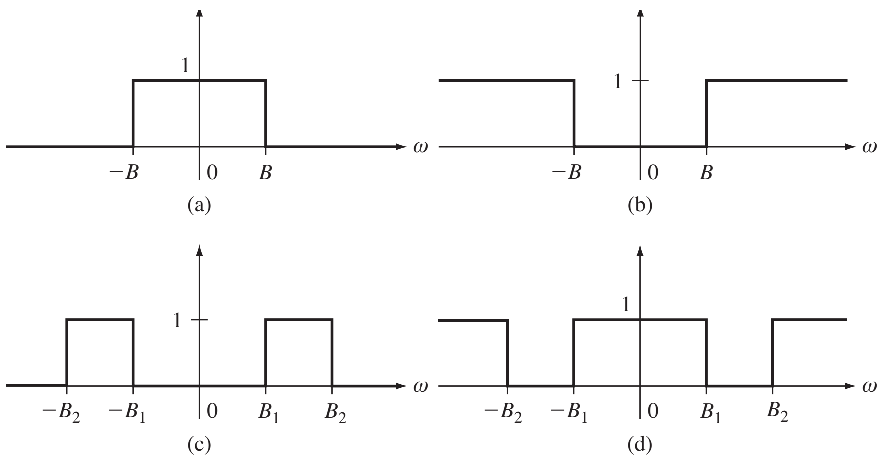


Figure 6.11: Ideal (a) Low pass (b) High pass (c) Band pass and (d) Band stop filters

Remark 6.2.1 (Ideal filters)
- *The low pass filter (a) passes frequencies below a threshold and blocks those above the threshold value.*

- *The high pass filter (b) passes those above the threshold and blocks those below.*

- *The band pass filter (c) passes all the frequencies within a certain band and blocks those outside.*

- *The band stop filter (d) (also called a* **notch** *filter) blocks those frequencies inside a certain range and passes all frequencies outside the range.*

- *The* **passband** *of filter is the set of all input frequencies passed by the filter without attenuation, and the* **stopband** *is the range of input frequencies that are completely attenuated.*

- *The width of the pass band is the* **bandwidth** *of the filter.*

- *The low pass filter is equal to the rectangular pulse function*

$$H(\omega) = p_{2B}(\omega)e^{-j\omega c} \tag{6.20}$$

for some $c \in \mathbf{R}$. We can obtain the impulse response of the filter as

$$h(t) = \frac{B}{\pi}\operatorname{sinc}\left(\frac{B(t-c)}{\pi}\right) \tag{6.21}$$

Since $h(t) \neq 0$ for $t < 0$, the filter is not causal, and hence cannot be physically realized, hence the name 'ideal filter'.

- *Building physically realizable filters that approximate ideal filters is an important problem area in signal processing.*

Definition 6.2.2 (Linear phase) *An LTI system is called a* **linear phase system** *if its frequency response H can be written as*

$$H(\omega) = K(\omega)e^{-j\omega t_d},$$

where K is a real-valued function and $t_d \in \mathbf{R}$ is a positive constant.

Note that then its phase response is a linear function of frequency, i.e.

$$\angle H(\omega) = -\omega t_d \text{ modulo } \pi.$$

Suppose that $x(t) = A_0 \cos(\omega_0 t + \Theta_0) + A_1 \cos(\omega_1 t + \Theta_1)$ is input to a linear phase system. Then the response is

$$y(t) = A_0 K(\omega_0)\cos(\omega_0(t - t_d) + \Theta_0) + A_1 K(\omega_1)\cos(\omega_1(t - t_d) + \Theta_1).$$

Thus linear phase systems are shifting input sinusoids of different frequencies by the *same* time constant t_d. We say that the filter has **no phase distortion**.

6.3 Frequency Response of Discrete-time Systems

The Fourier analysis of discrete-time systems is quite similar to that of continuous-time systems:
Recall that the output y from any input v can be expressed in terms of the convolution of the unit pulse response h as:

$$y[n] = (v \star h)[n] = \sum_{i=-\infty}^{\infty} h[i]v[n-i] \tag{6.22}$$

We need to assume for any LTI discrete-time system

Assumption 6.3.1 *The unit pulse response is absolutely summable:*

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty \tag{6.23}$$

Under this assumption the DTFT of h always exists

$$H(\Omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\Omega n} \tag{6.24}$$

Recall also that the DTFT of the convolution of two signals is the product of their DTFTs:

$$(x \star v)[n] \longleftrightarrow X(\Omega)V(\Omega) \tag{6.25}$$

Applying this to the unit pulse response $y[n] = (v \star h)[n]$ we get

$$Y(\Omega) = H(\Omega)V(\Omega) \tag{6.26}$$

where $V(\Omega)$ and $Y(\Omega)$ are the DTFT of the input v and output y, respectively.

Definition 6.3.1 *The equations*

$$y[n] = (v \star h)[n] \quad \textit{and} \quad Y(\Omega) = H(\Omega)V(\Omega) \tag{6.27}$$

are called the **time-domain** *and* **frequency-domain** *representations of the system.*

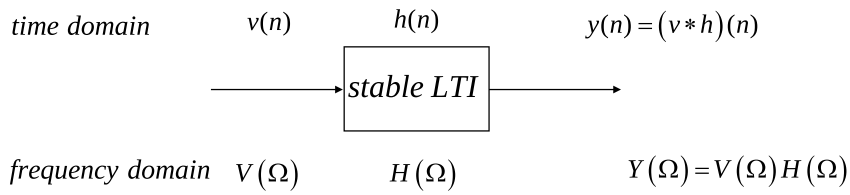


Figure 6.12: Equivalence of time and frequency domain representations of discrete-time systems

We may represent relationships visually as

Theorem 6.3.1 *For any input v, the* **amplitude spectrum** *of the output y is given by*

$$|Y(\Omega)| = |H(\Omega)||V(\Omega)| \tag{6.28}$$

and the **phase spectrum** *of the output y is given by*

$$\angle Y(\Omega) = \angle H(\Omega) + \angle V(\Omega) \tag{6.29}$$

The DTFT $H(\Omega)$ of h is called the **frequency response** of the system. It is periodic with period 2π.

Theorem 6.3.2 *Let $H(\Omega)$ be the frequency response of a stable LTI discrete-time system, and let $\Omega_0 \in \mathbf{R}$. Then the input $v[n] = e^{j\Omega_0 n}$ has output*

$$y[n] = e^{j\Omega_0 n} H(\Omega_0) \tag{6.30}$$

and the input $v[n] = \cos(\Omega_0 n)$ has output

$$y[n] = |H(\Omega_0)| \cos(\Omega_0 n + \angle H(\Omega_0)) \tag{6.31}$$

We may represent these relationships visually as

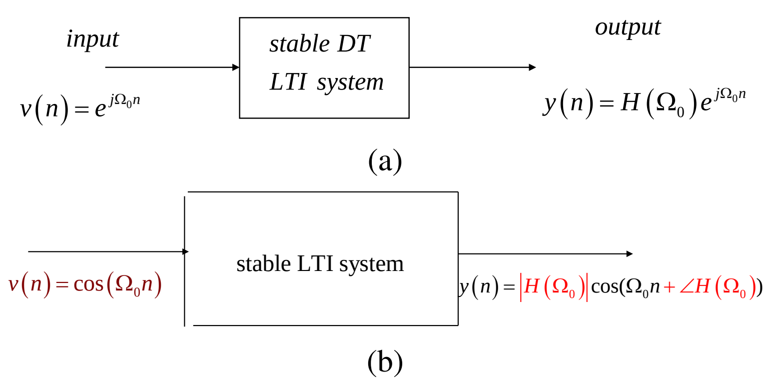


Figure 6.13: Stable LTI Discrete-time system response to inputs (a) $e^{j\Omega_0 n}$ and (b) $\cos(\Omega_0 n)$.

Example 6.3.1 (Moving Average filter) *Consider the N-point moving-average filter (MAF)*

$$y[n] = \frac{1}{N}\left[v[n] + v[n-1] + v[n-2] + \cdots + v[n-N+1]\right]$$

The unit impulse response is

$$h[n] = \frac{1}{N}\left[\delta[n] + \delta[n-1] + \delta[n-2] + \cdots + \delta[n-N+1]\right]$$

Taking the DTFT of both sides and using the time-shifting property of the DTFT gives

$$\begin{aligned} H(\Omega) &= \frac{1}{N}[1+e^{-j\Omega}+\cdots+e^{-j(N-1)\Omega}] \\ &= \frac{1}{N}\left[\frac{1-e^{-jN\Omega}}{1-e^{-j\Omega}}\right] \end{aligned}$$

Alternatively, we can apply the input $v[n]=e^{j\Omega n}$ *and obtain the output response* $y[n]=H(\Omega)e^{j\Omega n}$*. Substituting these gives*

$$\begin{aligned} H(\Omega)e^{j\Omega n} &= \frac{1}{N}[e^{j\Omega n}+e^{j\Omega(n-1)}+e^{j\Omega(n-2)}+\cdots+e^{j\Omega(n-N+1)}] \\ \Rightarrow H(\Omega) &= \frac{1}{N}[e^{j\Omega n}+e^{j\Omega(n-1)}+e^{j\Omega(n-2)}+\cdots+e^{j\Omega(n-N+1)}]e^{-j\Omega n} \\ &= \frac{1}{N}[1+e^{-j\Omega}+\cdots+e^{-j(N-1)\Omega}] \\ &= \frac{1}{N}\left[\frac{1-e^{-jN\Omega}}{1-e^{-j\Omega}}\right] \end{aligned}$$

It can be shown that a MVA filter has linear phase of over a range of frequencies - see the Problem Booklet.

Definition 6.3.2 (Cascaded Systems) *Let* S_1 *be a system that maps input* v *to outputs* w*, while* S_2 *maps inputs* w *to* y*:*

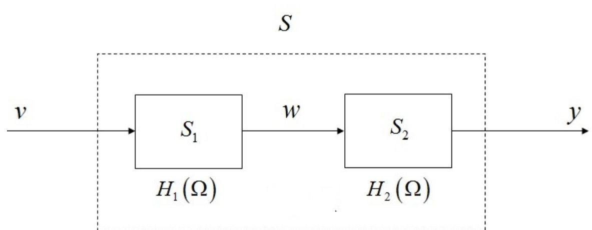


Figure 6.14: Cascaded systems

We say that the overall system S *that maps input* v *to* y *is the* **cascaded connection** *of the systems* S_1 *and* S_2.

Theorem 6.3.3 (Cascaded Systems) *If systems* S_1 *and* S_2 *have frequency responses* $H_1(\Omega)$ *and* $H_2(\Omega)$*, then the cascaded system* S *has frequency response*

$$H(\Omega)=H_1(\Omega)H_2(\Omega) \tag{6.32}$$

Definition 6.3.3 (System Inverse) *For cascaded systems* S_1 *and* S_2*, we say that system* S_2 *is the* **inverse system** *of* S_1 *if the cascaded system* S *is the* **identity system**, *i.e.*

$$Sv=v \tag{6.33}$$

Theorem 6.3.4 *Let systems* S_1 *and* S_2 *have frequency responses* $H_1(\Omega)$ *and* $H_2(\Omega)$*, and unit pulse responses* h_1 *and* h_2*, respectively. Then* S_2 *is the inverse of* S_1 *if*

$$H_1(\Omega)H_2(\Omega)=1 \tag{6.34}$$

Equivalently, S_2 *is the inverse of* S_1 *if*

$$(h_1\star h_2)[n]=\delta[n] \tag{6.35}$$

Remark 6.3.1 *Note that equations* (6.34) *and* (6.35) *are equivalent, by taking Fourier transforms, since* $\delta[n] \longleftrightarrow 1$. *If we want to show that two systems cancel each other out (i.e. their cascade is the identity system), we can use either equation* (6.34) *in frequency domain, or else equation* (6.35) *in time domain.*

6.4 Practice Problems

70. Let $H(\omega)$ be the frequency response of a continuous-time stable LTI system, and let $\omega_0 \in \mathbf{R}$.

    (a) Show that the output of the system from the input $v(t) = e^{j\omega_0 t}$ under zero initial conditions is

    $$y(t) = e^{j\omega_0 t} H(\omega_0)$$

    Hint: Use $y(t) = (v \star h)(t)$.

    (b) Show that the output of the system from the input $v(t) = \cos(\omega_0 t)$ under zero initial conditions is

    $$y(t) = |H(\omega_0)| \cos(\omega_0 t + \angle H(\omega_0))$$

    Hint: Use (a) and the fact that for any $z \in \mathbf{C}$ and $\theta \in \mathbf{R}$,

    $$\frac{1}{2}(z + \bar{z}) = Re(z) \quad \text{and} \quad Re(ze^{j\theta}) = |z| \cos(\theta + \angle z)$$

71. A linear time invariant continuous-time system has the frequency response

    $$H(\omega) = \begin{cases} 1, & 2 \leq |\omega| \leq 7 \\ 0, & \text{otherwise} \end{cases}$$

    Compute the output response y when the input v is

    (a) $v(t) = 2 + 3\cos(3t) - 5\sin(6t - \frac{\pi}{6}) + 4\cos(13t - \frac{\pi}{9})$

    (b) $v(t) = 1 + \sum_{k=1}^{\infty} \frac{1}{k} \cos(2kt)$

72. A linear time invariant continuous-time system has the frequency response

    $$H(\omega) = \frac{1}{j\omega + 1}$$

    Compute the output response y when the input v is

    (a) $v(t) = \cos(t)$, $-\infty < t < \infty$.

    (b) $v(t) = \cos(t + \frac{\pi}{4})$, $-\infty < t < \infty$.

73. A linear time invariant continuous-time system has the frequency response

    $$H(\omega) = \frac{j\omega}{j\omega + 2}$$

    (a) Obtain expressions for the amplitude and phase functions for H.

    (b) Plot the amplitude and phase response of H for $-\infty < \omega < \infty$.

74. A periodic signal v with period T has complex Fourier series

$$v(t) = \sum_{k=-\infty}^{\infty} c_k^v e^{jk\omega_0 t}$$

This signal is applied to the LTI continuous-time system with frequency response function

$$H(\omega) = \begin{cases} 10e^{-j5\omega}, & |\omega| > \frac{\pi}{T} \\ 0 & \text{otherwise} \end{cases}$$

(a) Let the resulting output response y have complex Fourier series

$$y(t) = \sum_{k=-\infty}^{\infty} c_k^y e^{jk\omega_0 t}$$

Express the coefficients c_k^y in terms of the coefficients c_k^v.

(b) Suppose $c_0^v = 2$. Find the constants a, b, and c such that

$$y(t) = av(t-b) + c$$

(c) Suppose the input signal is

$$v(t) = \begin{cases} 1, & -0.5 < t < 0.5 \\ 0, & 0.5 < t < 1.5 \end{cases} \qquad \text{and } v(t) = v(t+2)$$

Compute and plot $y(t)$ for this input signal.

75. Consider the RL series circuit below:

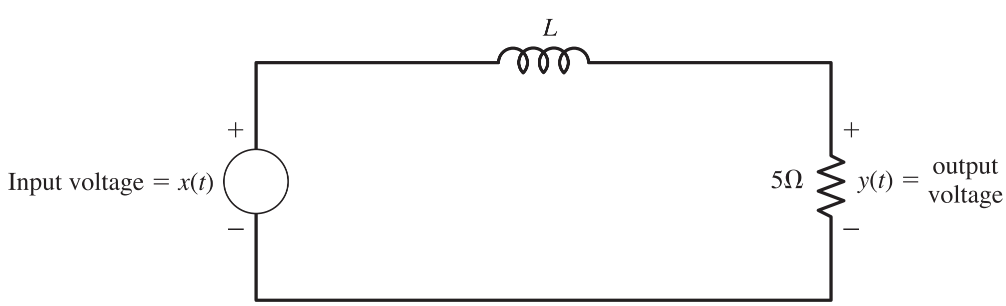


Figure 6.15: Diagram for Problem 75

(a) Use circuit laws to derive a differential equation for the output y in terms of R, L and the input x. Hence obtain $H(\omega)$, the frequency response of the circuit.

(b) Suppose the input x is the periodic signal

$$x(t) = 10|\sin(377t)|, \quad x(t) = x(t + \pi/377)$$

Obtain c_0^x, the constant term in the complex Fourier series for x.

(c) Let y be the output from this input x. Use the frequency response $H(\omega)$ to obtain c_0^y, the constant term in the complex Fourier series for y.

76. (a) Consider the signal x in Figure 6.16. The signal is applied to a system with frequency response

$$H_1(\omega) = 3e^{j2\omega}$$

Plot the graph of the output signal y.

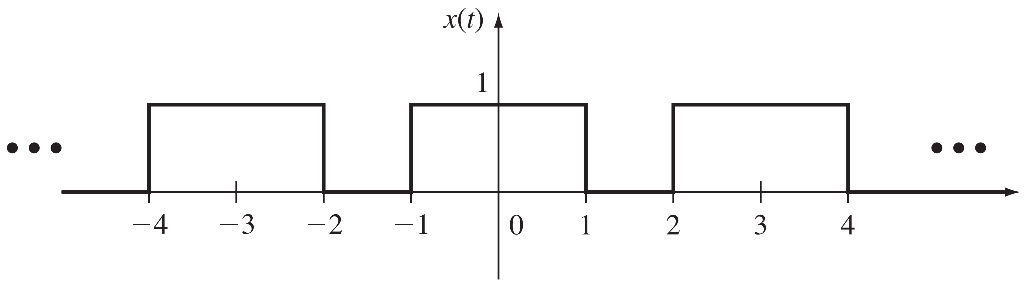


Figure 6.16: Diagram for Problem 76(a)

(b) When the signal x is applied to the system with frequency response

$$H_2(\omega) = a + be^{j\omega c}$$

the output y is shown in Figure 6.17. Determine the values of a, b and c. Are these values unique?
Hint: Express $y(t) = Ax(t) + Bx(t - C)$ for suitable values of A, B and C.

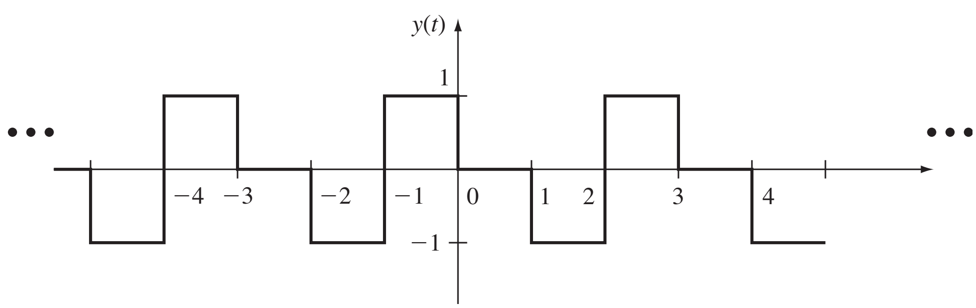


Figure 6.17: Diagram for Problem 76(b)

77. Consider the 2-point moving-average filter

$$y[n] = \frac{1}{2}[v[n] + v[n-1]]$$

(a) Show that the frequency response for this filter is given by

$$H(\Omega) = \frac{\sin(\Omega)}{2\sin(\Omega/2)} e^{-j\Omega/2}$$

(b) Show that the filter has linear phase for $0 \leq \Omega \leq \pi$.

78. An ideal lowpass digital filter has the frequency response function

$$H(\Omega) = \begin{cases} 1, & 0 \le |\Omega| \le \frac{\pi}{4} \\ 0, & \frac{\pi}{4} < |\Omega| < \pi \end{cases} \qquad \text{and } H(\Omega) = H(\Omega + 2\pi) \ .$$

(a) Determine the unit-pulse response $h[n]$ of the filter.

(b) Compute the output response y when the input x is given by

i. $x[n] = \cos(\pi n/8)$

ii. $x[n] = \cos(3\pi n/4) + \cos(\pi n/16)$

iii. $x[n] = \operatorname{sinc}(n/2)$

You may use the following DTFT pair, for any $B > 0$:

$$\frac{B}{\pi} \operatorname{sinc}\left(\frac{Bn}{\pi}\right) \longleftrightarrow \sum_{k=-\infty}^{\infty} p_{2B}(\Omega + 2\pi k).$$

79. Consider the discrete-time system given by the input-output difference equation

$$y[n+1] + 0.9y[n] = 1.9x[n+1].$$

(a) Use the input signal $x[n] = e^{j\Omega n}$ to determine the system's frequency response $H(\Omega)$.

(b) Obtain the unit pulse response for the system and use it to confirm your answer for part (a).

80. An LTI discrete-time system with unit pulse response $h_1[n] = (\frac{1}{3})^n u[n]$ is connected in cascade with another causal LTI discrete-time system with unit pulse response $h_2[n]$. The resulting cascaded connection has frequency response

$$H(\Omega) = \frac{3\sin(1.5\Omega)}{(3 - e^{-j\Omega})\sin(\Omega/2)}.$$

Find $h_2[n]$.

Chapter 7

Laplace and z-Transforms

In this chapter we introduce the Laplace and z-Transforms and apply them to the analysis of continuous-time and discrete-time systems. Recall that any absolutely integrable signal $x(t)$ has CTFT

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt \tag{7.1}$$

Similarly, the frequency response of a continuous-time LTI stable system is the CTFT of its impulse response

$$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\, dt \tag{7.2}$$

We used the CTFT (and also the DTFT) to analyse the spectral content of signals, and the frequency behaviour of systems. The essential assumption was that the signals involved were absolutely integrable (or absolutely summable). To analyse signals that are not absolutely integrable, and unstable systems, we will need more powerful tools - the Laplace transform (in continuous-time) and the z-transform (in discrete-time).

7.1 Laplace Transform: Definition

Definition 7.1.1 (Laplace transform) *For any continuous-time signal x we define the* **Laplace transform** *as*

$$X(s) = \int_{0}^{\infty} x(t)e^{-st}\, dt \tag{7.3}$$

where s is a complex variable.

- The LT (Laplace transform) depends only on the value of the signal for $t \geq 0$.
- The LT is particularly useful for analysing the behaviour of systems given in terms of differential equations with specified initial conditions.
- As for the CTFT, we write

$$x(t) \longleftrightarrow X(s) \tag{7.4}$$

  to indicate a Laplace transform pair.

Definition 7.1.2 (Region of convergence) *We define the* **region of convergence** *of the LT as the set of complex numbers s for which the transform $X(s)$ exists. This is*

$$RoC(x) = \{s = \sigma + j\omega \in \mathbf{C} : x(t)e^{-\sigma t} \textit{is absolutely integrable}\} \tag{7.5}$$

The RoC depends on the signal x and always has the form

$$RoC(x) = \{s \in \mathbf{C} : Re(s) > c\} \tag{7.6}$$

for some $c \in \mathbf{R}$.

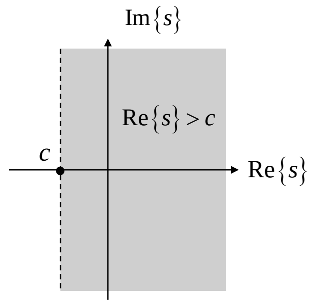


Figure 7.1: Region of convergence of the Laplace transform of x

Our next theorem describes the relationship between the Laplace and Fourier transforms.

Theorem 7.1.1 *Let x be a continuous-time signal such that $x(t) = 0$ for $t < 0$. Let X be its Laplace transform and assume that $s = 0$ is in its region of convergence. Then the CTFT of x is given by*

$$X(\omega) = X(s)|_{s=j\omega} \tag{7.7}$$

Example 7.1.1 *Let $x(t) = e^{-bt}u(t)$ for some $b \in \mathbf{R}$. Then*

$$\begin{aligned} X(s) &= \int_0^\infty x(t)e^{-st}\,dt \\ &= \int_0^\infty e^{-bt}e^{-st}\,dt \\ &= \int_0^\infty e^{-(b+s)t}\,dt \\ &= \frac{-1}{s+b}\left[e^{-(b+s)t}\right]_{t=0}^{t=\infty} \end{aligned}$$

Writing $s = \sigma + j\omega$, we need to find

$$\lim_{t\to\infty} e^{-(b+\sigma+j\omega)t} = \lim_{t\to\infty} e^{-j\omega t}e^{-(b+\sigma)t} = 0$$

provided $b + \sigma > 0$. Then

$$X(s) = \frac{1}{s+b}$$

with $RoC(x) = \{s \in \mathbf{C} : Re(s) > -b\}$. If $b > 0$, then the point $s = 0$ is in the region of convergence, and so the CTFT of x exists and is given by

$$X(\omega) = X(s)|_{s=j\omega} = \frac{1}{j\omega + b}$$

If $b \leq 0$, then $s = 0$ is not in the $RoC(x)$ and the CTFT of x does not exist.

7.2 Properties of the Laplace Transform

Similarly to the Fourier transform, the Laplace transform has many properties that allow us to compute transforms of signals without using the definition of the transform. Some of these are

Theorem 7.2.1 (Linearity) *If $x_1(t) \longleftrightarrow X_1(s)$ and $x_2(t) \longleftrightarrow X_2(s)$, and a and b are any two scalars, then*

$$ax_1(t)+bx_2(t) \longleftrightarrow aX_1(s)+bX_2(s) \tag{7.8}$$

Theorem 7.2.2 (Time shifting) *If $x(t) \longleftrightarrow X(s)$ and $c > 0$, then*

$$x(t-c)u(t-c) \longleftrightarrow X(s)e^{-cs} \tag{7.9}$$

Theorem 7.2.3 (Time scaling) *If $x(t) \longleftrightarrow X(s)$ and $a > 0$, then*

$$x(at) \longleftrightarrow \frac{1}{a}X\left(\frac{s}{a}\right) \tag{7.10}$$

Theorem 7.2.4 (Modulation) *Let $x(t) \longleftrightarrow X(s)$, let $a \in \mathbf{C}$, let $\omega \in \mathbf{R}$ and let N be a positive integer. Then*

$$x(t)e^{at} \quad \longleftrightarrow \quad X(s-a) \tag{7.11}$$

$$x(t)\cos(\omega t) \quad \longleftrightarrow \quad \frac{1}{2}[X(s+j\omega)+X(s-j\omega)] \tag{7.12}$$

$$x(t)\sin(\omega t) \quad \longleftrightarrow \quad \frac{j}{2}[X(s+j\omega)-X(s-j\omega)] \tag{7.13}$$

$$t^N x(t) \quad \longleftrightarrow \quad (-1)^N \frac{d^N}{ds^N}X(s) \tag{7.14}$$

Theorem 7.2.5 (Convolution) *If $x(t) \longleftrightarrow X(s)$ and $v(t) \longleftrightarrow V(s)$, then*

$$(x \star v)(t) \longleftrightarrow X(s)V(s) \tag{7.15}$$

Theorem 7.2.6 (Derivative) *If $x(t) \longleftrightarrow X(s)$, then*

$$\dot{x}(t) \quad \longleftrightarrow \quad sX(s)-x(0^-) \tag{7.16}$$

$$\ddot{x}(t) \quad \longleftrightarrow \quad s^2X(s)-sx(0^-)-\dot{x}(0^-) \tag{7.17}$$

where $x(0^-) = \lim_{t\to 0, t<0} x(t)$ and $\dot{x}(0^-) = \lim_{t\to 0, t<0} \dot{x}(t)$

Theorem 7.2.7 (Integral) *If $x(t) \longleftrightarrow X(s)$, then*

$$\int_0^t x(\lambda)\, d\lambda \longleftrightarrow \frac{X(s)}{s} \tag{7.18}$$

7.3 Laplace Transforms of special functions

Example 7.3.1 *Let $x(t) = \delta(t)$. Then its LT is*

$$\begin{aligned} X(s) &= \int_0^\infty x(t)e^{-st}\,dt \\ &= \int_0^\infty \delta(t)e^{-st}\,dt \\ &= e^0 \\ &= 1 \end{aligned}$$

by the Sifting theorem. Since this holds for any complex number s, we see that the RoC(x) is $\mathbf{C}$.

Example 7.3.2 *Let $x(t) = tu(t)$. Then its LT is*

$$\begin{aligned} X(s) &= \int_0^\infty te^{-st}\,dt \\ &= -\left[\frac{st+1}{s^2e^{st}}\right]_0^\infty \end{aligned}$$

using integration by parts. Writing $s = \sigma + j\omega$, we need to find

$$\lim_{t\to\infty} \frac{st+1}{s^2e^{st}} = \lim_{t\to\infty} \frac{(\sigma+j\omega)t+1}{(\sigma+j\omega)^2e^{(\sigma+j\omega)t}} = 0$$

provided $\sigma > 0$. (Here we have used $\lim_{t\to\infty} \frac{\alpha t}{e^{\alpha t}} = 0$ for $\alpha > 0$). Hence

$$X(s) = \frac{1}{s^2}$$

with $RoC(x) = \{s \in \mathbf{C} : Re(s) > 0\}$. Note that $s = 0$ is not in the $RoC(x)$ and so x does not have a CTFT.

Example 7.3.3 *Show that for any integer $N \geq 1$ and any $b \in \mathbf{R}$,*

$$t^N e^{-bt}u(t) \longleftrightarrow \frac{N!}{(s+b)^{N+1}}$$

Introduce

$$y(t) = e^{-bt}u(t), \quad Y(s) = \frac{1}{s+b}, \quad \text{and } x(t) = t^N y(t)$$

Then by modulation

$$\begin{aligned} X(s) &= (-1)^N \frac{d^N}{ds^N} Y(s) \\ &= (-1)^N \frac{d^N}{ds^N}\left(\frac{1}{s+b}\right) \\ &= \frac{N!}{(s+b)^{N+1}} \end{aligned}$$

Remark 7.3.1 *The following table lists some important Laplace Transform pairs:*

Table 7.1: Table of important Laplace transforms

$x(t)$	$X(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$t^N u(t)$	$\frac{N!}{s^{N+1}}$, $N = 1, 2, 3, \ldots$
$e^{-bt}u(t)$	$\frac{1}{s+b}$, $b \in C$
$\cos(\omega t)u(t)$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t)u(t)$	$\frac{\omega}{s^2+\omega^2}$

7.4 Inverse Laplace Transforms

In order to solve differential equations with the Laplace transform, we will need to be able to compute inverse transforms.

Definition 7.4.1 (Inverse Laplace transform) *For any continuous-time signal x with LT $X(s)$, we define the* **inverse Laplace transform** *as*

$$x(t) = \frac{1}{j2\pi} \int_{c-j\infty}^{c+j\infty} X(s)e^{st}\, ds \tag{7.19}$$

- The integral is evaluated on the contour (path) $s = c + j\omega$ in the complex plane, between $s = c - j\infty$ and $s = c + j\infty$. The contour must lie within the region of convergence of the LT of x.
- Evaluation of this integral involves techniques of complex integration, which are outside the scope of ELEN30012. To learn more about these, you should take a subject like MAST30021 Complex Analysis.
- In ELEN30012 we will use alternative methods to obtain inverse Laplace transforms. To use these methods, we will need to understand some properties of rational functions.

Definition 7.4.2 (Rational functions) *Let X be a complex function in the form*

$$X(s) = \frac{B(s)}{A(s)} \tag{7.20}$$

where B and A are real polynomials in the complex variable s, given by

$$B(s) = b_M s^M + b_{M-1}s^{M-1} + \ldots b_1 s + b_0 \tag{7.21}$$
$$A(s) = a_N s^N + a_{N-1}s^{N-1} + \ldots a_1 s + a_0 \tag{7.22}$$

where M and N are positive integers and the coefficients of B and A are all real numbers. We say that $X(s)$ is a **rational function of** *s. Assuming that $b_M \neq 0$ and $a_N \neq 0$, the* **degree** *of the polynomials B and A are M and N, respectively. We refer to B and A as the* **numerator polynomial** *and* **denominator polynomial** *of X, respectively. Finally we say that X is* **proper** *if $M \leq N$ and* **strictly proper** *if $M < N$.*

Definition 7.4.3 *Let $A(s)$ be a real polynomial in s and let $p \in \mathbf{C}$ be such that*

$$A(p) = 0 \tag{7.23}$$

We say that p **is a root** *or* **zero** *of A. We define the* **multiplicity** *of p to be the largest integer m such that*

$$A(s) = (s-p)^m H(s) \tag{7.24}$$

where H is a polynomial in s and $H(p) \neq 0$.

Example 7.4.1 *Let*

$$A(s) = (s-3)(s+j5)^3(s-j5)^3 \tag{7.25}$$

Then $s_1 = 3$, $s_2 = -j5$ and $s_3 = j5$ are roots of A with multiplicities $m_1 = 1$ and $m_2 = m_3 = 3$.

Theorem 7.4.1 (Fundamental Theorem of Algebra I) *Every real polynomial of degree $N \geq 1$ has a factorisation involving only linear factors in $\mathbf{C}$.*

This is equivalent to:

Theorem 7.4.2 (Fundamental Theorem of Algebra II) *Let A be a real polynomial of degree $N \geq 1$. Then A has factorisation*

$$A(s) = a_N(s-p_1)(s-p_2)\dots(s-p_N) \tag{7.26}$$

for some $a_N \in \mathbf{R}$ and some (not necessarily distinct) $p_1, p_2, \dots, p_N \in \mathbf{C}$.

Our next theorem says that complex roots of a real polynomial occur in complex-conjugate pairs:

Theorem 7.4.3 (Complex roots) *If A is a real polynomial and $p \in \mathbf{C}$ is such that $A(p) = 0$, then $A(\bar{p}) = 0$ also.*

Definition 7.4.4 (Poles of a rational function) *Let X be a rational function of s and let its denominator polynomial A have linear factorisation*

$$A(s) = a_N(s-p_1)(s-p_2)\dots(s-p_N) \tag{7.27}$$

Then the roots $p_1, p_2, \dots, p_N$ of A are said to be the **poles of** *X.*

Theorem 7.4.4 (Partial fractions with distinct poles) *Let X be a strictly proper rational function of s and assume its denominator polynomial has N distinct real or complex poles $p_1, p_2, \dots, p_N$. Then X has partial fraction expansion*

$$X(s) = \frac{c_1}{s-p_1} + \frac{c_2}{s-p_2} + \dots + \frac{c_N}{s-p_N} \tag{7.28}$$

where the constants c_i are given by

$$c_i = [(s-p_i)X(s)]_{s=p_i}, \quad i = 1, 2, \dots, N \tag{7.29}$$

Remark 7.4.1 (Partial fractions with distinct poles)

- *'X has distinct poles' means that the poles all have multiplicity of 1, which implies that $p_i \neq p_j$ whenever $i \neq j$.*

- *The constants c_i are called the* **residues** *of X. They are real or complex according to whether the corresponding pole p_i is real or complex, and $c_i = \bar{c}_j$ if $p_i = \bar{p}_j$.*

- *Taking the inverse LT we obtain the time-domain signal*

$$x(t) = c_1 e^{p_1 t} + c_2 e^{p_2 t} + \cdots + c_N e^{p_N t}, \quad t \geq 0 \tag{7.30}$$

Note: the solution for x depends only upon the poles of X.

- *For complex c_i, we may obtain a real expression for $x(t)$ using*

Theorem 7.4.5 *Let $p = \sigma + j\omega \in \mathbf{C}$ for some $\sigma \in \mathbf{R}$ and $\omega \in \mathbf{R}$. Let $c \in \mathbf{C}$ and $t \in \mathbf{R}$. Then*

$$ce^{pt} + \bar{c}e^{\bar{p}t} = 2|c|e^{\sigma t}\cos(\omega t + \angle c) \tag{7.31}$$

Example 7.4.2 (Partial Fractions with distinct poles) *Find x if its LT is the rational function*

$$X(s) = \frac{s^2 - 2s + 1}{s^3 + 3s^2 + 4s + 2}$$

Use the fact that

$$s^3 + 3s^2 + 4s + 2 = (s^2 + 2s + 2)(s+1) = (s+1-j)(s+1+j)(s+1)$$

The poles (roots) of X are $p_1 = -1 + j,\ p_2 = -1 - j,\ p_3 = -1$. We compute the residues of X:

$$\begin{aligned}
c_1 &= [(s-p_1)X(s)]_{s=p_1} \\
&= \left.\frac{s^2 - 2s + 1}{(s+1+j)(s+1)}\right|_{s=-1+j} \\
&= \frac{-3}{2} + j2 \\
c_2 &= \bar{c}_1 \\
c_3 &= [(s-p_3)X(s)]_{s=p_3}, \\
&= \left.\frac{s^2 - 2s + 1}{s^2 + 2s + 2}\right|_{s=-1} \\
&= 4 \\
\textit{Hence}\ \ X(s) &= \frac{c_1}{s-p_1} + \frac{\bar{c}_1}{s-\bar{p}_1} + \frac{c_3}{s-p_3}
\end{aligned}$$

Taking the inverse LT gives

$$\begin{aligned}
x(t) &= [c_1 e^{p_1 t} + \bar{c}_1 e^{\bar{p}_1 t} + c_3 e^{p_3 t}]u(t) \\
&= [2|c_1|e^{-t}\cos(t + \angle c_1) + c_3 e^{-t}]u(t) \\
&= [5e^{-t}\cos(t + 2.214) + 4e^{-t}]u(t)
\end{aligned}$$

Theorem 7.4.6 (Partial Fractions with repeated real poles) *Let X be a strictly proper rational function and assume its denominator polynomial has a real pole p_1 with multiplicity r, and the remaining $N - r$ poles (denoted by $p_{r+1}, p_{r+2}, \ldots, p_N$) of X are distinct. Then $X(s)$ has partial fraction expansion*

$$X(s) = \frac{c_1}{s-p_1} + \frac{c_2}{(s-p_1)^2} + \cdots + \frac{c_r}{(s-p_1)^r} + \frac{c_{r+1}}{s-p_{r+1}} + \cdots + \frac{c_N}{s-p_N} \tag{7.32}$$

where the residues c_i are given by

$$c_{r-i} = \frac{1}{i!}\left[\frac{d^i}{ds^i}\left[(s-p_1)^r X(s)\right]\right]_{s=p_1}, \quad i = 0,1,2,\ldots,r-1 \tag{7.33}$$

$$c_i = [(s-p_i)X(s)]_{s=p_i}, \quad i = r+1, r+2, \ldots, N \tag{7.34}$$

Remark 7.4.2 *Note that* $0! = 1$ *by definition.*

Example 7.4.3 (Partial Fractions with repeated real poles) *Find x if its LT is the rational function*

$$X(s) = \frac{5s-1}{(s-2)(s+1)^2}$$

The poles of X are $p_1 = -1$ *with multiplicity* $r = 2$*, and* $p_3 = 2$*. Compute residues:*

$$\begin{aligned}
c_1 &= \frac{1}{1!}\left[\frac{d}{ds}\left[(s-p_1)^r X(s)\right]\right]_{s=p_1} \\
&= \left[\frac{d}{ds}\left[(s+1)^2 X(s)\right]\right]_{s=-1} \\
&= \left[\frac{d}{ds}\left[\frac{5s-1}{s-2}\right]\right]_{s=-1} \\
&= \left.\frac{-9}{(s-2)^2}\right|_{s=-1} \\
&= -1 \\
c_2 &= \frac{1}{0!}\left[(s-p_1)^r X(s)\right]_{s=p_1} \\
&= \left[(s+1)^2 X(s)\right]_{s=-1} \\
&= \left.\frac{5s-1}{(s-2)}\right|_{s=-1} \\
&= 2 \\
c_3 &= \left[(s-p_3)X(s)\right]_{s=p_3} \\
&= \left[(s-2)X(s)\right]_{s=2} \\
&= \left.\frac{5s-1}{(s+1)^2}\right|_{s=2} \\
&= 1 \\
\textit{Hence } X(s) &= \frac{c_1}{s+1} + \frac{c_2}{(s+1)^2} + \frac{c_3}{(s-2)} \\
&= \frac{-1}{s+1} + \frac{2}{(s+1)^2} + \frac{1}{(s-2)}
\end{aligned}$$

Taking the inverse LT we obtain

$$x(t) = (-e^{-t} + 2te^{-t} + e^{2t})u(t)$$

We used the LT pairs

$$e^{-bt} \longleftrightarrow \frac{1}{s+b} \quad \textit{and} \quad t^N e^{-bt} \longleftrightarrow \frac{N!}{(s+b)^{N+1}}$$

7.5 z-Transform: Definition

Recall that any absolutely summable discrete-time signal $x[n]$ has DTFT

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} \tag{7.35}$$

Similarly, the frequency response of an LTI stable discrete-time system is the DTFT of its unit pulse response

$$H(\Omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\Omega n} \tag{7.36}$$

To analyse signals that are not absolutely summable, and unstable systems, we introduce the z-Transform:

Definition 7.5.1 (z-Transform) *For any discrete-time signal x we define the* **z-Transform** *(zT) as*

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} \tag{7.37}$$

where z is a complex variable.

Definition 7.5.2 (Region of convergence) *We define the* **region of convergence** *of the zT as the set of complex numbers z for which the transform $X(z)$ exists. This is*

$$RoC(x) = \{z \in \mathbf{C} \setminus \{0\} : x[n]z^{-n} \text{ is absolutely summable}\} \tag{7.38}$$

The RoC depends on the signal x and always has the form

$$RoC(x) = \{z \in \mathbf{C} : |z| > c\} \tag{7.39}$$

for some $c \geq 0$.

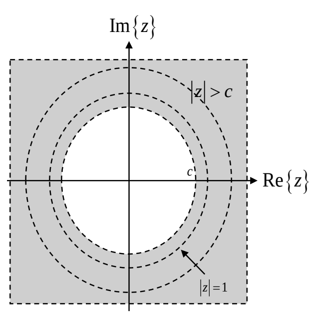


Figure 7.2: Region of convergence of the z-transform of x.

Our next theorem describes the relationship between the z and discrete Fourier transforms.

Theorem 7.5.1 *Let x be a discrete-time signal such that $x[n] = 0$ for $n < 0$. Let X be its z-Transform and assume that $z = 1$ is in its region of convergence. Then the DTFT of x is given by*

$$X(\Omega) = X(z)|_{z=e^{j\Omega}} \tag{7.40}$$

Example 7.5.1 *Let $x[n] = a^n u[n]$ for some $a \in \mathbf{R}$. Then*

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} x[n] z^{-n} \\ &= \sum_{n=0}^{\infty} a^n z^{-n} \\ &= \sum_{n=0}^{\infty} (az^{-1})^n \end{aligned}$$

We know that for a geometric series with common ratio r

$$\sum_{n=0}^{N} r^n = \frac{1-r^{N+1}}{1-r}, \quad \text{and} \quad \sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \tag{7.41}$$

provided $0 < |r| < 1$. Hence

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} (az^{-1})^n \\ &= \frac{1}{1-az^{-1}} \\ &= \frac{z}{z-a} \end{aligned}$$

provided $|az^{-1}| < 1$. This is equivalent to $|z| > |a|$, and hence $RoC(x) = \{z \in \mathbf{C} : |z| > |a|\}$. If $|a| < 1$, then the point $z = 1$ is in the region of convergence, and so the DTFT of x exists and is given by

$$\begin{aligned} X(\Omega) &= X(z)|_{z=e^{j\Omega}} \\ &= \frac{1}{1-ae^{-j\Omega}} \end{aligned}$$

Note that this result agrees with Example 5.1.2

7.6 Properties of the z-Transform

Similarly to the Laplace transform, the z-Transform has many properties that allow us to compute transforms of signals without using the definition of the transform. Some of these are

Theorem 7.6.1 (Linearity) *If $x_1[n] \longleftrightarrow X_1(z)$ and $x_2[n] \longleftrightarrow X_2(z)$, and a and b are any two scalars, then*

$$ax_1[n] + bx_2[n] \longleftrightarrow aX_1(z) + bX_2(z) \tag{7.42}$$

Theorem 7.6.2 (Right Time shift) *If $x[n] \longleftrightarrow X(z)$ and $q > 0$, then*

$$x[n-q]u[n-q] \longleftrightarrow X(z)z^{-q} \tag{7.43}$$

Theorem 7.6.3 (Modulation) *Let $x[n] \longleftrightarrow X(z)$, let $a \in \mathbf{C}$, and let $\Omega \in \mathbf{R}$. Then*

$$x[n]a^n \longleftrightarrow X\left(\frac{z}{a}\right) \tag{7.44}$$

$$x[n]\cos(\Omega n) \longleftrightarrow \frac{1}{2}[X(ze^{j\Omega}) + X(ze^{-j\Omega})] \tag{7.45}$$

$$x[n]\sin(\Omega n) \longleftrightarrow \frac{j}{2}[X(ze^{j\Omega}) - X(ze^{-j\Omega})] \tag{7.46}$$

$$nx[n] \longleftrightarrow -z\frac{d}{dz}X(z) \tag{7.47}$$

Theorem 7.6.4 (Convolution) *If $x[n] \longleftrightarrow X(z)$ and $v[n] \longleftrightarrow V(z)$, then*

$$(x \star v)[n] \longleftrightarrow X(z)V(z) \tag{7.48}$$

Theorem 7.6.5 (Left Time Shift) *If $x[n] \longleftrightarrow X(z)$, and $q > 0$, then*

$$x[n+q] \longleftrightarrow z^q X(z) - x[0]z^q - x[1]z^{q-1} - \cdots - x[q-1]z \tag{7.49}$$

7.7 z-Transforms of special functions

Example 7.7.1 *Let $x[n] = \delta[n]$. Then its zT is*

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} x[n]z^{-n} \\ &= \sum_{n=0}^{\infty} \delta[n]z^{-n} \\ &= z^0 \\ &= 1 \end{aligned}$$

by the Sifting theorem. Since this holds for any complex number z, we see that the RoC(x) is $\mathbf{C} \setminus \{0\}$.

Example 7.7.2 *Let $x[n] = a^n p_3[n]$ for some $a \in \mathbf{R}$. To find its zT, we first recall that*

$$p_3[n] = \begin{cases} 1, & -1 \le n \le 1 \\ 0, & \text{otherwise} \end{cases} = u[n+1] - u[n-2]$$

We saw earlier that zT of $u[n]$ is $U(z) = \frac{1}{1-z^{-1}}$. Hence by left and right time-shifting

$$\begin{aligned} P_3(z) &= zU(z) - u[0]z - U(z)z^{-2} \\ &= \frac{z}{1-z^{-1}} - z - \frac{z^{-2}}{1-z^{-1}} \\ &= 1 + z^{-1} \end{aligned}$$

Since $x[n] = a^n p_3[n]$, by modulation we have

$$X(z) = P_3\left(\frac{z}{a}\right) = 1 + \frac{a}{z}$$

Example 7.7.3 *Show that any $p \in \mathbf{R}$,*

$$np^n u[n] \longleftrightarrow \frac{pz}{(z-p)^2}$$

Introduce

$$y[n] = p^n u[n] \quad \text{and } x[n] = ny[n]$$

Recall that $u[n] \longleftrightarrow \frac{1}{1-z^{-1}} = U(z)$. Then modulating u with p^n gives

$$\begin{aligned} Y(z) &= U\left(\frac{z}{p}\right) = \frac{z}{z-p} \\ \Rightarrow X(z) &= -z\frac{d}{dz}\left(\frac{z}{z-p}\right) \quad \text{by modulating } y \text{ with } n \\ &= -z\left[\frac{z-p-z}{(z-p)^2}\right] \\ &= \frac{pz}{(z-p)^2} \end{aligned}$$

Remark 7.7.1 *The following table lists some important z-Transform pairs:*

Table 7.2: Table of important z-Transforms

$x[n]$	$X(z)$
$\delta[n]$	1
$\delta[n-q]$	$z^{-q}, \quad q \in \mathbf{Z}^+$
$a^n u[n]$	$\frac{z}{z-a}, \quad a \in \mathbf{R}$
$na^n u[n]$	$\frac{az}{(z-a)^2}, \quad a \in \mathbf{R}$

7.8 Inverse z-Transforms

In order to solve difference equations with the z-Transform, we will need to be able to compute inverse transforms.

Definition 7.8.1 (Inverse z-Transform) *For any discrete-time signal x with z-Transform X, we define the* **inverse z-transform** *as*

$$x[n] = \frac{1}{j2\pi} \int_C X(z) z^{n-1}\, dz \tag{7.50}$$

- The integral is evaluated on any closed contour C within the region of convergence of X.
- As with the inverse LT, evaluation of this complex integral is outside the scope of ELEN30012.
- Hence we will use partial fractions to obtain inverse z-transforms.

Theorem 7.8.1 (Partial fractions with distinct nonzero poles) *Let X be a proper rational function of z and assume its denominator polynomial has N distinct nonzero poles $p_1, p_2, \ldots, p_N$. Then X has partial fraction expansion*

$$\frac{X(z)}{z} = \frac{c_0}{z} + \frac{c_1}{z-p_1} + \frac{c_2}{z-p_2} + \cdots + \frac{c_N}{z-p_N} \tag{7.51}$$

where the residues c_i are given by

$$c_0 = X(0), \quad c_i = \left[(z-p_i) \frac{X(z)}{z} \right]_{z=p_i}, \quad i = 1, 2, \ldots, N \tag{7.52}$$

Remark 7.8.1

- *Unlike the LT, in this theorem we have not required that X be a strictly proper rational function. We accommodate the case where the numerator and denominator polynomials of X have the same degree by working with $X(z)/z$ instead of $X(z)$.*
- *Multiplying through by z gives*

$$X(z) = c_0 + \frac{c_1 z}{z-p_1} + \frac{c_2 z}{z-p_2} + \cdots + \frac{c_N z}{z-p_N} \tag{7.53}$$

- *Taking the inverse zT we obtain the time-domain signal x with*

$$x[n] = c_0 \delta[n] + (c_1 p_1^n + c_2 p_2^n + \cdots + c_N p_N^n) u[n] \tag{7.54}$$

- *For complex c_i, we obtain a real expression for $x[n]$ using*

Theorem 7.8.2 *Let $p = \sigma e^{j\Omega} \in \mathbf{C}$ for some real numbers σ and Ω, and let $c \in \mathbf{C}$. Then*

$$cp^n + \bar{c}\bar{p}^n = 2|c|\sigma^n \cos(\Omega n + \angle c) \tag{7.55}$$

Example 7.8.1 (Partial fractions with distinct poles) *Find x if its zT is the rational function*

$$X(z) = \frac{z^3+1}{(z+1/2+j\sqrt{3}/2)(z+1/2-j\sqrt{3}/2)(z-2)}$$

The roots (poles) of X are $p_1 = -1/2 - j\sqrt{3}/2,\ p_2 = -1/2 + j\sqrt{3}/2,\ p_3 = 2$. We compute the residues of $X(z)/z$:

$$\begin{aligned}
c_0 &= X(0) \\
&= -1/2 \\
c_1 &= [(z-p_1)X(z)/z]_{z=p_1} \\
&= 0.429 + j0.0825 \\
c_2 &= \bar{c}_1 \\
c_3 &= [(z-p_3)X(z)/z]_{z=p_3} \\
&= 0.643 \\
\text{Hence } \frac{X(z)}{z} &= \frac{c_0}{z} + \frac{c_1}{z-p_1} + \frac{\bar{c}_1}{z-\bar{p}_1} + \frac{c_3}{z-p_3} \\
\Rightarrow X(z) &= c_0 + \frac{c_1 z}{z-p_1} + \frac{\bar{c}_1 z}{z-\bar{p}_1} + \frac{c_3 z}{z-p_3} \\
\text{Also } |p_1| &= 1 = \sigma, \quad \angle p_1 = \pi + \tan^{-1}\left(\frac{0.866}{0.5}\right) = \frac{4\pi}{3} \text{ rad} \\
|c_1| &= 0.437, \quad \angle c_1 = \tan^{-1}\left(\frac{0.0825}{0.429}\right) = 0.19 \text{ rad}
\end{aligned}$$

Hence taking the inverse zT gives

$$\begin{aligned}
x[n] &= c_0\delta[n] + (c_1 p_1^n + \bar{c}_1 \bar{p}_1^n + c_3 p_3^n)u[n] \\
&= c_0\delta[n] + (2|c_1|\sigma^n \cos(\Omega n + \angle c_1) + c_3 p_3^n)u[n] \\
&= -0.5\delta[n] + (0.874\cos(4\pi n/3 + 0.19) + 0.643(2)^n)u[n]
\end{aligned}$$

Theorem 7.8.3 (Partial fractions with repeated real poles) *Let X be a rational function of z and assume its denominator polynomial has a real nonzero pole p_1 with multiplicity r, and the remaining $N-r$ poles (denoted by $p_{r+1}, p_{r+2}, \ldots, p_N$) of X are distinct and nonzero. Then $X(z)/z$ has partial fraction expansion*

$$\frac{X(z)}{z} = \frac{c_0}{z} + \frac{c_1}{z-p_1} + \frac{c_2}{(z-p_1)^2} + \cdots + \frac{c_r}{(z-p_1)^r} + \frac{c_{r+1}}{z-p_{r+1}} + \cdots + \frac{c_N}{z-p_N} \tag{7.56}$$

where the residues c_i are given by

$$c_0 = X(0) \tag{7.57}$$

$$c_{r-i} = \frac{1}{i!}\left[\frac{d^i}{dz^i}\left[(z-p_1)^r \frac{X(z)}{z}\right]\right]_{z=p_1}, \quad i = 0, 1, 2, \ldots, r-1 \tag{7.58}$$

$$c_i = \left[(z-p_i)\frac{X(z)}{z}\right]_{z=p_i}, \quad i = r+1, r+2, \ldots, N \tag{7.59}$$

Example 7.8.2 (Partial fractions with repeated real poles) *Find x given that its zT is such that*

$$\frac{X(z)}{z} = \frac{6z^2+2z-1}{(z+1)(z-1)^2}$$

The roots (poles) of X are $p_1 = 1$ with multiplicity $r = 2$, and $p_3 = -1$. Compute residues:

$$\begin{aligned}
c_0 &= X(0) = 0 \\
c_1 &= \frac{1}{1!}\left[\frac{d}{dz}\left[(z-p_1)^r X(z)/z\right]\right]_{z=p_1} \\
&= \left[\frac{d}{dz}\left[(z-1)^2 X(z)/z\right]\right]_{z=1} \\
&= \left[\frac{d}{dz}\left[\frac{6z^2+2z-1}{(z+1)}\right]\right]_{z=1} \\
&= \left.\frac{(z+1)(12z+2)-(6z^2+2z-1)(1)}{(z+1)^2}\right|_{z=1} \\
&= 5.25 \\
c_2 &= \frac{1}{0!}\left[(z-p_1)^r X(z)/z\right]_{z=p_1} \\
&= \left[(z-1)^2 X(z)/z\right]_{z=1} \\
&= \left.\frac{6z^2+2z-1}{z+1}\right|_{z=1} \\
&= 3.5 \\
c_3 &= \left[(z-p_3)X(z)/z\right]_{z=p_3} \\
&= \left[(z+1)X(z)/z\right]_{z=-1} \\
&= \left.\frac{6z^2+2z-1}{(z-1)^2}\right|_{z=-1} \\
&= 0.75 \\
\textit{Hence } \frac{X(z)}{z} &= \frac{c_0}{z} + \frac{c_1}{z-p_1} + \frac{c_2}{(z-p_1)^2} + \frac{c_3}{z-p_3} \\
\Rightarrow X(z) &= \frac{5.25z}{z-1} + \frac{3.5z}{(z-1)^2} + \frac{0.75z}{z+1}
\end{aligned}$$

Hence taking the inverse zT gives

$$\begin{aligned}
x[n] &= [5.25(1)^n + 3.5n(1)^n + 0.75(-1)^n]u[n] \\
&= [5.25 + 3.5n + 0.75(-1)^n]u[n]
\end{aligned}$$

We used the zT pairs

$$a^n u[n] \longleftrightarrow \frac{z}{z-a} \quad \textit{and} \quad na^n u[n] \longleftrightarrow \frac{az}{(z-a)^2}$$

7.9 Practice Problems

81. (a) Use the definition of Laplace transform to show that

$$te^{-bt}u(t) \longleftrightarrow \frac{1}{(s+b)^2}$$

is a Laplace transform pair, for any $b \in \mathbf{R}$.

(b) Hence use the definition of the Laplace transform and Mathematical Induction to show that, for any integer $N \geq 1$,

$$t^N e^{-bt}u(t) \longleftrightarrow \frac{N!}{(s+b)^{N+1}}$$

is a Laplace transform pair.

82. A continuous-time signal x has Laplace transform

$$X(s) = \frac{s+1}{s^2+5s+7}$$

In each case, find the LT of v:

(a) $v(t) = x(3t-4)u(3t-4)$

(b) $v(t) = tx(t)$

(c) $v(t) = \frac{d^2x}{dt^2}$, assuming $x(0^-) = 1$, $\dot{x}(0^-) = -4$.

(d) $v(t) = x(t)\sin(2t)$

83. Let $p = \sigma + j\omega \in \mathbf{C}$ for some real numbers σ and ω. Let $c \in \mathbf{C}$ and $t \in \mathbf{R}$. Show that

$$ce^{pt} + \bar{c}e^{\bar{p}t} = 2|c|e^{\sigma t}\cos(\omega t + \angle c)$$

84. Let $x(t)$ and $X(s)$ be an LT pair. Using the definition of the Laplace transform to prove the following LT pairs:

$$\begin{array}{lcllr} x(t-c)u(t-c) & \longleftrightarrow & X(s)e^{-cs}, & c>0 & \text{(Time shifting)} \\ x(at) & \longleftrightarrow & \frac{1}{a}X\left(\frac{s}{a}\right), & a>0 & \text{(Time Scaling)} \\ tx(t) & \longleftrightarrow & -\frac{dX}{ds} & & \text{(Modulation)} \\ e^{at}x(t) & \longleftrightarrow & X(s-a) & & \text{(Modulation)} \end{array}$$

85. Use partial fractions to obtain inverse Laplace transforms for the following functions.

(a) $X(s) = \dfrac{s+2}{s^2+7s+12}$

(b) $X(s) = \dfrac{s+1}{s(s+2.5-j\frac{\sqrt{3}}{2})(s+2.5+j\frac{\sqrt{3}}{2})}$

(c) $X(s) = \dfrac{3s^2+2s+1}{(s+1)(s+2)^2}$

(d) $X(s) = \dfrac{s^2-2s+1}{s(s-j2)(s+j2)}$

86. Consider the discrete-time signal x

$$x[n] = \begin{cases} b^n, & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases}$$

where $b \in \mathbf{R}$ and N is a positive integer. Use the definition of z-Transform to find the transform of x. What is its region of convergence?

87. A discrete-time signal x has z-Transform

$$X(z) = \frac{z}{8z^2 - 2z - 1}$$

In each case, find the z-transform of v:

(a) $v[n] = x[n-4]u[n-4]$

(b) $v[n] = x[n+2]$, assuming $x[0] = 0$, $x[1] = 1/8$.

(c) $v[n] = (x \star x)[n]$

(d) $v[n] = nx[n]$

88. Compute the z-Transform of the convolution $y[n] = (x \star v)[n]$ for each of the following pairs of signals.

(a) $x[n] = u[n] + 3\delta[n-1]$, $v[n] = u[n-2]$

(b) $x[n] = u[n]$, $v[n] = nu[n]$

89. Let $x[n]$ and $X(z)$ be a zT pair. Using the definition of the z-Transform to prove the following zT pairs:

$$\begin{array}{lcl}
x[n-q]u[n-q] & \longleftrightarrow & X(z)z^{-q}, \quad q > 0 \\
x[n]a^n & \longleftrightarrow & X\left(\frac{z}{a}\right), \quad a \in \mathbf{C} \\
x[n]e^{j\Omega n} & \longleftrightarrow & X(ze^{-j\Omega})], \quad \Omega \in \mathbf{R} \\
x[n]\cos(\Omega n) & \longleftrightarrow & \frac{1}{2}[X(ze^{j\Omega}) + X(ze^{-j\Omega})], \quad \Omega \in \mathbf{R} \\
x[n]\sin(\Omega n) & \longleftrightarrow & \frac{j}{2}[X(ze^{j\Omega}) - X(ze^{-j\Omega})] \quad \Omega \in \mathbf{R} \\
nx[n] & \longleftrightarrow & -z\frac{d}{dz}X(z)
\end{array}$$

90. Use Modulation and the z-Transform pair $u[n] \longleftrightarrow \dfrac{z}{z-1}$ to derive the z-Transform pairs

$$\cos(\Omega n)u[n] \longleftrightarrow \frac{z^2 - z\cos(\Omega)}{z^2 - 2z\cos(\Omega) + 1}, \quad \sin(\Omega n)u[n] \longleftrightarrow \frac{z\sin(\Omega)}{z^2 - 2z\cos(\Omega) + 1}$$

Hence find the inverse z-transforms of

$$\text{(a) } X_1(z) = \frac{z^2}{z^2+1} \quad \text{(b) } X_2(z) = \frac{z}{z^2+1}$$

91. Let $p = \sigma e^{j\Omega} \in \mathbf{C}$ for some real numbers σ and Ω. Let $c \in \mathbf{C}$ and $n \in \mathbf{R}$. Show that

$$cp^n + \bar{c}\bar{p}^n = 2|c|\sigma^n \cos(\Omega n + \angle c)$$

92. Use partial fractions to obtain inverse z-Transforms for the following functions.

   (a) $X(z) = \dfrac{z+2}{(z-1)(z^2+1)}$

   (b) $X(z) = \dfrac{z+0.3}{z^2+0.75z+0.125}$

   (c) $X(z) = \dfrac{4z+1}{z^2-z+0.5}$

   (d) $X(z) = \dfrac{2z+1}{z(10z^2-z-2)}$

Chapter 8

State Representations of Systems

In this chapter we introduce the concept of a state vector and see how to express systems in their state-space form. We then develop methods for obtaining solutions for the outputs of systems given in state-space form.

8.1 Defining state representations

Recall that linear time-invariant (LTI) discrete-time systems can be expressed in their **input-output form** with a difference equation

$$y[n]+\sum_{i=1}^{N} a_i y[n-i]=c_0 v[n-N], \quad \text{for } n\geq 0 \tag{8.1}$$

An alternative equivalent form is

$$y[n+N]+\sum_{i=0}^{N-1} b_i y[n+i]=c_0 v[n], \quad \text{for } n\geq -N \tag{8.2}$$

where $v[n]$ and $y[n]$ are the inputs and outputs, and the coefficients a_i, b_i, and c_0 are constants.
Similarly an LTI continuous-time system can expressed as a differential equation

$$\frac{dy^N}{dt^N}+a_{N-1}\frac{dy^{N-1}}{dt^{N-1}}+\ldots a_1\frac{dy}{dt}+a_0 y(t)=c_0 v(t) \quad \text{for } t\geq 0 \tag{8.3}$$

Definition 8.1.1 *A* **state vector** *for a system is a vector of the form*

$$x(t)=\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{bmatrix}, \quad or \quad x[n]=\begin{bmatrix} x_1[n] \\ x_2[n] \\ \vdots \\ x_N[n] \end{bmatrix} \tag{8.4}$$

- *Together with a known input $v(t)$, the state vector contains sufficient information at any time t (or n) to enable the future outputs $y(t)$ (or $y[n]$) to be uniquely determined.*

- *The variables $x_1,\ldots,x_N$ are called the* **state variables** *of the system, and N is the* **dimension** *of the state.*

- *The trace $\{x(t):t\in\mathbf{R}\}$ (or $\{x[n]:n\in\mathbf{Z}\}$) is the* **state trajectory** *of the system.*

Definition 8.1.2 *The* **state equations** *for a linear time-invariant system with an N-dimensional state vector are*

$$\begin{aligned} \dot{x}(t) &= Ax(t)+Bv(t) \\ y(t) &= Cx(t)+Dv(t) \end{aligned} \quad \text{or} \quad \begin{aligned} x[n+1] &= Ax[n]+Bv[n] \\ y[n] &= Cx[n]+Dv[n] \end{aligned} \tag{8.5}$$

where

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_N(t) \end{bmatrix}, \quad \text{and} \quad x[n+1] = \begin{bmatrix} x_1[n+1] \\ x_2[n+1] \\ \vdots \\ x_N[n+1] \end{bmatrix} \tag{8.6}$$

and A is $N \times N$ matrix, B is a $N \times 1$ vector, C is a $1 \times N$ vector, and D is a scalar.

Definition 8.1.3 (Obtaining state representations for continuous-time systems) *To obtain a state space model for the continuous-time system*

$$\frac{dy^N}{dt^N} + a_{N-1}\frac{dy^{N-1}}{dt^{N-1}} + \ldots a_1\frac{dy}{dt} + a_0 y(t) = c_0 v(t) \quad \text{for } t \geq 0 \tag{8.7}$$

1. *Define a state vector $x(t)$ with state variables*

$$x_1(t) = \frac{1}{c_0}y(t),\ x_2(t) = \dot{x}_1(t),\ x_3(t) = \dot{x}_2(t), \ldots, x_N(t) = \dot{x}_{N-1}(t) \tag{8.8}$$

2. *Then the state equations are*

$$\dot{x}_1(t) = x_2(t) \tag{8.9}$$
$$\dot{x}_2(t) = x_3(t) \tag{8.10}$$
$$\vdots$$
$$\dot{x}_N(t) = -a_0 x_1(t) - a_1 x_2(t) - \cdots - a_{N-1} x_N(t) + v(t) \tag{8.11}$$
$$y(t) = c_0 x_1(t) \tag{8.12}$$

If we write these in matrix form we obtain the **state representation**

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{N-1} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} v(t) \tag{8.13}$$

$$y(t) = \begin{bmatrix} c_0 & 0 & 0 & \ldots & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{bmatrix} \tag{8.14}$$

This is known as the **controller canonical form**. *It is not unique; later we will see other ways to achieve state space representations.*

Example 8.1.1 (State model for the Mass Spring Damper) *A Mass Spring Damper system consists of a mass M that is pulled upwards by an applied force v, while restrained by a spring K and a damper D.*

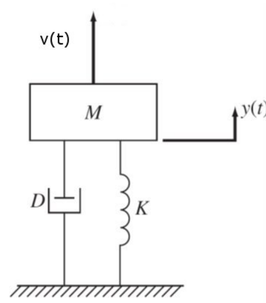


Figure 8.1: Mass-spring damper system

The system is described by the second-order differential equation

$$\frac{d^2y}{dt^2}+\frac{D}{M}\frac{dy}{dt}+\frac{K}{M}y(t)=v(t) \tag{8.15}$$

To obtain a state space model we define a state vector $x(t)$ with two state variables

$$x_1(t)=y(t), \quad x_2(t)=\dot{y}(t)$$

Then the state equations are

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= -\frac{K}{M}x_1(t)-\frac{D}{M}x_2(t)+v(t)\\ y(t) &= x_1(t)\end{aligned}$$

and writing these in matrix form we obtain the **state representation**

$$\begin{aligned}\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} &= \begin{bmatrix}0 & 1\\ -\frac{K}{M} & -\frac{D}{M}\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}v(t)\\ y(t) &= \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}\end{aligned}$$

Definition 8.1.4 (Obtaining state representations for discrete-time systems) *To obtain a state space model for the discrete-time system*

$$y[n+N]+\sum_{i=0}^{N-1}a_iy[n+i]=c_0v[n], \quad \textit{for } n\geq -N \tag{8.16}$$

1. *Define a state vector $x[n]$ with state variables*

$$x_1[n]=\frac{1}{c_0}y[n],\ x_2[n]=x_1[n+1],\ \ldots,x_N[n]=x_{N-1}[n+1] \tag{8.17}$$

2. *Then the state equations are*

$$x_1[n+1] = x_2[n] \tag{8.18}$$
$$x_2[n+1] = x_3[n] \tag{8.19}$$
$$\vdots$$
$$x_N[n+1] = -a_0x_1[n]-a_1x_2[n]-\cdots-a_{N-1}x_N[n]+v[n] \tag{8.20}$$
$$y[n] = c_0x_1[n] \tag{8.21}$$

The state matrices are the same as for continuous-time systems.

Definition 8.1.5 (Multiple-input multiple output systems) *A continuous-time system in state space form*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bv(t) \\ y(t) &= Cx(t) + Dv(t) \end{aligned} \tag{8.22}$$

is a **multiple-input multiple output system***, or* **MIMO system***, if A is an $N \times N$ matrix, B is a $N \times m$ matrix, C is a $p \times N$ matrix and D is a $p \times m$ matrix, where N, m, and p are all integers greater than or equal to 2. Systems where $p = m = 1$ are called* **scalar-input scalar output systems***, or* **SISO** *systems.*

MIMO systems have more than one input variable, and more than one output variable. The state space equations for a discrete-time MIMO system are very similar:

$$\begin{aligned} x[n+1] &= Ax[n] + Bv[n] \\ y[n] &= Cx[n] + Dv[n] \end{aligned} \tag{8.23}$$

Example 8.1.2 (Coupled two-car MIMO system) *Two cars driving along a level surface may be described by the equations*

$$\begin{aligned} \ddot{d}_1(t) + \frac{k_f}{M}\dot{d}_1(t) &= \frac{1}{M}f_1(t) \\ \ddot{d}_2(t) + \frac{k_f}{M}\dot{d}_2(t) &= \frac{1}{M}f_2(t) \\ w(t) &= d_2(t) - d_1(t) \end{aligned}$$

where d_1 and d_2 are the positions of the first and second cars, and M and k_f are constants.

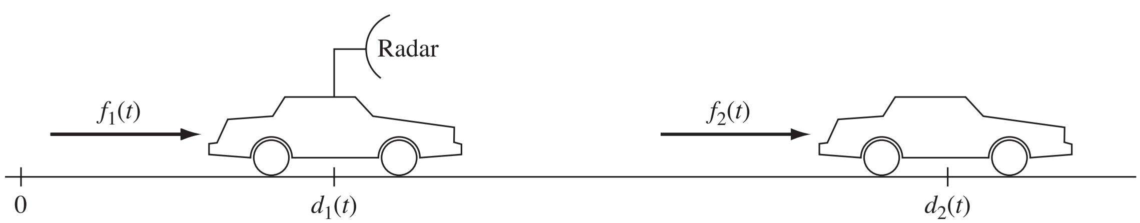


Figure 8.2: Coupled two-car system

If we introduce state variables

$$x_1(t) = \dot{d}_1(t), \quad x_2(t) = \dot{d}_2(t), \quad x_3(t) = w(t)$$

and take

$$y(t) = \begin{bmatrix} \dot{d}_1(t) \\ w(t) \end{bmatrix}$$

as the output, then the state model is

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} \frac{-k_f}{M} & 0 & 0 \\ 0 & -\frac{K}{M} & 0 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{M} & 0 \\ 0 & \frac{1}{M} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} \\ y(t) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \end{aligned}$$

Definition 8.1.6 (Integrator realization) *An* **integrator realization** *is a diagram that represents a continuous-time system in terms of integrators, summers and scalar multipliers.*

The steps of drawing the realization diagram are as follows:

1. For each state variable x_i, construct an integrator and define the output of the integrator to be x_i. Hence the input to the integrator is $\dot{x}_i$. A system with N states will need N integrators.

2. Put a summer in front of each integrator. Feed into the summer scalar multiples of the state variables according to the i-th state equation $\dot{x}_i(t) = A_i x(t) + B_i v(t)$, where A_i and B_i are the i-th rows of A and B, respectively.

3. Put scalar multiples of the state variables into a summer to realize the output equation $y_i(t) = C_i x(t) + D_i v(t)$, where C_i and D_i are the i-th rows of C and D, respectively.

Example 8.1.3 (Integrator realization) *Draw the integrator realization for the system*

$$\begin{aligned} \dot{x}_1(t) &= -x_1(t) - 3x_2(t) + v(t) \\ \dot{x}_2(t) &= x_1(t) + 2v(t) \\ y(t) &= x_1(t) + x_2(t) + 2v(t) \end{aligned}$$

1. *Integrators are drawn as a box containing the $\int$ symbol. Place these in the middle of your diagram. Draw one arrow entering the box (for $\dot{x}_i$) and one arrow leaving the box (for x_i).*

2. *A summer is drawn as a circle with arrows entering it, with $+/-$ used to indicate add/subtract.*

3. *Scalar multipliers are drawn as a box with the scalar inside.*

4. *The input $v(t)$ is drawn as an arrow on the left of the diagram, and the output $y(t)$ is an arrow on the right of the diagram.*

- *The summer on the left creates the state equation $\dot{x}_1(t) = -x_1(t) - 3x_2(t) + v(t)$*
- *The middle summer creates the state equation $\dot{x}_2(t) = x_1(t) + 2v(t)$.*
- *The summer on the right creates the output equation $y(t) = x_1(t) + x_2(t) + 2v(t)$*

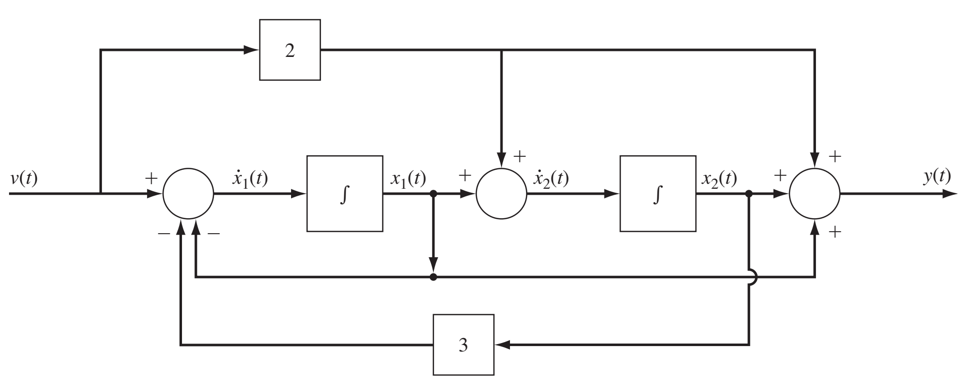


Figure 8.3: Integrator realization

Definition 8.1.7 (Unit delay realizations) *A* **unit delay realization** *is a diagram that represents a discrete-time system in terms of unit delays, summers and scalar multipliers.*

The process for drawing the diagram is the same as for building integrator realizations for continuous-time systems, except that unit delays are used in place of integrators.

Example 8.1.4 (Unit delay realization) *Draw the unit delay realization for the third-order system*

$$\begin{aligned} x_1[n+1] &= -x_2[n]+v_1[n]+v_3[n] \\ x_2[n+1] &= x_1[n]+v_2[n] \\ x_3[n+1] &= x_2[n]+v_3[n] \\ y_1[n] &= x_2[n] \\ y_2[n] &= x_1[n]+x_3[n]+v_2[n] \end{aligned}$$

1. *We create a unit delay block containing the D symbol for each variable, with $x_i[n+1]$ entering and $x_i[n]$ leaving the block. Place these in the middle of your diagram.*

2. *We have three input variables v_1, v_2 and v_3 drawn on the left, and two output variables y_1 and y_2 drawn on the right side of the diagram.*

3. *Summers are used to implement each state equation.*

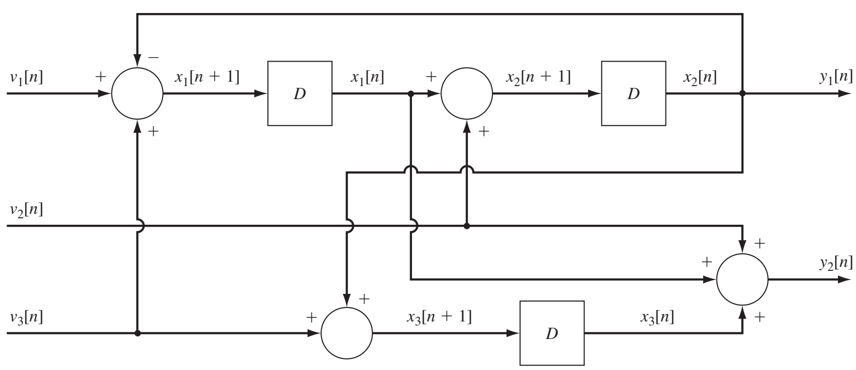


Figure 8.4: Unit delay realization

8.2 Matrix Exponentials

Our next task will be to use state representations to find solutions to linear time-invariant systems. First we require some results from matrix analysis.

Definition 8.2.1 (Matrix Exponential) *Let A be any real $N \times N$ matrix. For any $t \in \mathbf{R}$, the* **matrix exponential** *of A is the matrix function defined by the matrix power series*

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \frac{A^4t^4}{4!} + \dots \tag{8.24}$$

where I is the $N \times N$ identity matrix.

Theorem 8.2.1 (Properties of the Matrix Exponential)

- *For any two real numbers t and s,*
$$e^{A(t+s)} = e^{At}e^{As} \tag{8.25}$$
- *The matrix inverse of e^{At} is e^{-At}, and hence $e^{At}e^{-At} = I$.*
- *The time derivative of e^{At} is given by*
$$\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A \tag{8.26}$$

Theorem 8.2.2 (Matrix Exponential of a Diagonal Matrix) *Let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be real or complex numbers and let Λ be the diagonal matrix*

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & \ldots & 0 & \lambda_N \end{bmatrix} \tag{8.27}$$

Then the matrix exponentials of Λ are, for $t \in \mathbf{R}$ and integer n,

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \ldots & 0 \\ 0 & e^{\lambda_2 t} & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & \ldots & 0 & e^{\lambda_N t} \end{bmatrix}, \quad \Lambda^n = \begin{bmatrix} {\lambda_1}^n & 0 & \ldots & 0 \\ 0 & {\lambda_2}^n & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & \ldots & 0 & {\lambda_N}^n \end{bmatrix} \tag{8.28}$$

Theorem 8.2.3 (Eigenvalues of similar matrices) *Let A and $\bar{A}$ be square matrices, and assume there exists an invertible matrix P such that*
$$\bar{A} = PAP^{-1} \tag{8.29}$$
Then A and $\bar{A}$ have the same eigenvalues. We say that A and $\bar{A}$ are **similar matrices**.

Definition 8.2.2 (Diagonalizability) *For any $N \times N$ matrix A, we say that A is* **diagonalizable** *if there exists an invertible matrix P and a diagonal matrix Λ such that*

$$\Lambda = PAP^{-1} \tag{8.30}$$

Theorem 8.2.4 *Assume A has N distinct eigenvalues $\{\lambda_1, \ldots, \lambda_N\}$ and let $\{v_1, \ldots, v_N\}$ be the corresponding eigenvectors. Then A is diagonalizable with*

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & \ldots & 0 & \lambda_N \end{bmatrix}, \quad P = [v_1 \ldots v_N]^{-1} \tag{8.31}$$

Note that Λ and P will be complex matrices if A has complex eigenvalues. Also Λ and A are similar matrices, and hence they have the same eigenvalues.

For diagonalizable matrices, there is a convenient way to compute their matrix exponentials.

Theorem 8.2.5 *Let A be a diagonalizable matrix, and let P and Λ be such that*

$$\Lambda = PAP^{-1} \tag{8.32}$$

where P is an invertible matrix and Λ is a diagonal matrix. Then for any $t \in \mathbf{R}$,

$$e^{At} = P^{-1}e^{\Lambda t}P \tag{8.33}$$

and for any integer n,

$$A^n = P^{-1}\Lambda^n P \tag{8.34}$$

Example 8.2.1 *Find e^{At} if*

$$A = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$$

Since A is an upper triangular matrix, the eigenvalues appear on the leading diagonal: $\lambda_1 = 0$ and $\lambda_2 = 1$. We find that the corresponding eigenvectors are $v_1 = [1\ 0]^T$, $v_2 = [-1\ 1]^T$. Hence a diagonalization for A is given by

$$\Lambda = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Then $A = P^{-1}\Lambda P$ and

$$\begin{aligned} e^{At} &= P^{-1}e^{\Lambda t}P \\ &= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^t \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1-e^t \\ 0 & e^t \end{bmatrix} \end{aligned}$$

8.3 System responses using state representations

In this section we can use our results for matrix analysis to obtain formulae for the system response using state representations.

Theorem 8.3.1 *The continuous-time system*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bv(t) \\ y(t) &= Cx(t) + Dv(t) \end{aligned} \tag{8.35}$$

has solution

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\lambda)}Bv(\lambda)\, d\lambda, \quad \textit{for } t \geq 0 \tag{8.36}$$

and output

$$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\lambda)}Bv(\lambda)\, d\lambda + Dv(t), \quad \textit{for } t \geq 0 \tag{8.37}$$

The exponential matrix e^{At} is known as the **state-transition matrix** *of the system.*

Definition 8.3.1 (Zero-input solutions and responses) *When the zero input ($v(t) = 0$) is applied to a continuous-time system, the solution and output are*

$$x_{zi}(t) = e^{At}x(0), \quad y_{zi}(t) = Ce^{At}x(0), \quad \textit{for } t \geq 0 \tag{8.38}$$

They are called the **zero-input solution** *(or* **unforced solution***) and the* **zero-input response** *of the system.*

Definition 8.3.2 (Zero-state solutions and responses) *When the initial state is zero ($x(0) = 0$), the solution and output are*

$$x_{zs}(t) = \int_0^t e^{A(t-\lambda)} Bv(\lambda)\, d\lambda, \text{ for } t \geq 0 \tag{8.39}$$

$$y_{zs}(t) = \int_0^t Ce^{A(t-\lambda)} Bv(\lambda)\, d\lambda + Dv(t), \text{ for } t \geq 0 \tag{8.40}$$

They are called the **zero-state solution** *and the* **zero-state response** *of the system.*

Theorem 8.3.2 *The discrete-time system*

$$\begin{aligned} x[n+1] &= Ax[n] + Bv[n] \\ y[n] &= Cx[n] + Dv[n] \end{aligned} \tag{8.41}$$

has solution

$$x[n] = A^n x[0] + \sum_{i=0}^{n-1} A^{n-i-1} Bv[i], \quad \text{for } n \geq 1 \tag{8.42}$$

and output

$$y[n] = CA^n x[0] + \sum_{i=0}^{n-1} CA^{n-i-1} Bv[i] + Dv[n], \quad \text{for } n \geq 1 \tag{8.43}$$

The matrix A^n is known as the **state-transition matrix** *of the system.*

Definition 8.3.3 (Zero-input solutions and responses) *When the zero input ($v[n] = 0$) is applied to a discrete-time system, we obtain the* **zero-input solution** *and the* **zero-input response** *of the system:*

$$x_{zi}[n] = A^n x[0], \quad y_{zi}[n] = CA^n x[0], \quad \text{for } n \geq 1 \tag{8.44}$$

Definition 8.3.4 (Zero-state solutions and responses) *When the initial state is zero ($x[0] = 0$), we obtain the* **zero-state output** *and the* **zero-state response** *of the system:*

$$x_{zs}[n] = \sum_{i=0}^{n-1} A^{n-i-1} Bv[i], \quad \text{for } n \geq 1 \tag{8.45}$$

$$y_{zs}[n] = \begin{cases} Dv[0], & n = 0 \\ \sum_{i=0}^{n-1} CA^{n-i-1} Bv[i] + Dv[n], & n > 0 \end{cases} \tag{8.46}$$

8.4 Equivalent state representations

In this section we explore how to obtain different state representations for a system. We will see that alternative representations can have computational advantages.

Definition 8.4.1 (Coordinate transformations) *Let x be a state vector in $\mathbf{R}^n$ and let P be an invertible $n \times n$ matrix A. Then we can define a new state vector $\bar{x}$ by*

$$\bar{x} = Px \tag{8.47}$$

We say that P is a **coordinate transformation** *matrix.*

Definition 8.4.2 (Equivalent state representations) *Let (A,B,C,D) be the state matrices of either a continuous-time or discrete-time system*

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t)+Bv(t) \\ y(t) & = & Cx(t)+Dv(t) \end{array} \quad or \quad \begin{array}{rcl} x[n+1] & = & Ax[n]+Bv[n] \\ y[n] & = & Cx[n]+Dv[n] \end{array} \tag{8.48}$$

and let $\bar{x} = Px$, for some coordinate transformation P.

Definition 8.4.3 (Equivalent state representations) *Introduce state matrices $(\bar{A},\bar{B},\bar{C},\bar{D})$ where*

$$\bar{A} = PAP^{-1},\ \bar{B} = PB,\ \bar{C} = CP^{-1}, \bar{D} = D \tag{8.49}$$

Then the state representation

$$\begin{array}{rcl} \dot{\bar{x}}(t) & = & \bar{A}\bar{x}(t)+\bar{B}v(t) \\ y(t) & = & \bar{C}\bar{x}(t)+\bar{D}v(t) \end{array} \quad or \quad \begin{array}{rcl} x[n+1] & = & \bar{A}\bar{x}[n]+\bar{B}v[n] \\ y[n] & = & \bar{C}\bar{x}[n]+\bar{D}v[n] \end{array} \tag{8.50}$$

is **equivalent** *to the state representation*

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t)+Bv(t) \\ y(t) & = & Cx(t)+Dv(t) \end{array} \quad or \quad \begin{array}{rcl} x[n+1] & = & Ax[n]+Bv[n] \\ y[n] & = & Cx[n]+Dv[n] \end{array} \tag{8.51}$$

We say that the quadruples of state matrices (A,B,C,D) and $(\bar{A},\bar{B},\bar{C},\bar{D})$ are **equivalent state representations**

Theorem 8.4.1 (Equivalent state representations) *Let (A,B,C,D) and $(\bar{A},\bar{B},\bar{C},\bar{D})$ be equivalent state space representations for an LTI system. For any input signal v, the output y can be obtained using either equation 8.50 with $(\bar{A},\bar{B},\bar{C},\bar{D})$ or equation 8.51 with (A,B,C,D).*

Example 8.4.1 (Equivalent state representations) *Consider the system with state space representation*

$$\begin{array}{rcl} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} & = & \begin{bmatrix} -3 & 2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} v(t) \\ y(t) & = & \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{array}$$

Introduce

$$P = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad and \quad \bar{x}(t) = Px$$

Note that $\det(P) = 1$*, so P is invertible. Introduce new state matrices*

$$\bar{A} = PAP^{-1} = \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix}, \bar{B} = PB = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \bar{C} = CP^{-1} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Then the state space representation

$$\begin{array}{rcl} \begin{bmatrix} \dot{\bar{x}}_1(t) \\ \dot{\bar{x}}_2(t) \end{bmatrix} & = & \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v(t) \\ y(t) & = & \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix} \end{array}$$

is an equivalent state representation for the system.

Observe that the system is in now controller canonical form. Next we explore how coordinate transformations can convert systems into a useful diagonal form.

Example 8.4.2 (Diagonal state representations) *Consider the system with state space representation*

$$\begin{aligned} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{aligned}$$

Simple computations reveal that $\lambda_1 = -1$ and $\lambda_2 = -2$ are the eigenvalues of A, with corresponding eigenvectors $v_1 = [1\ -1]^T$ and $v_2 = [1\ -2]^T$. Hence the matrix A has distinct eigenvalues and is diagonalizable. The diagonal matrix Λ and coordinate transformation P are

$$\Lambda = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

So we may use P to obtain an alternative state representation for the system in which A is replaced by a diagonal matrix. We define new coordinates $\bar{x} = Px$ and new state matrices

$$\bar{A} = PAP^{-1} = \Lambda, \ \bar{B} = PB = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ \bar{C} = CP^{-1} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

Then the diagonal state representation in $\bar{x}$-coordinates is

$$\begin{aligned} \begin{bmatrix} \dot{\bar{x}}_1(t) \\ \dot{\bar{x}}_2(t) \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} v(t) \\ y(t) &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix} \end{aligned}$$

is an equivalent state representation for the system. Suppose the initial state is $x(0) = [2 - 2]^T$. Then

$$\bar{x}(0) = Px(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

The zero-input solution in $\bar{x}$-coordinates is

$$\begin{aligned} \bar{x}_{zi}(t) &= e^{\Lambda t}\bar{x}(0) \\ &= \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 2e^{-t} \\ 0 \end{bmatrix} \end{aligned}$$

Next suppose the input $v(t) = u(t)$, the step input. Then zero-state solution in $\bar{x}$-coordinates is

$$\begin{aligned} \bar{x}_{zs}(t) &= \int_0^t e^{\Lambda(t-\lambda)}\bar{B}v(\lambda)\, d\lambda \\ &= \int_0^t \begin{bmatrix} e^{-(t-\lambda)} & 0 \\ 0 & e^{-2(t-\lambda)} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(\lambda)\, d\lambda \\ &= \int_0^t \begin{bmatrix} e^{-(t-\lambda)} \\ -e^{-2(t-\lambda)} \end{bmatrix} u(\lambda)\, d\lambda \\ &= \begin{bmatrix} 1 - e^{-t} \\ \frac{-1}{2}(1 - e^{-2t}) \end{bmatrix} \end{aligned}$$

Hence the zero-input response and the zero-state response are

$$\begin{aligned}
y_{zi}(t) &= \bar{C}\bar{x}_{zi}(t) \\
&= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2e^{-t} \\ 0 \end{bmatrix} \\
&= 2e^{-t} \\
y_{zs}(t) &= \bar{C}\bar{x}_{zs}(t) \\
&= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1-e^{-t} \\ \frac{-1}{2}(1-e^{-2t}) \end{bmatrix} \\
&= \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}
\end{aligned}$$

Thus the system output from the given initial value $x(0)$ is

$$y(t) = y_{zi}(t) + y_{zs}(t) = \frac{1}{2} + e^{-t} + \frac{1}{2}e^{-2t}$$

8.5 Practice Problems

93. Use Kirchoff's laws to find the state space model for the circuit below. Use state variables $x_1(t) = i_L(t)$, $x_2(t) = v_C(t)$ and with the output defined as $y(t) = i_L(t) + V_C(t)$.

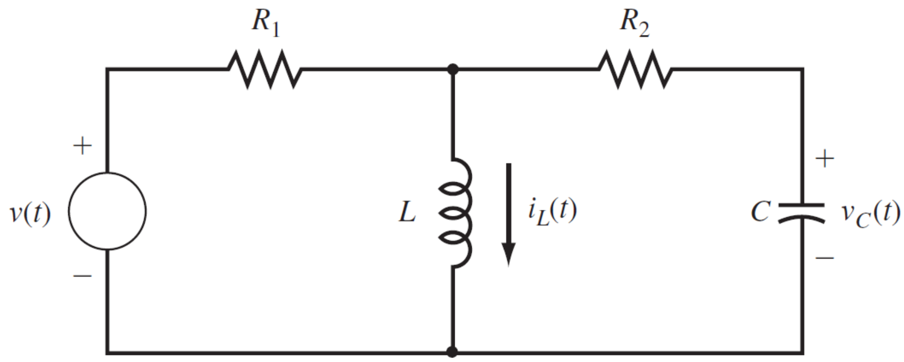


Figure 8.5: Diagram for Problem 93

94. A system is described by the second order differential equation

$$\ddot{y}(t) + 6\dot{y}(t) + 9y(t) = -v(t)$$

   (a) Obtain the state matrices (A, B, C, D) for its controller canonical form state representation:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bv(t) \\ y(t) &= Cx(t) + Dv(t). \end{aligned}$$

   (b) Draw the block diagram for the integrator realization for this system.

95. A system is described by the difference equation

$$y[n+3] + 5y[n+2] + 9y[n] = 3v[n].$$

   (a) Obtain the state matrices (A, B, C, D) for its controller canonical form state representation

$$\begin{aligned} x[n+1] &= Ax[n] + Bv[n] \\ y[n] &= Cx[n] + Dv[n]. \end{aligned}$$

   (b) Draw the block diagram for the unit delay realization for this system.

96. (a) For the two-input two-output continuous-time system in Figure 8.6 shown below, find a state model that has the least number of state variables.

   (b) Consider the corresponding discrete-time system, in which the integrator blocks are replaced by unit delays, and all the inputs and outputs are discrete-time signals. Find a state model for this discrete-time system.

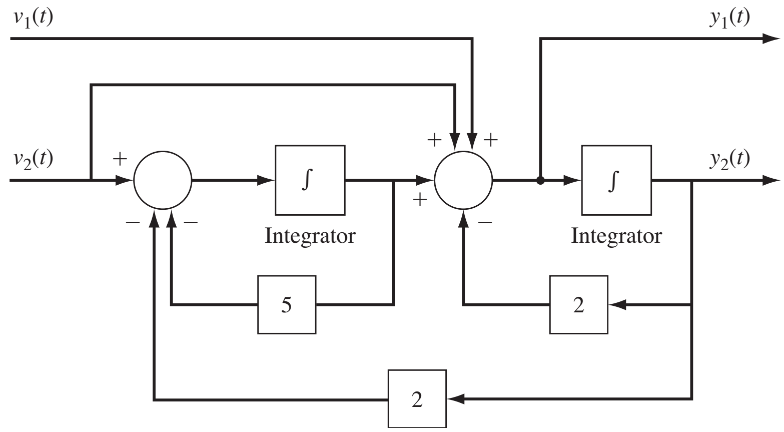


Figure 8.6: Diagram for Problem 96

97. A two-car system has state space representation

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} \frac{-k_f}{M} & 0 & 0 \\ 0 & \frac{-k_f}{M} & 0 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{M} & 0 \\ 0 & \frac{1}{M} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$$

Assume $k_f = 10$, $M = 1000$ and $x(0) = [60\ 60\ 100]^T$. Find the forces $f_1(t)$ and $f_2(t)$ that must be applied to the car to obtain $x(t) = [60\ 60\ 100]^T$ for all $t \geq 0$.

98. A linear time-invariant discrete-time system is given by the state-space model

$$\begin{aligned} \begin{bmatrix} x_1[n+1] \\ x_2[n+1] \end{bmatrix} &= \begin{bmatrix} -1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \end{bmatrix} + \begin{bmatrix} 0.5 & 1 \\ -1 & -0.5 \end{bmatrix} \begin{bmatrix} v_1[n] \\ v_2[n] \end{bmatrix} \\ \begin{bmatrix} y_1[n] \\ y_2[n] \end{bmatrix} &= \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \end{bmatrix} \end{aligned}$$

(a) Compute $x[1]$, $x[2]$ and $x[3]$ when $x[0] = [1\ 1]^T$ and $v[n] = [n\ n]^T$.

(b) Suppose $x[0] = [0\ 0]^T$. Find an input sequence $v[0]$, $v[1]$ that will drive the system state to $x[2] = [-1\ 2]^T$.

(c) Suppose $x[0] = [1\ -2]^T$. Find an input sequence $v[0]$, $v[1]$ that will drive the system state to $x[2] = [0\ 0]^T$.

99. A discrete-time system with state matrices (A, B, C) is such that

$$CB = \begin{bmatrix} 6 \\ 3 \end{bmatrix}, \quad CAB = \begin{bmatrix} 22 \\ 11 \end{bmatrix}$$

With $x[0] = [0\ 0]^T$, find an input sequence $v[0]$, $v[1]$ that will drive the system outputs to $y[1] = [6\ 3]^T$ and $y[2] = [4\ 2]^T$.

100. A continuous-time system has state space model (A,B) where

$$A = \begin{bmatrix} 3 & -2 \\ 9 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Determine if there is coordinate transformation $\bar{x} = Px$ that will bring A into diagonal form. If so, give P and the corresponding diagonalized system $(\bar{A}, \bar{B})$.

101. A linear time-invariant continuous-time system has a state space representation (A,B,C) where

$$A = \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 3 \end{bmatrix}$$

(a) Compute the state transition matrix e^{At}.

(b) Compute the response y_{zi} arising from the zero input and initial state $x(0) = [1\ -1]^T$.

102. (a) Show that for a scalar-input scalar-output continuous-time system, the system impulse response h is given by

$$h(t) = Ce^{At}B + D\delta(t), \quad \text{for } t \geq 0$$

(b) Show that for a scalar-input scalar-output discrete-time system, the system unit pulse response h is given by

$$h[n] = \begin{cases} D, & n = 0 \\ CA^{n-1}B, & n \geq 1 \end{cases}$$

103. The motion of a car moving on a level surface may be described by the differential equation

$$\frac{d^2y(t)}{dt} + \frac{k_f}{M}\frac{dy(t)}{dt} = \frac{1}{M}v(t)$$

where $y(t)$ is the position of the car at time t, $v(t)$ is the applied force and M and k_f are constants. Assume $M = 1$ and $k_f = 0.1$.

(a) Obtain a state representation (A,B,C) for the system in controller canonical form.

(b) Obtain the zero-input solution when the initial state is $x(0) = [y_0\ v_0]^T$, for some constants y_0 and v_0.

(c) The motion of the car is such that $y(10) = 0$ and $\dot{y}(10) = 55$. Find $y(0)$ and $\dot{y}(0)$.

(d) A force of $v(t) = 1$ is applied to the car for time $0 \leq t \leq 5$. The state $x(5) = [50\ 20]^T$. Find $x(0)$.

Note: parts (c) and (d) are unrelated questions.

104. (a) Let A and $\bar{A}$ and be square matrices such that

$$A = P\bar{A}P^{-1}$$

for some invertible matrix P. Show that A and $\bar{A}$ have the same eigenvalues.

(b) Let A and $\bar{A}$ be given by

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}$$

Assume there exists a transformation P satisfying

$$A = P\bar{A}P^{-1}$$

Determine the values of a_0, a_1 and a_2.

(c) Find the matrix P.

105. Let (A,B,C) be a state representation for an LTI continous-time system, with

$$A = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

(a) Obtain a coordinate transformation $\bar{x} = Px$ that will bring A into diagonal form. What is the corresponding diagonalized state representation $(\bar{A}, \bar{B}, \bar{C})$? for the system?

(b) Use the diagonalized representation to obtain the zero-input response y_{zi}, if $x(0) = [1\ 0]^T$.

106. A combined bank loan and savings account can be modelled by two-input two-output difference equation

$$\begin{bmatrix} x_1[n+1] \\ x_2[n+1] \end{bmatrix} = \begin{bmatrix} 1+\frac{I_1}{12} & 0 \\ 0 & 1+\frac{I_2}{12} \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} v_1[n] \\ v_2[n] \end{bmatrix}$$

where x_1 and x_2 denote the balance of the loan account and savings account at the end of the n-th month; thus $x_1[0]$ and $x_2[0]$ represent the initial values of the loan, and initial amount of the savings account. Also I_1 and I_2 are the (annual) interest rates on the loan and savings accounts, respectively, while v_1 and v_2 are the monthly payments on the loan and deposits into the savings account, made at the end of each month.

Note: You may use MATLAB for the following calculations.

(a) Assume the initial amount in the savings account is \$20,000, and assume monthly payments and deposits of $p[n] = d[n] = \$2,000$. Assume interest rates of $I_1 = 6\%$ and $I_2 = 3\%$. Determine the number of months required to pay off a loan of \$300,000.

(b) Determine the amount in the savings account at the end of this time.

(c) Find the 'interest only' loan payment, i.e. the constant monthly payment $p[n]$ that will maintain the loan balance at \$300,000.

(d) For the loan payment obtained in part (c), determine the smallest constant monthly deposits $d[n]$ required to maintain the savings account balance at \$20,000.

Chapter 9

Transfer functions

In this chapter we introduce the concept of a transfer function of a system and use it to analyse the input-output behaviour of continuous-time and discrete-time systems.

9.1 Continuous-time Transfer functions

Definition 9.1.1 (Transfer function of a continuous-time system) *Consider the N-th order LTI continuous-time system described by the differential equation*

$$\frac{d^N y(t)}{dt^N} + \sum_{i=0}^{N-1} a_i \frac{d^i y(t)}{dt^i} = \sum_{i=0}^{M} b_i \frac{d^i v(t)}{dt^i} \tag{9.1}$$

where $M < N$ and the coefficients a_i and b_i are real numbers. Assume that $v^{(i)}(0^-) = 0$ for all $i = 0,\ 1,2,\ldots,M-1$ and $y^{(i)}(0^-) = 0$ for all $i = 0,1,2,\ldots,N-1$. Taking Laplace Transforms of both sides, we obtain

$$(s^N + a_{N-1}s^{N-1} + \ldots a_1 s + a_0)Y(s) = (b_M s^M + b_{M-1}s^{M-1} + \ldots b_1 s + b_0)V(s) \tag{9.2}$$

where Y and V are the Laplace transforms of y and v. Then we define the **input-output transfer function** *H for the system to be the rational function*

$$H(s) = \frac{Y(s)}{V(s)} = \frac{b_M s^M + b_{M-1}s^{M-1} + \ldots b_1 s + b_0}{s^N + a_{N-1}s^{N-1} + \ldots a_1 s + a_0} \tag{9.3}$$

Let A and B be the real polynomials

$$\begin{aligned} B(s) &= b_M s^M + b_{M-1}s^{M-1} + \ldots b_1 s + b_0 && (9.4) \\ A(s) &= s^N + a_{N-1}s^{N-1} + \ldots a_1 s + a_0 && (9.5) \end{aligned}$$

Then B and A are the **numerator and denominator polynomials of** *H.*

Remark 9.1.1 *Let H be the input-output transfer function of an LTI continuous-time system, let v and y be inputs and outputs to the system and let $V(s)$ and $Y(s)$ be their Laplace transforms, respectively. Then*

$$Y(s) = H(s)V(s) \tag{9.6}$$

Note that the transfer function assumes zero initial conditions for both the input v and the output y. Thus transfer functions can only be used for obtaining the zero-state response of the system.

Example 9.1.1 (Example) *An RC circuit has differential equation*

$$\frac{dy}{dt} + \frac{1}{RC}y(t) = \frac{1}{RC}v(t) \tag{9.7}$$

where the input v is the applied voltage and the output y is the capacitor voltage. Taking Laplace transforms of both sides and using the LT pair

$$\frac{dy}{dt} \longleftrightarrow sY(s) - y(0^-)$$

we obtain

$$Y(s) = \frac{1/RC}{s+1/RC}V(s)$$

assuming $y(0^-) = 0$. *Hence the input-output transfer function of the system is*

$$H(s) = \frac{Y(s)}{V(s)} = \frac{1/RC}{s+1/RC}$$

Note that we could have obtained the transfer function directly from the differential equation in 9.7. Comparing with equation 9.1, we see that $N = 1$, $M = 0$, $a_0 = \frac{1}{RC}$, *and* $b_0 = \frac{1}{RC}$. *Hence*

$$B(s) = 1/RC, \quad A(s) = s + 1/RC$$

and so

$$H(s) = \frac{B(s)}{A(s)} = \frac{1/RC}{s+1/RC}$$

Now suppose the input v is u, the unit step input. Recall the LT pair

$$u(t) \longleftrightarrow \frac{1}{s}$$

The LT of the output from v is

$$\begin{aligned} Y(s) &= H(s)V(s) \\ &= \left(\frac{1/RC}{s+1/RC}\right)\left(\frac{1}{s}\right) \\ &= \frac{1}{s} - \frac{1}{s+1/RC} \\ \Rightarrow \quad y(t) &= (1 - e^{-t/RC})u(t), \end{aligned}$$

Definition 9.1.2 (Convergent, bounded and divergent signals) *For any continuous time signal x, we say that*

(a) x **converges to zero** *if* $|x(t)| \to 0$ *as* $t \to \infty$.

(b) x *is* **bounded** *if there exists a constant* $c \geq 0$ *such that*

$$|x(t)| \leq c \qquad \text{for all } t \geq 0 \tag{9.8}$$

(c) x *is* **unbounded** *or* **divergent** *if* $|x(t)| \to \infty$ *as* $t \to \infty$.

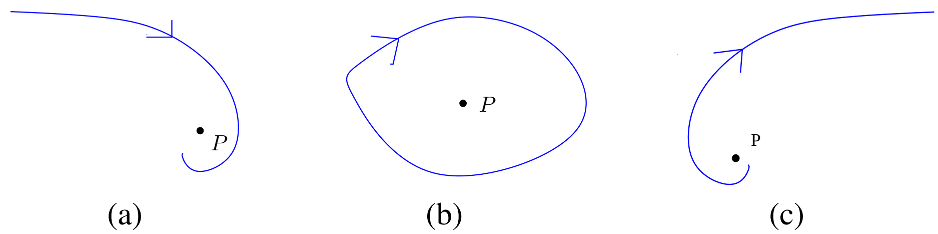


Figure 9.1: (a) convergent signal (b) bounded signal (c) unbounded signal

Definition 9.1.3 (Stability of a continuous-time system)

- *An LTI system is said to be* **bounded-input bounded-output stable** *(BIBO stable) if bounded inputs lead to bounded outputs.*
- *An LTI system is BIBO stable if and only if its unit impulse response h is absolutely integrable:*

$$\int_0^\infty |h(t)|\, dt < \infty \tag{9.9}$$

- *If the impulse response is unbounded, then the system is said to be* **unstable**. *This means that bounded input signals produce unbounded outputs.*
- *An LTI system is said to be* **marginally stable** *if its unit impulse response h is bounded but not absolutely integrable. Marginal stability means that there exists at least one bounded input signal that yields an unbounded output.*

Theorem 9.1.1 (Stable transfer functions) *Let H be the transfer function of an LTI continuous-time system with numerator and denominator polynomials B and A, and assume that if B and A have any common factors, then these have been canceled. Then*

1. *The system is* **BIBO stable** *if, for all poles p of H, $Re(p) < 0$, and*
2. *The system is* **marginally stable** *if, for all poles p of H,*
   (a) *$Re(p) \leq 0$, and*
   (b) *there is at least one pole p such that $Re(p) = 0$, and p is a non-repeated pole.*

Note that the set of complex numbers p such that $Re(p) < 0$ is the open left-hand complex plane (LHP) (not including the imaginary axis):

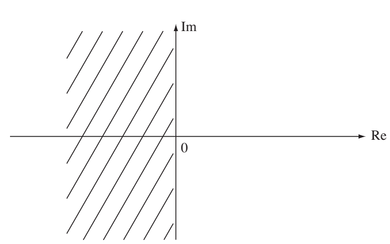


Figure 9.2: The open left-hand complex plane (LHP)

Example 9.1.2 (Stable transfer functions) *Consider the system with transfer function*

$$H_1(s) = \frac{(s-1)(s-2)}{(s+4)^3(s-2)(s^2+9)}$$

First we cancel any common factors:

$$H_1(s) = \frac{s-1}{(s+4)^3(s^2+9)}$$

The poles of this transfer function are given by the roots of the denominator polynomial are

$$\{-4, -4, -4, \pm j3\}$$

The system has repeated poles in the left-hand complex plane, and a pair of non-repeated poles on the imaginary axis. So by Theorem 9.1.1, the system is **marginally stable**. *Most bounded inputs lead to bounded outputs, but there exist some bounded inputs that yield unbounded outputs.*
To see directly why the system is marginally stable, we can use partial fractions:

$$H_1(s) = \frac{c_1}{s+4} + \frac{c_2}{(s+4)^2} + \frac{c_3}{(s+4)^3} + \frac{c_4}{s+j3} + \frac{\bar{c}_4}{s-j3}$$

Taking inverses we obtain the impulse response of the system:

$$h_1(t) = c_1 e^{-4t} + c_2 t e^{-4t} + \frac{1}{2} c_3 t^2 e^{-4t} + 2|c_4| \cos(3t + \angle c_4)$$

- *For $t \to \infty$, the terms involving e^{-4t} will vanish (converge to zero), while the $\cos(3t + \angle c_4)$ term is bounded.*
- *Hence the impulse response is bounded, but not convergent, and the system is* **marginally stable**.

Next we consider the transfer function

$$H_2(s) = \frac{s-1}{(s+4)(s^2+9)^2}$$

In this case the poles of the denominator polynomial are $\{-4, \pm j3, \pm j3\}$. Since the system has a pair of repeated poles on the imaginary axis, it is unstable. To see why, we use partial fractions:

$$H_2(s) = \frac{c_1}{s+4} + \frac{c_2}{s+j3} + \frac{\bar{c}_2}{s-j3} + \frac{c_3}{(s+j3)^2} + \frac{\bar{c}_3}{(s-j3)^2}$$

Taking inverses we obtain the impulse response of the system:

$$h_2(t) = c_1 e^{-4t} + d_1 \cos(3t + \phi_1) + d_2 t \cos(3t + \phi_2)$$

For $t \to \infty$, the term $d_2 t \cos(\omega t + \phi_2)$ becomes arbitrarily large, so h_2 is unbounded and the system is **unstable**.

9.2 Discrete-time transfer functions

Definition 9.2.1 (Transfer function of a discrete-time system) *Consider the N-th order LTI discrete-time system described by the difference equation*

$$y[n+N]+\sum_{i=0}^{N-1} a_i y[n+i]=\sum_{i=0}^{M} b_i v[n+i], \tag{9.10}$$

where $M<N$ and the coefficients a_i and b_i are real numbers. Assume that $v[i]=0$ for all $i=0,\ 1,2,\ldots,M-1$, and $y[i]=0$ for all $i=0,1,2,\ldots,N-1$. Taking z-Transforms of both sides, we obtain

$$(z^N+a_{N-1}z^{N-1}+\ldots a_1z+a_0)Y(z)=(b_Mz^M+b_{M-1}z^{M-1}+\ldots b_1z+b_0)V(z) \tag{9.11}$$

where Y and V are the z-Transforms of y and v. Then we define the **input-output transfer function** *H for the system to be the rational function*

$$H(z)=\frac{Y(z)}{V(z)}=\frac{b_Mz^M+b_{M-1}z^{M-1}+\ldots b_1z+b_0}{z^N+a_{N-1}z^{N-1}+\ldots a_1z+a_0} \tag{9.12}$$

Let A and B be the real polynomials

$$B(z) = b_Mz^M+b_{M-1}z^{M-1}+\ldots b_1z+b_0 \tag{9.13}$$
$$A(z) = z^N+a_{N-1}z^{N-1}+\ldots a_1z+a_0 \tag{9.14}$$

Then B and A are the **numerator and denominator polynomials of** *H.*

Remark 9.2.1 *Let H be the input-output transfer function of an LTI discrete-time system, let v and y be inputs and outputs to the system and let V and Y be their z-Transforms, respectively. Then*

$$Y(z)=H(z)V(z) \tag{9.15}$$

Note that the transfer function assumes zero initial conditions for both the input v and the output y. Thus transfer functions can only be used for obtaining the zero-state response of the system.

Definition 9.2.2 (Stability of a discrete-time system)

- *An LTI system is said to be* **bounded-input bounded-output stable** *(BIBO stable) if bounded inputs lead to bounded outputs.*
- *An LTI system is BIBO stable if and only if its unit pulse response h is absolutely summable:*

$$\sum_{n=0}^{\infty} |h[n]|<\infty \tag{9.16}$$

- *An LTI system is said to be* **marginally stable** *if its unit pulse response h is bounded but not absolutely summable.*
- *An LTI system is said to be* **unstable** *if its unit pulse response h is unbounded.*

Theorem 9.2.1 (Stable transfer functions) *Let H be the transfer function of an LTI discrete-time system with numerator and denominator polynomials B and A, and assume that any common factors of B and A have been canceled. Then*

1. *The system is* **BIBO stable** *if, for every pole p of H, $|p| < 1$, and*
2. *The system is* **marginally stable** *if, for all poles p of H, either*
   - *(a) $|p| \leq 1$, and*
   - *(b) there is at least one pole p such that $|p| = 1$, and p is a non-repeated pole.*

Note that the set of complex numbers p such that $|p| < 1$ is the open unit disc in complex plane (not including the boundary $|p| = 1$):

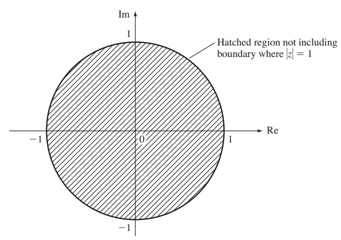


Figure 9.3: The open unit disk in the complex plane

Example 9.2.1 *Consider the difference equation*

$$y[n+4] + \frac{11}{2}y[n+3] + \frac{47}{18}y[n+2] + \frac{11}{18}y[n+1] + \frac{5}{18}y[n] = v[n+2] + v[n+1]$$

Assuming zero initial conditions, the left shift theorem gives the zT pair

$$x[n+q] \longleftrightarrow z^q X(z)$$

Applying this to the difference equation gives

$$Y(z)\left[z^4 + \frac{11}{2}z^3 + \frac{47}{18}z^2 + \frac{11}{18}z + \frac{5}{18}\right] = V(z)[z^2 + z]$$

and so we obtain input-output transfer function

$$H(z) = \frac{Y(z)}{V(z)} = \frac{z^2 + z}{z^4 + \frac{11}{2}z^3 + \frac{47}{18}z^2 + \frac{11}{18}z + \frac{5}{18}}$$

Alternatively, we could have obtained the transfer function directly from the difference equation in 9.7. Comparing with equation 9.10, we see that $N = 4$, $M = 2$, $a_0 = \frac{5}{18}$, $a_1 = \frac{11}{18}$, $a_2 = \frac{47}{18}$, $a_3 = \frac{11}{2}$, and $b_0 = 0$, $b_1 = 1$, $b_2 = 1$. Hence

$$B(z) = z^2 + z, \quad A(z) = z^4 + \frac{11}{2}z^3 + \frac{47}{18}z^2 + \frac{11}{18}z + \frac{5}{18}$$

and so

$$H(z) = \frac{B(z)}{A(z)} = \frac{z^2 + z}{z^4 + \frac{11}{2}z^3 + \frac{47}{18}z^2 + \frac{11}{18}z + \frac{5}{18}}$$

Dividing by z and factorising the denominator polynomial gives

$$\frac{H(z)}{z} = \frac{z+1}{(z+.5)(z+5)(z^2+1/9)}$$

The poles of the denominator polynomial are $\{-0.5, -5, \pm j/3\}$. *The pole at* $z = -5$ *is outside the unit disc, and hence by Theorem 9.2.1, the system is* **unstable**. *To see why, we use partial fractions:*

$$\begin{aligned}
\frac{H(z)}{z} &= \frac{c_0}{z} + \frac{c_1}{z+5} + \frac{c_2}{z+.5} + \frac{c_3}{z+j/3} + \frac{\bar{c}_3}{z-j/3} \\
H(z) &= c_0 + \frac{c_1 z}{z+5} + \frac{c_2 z}{z+.5} + \frac{c_3 z}{z+j/3} + \frac{\bar{c}_3 z}{z-j/3} \\
\Rightarrow h[n] &= c_0\delta[n] + (c_1(-5)^n + c_2(-0.5)^n + 2|c_3|(\frac{1}{3})^n \cos(\frac{\pi n}{2} + \angle c_3))u[n] \qquad (9.17)
\end{aligned}$$

As $n \to \infty$, $|c_1(-5)^n| \to \infty$, *so h is unbounded and the system is* **unstable**.

9.3 Step responses

Theorem 9.3.1 (Step response of a continuous-time system) *Let H be the transfer function of an LTI continuous-time system with numerator and denominator polynomials B and A, and assume that any common factors of B and A have been canceled. Let v be the unit-step function, so that* $V(s) = 1/s$. *Let y be the output from the input v so that*

$$Y(s) = \frac{B(s)V(s)}{A(s)} = \frac{B(s)}{sA(s)} \qquad (9.18)$$

Let E be a polynomial function in s such that

$$Y(s) = \frac{E(s)}{A(s)} + \frac{H(0)}{s} \qquad (9.19)$$

Then the system **step response** *is given by*

$$y(t) = y_1(t) + H(0), \quad t \geq 0. \qquad (9.20)$$

where y_1 *is the inverse LT of* $E(s)/A(s)$.

Remark 9.3.1

- *For stable systems, the term* $y_1(t)$ *converges to zero. We say that* y_1 *is the* **transient response** *of the system. The constant* $H(0)$ *is the* **steady-state** *value of the step response.*

- *Recall the RC circuit example had transfer function*

$$H(s) = \frac{1/RC}{s+1/RC} = \frac{Y(s)}{U(s)} \qquad (9.21)$$

so $H(0) = 1$. *The step input u with* $U(s) = 1/s$ *yields output*

$$\begin{aligned}
Y(s) &= \frac{1}{s} - \frac{1}{s+1/RC} \\
\Rightarrow y(t) &= 1 - e^{-t/RC} \\
&= H(0) + y_1(t) \qquad (9.22)
\end{aligned}$$

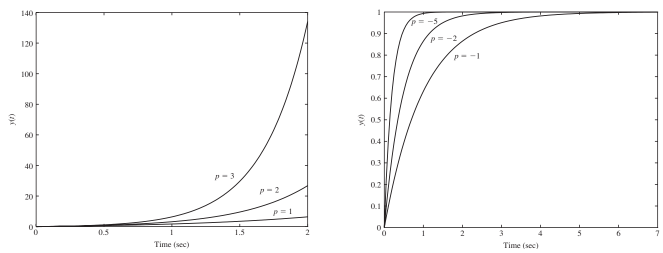
Hence $H(0) = 1$ *is the steady-state response, and* $y_1(t) = -e^{-t/RC}$ *is the transient response.*

Case 9.3.1 (First-order step response) *For a general first-order transfer function $H(s) = \frac{k}{s-p}$, the step response is $y(t) = -\frac{k}{p}(1 - e^{pt})$. The response is shown for the cases*

(a) the poles are unstable, e.g. $p = 1$, $p = 2$ and $p = 3$.

(b) the poles are stable, e.g. $p = -1$, $p = -2$ and $p = -5$.

For the unstable poles, the output becomes arbitrarily large (unbounded). For the stable poles, the output converges to $H(0) = -\frac{k}{p}$.



(a) (b)

Figure 9.4: step response from (a) unstable poles and (b) stable poles

Case 9.3.2 (Second-order step response) *Consider the second-order transfer function*

$$H(s) = \frac{k}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{9.23}$$

The real parameters ζ and ω_n are respectively called the **damping ratio** *and* **natural frequency**. *Assuming $\zeta > 0$ and $\omega_n > 0$ implies the system is stable. The poles are*

$$p_1, p_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

- *When $\zeta > 1$, the poles are real and distinct (Overdamped).*
- *When $\zeta = 1$, the poles are real and repeated (Critically damped).*
- *When $\zeta < 1$, the poles are complex-conjugate pairs (Underdamped).*

Case 9.3.3 (Overdamped response: $\zeta > 1$) *With $\zeta > 1$ we have two distinct real poles p_1, p_2 and the transform of the step response is*

$$\begin{aligned} Y(s) &= \frac{k}{s(s-p_1)(s-p_2)} \\ \Rightarrow y(t) &= \frac{k}{p_1 p_2}(k_1 e^{p_1 t} + k_2 e^{p_2 t} + 1), \quad t \geq 0 \end{aligned}$$

Here

$$\begin{aligned} y_{tr}(t) &= \frac{k}{p_1 p_2}(k_1 e^{p_1 t} + k_2 e^{p_2 t}) \quad \textit{is the transient response} \\ y_{ss} &= \frac{k}{p_1 p_2} \quad \textit{is the steady-state value} \end{aligned}$$

Example 9.3.1 **(**$\zeta > 1$ **)** *Suppose the LT of the step response is*

$$Y(s) = \frac{2}{s(s+1)(s+2)} = \frac{-2}{s+1} + \frac{1}{s+2} + \frac{1}{s}$$

with system poles $p = -1$ *and* $p = -2$*. Then the step response is*

$$y(t) = -2e^{-t} + e^{-2t} + 1, \quad t \geq 0$$

where

$$\begin{aligned} y_{tr}(t) &= -2e^{-t} + e^{-2t} \quad \text{is the transient response} \\ y_{ss} &= 1 \quad \text{is the steady-state value} \end{aligned}$$

Note that $e^{-2t} << e^{-t}$ *so that the contribution to the output from the pole* $p_2 = -2$ *is much smaller than from pole* $p_1 = -1$*. As a result the output looks similar to the graph of a step response from a first order system:*

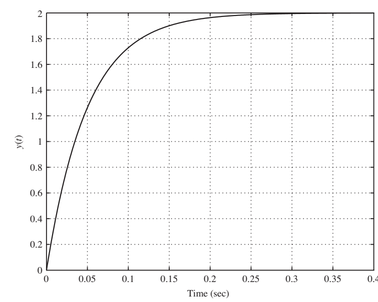


Figure 9.5: Overdamped response

We say that the pole at $p_2 = -1$ *is the* **dominant pole***. The dominant pole is the one that is closer to the imaginary axis, as it has the larger time constant.*

Case 9.3.4 (Critically damped response: **(**$\zeta = 1$**)** *With* $\zeta = 1$ *we have repeated real poles* $p_1 = p_2$ *and the transform of the step response is*

$$\begin{aligned} Y(s) &= \frac{k}{s(s+\omega_n)^2} \\ \Rightarrow y(t) &= \frac{k}{\omega_n^2}(1 - (1+\omega_n t)e^{-\omega_n t}), \quad t \geq 0 \end{aligned}$$

Here

$$\begin{aligned} y_{tr}(t) &= -\frac{k}{\omega_n^2}(1+\omega_n t)e^{-\omega_n t} \quad \text{is the transient response} \\ y_{ss} &= \frac{k}{\omega_n^2} \quad \text{is the steady-state value} \end{aligned}$$

Example 9.3.2 **(**$\zeta = 1$ **)** *Suppose the LT of the step response is*

$$Y(s) = \frac{4}{s(s+2)^2} = \frac{-1}{s+2} + \frac{-2}{(s+2)^2} + \frac{1}{s}$$

with system repeated poles at $p = -2$*. Then the step response is*

$$y(t) = 1 - (1+2t)e^{-2t}, \quad t \geq 0$$

The response is similar to a first-order system with pole at $p = -2$*.*

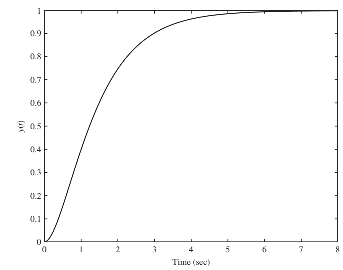


Figure 9.6: Critically damped response

Case 9.3.5 (Underdamped response: $0 < \zeta < 1$**)** *With* $0 < \zeta < 1$ *we have complex-conjugate poles* $p_1, p_2 = -\zeta\omega_n \pm j\omega_d$*, where* $\omega_d = \omega_n\sqrt{1-\zeta^2}$*. The transform of the step response is*

$$\begin{aligned} Y(s) &= \frac{k}{s((s+\zeta\omega_n)^2+\omega_d^2)} \\ \Rightarrow y(t) &= \frac{k}{\omega_n^2}\left[1 - \frac{\omega_n}{\omega_d}e^{-\zeta\omega_n t}\sin(\omega_d t + \phi)\right], \quad t \geq 0 \end{aligned}$$

where $\phi = \tan^{-1}(\omega_d/\zeta\omega_n)$.

- *The steady-state value is* $y_{ss} = \frac{k}{\omega_n^2}$.
- *The transient response is* $y_{tr}(t) = -\frac{k}{\omega_n^2}\frac{\omega_n}{\omega_d}e^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$*. This is an exponentially decaying sinusoid.*

Example 9.3.3 ($0 < \zeta < 1$) *Suppose the LT of the step response is*

$$Y(s) = \frac{17}{s(s^2+2s+17)}$$

with system poles $p_1, p_2 = -1 \pm j4$*. The step response is*

$$y(t) = 1 - \frac{\sqrt{17}}{4}e^{-2t}\sin(4t + 1.326), \quad t \geq 0$$

The response is an oscillating and decaying sinusoid. The dashed line shows the exponential part of the transient.

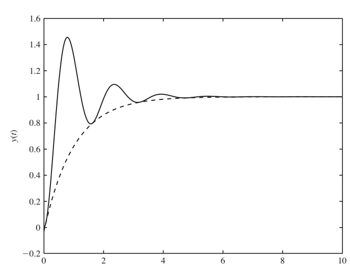


Figure 9.7: Underdamped response

Case 9.3.6 (Undamped response: $\zeta = 0$**)** *With* $\zeta = 0$ *we have purely imaginary poles* $p_1, p_2 = \pm j\omega_n$*. The transform of the step response is*

$$\begin{aligned} Y(s) &= \frac{k}{s(s^2+\omega_n^2)} \\ \Rightarrow y(t) &= \frac{k}{\omega_n^2}\left[1 - \cos(\omega_n t)\right], \quad t \geq 0 \end{aligned}$$

Since the system poles lie on the imaginary axis, the system is **marginally stable**. *The step response is sinusoidal and hence bounded but not convergent.*

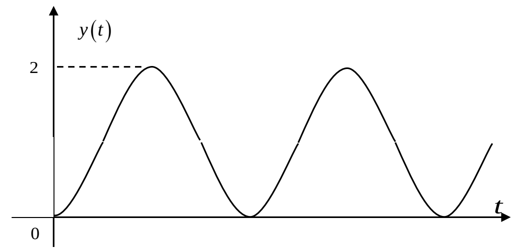


Figure 9.8: Undamped response

Case 9.3.7 (ζ, ω_n and the pole locations) *For $0 \leq \zeta < 1$, the poles are complex-conjugate pairs and their locations are shown in the diagram:*

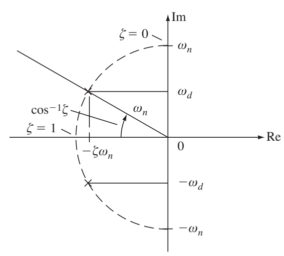
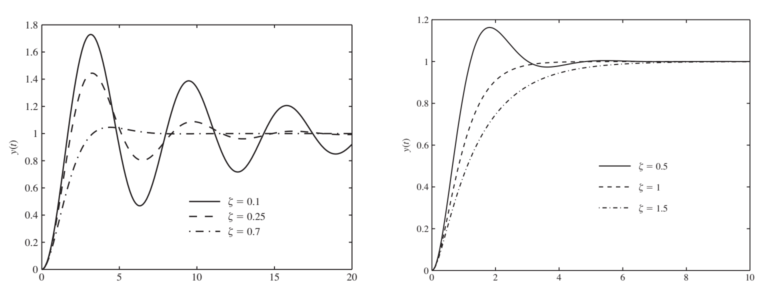


Figure 9.9: Pole locations

- *The distance of the poles from the origin is ω_n, and the phase of the poles is $\pi \pm \cos^{-1}(\zeta)$.*
- *Smaller values of ζ yield poles closer to the imaginary axis. This corresponds to a slower transient response, and explains why ζ is called the* **damping ratio**.

Case 9.3.8 (ζ and the oscillations in the transient response) *The graphs in Figure 9.10 show how the oscillations in the transient vary with ζ:*

- *For $0 \leq \zeta < 1$, there are oscillations in the transient, with smaller ζ giving rise to larger oscillations.*
- *For $\zeta \geq 1$, the poles are real and there is no overshoot of the steady-state value.*



(a) (b)

Figure 9.10: (a) Oscillatory response from $0 \leq \zeta < 1$ (b) Nonovershooting response from $\zeta \geq 1$.

Remark 9.3.2 (Limitations of the transient analysis)
The above discussion assumed the transfer function was of the form

$$H(s) = \frac{k}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The analysis is not applicable if the numerator is not constant. Consider

$$H(s) = \frac{s-2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The numerator polynomial $B(s) = s-2$ has a zero at $s = 2$. Such systems always exhibit **undershoot** *in their transient response, as shown in Figure 9.11. This means the output first moves away from the steady-state value, before converging to it.*

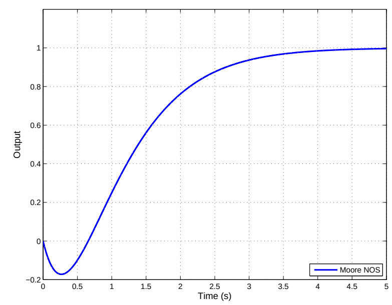


Figure 9.11: Undershoting response

9.4 Sinusoidal response

Theorem 9.4.1 (Sinusoidal response of a continuous time system) *Let H be the transfer function of an LTI continuous-time system with numerator and denominator polynomials B and A, and assume that B and A have no common factors. Let $v(t) = \cos(\omega_0 t)u(t)$ be a sinusoidal input of arbitrary frequency ω_0. Let y be the output from input v. Then*

$$Y(s) = \frac{B(s)V(s)}{A(s)} = \frac{sB(s)}{A(s)(s^2+\omega_0^2)} \tag{9.24}$$

Let γ be a polynomial in s such that

$$Y(s) = \frac{\gamma(s)}{A(s)} + \frac{c}{s-j\omega_0} + \frac{\bar{c}}{s+j\omega_0} \tag{9.25}$$

Then the system **sinusoidal response** *is given by*

$$y(t) = y_1(t) + |H(j\omega_0)|\cos(\omega_0 t + \angle H(j\omega_0)), \quad t \geq 0 \tag{9.26}$$

where y_1 is the inverse LT of $\gamma(s)/A(s)$.

Remark 9.4.1

- *For stable systems, the* **transient response** *term $y_1(t)$ converges to zero. The* **steady-state response** *is*

$$y_{ss}(t) = |H(j\omega_0)|\cos(\omega_0 t + \angle H(j\omega_0)), \quad t \geq 0$$

- *The steady-state response is also a sinusoidal signal, of the same frequency as the input. It is scaled in magnitude by the amount* $|H(j\omega_0)|$ *and phase-shifted by* $\underline{/H(j\omega_0)}$.
- *Recall that if* $H(\omega)$ *is the frequency response of a stable LTI system, then the input* $v(t) = \cos(\omega_0 t)$ *has output*
$$y(t) = |H(\omega_0)|\cos(\omega_0 t + \underline{/H(\omega_0)})$$
These results are in agreement because $H(\omega) = H(s)|_{s=j\omega} = H(j\omega)$ *when* H *is a stable LTI system There is no transient term because* $\cos(\omega_0 t)$ *is applied from* $t = -\infty$.

Example 9.4.1 (Resonance) *Consider the system with transfer function*

$$H(s) = \frac{64}{s^2+64}$$

The system is marginally stable with purely imaginary poles $p = \pm j8$. *Applying a sinusoidal input* $v(t) = \cos(8t)u(t)$, *the output has LT*

$$\begin{aligned} Y(s) &= \frac{64}{(s^2+64)}\frac{s}{(s^2+64)} \\ &= \frac{-j8}{4}\left(\frac{1}{(s-j8)^2} - \frac{1}{(s+j8)^2}\right) \\ \Rightarrow y(t) &= -j2\left(te^{j8t} - te^{-j8t}\right) \\ &= 4t\sin(8t), \quad t \geq 0 \end{aligned}$$

Clearly $y(t) \to \infty$ *as* $t \to \infty$. *This is called* **resonance**, *where a bounded input produces an unbounded output. This can only occur because the system is marginally stable.*

Theorem 9.4.2 (Response of a discrete-time system) *Let* H *be the transfer function of a stable LTI discrete-time system.*

1. *The system* **step response** *to* $v[n] = u[n]$ *is*
$$y[n] = y_{tr}[n] + H(1), \quad n \geq 0. \tag{9.27}$$
where y_{tr} *is the* **transient response**, *and* $H(1)$ *is the* **steady-state value**.
2. *The system* **sinusoidal response** *to* $v[n] = \cos(\Omega_0 n)u[n]$ *is*
$$y[n] = y_{tr}[n] + y_{ss}[n], \quad n \geq 0. \tag{9.28}$$
where y_{tr} *is the* **transient response**, *and the* **steady-state response** *is*
$$y_{ss}[n] = |H(e^{j\Omega_0})|\cos(\Omega_0 n + \underline{/H(e^{j\Omega_0})}), \quad n \geq 0 \tag{9.29}$$

Example 9.4.2 (Step response) *Consider the discrete-time system with transfer function*

$$H(z) = \frac{1}{z-a}, \quad \text{where } |a| < 1$$

Suppose the input is the unit step input. Recalling the zT pair $u[n] \longleftrightarrow \frac{z}{z-1}$*, the zT of the output is*

$$\begin{aligned} Y(z) &= H(z)U(z) \\ &= \left(\frac{1}{z-a}\right)\left(\frac{z}{z-1}\right) \\ &= \frac{1}{1-a}\left(\frac{-z}{z-a}+\frac{z}{z-1}\right) \\ \Rightarrow \quad y[n] &= \frac{1}{1-a}-\frac{a^n}{1-a}, \quad \text{for } n \geq 0 \\ &= y_{ss}+y_{tr}[n] \end{aligned}$$

9.5 State representations

Consider the N-th order LTI continuous-time system described by the differential equation

$$\frac{d^N y}{dt^N}+\sum_{i=0}^{N-1} a_i \frac{d^i y}{dt^i}=\sum_{i=0}^{N-1} c_i \frac{d^i v}{dt^i} \tag{9.30}$$

where the coefficients a_i and c_i are real numbers. Recall that a **state vector** for a system is a vector of the form

$$x(t)=\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{bmatrix} \tag{9.31}$$

- Together with a known input $v(t)$, the state vector contains sufficient information at any time t to enable the future outputs $y(t)$ to be uniquely determined.
- The variables $x_1,\ldots,x_N$ are called the **state variables** of the system, and N is the **dimension** of the state.
- In Chapter 8 we discussed the state representation of systems in the form (9.30) with $c_i=0$ for all $i \geq 1$, see (8.3). Here we consider the more general case where the input also includes time derivatives of the input signal v.

Definition 9.5.1 (State representation for a strictly proper N-th order continuous-time system) *Let H be the input-output transfer function of a LTI continuous-time system defined by equation 9.30. Let A and C be the real polynomials*

$$\begin{aligned} C(s) &= c_{N-1}s^{N-1}+\cdots+c_1 s+c_0 \\ A(s) &= s^N+a_{N-1}s^{N-1}+\cdots+a_1 s+a_0 \end{aligned}$$

Then the transfer function H can be expressed as the rational function

$$H(s)=\frac{C(s)}{A(s)} \tag{9.32}$$

Introduce the implicit variable w such that $C(s)W(s)=Y(s)$:

$$c_0 w(t)+c_1\frac{dw}{dt}+\cdots+c_{N-1}\frac{d^{N-1}w}{dt^{N-1}}=y(t) \tag{9.33}$$

and also define state variables

$$
\begin{aligned}
x_i(t) &= \frac{dw^{i-1}}{dt^{i-1}}, \quad i = 1, 2, \ldots, N && (9.34)\\
\text{then } x_{i+1}(t) &= \frac{dx_i}{dt}, \quad i = 1, 2, \ldots, N-1 \\
\text{Hence } y(t) &= c_0 x_1(t) + c_1 x_2(t) + \cdots + c_{N-1} x_N(t) && (9.35)
\end{aligned}
$$

$$
\begin{aligned}
\text{Also } V(s) &= \frac{V(s)Y(s)}{Y(s)} \\
&= \frac{A(s)Y(s)}{C(s)} \\
&= A(s)W(s) && (9.36)\\
\text{Hence } v(t) &= a_0 w(t) + a_1 \frac{dw}{dt} + \cdots + a_{N-1} \frac{d^{N-1}w}{dt^{N-1}} + \frac{d^N w}{dt^N} \\
&= a_0 x_1(t) + a_1 x_2(t) + \cdots + a_{N-1} x_N(t) + \dot{x}_N(t) && (9.37)
\end{aligned}
$$

If we write these in matrix form we obtain the **state representation in controller canonical form***:*

$$
\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_N(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{N-1} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} v(t) \tag{9.38}
$$

$$
y(t) = \begin{bmatrix} c_0 & c_1 & \ldots & c_{N-1} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{bmatrix} \tag{9.39}
$$

If we introduce state matrices

$$
A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{N-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} c_0 & c_1 & \ldots & c_{N-1} \end{bmatrix}
$$

then we can write the system representation in matrix form as

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bv(t) \\
y(t) &= Cx(t)
\end{aligned} \tag{9.40}
$$

For brevity we sometimes use (A, B, C) to denote the state matrices in the system representation.

Example 9.5.1 *Consider the continuous-time system described by*

$$
\frac{d^2y}{dt^2} + 4\frac{dy}{dt} - 8y(t) = -\frac{dv}{dt} + v(t)
$$

Assuming zero initial conditions we have the Laplace transform pairs

$$\frac{dx}{dt} \longleftrightarrow sX(s), \quad \frac{d^2x}{dt^2} \longleftrightarrow s^2X(s)$$

which gives

$$Y(s)[s^2+4s-8] = V(s)[-s+1]$$

so the input-output transfer function for the system is

$$H(s) = \frac{Y(s)}{V(s)} = \frac{-s+1}{s^2+4s-8}$$

To obtain a state representation in controller canonical form, we introduce polynomials

$$\begin{aligned} C(s) &= c_0 + c_1 s = 1 - s \\ A(s) &= a_0 + a_1 s + s^2 = -8 + 4s + s^2 \end{aligned}$$

and let the implicit variable $w(t)$ be such that $C(s)W(s) = Y(s)$, which is equivalent to

$$w(t) - \frac{dw}{dt} = y(t)$$

Next we define state variables

$$x_1(t) = w(t),\ x_2(t) = \dot{x}_1(t),$$

Hence

$$y(t) = x_1(t) - x_2(t)$$

Also $V(s) = A(s)W(s)$, so

$$\begin{aligned} v(t) &= a_0 w(t) + a_1 \dot{w}(t) + \ddot{w}(t) \\ &= a_0 x_1(t) + a_1 x_2(t) + \dot{x}_2(t) \end{aligned}$$

which yields

$$\dot{x}_2(t) = v(t) + 8x_1(t) - 4x_2(t)$$

Hence the state representation in controller form is

$$\begin{aligned} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 8 & -4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v(t) \\ y(t) &= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{aligned}$$

Definition 9.5.2 (State representation for a strictly proper N-th order discrete-time system) *Consider the N-th order LTI discrete-time system described by the difference equation*

$$y[n+N] + \sum_{i=0}^{N-1} a_i y[n+i] = \sum_{i=0}^{N-1} c_i v[n+i], \quad \text{for } n \geq -N \tag{9.41}$$

where the coefficients a_i and c_i are real numbers. Let H be the transfer function from the input y to the output v. Let A and C be the real polynomials

$$C(z) = c_{N-1}z^{N-1} + \ldots c_1 z + c_0 \tag{9.42}$$

$$A(z) = z^N + a_{N-1}z^{N-1} + \ldots a_1 z + a_0 \tag{9.43}$$

The transfer function H can be expressed as the rational function

$$H(z) = \frac{C(z)}{A(z)} \tag{9.44}$$

Introduce the implicit variable w such that $C(z)W(z) = Y(z)$*:*

$$c_{N-1}w[n+N-1] + \ldots c_1 w[n+1] + c_0 w[n] = y[n] \tag{9.45}$$

and also define state variables

$$\begin{aligned} x_i[n] &= w[n+i-1], \quad i = 1,2,\ldots,N & (9.46)\\ \text{then } x_i[n+1] &= x_{i+1}[n], \quad i = 1,2,\ldots,N-1 & (9.47)\end{aligned}$$

It can be shown that $V(z) = A(z)W(z)$*, and hence*

$$v[n] = a_0 x_1[n] + a_1 x_2[n] + \cdots + a_{N-1}x_N[n] + x_N[n+1] \tag{9.48}$$

Writing these in matrix form we obtain the **state representation in controller canonical form***:*

$$x[n+1] = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{N-1} \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \\ \vdots \\ x_N[n] \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} v[n] \tag{9.49}$$

$$y[n] = \begin{bmatrix} c_0 & c_1 & \ldots & c_{N-1} \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \\ \vdots \\ x_N[n] \end{bmatrix} \tag{9.50}$$

Example 9.5.2 *Obtain a state representation for LTI discrete-time system described by the difference equation*

$$y[n+3] = 8v[n+2] - 4v[n+1], \quad \textit{for } n \geq -3$$

Let A and C be the real polynomials

$$\begin{aligned} C(z) &= 8z^2 - 4z \\ A(z) &= z^3 \end{aligned}$$

Then the input-output transfer function for the system is

$$H(z) = \frac{Y(z)}{V(z)} = \frac{8z^2 - 4z}{z^3}$$

Introduce the implicit variable w such that $C(z)W(z) = Y(z)$*:*

$$8w[n+2] - 4w[n+1] = y[n]$$

Also introduce the state variables

$$\begin{aligned} x_1[n] &= w[n], \quad x_2[n] = w[n+1], \quad x_3[n] = w[n+2] \\ \textit{then} \quad x_1[n+1] &= x_2[n] \quad \textit{and } x_2[n+1] = x_3[n] \\ y[n] &= -4x_2[n] + 8x_3[n] \\ V(z) &= A(z)W(z) \\ \Rightarrow v[n] &= w[n+3] = x_3[n+1] \end{aligned}$$

Hence we obtain the state representation

$$\begin{aligned} \begin{bmatrix} x_1[n+1] \\ x_2[n+1] \\ x_3[n+1] \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \\ x_3[n] \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} v[n] \\ y[n] &= \begin{bmatrix} 0 & -4 & 8 \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \\ x_3[n] \end{bmatrix} \end{aligned}$$

Next we consider how to obtain the transfer function from any given state representation. Suppose the system is

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bv(t) \\ y(t) &= Cx(t) \end{aligned} \tag{9.51}$$

Taking Laplace transforms in the first state equation, and assuming zero initial conditions, gives

$$\begin{aligned} sX(s) &= AX(s) + BV(s) \\ \Rightarrow (sI - A)X(s) &= BV(s) \\ \Rightarrow X(s) &= (sI - A)^{-1}BV(s) \end{aligned} \tag{9.52}$$

where I denotes the $N \times N$ identity matrix. Taking Laplace transforms in the second state equation gives

$$\begin{aligned} Y(s) &= CX(s) \\ &= C(sI - A)^{-1}BV(s) \end{aligned} \tag{9.53}$$

The above analysis leads to:

Theorem 9.5.1 *Consider an LTI system with either continuous-time or discrete-time state representation given by*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bv(t) \\ y(t) &= Cx(t) \end{aligned} \quad \text{or} \quad \begin{aligned} x[n+1] &= Ax[n] + Bv[n] \\ y[n] &= Cx[n] \end{aligned} \tag{9.54}$$

Then the corresponding input-output transfer function for the system is

$$H(s) = C(sI - A)^{-1}B \quad \text{or} \quad H(z) = C(zI - A)^{-1}B \tag{9.55}$$

Note that while an LTI system has many different state representations, its transfer function is unique.
To understand the stability of systems in state representation, we need a technical lemma:

Theorem 9.5.2 *Let W be an invertible matrix. Then its inverse is given by*

$$W^{-1} = \frac{\mathrm{cof}(W)^T}{\det(W)} \tag{9.56}$$

where the (i, j)-th entry of the "cofactor" matrix $\mathrm{cof}(W)$ *is defined as* $(-1)^{i+j}\det(W_{ij})$*, where W_{ij} is the matrix W with the i-th row and the j-th column removed.*

Applying this lemma to the expression for the transfer function, we see that

$$H(s) = \frac{C\,\mathrm{cof}(sI - A)^T B}{\det(sI - A)} \tag{9.57}$$

Since the poles of the system are the roots of the denominator polynomial of the transfer function, we obtain:

Theorem 9.5.3 *The poles of a LTI continuous-time or discrete-time system are those $s \in \mathbf{C}$ (or $z \in \mathbf{C}$) such that*

$$\det(sI - A) = 0 \quad or \quad \det(zI - A) = 0 \tag{9.58}$$

Recall the definition of an eigenvalue of a square matrix:

Definition 9.5.3 *Let A be a square matrix. Then $\lambda \in \mathbf{C}$ is an eigenvalue of A if and only if*

$$\det(\lambda I - A) = 0 \tag{9.59}$$

Hence we see that

Theorem 9.5.4 *For any LTI continuous-time or discrete-time system with state representation (A,B,C), the poles of the system are equal to the eigenvalues of A.*

Example 9.5.3 (Transfer function from the state representation) *To obtain the transfer function of the system*

$$\begin{array}{rrcl} & \dot{x}(t) & = & \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} v(t) \\ & y(t) & = & \begin{bmatrix} 3 & 4 \end{bmatrix} x(t) \\ \textit{we compute} & H(s) & = & C(sI-A)^{-1}B \\ & & = & \begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ & & = & \begin{bmatrix} 3 & 4 \end{bmatrix} \dfrac{1}{\det(sI-A)} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ & & = & \begin{bmatrix} 3 & 4 \end{bmatrix} \dfrac{1}{\det(sI-A)} \begin{bmatrix} s+5 \\ -2+2s \end{bmatrix} \\ & & = & \dfrac{11s+7}{(s+1)(s+2)} \end{array}$$

We see that the poles of the system are $p_1 = -1$ and $p_2 = -2$. To compare with the eigenvalues of A, we compute

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda & -1 \\ 2 & \lambda+3 \end{bmatrix} = (\lambda+1)(\lambda+2)$$

As expected, the eigenvalues are -1 and -2.

Theorem 9.5.5

- *An LTI continuous-time system with state representation (A,B,C) is* **stable** *if and only if all the eigenvalues of A are in the left-hand complex plane.*
- *An LTI discrete-time system with state representation (A,B,C) is* **stable** *if and only if all the eigenvalues of A are inside the unit disc.*

9.6 Practice Problems

107. Determine the stability of the following LTI systems.

 (a) $H(s) = \dfrac{s+3}{s^2+3}$

 (b) $H(s) = \dfrac{4s+8}{(s^2+4s+13)(s+4)}$

 (c) $D\dfrac{d^3y}{dt^3} + E\dfrac{d^2y}{dt^2} + C\dfrac{dy}{dt} = kx(t)$, where the constants D, E, C and k are all positive.

108. Determine the stability of the LTI systems with the following impulse response functions.

 (a) $h(t) = [2t^3 - 2t^2 + 3t - 2](u(t) - u(t-10))$

 (b) $h(t) = \sin(2t)$

 (c) $h(t) = e^{-t}\sin(2t)$

 (d) $h(t) = e^{t}\sin(2t)$

109. Suppose a system has input-output transfer function

$$H(s) = \frac{8}{s+4}$$

 (a) Is this system BIBO stable? Give a reason.

 (b) Compute the system zero-state response to the following inputs. In each case, identify the transient and steady-state components of the response.

  i. $v(t) = u(t)$.

  ii. $v(t) = tu(t)$

110. Suppose y is the output to an LTI continuous-time system with input x, with zero initial conditions. Use the input-output transfer function of the system to show that the output of the system to the input $\dot{x}(t)$ is $\dot{y}(t)$.

111. Suppose a system has input-output transfer function

$$H(s) = \frac{32}{s^2+8s+16}$$

 (a) Give an input-output differential equation that describes the system. Use v and y as the input and output variables, respectively.

 (b) Determine whether the system is critically damped, overdamped or underdamped.

 (c) Find the steady-state value of the step response. Hence plot the graph of the step response of the system assuming zero initial conditions. (You do not need to solve for $y(t)$).

 (d) Find the steady-state sinusoidal response when the input is $v(t) = \cos(4t + \pi)$.

112. Repeat Question 111 for the system with input-output transfer function

$$H(s) = \frac{32}{s^2 + 10s + 16}$$

113. Consider the system with input-output transfer function

$$H(s) = \frac{242.5(s+8)}{(s+2)[(s+4)^2+81](s+10)}$$

(a) Identify the poles of the system.

(b) Without computing the actual response, give the general form of the step response.

(c) Determine y_{ss}, the steady-state value of the step response.

114. When the input $v[n] = u[n] - 2u[n-2] + u[n-4]$ is applied to an LTI discrete-time system, the resulting response from zero initial conditions is $y[n] = nu[n] - nu[n-4]$.

(a) Express the signals v and y in terms of the unit pulse function $\delta[n]$.

(b) Use the definition of z-Transform to show that for any integer $q > 0$, $\delta[n-q] \longleftrightarrow z^{-q}$ is a z-transform pair.

(c) Hence find the system transfer function $H(z)$.

115. An LTI discrete-time system is described by the difference equation

$$y[n+2] + y[n] = 2v[n+1] - v[n]$$

(a) Assuming zero initial conditions, obtain the system input-output transfer function $H(z)$ and hence determine whether the system is BIBO stable.

(b) Find the steady-state value of the system step response.

(c) Use z-Transforms to compute the step response and hence verify your answer to (b).

(d) Give an example of a bounded input signal v that yields an unbounded output y. Explain your choice of v.

116. A discrete-time system has unit pulse response given by

$$h[n] = \begin{cases} 0, & n \leq 0 \\ 1, & n = 1 \\ h[n-2] + h[n-1], & n \geq 2 \end{cases}$$

(a) Use the definition of z-Transform to show that the transfer function H of the system satisfies

$$H(z) = z^{-1} + H(z)z^{-2} + H(z)z^{-1}$$

and hence solve for $H(z)$.

(b) Obtain the difference equation that describes this system.

117. An LTI continuous-time system has input-output transfer function

$$H(s) = \frac{1}{s^2 - 3s - 4}$$

(a) Determine the stability of the system.

(b) Introduce suitable state variables to obtain a state representation for this system in controller canonical form.

(c) Obtain a diagonal state representation for the system.

(d) Show that input-output transfer function obtained from the diagonal state representation is $H(s)$.

(e) Use your state representation to confirm your answer for the stability of the system.

118. An LTI continuous-time system is described by the differential equation

$$\frac{d^3y}{dt^3} = 2\frac{d^2v}{dt^2} + 4v(t) \quad \text{for } t \geq 0$$

(a) Obtain the input-output transfer function $H(s)$. Is this system stable?

(b) Introduce state variables to obtain a state representation in controller canonical form for this system.

(c) Use your state representation to verify your answer for $H(s)$. You may use the fact that for any real number $s \neq 0$,

$$\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 0 & 0 & s \end{bmatrix}^{-1} = \begin{bmatrix} 1/s & 1/s^2 & 1/s^3 \\ 0 & 1/s & 1/s^2 \\ 0 & 0 & 1/s \end{bmatrix}$$

119. An LTI discrete-time system has input-output transfer function

$$H(z) = \frac{z+2}{z^2 + 3z}$$

(a) Determine the stability of the system.

(b) Introduce suitable state variables to obtain a state representation for this system in controller canonical form.

(c) Obtain a diagonal state representation for the system.

(d) Show that input-output transfer function obtained from the diagonal state representation is $H(z)$.

(e) Use your state representation to confirm your answer for the stability of the system.

120. We are given two system with state models

$$\begin{aligned} \dot{x}(t) &= A_1 x(t) + B_1 v(t), \quad y(t) = C_1 x(t) \\ \dot{\bar{x}}(t) &= A_2 \bar{x}(t) + B_2 v(t), \quad y(t) = C_2 \bar{x}(t) \end{aligned}$$

where the state matrices are in each case

$$\begin{aligned} A_1 &= \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \, B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \, C_1 = [1 \;\; 2] \\ A_2 &= \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix}, \, B_2 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}, \, C_2 = [0 \;\; 1] \end{aligned}$$

Decide whether the systems are input-output equivalent, i.e does there exist a coordinate transformation P such that $\bar{x} = Px$?

Appendix A

Numerical Answers to Practice Problems

1. (a) $x(t) = \begin{cases} 1, & -1 \le t < 1 \\ -1, & 1 \le t < 3 \\ 0, & \text{elsewhere} \end{cases}$ (b) $x(t) = \begin{cases} -t, & 0 \le t < 1 \\ 1, & 1 \le t < 2 \\ 0, & \text{elsewhere} \end{cases}$

2. (a) $v(t) = t^2 + 3t + 3$. (b) $v_1(t) = (t^2 + 3t + 3)u(t + 0.5)$.

3. (a) $x[n] = \begin{cases} -0.707, & n = -3, -1 \\ -1, & n = -2 \\ 0.707, & n = 1, 3 \\ 1, & n = 2 \\ 0, & n = -4,\ 0,\ 4 \end{cases}$ (b) $x[n] = \begin{cases} 0 & n < 0 \\ 2^n, & n \ge 0 \end{cases}$

(c) $x[n] = \begin{cases} 1, & n = -1, 1 \\ 2, & n = 2 \\ 3, & n = 3 \\ 0, & \text{elsewhere} \end{cases}$ (d) $x[n] = \begin{cases} 2, & n = -1 \\ 1, & n = 1 \\ 0, & \text{elsewhere} \end{cases}$

4. (a) $y[n] = \begin{cases} \frac{1}{N}, & n = 0, 1, 2, \ldots, N-1 \\ 0 & \text{otherwise} \end{cases}$ (b) $y[n] = \begin{cases} 0, & n < 0 \\ \frac{n+1}{N}, & n = 0, 1, 2, \ldots, N-1 \\ 1 & n \ge N \end{cases}$

(c) $y[n] = \begin{cases} 0, & n < 0 \\ \frac{n(n+1)}{2N}, & n = 0, 1, 2, \ldots, N-1 \\ \frac{2n-N+1}{2} & n \ge N \end{cases}$

5. (a) Not periodic (b) Not periodic (c) Not periodic (d) Periodic, $T = 4$.

6. (a) Periodic, $T = 2\pi$ (b) Periodic, $T = \frac{2\pi}{\sqrt{5}}$ (c) Not periodic (d) Periodic, $T = 4$.

7. $x(t) = \begin{cases} 1, & -4 \le t < -2 \text{ and } 0 \le t < 2 \text{ and } 4 \le t < 6 \\ 0, & -6 \le t < -4 \text{ and } -2 \le t < 0 \text{ and } 2 \le t < 4 \end{cases}$

8. (a) Causal, memoryless, nonlinear, time-invariant (b) Causal, memoryless, linear, time-varying (c) Causal, has memory, linear, time-varying (d) Causal, has memory, linear, time-varying

9. (a) Linear (b) Linear (c) Nonlinear (d) Linear (e) Nonlinear

10. (a) Linear (b) Nonlinear (c) Nonlinear (d) Linear

11. $M = -1$, $n_0 = -3$

14. (a) $f_1 = 5\ Hz$. (c) $T_s = 0.1\ s$

15. (a) $y_1(t) = \cos(3t)$ (b) $y_2(t) = \cos(3t-1)$ (c) Not time-invariant

16. (a) Time-invariant (b) Time-varying (c) Time-invariant

19. $c = \dfrac{iy[0]}{[1-(1+i)^{-N}]}$

20. (a) $h[0] = 0,\ h[1] = h[3] = 2,\ h[2] = -2.$
(b) $h[0] = 0 = h[1],\ h[2] = 1, h[3] = -1.5.$
(c) $h[0] = 0,\ h[1] = 1, h[2] = -3/2,\ h[3] = 1/2.$

22. (a) $a_1 = -0.7$ and $b_0 = 0.3.$ (b) $y[n] = 0.3\delta[n] + 0.51\delta[n-1] + 1.34(0.7)^n u[n-2]$

23. (a) $(y[0], y[1], y[2], y[3], y[4], y[5]) = (4, -7, 9, -11, -7, 4)$; $y[n] = 0$ otherwise.
(b) $y[n] = \frac{5}{4}(2^n)$ for $n \le 3$, $y[4] = -12$, $y[5] = 24$, $y[6] = 48$, $y[n] = 0$ otherwise.
(c) $y[n] = \cos(\pi n/3)u[n] - \cos(\pi(n-2)/3)u[n-2]$

24. (a) $(u \star u)[n] = \begin{cases} 0, & n < 0 \\ n+1, & n \ge 0 \end{cases}$ (b) $(x \star x)[n] = \begin{cases} 0, & n < 0 \\ n+1, & 0 \le n \le q \\ 2q-n+1 & q < n \le 2q \\ 0, & n > 2q \end{cases}$ (c) $n \le q$.

25. $y[n] = (5(0.5)^n - 9(0.25)^n)u[n-1]$

26. (a) 1 (b) $e^{-2(t-2)}$.

27. (a) $(x \star v)(t) = \begin{cases} 2t, & 0 \le t \le 1 \\ 2, & 1 \le t \le 3 \\ -2t+8 & 3 \le t \le 4 \\ 0, & t \ge 4 \end{cases}$ (b) $(x \star v)(t) = \begin{cases} -t^2+4t, & 0 \le t \le 2 \\ t^2-8t+16, & 2 \le t \le 4 \\ 0, & t \ge 4 \end{cases}$
(c) $(x \star v)(t) = \begin{cases} 2e^{-t}(1-e^{-t}), & 0 \le t \le 2 \\ 2e^{-t}(1-e^{-2}), & t \ge 2 \end{cases}$

28. (a) $(x \star h)(t) = \begin{cases} 2t, & 1 \le t < 2 \\ 5, & t = 2 \\ 5-t, & 2 < t \le 3 \\ 4-t, & 3 < t \le 4 \\ 0, & \text{elsewhere} \end{cases}$ (b) $(x \star h)(t) = \begin{cases} t+3, & -2 \le t < -1 \\ 4, & t = -1 \\ t+4, & -1 < t \le 0 \\ 2-2t, & 0 < t \le 1 \\ 0, & \text{elsewhere} \end{cases}$

29. (a) $h(t) = \begin{cases} t+2, & t \ge 0 \\ 0, & t < 0 \end{cases}$ (b) For $1 \le t \le 2$, $y(t) = -\frac{t^2}{2} + 3$

30. $y(t) = \frac{-1}{2}(1 - 2e^{-t} + e^{-2t})u(t)$

33. $I\theta[n+2] + (Tk_d - 2I)\theta[n+1] + (I - Tk_d)\theta[n] = T^2\tau[n]$

34. (a) $1.293\cos(3t) - 0.707\sin(3t)$ (b) $0.293\cos(2t) + 2.439\sin(2t)$ (c) $9\cos(\pi t) - 1.732\sin(\pi t)$

35. (a) $1.473\cos(3t+0.5)$ (b) $2.457\cos(2t-1.451)$ (c) $9.165\cos(\pi t + 0.19)$

36. (c) $a_0 = 1$, $a_k = \frac{3}{k\pi}\sin\left(\frac{4\pi k}{3}\right)$, $b_k = \frac{3}{k\pi}[1 - \cos\left(\frac{4\pi k}{3}\right)]$, $k = 1, 2, \ldots$.
$a_1 = -0.8270,\ b_1 = 1.4324,\ a_2 = 0.4135,\ b_2 = 0.7162.$

37. (c) $a_0 = \frac{1}{2}$, $a_k = \begin{cases} 0, & \text{if } k \text{ is even} \\ \frac{4}{k^2\pi^2}, & \text{if } k \text{ is odd} \end{cases}$, $b_k = 0$, $k = 1, 2, \ldots$. $a_1 = 0.4053$, $a_2 = 0$.

38 (c) $a_0 = \frac{\pi}{4}$, $a_k = \frac{(-1)^k - 1}{k^2\pi}$, $k = 1, 2, \ldots$. $b_k = \frac{(-1)^{k+1}}{k}$, $k = 1, 2, \ldots$.
$a_1 = -0.6366,\ a_2 = 0,\ b_1 = 1,\ b_2 = 1.$

39. (a) $f_1(t) = \begin{cases} 0, & t = 0 \\ \frac{1}{t}, & 0 < t < T \end{cases}$, $f_1(t+T) = f_1(t)$.

(b) (a) $f_2(t) = \begin{cases} 0, & t = 0 \\ \cos(\frac{1}{t}), & 0 < t < T \end{cases}$, $f_2(t+T) = f_2(t)$.

(c) $f_3(t) = \begin{cases} 0, & \text{if } t \text{ is a rational number} \\ 1, & \text{if } t \text{ is an irrational number} \end{cases}$, $f_3(t+T) = f_3(t)$.

40. (a) True (b) False (c) False

47. (a) ω_0 in both cases. (b) $\tilde{c}_k = e^{-jk\omega_0 t_0} c_k$, $\hat{c}_k = c_{-k}$.

48. (a) $F_1(\omega) = -\operatorname{sinc}(\dfrac{\omega}{2\pi})e^{-j\omega/2} + \operatorname{sinc}(\dfrac{\omega}{2\pi})e^{-j3\omega/2}$

(b) $F_2(\omega) = 6\operatorname{sinc}(\dfrac{3\omega}{2\pi})e^{-j\omega/2} - 2\operatorname{sinc}^2(\dfrac{\omega}{2\pi})e^{-j\omega}$

(c) $F_3(\omega) = \dfrac{1}{2}[\operatorname{sinc}(\dfrac{\omega+\pi}{2\pi}) + \operatorname{sinc}(\dfrac{\omega-\pi}{2\pi})]$

(d) $F_4(\omega) = \dfrac{2 - 2e^{-1}[\cos(\omega) - \omega\sin(\omega)]}{1+\omega^2}$.

49. $F(\omega) = \frac{1}{\omega^2}[(1 + j\pi\omega)e^{-j\pi\omega} - 1]$

50. (a) $x_1(t) = \frac{1}{2}[\delta(t+4) + \delta(t-4)]$ (b) $x_2(t) = \frac{1}{4}[2\delta(t) - \delta(t+6) - \delta(t-6)]$
(c) $x_3(t) = 1 + \cos(4\pi t)$

51. $G(\omega) = \frac{1}{\omega^2}[(1 + j\pi\omega)e^{-j2\pi\omega} - e^{-j\pi\omega}]$

52. $H(\omega) = \frac{2}{\omega^2}[(1 + j\pi\omega/2)e^{-j\pi\omega/2} - 1]$

53. (a) $x(t) = \frac{1}{2}e^{jt} + \frac{1}{2}e^{-jt} + \frac{1+j\sqrt{3}}{8}e^{j4t} + \frac{1-j\sqrt{3}}{8}e^{-j4t}$
(b) $X(\omega) = \pi[\delta(\omega+1) + \delta(\omega-1)] + \frac{\pi}{4}[(1+j\sqrt{3})\delta(\omega-4) + (1-j\sqrt{3})\delta(\omega+4)]$
(c) $X(\omega) = 2\pi c_1\delta(\omega-1) + 2\pi c_{-1}\delta(\omega+1) + 2\pi c_4\delta(\omega-4) + 2\pi c_{-4}\delta(\omega+4)$

54. (a) $\longleftrightarrow$ (5), (b) $\longleftrightarrow$ (1), (c) $\longleftrightarrow$ (6), (d) $\longleftrightarrow$ (4),
(e) $\longleftrightarrow$ (3), (f) $\longleftrightarrow$ (7), (g) $\longleftrightarrow$ (8), (h) $\longleftrightarrow$ (2).

55. (a) $X_1(\omega) = \frac{e^{-j4\omega/5}}{j\omega+5b}$ (b) $X_2(\omega) = \frac{1}{j(\omega-2)+b}$ (c) $X_3(\omega) = \frac{1}{(j\omega+b)^2}$ (d) $X_4(\omega) = -2\pi e^{-b\omega}u(\omega)$

57. (a) $X_1(\omega) = 2X(-\omega)\cos(\omega)$ (b) $X_2(\omega) = \frac{1}{3}e^{-j2\omega}X(\omega/3)$

58. $X(\Omega) = 2\left[\dfrac{\sin(5\Omega/2)}{\sin((\Omega/2)}e^{-j2\Omega} - \dfrac{\sin(3\Omega/2)}{\sin((\Omega/2)}e^{-j6\Omega}\right]$

60. (a) For $|a| < 1$ (b) $F(\Omega) = \dfrac{1}{1 - ae^{-j\Omega}}$ (c) $|F(\Omega)| = \dfrac{1}{\sqrt{1 - 2a\cos(\Omega) + a^2}}$

62. (a) $V_1(\Omega) = \dfrac{e^{-j5\Omega}}{e^{j\Omega} + b}$ (b) $V_2(\Omega) = \dfrac{1}{e^{-j\Omega} + b}$

(c) $V_3(\Omega) = \dfrac{1}{(e^{j\Omega} + b)^2}$ (d) $V_4(\Omega) = \dfrac{1}{e^{j(\Omega-2)} + b}$

63. (a) $x_1[n] = \frac{1}{j2}(\delta[n+1] - \delta[n-1])$ (b) $x_2[n] = \frac{1}{2}(\delta[n+1] + \delta[n-1])$
(c) $x_3[n] = \frac{1}{4}(\delta[n+2] + 2\delta[n] + \delta[n-2])$ (d) $x_4[n] = \frac{1}{j4}(\delta[n+2] - \delta[n-2])$

64. (b) Even function (d) $P_x(\Omega) = |X(\Omega)|^2$ (e) $P_x(0) = |\sum_{n=-\infty}^{\infty} x[n]|^2$

65. (a) $X_0 = 2, X_1 = 0, X_2 = 2, X_3 = 0$. (b) $X_0 = 0, X_1 = 2 - j2, X_2 = 0, X_3 = 2 + j2$

66. $x[0] = -1, x[1] = 0, x[2] = 1, x[3] = 2, x[n] = 0$, otherwise.

68 (a) $X(\Omega) = 1 + e^{-j\Omega}$ (b) 2 (c) 2 (d) (i) $X_k = 1 + e^{-j2\pi k/N}$ (ii) $|X_k| = |2\cos(\frac{\pi k}{N})|$. As $N \to \infty$, the graph of $|X_k|$ approximates the graph of $|2\cos(\Omega/2)|$ for $0 \le \Omega \le 2\pi$.

71. (a) $y(t) = 3\cos(3t) - 5\sin(6t - \frac{\pi}{6})$ (b) $y(t) = \cos(2t) + \frac{1}{2}\cos(4t) + \frac{1}{3}\cos(6t)$

72. (a) $y(t) = 0.707\cos(t - \frac{\pi}{4})$. (b) $y(t) = 0.707\cos(t)$

73. (a) $|H(\omega)| = \begin{cases} \frac{\omega}{\sqrt{\omega^2+4}}, & \omega \ge 0 \\ \frac{-\omega}{\sqrt{\omega^2+4}}, & \omega < 0 \end{cases}$ (b) $\underline{/H(\omega)} = \begin{cases} \frac{\pi}{2} - \tan^{-1}\left(\frac{\omega}{2}\right), & \omega \ge 0 \\ \frac{-\pi}{2} - \tan^{-1}\left(\frac{\omega}{2}\right), & \omega < 0 \end{cases}$

74. (a) $c_k^y = \begin{cases} 10e^{-j5k\omega_0}c_k^v, & k \ne 0 \\ 0, & k = 0 \end{cases}$ (b) $a = 10, b = 5, c = -20$ (c) $y(t) = 10v(t-5) - 5$

75. (a) $H(\omega) = \dfrac{5}{j\omega L + 5}$ (b) $c_0^x = \frac{20}{\pi}$ (c) $c_0^y = \frac{20}{\pi}$

76. (a) $y(t) = 3x(t+2)$ (b) $a = 1, b = -1, c = -1$. Alternative: $a = 1, b = -1, c = 2$.

78 . (a) $h[n] = \frac{1}{4}\,\text{sinc}(\frac{n}{4})$ (b) (i) $y[n] = \cos(\frac{\pi n}{8})$ (ii) $y[n] = \cos(\frac{\pi n}{16})$ (iii) $y[n] = \frac{1}{2}\,\text{sinc}(\frac{n}{4})$

79. (a) $H(\Omega) = \frac{1.9e^{j\Omega}}{0.9+e^{j\Omega}}$ (b) $h[n] = 1.9(-0.9)^n u[n]$

80. $h_2[n] = p_3[n]$

82. (a) $V(s) = \dfrac{(s+3)e^{-4s/3}}{s^2+15s+63}$ (b) $V(s) = \dfrac{s^2+2s-2}{(s^2+5s+7)^2}$
(c) $V(s) = \dfrac{13s+28}{s^2+5s+7}$ (d) $V(s) = \dfrac{2s^2+4s+4}{s^4+10s^3+47s^2+110s+109}$

85. (a) $x(t) = (-e^{-3t} + 2e^{-4t})u(t)$ (b) $x(t) = (\frac{1}{7} + \frac{2}{\sqrt{7}}e^{-2.5t}\cos(\frac{\sqrt{3}t}{2} - 1.761))u(t)$
(c) $x(t) = (2e^{-t} + e^{-2t} - 9te^{-2t})u(t)$ (d) $x(t) = (\frac{1}{4} + 1.25\cos(2t + 0.9273))u(t)$

86. $X(z) = \dfrac{1 - \left(\frac{b}{z}\right)^N}{1 - \frac{b}{z}}$ $RoC(x) = \mathbf{C} \setminus \{0\}$

87. (a) $V(z) = \dfrac{1}{8z^5 - 2z^4 - z^3}$ (b) $V(z) = \dfrac{\frac{1}{8}(2z^2+z)}{8z^2-2z-1}$
(c) $V(z) = \dfrac{z^2}{(8z^2-2z-1)^2}$ (d) $V(z) = \dfrac{8z^3+z}{(8z^2-2z-1)^2}$

88. (a) $Y(z) = \dfrac{1}{(z-1)^2} + \dfrac{3z^{-2}}{z-1}$ (b) $Y(z) = \dfrac{z^2}{(z-1)^3}$

90. (a) $x_1[n] = \cos(\pi n/2)u[n]$ (b) $x_2[n] = \sin(\pi n/2)u[n]$

92. (a) $x[n] = (-2\delta[n] + \frac{3}{2} + \frac{\sqrt{10}}{2}\cos(\pi n/2 + 1.249))u[n]$
(b) $x[n] = (2.4\delta[n] - 1.6(-.5)^n - 0.8(-0.25)^n)u[n]$

(c) $x[n] = (2\delta[n] + 10.2(0.707)^n \cos(\pi n/4 - 1.7628))u[n]$

(d) $x[n] = (-0.75\delta[n] - 0.5\delta[n-1] + \frac{8}{9}(.5)^n - 0.139(-0.4)^n)u[n]$

93. $\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \frac{-R_1R_2}{L(R_1+R_2)} & \frac{R_1}{L(R_1+R_2)} \\ \frac{-R_1}{C(R_1+R_2)} & \frac{-1}{C(R_1+R_2)} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \frac{R_2}{L(R_1+R_2)} \\ \frac{1}{C(R_1+R_2)} \end{bmatrix} v,\ y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

94. (a) $A = \begin{bmatrix} 0 & 1 \\ -9 & -6 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = [-1\ 0].$

95. (a) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -9 & 0 & -5 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, C = [3\ 0\ 0].$

96. (a) Let x_1 and x_2 be the outputs from the left and right integrators, respectively. Then

$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -5 & -2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix},\ y(t) = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}$

97. $f_1(t) = f_2(t) = 600$, for all $t \geq 0$.

98. (a) $x[1] = \begin{bmatrix} 0 \\ -3 \end{bmatrix}, x[2] = \begin{bmatrix} -1.5 \\ 4.5 \end{bmatrix}, x[3] = \begin{bmatrix} 9 \\ -10.5 \end{bmatrix}.$

(b) $v[0] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, v[1] = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$ is one possible solution.

(c) $v[0] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, v[1] = \begin{bmatrix} 0 \\ -6 \end{bmatrix}$ is one possible solution.

99. $v[0] = 1, v[1] = -3.$

100. $P = \begin{bmatrix} 1 & \frac{-1}{3} \\ 1 & \frac{-2}{3} \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} \frac{1}{3} \\ \frac{-1}{3} \end{bmatrix}.$

Note: The matrices $\bar{A}$ and $\bar{B}$ are not unique.

101. (a) $e^{At} = \begin{bmatrix} 1 & 2(1-e^{-t}) \\ 0 & e^{-t} \end{bmatrix}$ (b) $y(t) = -1 - e^{-t}$ for $t \geq 0$.

103. (a) $A = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = [1\ 0].$

(b) $x(t) = \begin{bmatrix} y_0 + 10v_0(1 - e^{-0.1t}) \\ v_0 e^{-0.1t} \end{bmatrix}$ for $t \geq 0$.

(c) $y(0) = 550(1-e), \dot{y}(0) = 55e.$

(d) $x(0) = \begin{bmatrix} -64.87 \\ 26.49 \end{bmatrix}.$

104. (b) $a_0 = -2, a_1 = -1, a_2 = 2.$ (c) $P = \begin{bmatrix} -1 & -1 & -1 \\ 2 & 1 & -1 \\ -4 & -1 & -1 \end{bmatrix}^{-1}.$

105. (a) $P = \begin{bmatrix} j & \frac{1}{3} \\ -j & \frac{1}{3} \end{bmatrix},\ \bar{A} = \begin{bmatrix} j3 & 0 \\ 0 & -j3 \end{bmatrix},\ \bar{B} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}, \bar{C} = [-j/2\ j/2],$ (b) $y_{zi}(t) = \cos(3t)$.

106. (a) 278 months (b) \$40,039 (c) \$1,500 (d) \$1.450

107. (a) marginally stable (b) stable (c) marginally stable

108. (a) stable (b) marginally stable (c) stable (d) unstable

109. (a) Yes. The pole at $s=-4$ is in the Left-Hand complex plane.
(b)(i) $y(t)=2-2e^{-4t}$, where $y_{ss}(t)=2$, $y_{tr}(t)=-2e^{-4t}$
(b)(ii) $y(t)=2t-\frac{1}{2}+\frac{1}{2}e^{-4t}$, where $y_{ss}(t)=2t-\frac{1}{2}$, $y_{tr}(t)=\frac{1}{2}e^{-4t}$

111. (a) $\frac{d^2y}{dt^2}+8\frac{dy}{dt}+16y(t)=32v(t)$ (b) critically damped
(c) $y_{ss}=2$ (d) $y_{ss}(t)=\cos(4t+\frac{\pi}{2})$

112. (a) $\frac{d^2y}{dt^2}+10\frac{dy}{dt}+16y(t)=32v(t)$ (b) overdamped
(c) $y_{ss}=2$ (d) $y_{ss}(t)=0.8\cos(4t+\frac{\pi}{2})$

113 (a) $-2,\ -4\pm j9,\ -10$ (b) $y(t)=c_1+c_2e^{-2t}+c_3e^{-4t}\cos(9t)+c_4e^{-4t}\sin(9t)+c_5e^{-10t}$
(c) $y_{ss}=1$

114. (a) $v[n]=\delta[n]+\delta[n-1]-\delta[n-2]-\delta[n-3]$, $y[n]=\delta[n-1]+2\delta[n-2]+3\delta[n-3]$.
(c) $H(z)=\dfrac{z^2+2z+3}{z^3+z^2-z-1}$

115. (a) $H(z)=\dfrac{2z-1}{z^2+1}$, not BIBO stable (it is marginally stable). (b) $y_{ss}[n]=\frac{1}{2}$.
(c) $y[n]=\frac{1}{2}+\frac{3}{2}\sin\left(\frac{\pi n}{2}\right)-\frac{1}{2}\cos\left(\frac{\pi n}{2}\right)$, $n\geq 0$
(d) $v[n]=\sin\left(\frac{\pi n}{2}\right)$ or alternatively, $v[n]=\cos\left(\frac{\pi n}{2}\right)$.

116. (a) $H(z)=\dfrac{z}{z^2-z-1}$ (b) $y[n+2]-y[n+1]-y[n]=v[n+1]$

117. (a) unstable (b) State variables: $w(t)=y(t)$, $x_1(t)=w(t)$, $x_2(t)=\frac{dw}{dt}$

$$\begin{aligned}\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} &= \begin{bmatrix}0 & 1\\ 4 & 3\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}v(t)\\ y(t) &= \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}\end{aligned}$$

(c)
$$\begin{aligned}\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} &= \begin{bmatrix}-1 & 0\\ 0 & 4\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}+\begin{bmatrix}\frac{1}{5}\\ \frac{1}{5}\end{bmatrix}v(t)\\ y(t) &= \begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}\end{aligned}$$

118. (a) $H(s)=\frac{2s^2+4}{s^3}$ which is unstable.
(b) Define $w(t)$ such that $2\ddot{w}(t)+4w(t)=y(t)$ and $x_1(t)=w(t)$, $x_2(t)=\dot{w}(t)$, $x_3(t)=\ddot{w}(t)$

$$\dot{x}(t)=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}v(t),\quad y(t)=\begin{bmatrix}4 & 0 & 2\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix}$$

119. (a) unstable (b) State variables: $w[n+1]+2w[n]=y[n],\ x_1[n]=w[n],\ x_2[n]=w[n+1]$

$$\begin{aligned}\begin{bmatrix} x_1[n+1] \\ x_2[n+1]\end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 0 & -3\end{bmatrix}\begin{bmatrix} x_1[n] \\ x_2[n]\end{bmatrix} + \begin{bmatrix} 0 \\ 1\end{bmatrix} v[n] \\ y[n] &= \begin{bmatrix} 2 & 1\end{bmatrix}\begin{bmatrix} x_1[n] \\ x_2[n]\end{bmatrix}\end{aligned}$$

(c) Using the coordinate transformation $\bar{x}=Px$ where $P=\begin{bmatrix} 1 & 1/3 \\ 0 & 1/3\end{bmatrix}$ we obtain

$$\begin{aligned}\begin{bmatrix} \bar{x}_1[n+1] \\ \bar{x}_2[n+1]\end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & -3\end{bmatrix}\begin{bmatrix} \bar{x}_1[n] \\ \bar{x}_2[n]\end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3}\end{bmatrix} v[n] \\ y[n] &= \begin{bmatrix} 2 & 1\end{bmatrix}\begin{bmatrix} \bar{x}_1[n] \\ \bar{x}_2[n]\end{bmatrix}\end{aligned}$$

Remark: In (c) the matrices $\bar{B}$ and $\bar{C}$ are not unique.

120. $H_1(s)=\dfrac{3s-3}{s^2-3s+2}$ while $H_2(s)=\dfrac{3s-27}{s^2-3s+2}$, hence the systems are not input-output equivalent.