GR-M1-1.pdf

M1 GR + Cosmology - 1 - Review of Special Relativity

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Contents

1 Course Preliminaries 3
1.1 Goals: 3
1.2 Recommended textbooks 3

2 Founding principles of SR and essential physical implications 4
2.1 The founding principles 4
2.2 Physical implications 4
2.2.1 Time dilation 4
2.2.2 Lorentz-Fitzgerald Length contraction 5
2.2.3 The 'barn and pole' paradox 6
2.2.4 No universal simultaneity 7

3 Formalism of special relativity 8
3.1 Observers, reference frames and events 8
3.2 The prototypical 4-vector: the displacement vector 8
3.2.1 The components of a displacement 4-vector 8
3.2.2 The displacement 4-vector itself 8
3.3 The Lorentz transformation matrix 9
3.4 Transformation of the $x - ct$ axes 11
3.5 Invariance of space-time 4-volume 11
3.6 The invariant squared interval 11
3.7 Notation: the metric, summation convention, and index raising and lowering 12
3.7.1 The Minkowski metric 12
3.7.2 The Einstein summation convention 12
3.7.3 Covariant and contravariant vectors – metric as index raising/lowering operator 13
3.7.4 The scalar product of two 4-vectors 13
3.8 What causes length contraction? 13

4 The 4-velocity and 4-momentum 15
4.1 The 4-velocity 15
4.2 The 4-momentum 15
4.3 The 'cricketers on trains' thought experiment: 16
4.4 Equivalence of mass and energy 17
4.4.1 The relativistic energy-momentum relation 18
4.4.2 Conservation of total 4-momentum and the invariant mass 18
4.4.3 The 4-momentum for massless particles 19

5 Photons and electromagnetic wave packets in special relativity 20
5.1 The 4-gradient of a scalar as a covariant 4-vector 20
5.2 The Doppler Shift 21
5.3 Aberration 22
5.4 Compton scattering 22

6 Transformation of volumes and densities **23**

6.1 Spatial volumes and space-density of particles . . . 23
6.2 Transformation of momentum space volume . . . 24
6.3 Phase-space density and phase-space volume invariance . . . 24

7 Continuity of particle number, energy and momentum **26**

7.1 Introductory remarks . . . 26
7.2 Particle number continuity equation . . . 27
7.3 The stress tensor and continuity of energy and momentum . . . 28
7.4 Transformation of the stress tensor under a boost . . . 31

8 Perfect fluids **31**

8.1 Stress-energy for a perfect fluid . . . 31
8.2 ${T^{0\alpha}}_{,\alpha} = 0$ is the first law of thermodynamics . . . 31

9 Relativistic scalar fields **32**

9.1 Review of some elements of mechanics . . . 32
9.1.1 The Lagrangian and the Euler-Lagrange equations: . . . 32
9.1.2 Energy conservation: . . . 33
9.1.3 Hamilton's equations: . . . 34
9.2 The 'scalar elasticity' model for a scalar field: . . . 36
9.2.1 The discrete lattice model: . . . 36
9.2.2 The continuum limit: . . . 36
9.2.3 Time translational invariance: . . . 37
9.2.4 Spatial translational invariance: . . . 39
9.2.5 Some questions: . . . 40
9.3 Transition to a relativistic scalar field . . . 41
9.3.1 Lagrangian, action, equations of motion and stress-tensor . . . 41
9.3.2 What does it mean? . . . 42
9.3.3 More general relativistic field theories . . . 43
9.3.4 Applications of the scalar field . . . 43

10 Additional material: **44**

10.1 Liouville's theorem . . . 44
10.2 Invariance of power . . . 45
10.3 Summary of relativistic invariants . . . 45
10.4 From Hamilton and Jacobi to Dirac and Feynman . . . 45

List of Figures

1 Time dilation . . . 5
2 Length contraction . . . 6
3 The barn and pole paradox . . . 7
4 Time-like, space-like and null 4-vectors . . . 12
5 The trains paradox . . . 14
6 Conservation of relativistic 3-momentum . . . 17
7 Mass-energy equivalence . . . 18
8 Relativistic energy-momentum relation . . . 19
9 A null wave packet in space-time . . . 21
10 Why isn't the transverse Doppler effect a blue-shift? (1) . . . 22
11 Why isn't the transverse Doppler effect a blue-shift? (2) . . . 22
12 Relativistic beaming . . . 23
13 Transformation of spatial volumes and space-density . . . 24
14 Transformation of 3-momentum volume elements . . . 25
15 The collisional Boltzmann equation . . . 26
16 Hamiltonian dynamics for a system with 1 degree of freedom . . . 33

GR-M1-1b.pdf

M1 GR + Cosmology - 1b - Covariance and Gauge Invariance of Electromagnetism

Nick Kaiser

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Contents

1 Outline **2**

2 Maxwell's equations in terms of E and B. **3**

3 Comments on transformation of charge and current densities **4**

4 Gauge Invariance in electromagnetism and the gauge principle **4**
- 4.1 The electromagnetic 4-potential . . . 4
- 4.2 Invariance of electromagnetic fields under a gauge transformation . . . 5

5 Classical particle electrodynamics **6**
- 5.1 The Lagrangian and the action . . . 6
- 5.2 The canonical and mechanical 3-momenta . . . 6
- 5.3 The Euler-Lagrange equation . . . 6
- 5.4 $d\mathbf{p}/dt$ and the Faraday tensor . . . 7
- 5.5 The Hamiltonian and the energy-momentum relation . . . 7
- 5.6 Hamilton's equations and dH/dt . . . 7
- 5.7 Covariant equations of motion for the canonical and mechanical 4-momenta . . . 8
- 5.8 Gauge invariance of particle electrodynamics . . . 8
- 5.9 The Hamilton-Jacobi equation . . . 8

6 Wave-mechanics of a charged field **9**
- 6.1 The quantum mechanical wave-function from Hamilton and Jacobi . . . 10
- 6.2 The gauge-covariant derivative . . . 10
- 6.3 Classical wave electrodynamics . . . 10

7 Covariant vs. non-covariant formulation of particle electrodynamics **11**
- 7.1 The components of the Faraday tensor . . . 11
- 7.2 The Lorentz force law and the work equation . . . 11
- 7.3 Maxwell's equations in terms of the Faraday tensor . . . 12
- 7.4 Maxwell's equations in the Lorentz gauge . . . 12
- 7.5 Conservation of electric charge . . . 13
- 7.6 Useful ways to express the 4-current density . . . 13
- 7.7 Transformation of the 4-current under a Lorentz boost . . . 14

8 Liouville's theorem for charged particles **14**

9 The stress-energy tensor in electromagnetism **15**
- 9.1 The stress-energy tensor for charged particles . . . 15
  - 9.1.1 Stress-energy in terms of the 3, 4 or 6 dimensional particle densities . . . 15
  - 9.1.2 Continuity of energy and momentum . . . 15
- 9.2 The stress-energy tensor for EM radiation . . . 16
  - 9.2.1 Energy density of the electromagnetic field . . . 16

9.2.2 Poynting's theorem . . . 16
9.2.3 Maxwell's electromagnetic stress tensor . . . 17
9.2.4 The momentum density of the electromagnetic field . . . 19
9.2.5 The Lagrangian for electromagnetism in the presence of charges . . . 20
9.2.6 The canonical stress-energy tensor for the radiation . . . 21
9.2.7 The symmetric stress-energy tensor for the radiation . . . 21

A The 4-current density in terms of 3, 4 and 6 dimensional particle densities 22
A.1 The 4-current density in terms of the density in 3D space . . . 22
A.2 The 4-current density and the density in 4D spacetime . . . 23
A.3 The 4-current density in terms of the density in 6D phase-space . . . 25

B Continuity of 4-momentum in terms of 3, 4 and 6 dimensional particle densities 25
B.1 Continuity equation in terms of the 3D density . . . 25
B.2 Continuity equation in terms of the 4D density . . . 25
B.3 Continuity equation in terms of the 6D density . . . 26

C The radiation Lagrangian and stress tensor in terms of E and B 26

List of Figures

1 Maxwell's equations . . . 4
2 Charge and current transformation . . . 5
3 The Hamilton-Jacobi equations . . . 9
4 The Aharonov-Bohm effect . . . 11
5 The energy density of electromagnetic fields . . . 17
6 The stress 3-tensor for an electric field . . . 18
7 The stress 3-tensor for a magnetic field . . . 19
8 The Feynman disk paradox . . . 20

1 Outline

Here we continue with our review of special relativity, but now looking at Maxwell's electromagnetism. In outline:

- we start with Maxwell's equations and discuss how the source of the fields – the charge and current densities – transform under Lorentz boosts
- we introduce the electromagnetic 4-potential (a 4-vector) $\vec{A} \longrightarrow A^\mu = (\phi/c, \mathbf{A})$ from which the **E** and **B** fields may be obtained
- we also discuss the ambiguity of $\vec{A}$ coming from invariance of EM under a *gauge transformation*
  - this is useful as we will later see something mathematically very similar, though physically very different, in GR
- we then develop the classical dynamics of relativistic charged particles, starting with their Lagrangian, introducing the *canonical momentum* and the *canonical energy* (the Hamiltonian) and contrast these with their 'mechanical' counterparts
- we obtain the equations of motion (Euler-Lagrange equations); the energy-momentum relation; Hamilton's equations and the 'work equation' and finally the Hamilton-Jacobi equation.
- we next consider some aspects of electromagnetic *wave mechanics*
  - we review how Hamilton-Jacobi leads, following Dirac and Feynman's identification of the classical particle action with the phase of the QM wave-function to the relativistic (and, in the appropriate limit, the non-relativistic) Schrödinger equation for an electrically charged particle

GR-M1-2.pdf

M1 GR + Cosmology - 2 - Historical sketch of GR

Nick Kaiser

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Contents

1 Newtonian gravity **3**
1.1 Newtonian reference frames 3
1.1.1 Inertial reference frames 3
1.1.2 Newtonian frames with gravity 3
1.2 The inverse-square gravitational force 3
1.2.1 Passive, active and inertial mass 4
1.3 The kinetic energy T 4
1.4 The gravitational binding energy 5
1.4.1 The gravitational binding energy is $U = \sum_i \mathbf{r}_i \cdot \mathbf{F}_i$ 5
1.4.2 The gravitational binding energy is $U = -\frac{1}{2}\sum_i \sum_{j\neq i} Gm_i m_j / |\mathbf{r}_i - \mathbf{r}_j|$ 6
1.4.3 The gravitational potential and acceleration fields $\phi(\mathbf{r})$ and $\mathbf{g}(\mathbf{r})$ 6
1.4.4 Potential and gravity for a continuous density field $\rho(\mathbf{r})$ 6
1.5 Poisson's equation and Gauss's law 7
1.6 The potential energy in terms of the gravity 8
1.7 The Newtonian gravitational stress tensor 9
1.8 The tidal field tensor: $\boldsymbol{\nabla}\mathbf{g} = -\boldsymbol{\nabla}\boldsymbol{\nabla}\phi$ 10
1.9 Gravity vs. electrostatics 10
1.10 Measuring the gravitational field 11

2 1905: Special Relativity (SR) **12**
2.1 Principles and main implications 12
2.2 Space-time; events; intervals; Lorentz transformation and the light-cone structure 12
2.3 Scalars, vectors, tensors and the laws of physics 13
2.3.1 Lorentz scalars, 4-vectors and 4-tensors 13
2.3.2 Equations of motion 13
2.3.3 Conservation of continuity laws 13

3 1907-1910: Einstein's "happiest thought" **15**
3.1 What was he thinking about? 15
3.2 The gravitational redshift 15
3.2.1 The tower thought experiment 15
3.2.2 The 'rocket' (or elevator) thought experiment 16
3.3 Einstein's principle of equivalence 16
3.4 Gravitational light-bending 17
3.5 Gravitational time dilation 18
3.6 The parable of the apple 18
3.7 Motion of particles and matter waves in a gravitational field 19
3.7.1 Particle trajectories as extremal paths 19
3.7.2 Matter waves in a gravitational potential 21

4 1915: Einstein's theory of gravity 21
4.1 The space-time manifold 21
4.2 The metric 22
4.3 The covariant derivative 23
4.3.1 Curvilinear coordinates 23
4.3.2 Derivative of a vector field 23
4.4 The curvature tensor 24
4.5 The field equations of GR 25
4.6 Some comments and questions 26

5 Road-Map 27

A Matter-wave packets 28

List of Figures

1 Gravitational binding energy as a sum over pairs of particles 5
2 Proof that $\nabla \cdot \mathbf{g} = 0$ outside a point mass 7
3 The integral form of Gauss's law 8
4 Binding energy in terms of the gravitational acceleration 9
5 Einstein's tower thought experiment and the Pound-Rebka measurement 15
6 Einstein's rocket thought experiment 16
7 Einstein's principle of equivalence 17
8 Gravitational time dilation 19
9 Extremal particle paths 20
10 A scalar field in a potential well 22
11 Curvilinear coordinates 23
12 Riemann curvature tensor 25
13 Matter wave packets 29
14 Phase and group velocities 30

GR-M1-3.pdf

ENS M1 General Relativity - Lecture 3 - Tensor Calculus

Nick Kaiser

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Contents

1 Introduction **2**

2 Tensor calculus in rectilinear coordinates **2**
2.1 Frames of reference 2
2.2 Lorentz scalars 2
2.3 Vectors 2
2.3.1 Basis vectors 3
2.3.2 Transformation of coordinates and basis vectors 3
2.3.3 The norm and scalar product of vectors 4
2.3.4 The scalar products of the basis vectors 5
2.4 1-forms 5
2.4.1 Definition of 1-forms and the contraction 5
2.4.2 Visualising 1-forms and vectors 5
2.4.3 The basis for 1-forms 6
2.4.4 Transformation of 1-form components and basis 1-forms 7
2.4.5 Magnitude and scalar products of 1-forms 7
2.5 Tensors 7
2.5.1 The definition of a $\binom{0}{N}$ tensor 8
2.5.2 Outer products and bases for tensors 8
2.5.3 The metric as a mapping of a vector to a 1-form and vice versa 8
2.5.4 The definition of a $\binom{M}{0}$ tensor 9
2.5.5 The definition of a $\binom{M}{N}$ tensor 9
2.5.6 Raising and lowering tensors with $\mathbf{g}(\ ,\ )$ and $\mathbf{g}^{-1}(\ ,\ )$ 9
2.5.7 Symmetries 10
2.6 Derivatives and path derivatives of vectors and tensors 10

3 Tensor calculus in curvilinear coordinates **11**
3.1 Vectors and 1-forms in curvilinear coordinates 11
3.1.1 Displacement vectors 12
3.1.2 1-forms as gradients of scalar fields 12
3.1.3 Curves and path derivatives of scalars 12
3.1.4 The coordinate basis vectors 13
3.1.5 Coordinate basis 1-forms 14
3.1.6 Visualising basis vectors and basis 1-forms: 14
3.1.7 Orthogonality of the basis vectors and basis 1-forms 15
3.2 The metric in curvilinear coordinates 16
3.2.1 The metric as a way to measure magnitudes of vectors 16
3.2.2 The metric as a mapping between vectors and 1-forms 16
3.2.3 The metric in terms of transformation matrices 17
3.3 The covariant derivative 17
3.3.1 The derivative of the basis vectors: the connection $\Gamma^{\mu}{}_{\alpha\nu}$ 17
3.3.2 The covariant derivative of a vector 18
3.3.3 Parallel transport of a tangent vector along a curve 19

3.3.4 Covariant derivatives of scalars, 1-forms and other tensors 20
3.3.5 Computing the Christoffel symbols from the metric 21
3.3.6 Parallel transport of the 4-momentum 1-form 22
3.4 Useful formulae for the Laplacian and divergence 22
3.5 Concluding comments 23

A An example: Polar coordinates **23**

B Transformation of the covariant derivative of a vector $\nabla\vec{V}$ **24**

C Covariant derivative vs. gauge covariant derivative **24**

List of Figures

1 The 4-momentum 1-form 6
2 Basis vectors and vector components 13
3 Coordinate basis 1-forms 15
4 Basis vectors 1-forms in polar coordinates 18

1 Introduction

This lecture develops the formalism of 'tensor calculus'. We closely follow the approach and terminology of Bernard Schutz's textbook.

The first section deals with special relativity in ordinary rectilinear coordinates. The goal here is to make more precise concepts we have already introduced. We introduce basis vectors and their equivalents for 1-forms and tensors and we show how these things transform under boosts and/or rotations. We also introduce the concept of 1-forms and tensors as scalar valued functions of vectors. We briefly discuss symmetries and finish with the definition the derivatives of vector and tensor fields, along with path derivatives, and the various different notations for these things.

The second section extends this to arbitrary curvilinear coordinate systems, where, in the definition of the derivative – the so-called 'covariant derivative' – we have to worry about how the basis vectors vary with position. We define the 'connection' appearing in the covariant derivative and we show how this is related to, and can be determined explicity from, the metric tensor. All of this is still in the domain of flat (i.e. Minkowskian) space-time. All of the results and constructs, however, are still valid on a curved manifold, and it is formally then a relatively straightforward jump – though a conceptually massive one – to differential geometry in curved space-time.

2 Tensor calculus in rectilinear coordinates

2.1 Frames of reference

We concentrated before on frames of reference of observers that are boosted with respect to each other. More generally, reference frames can be rotated with respect to each other also. So these reference frames form a 6-parameter family determined by the boost velocity $\mathbf{v}$ and the three 'Euler angles' defining the spatial rotations.

The physical quantities we deal with may be classified as scalars, vectors or tensors according to how they transform under such changes of coordinate system.

2.2 Lorentz scalars

Lorentz scalars are measurable quantities whose values are independent of the observer's coordinate frame.

2.3 Vectors

The *prototype 4-vector* is the displacement vector – the space-time separation $\vec{\Delta x}$ between two events – with components, in some observer's frame, $\{\Delta x^\alpha\} = (c\Delta t, \Delta x, \Delta y, \Delta z)$.

ENS M1 General Relativity - Lecture 4 - Space-time curvature

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Contents

1 Space-time curvature and gravity **2**
- 1.1 The manifold: the arena for gravity . . . 2
- 1.2 Coordinates, foliation and distances . . . 2
- 1.3 Locally flat coordinates on a 2D manifold . . . 3
- 1.4 Inertial observers and locally inertial coordinates . . . 4
- 1.5 Measurement of the metric . . . 6
- 1.6 Light-cone structure and its orientability . . . 6

2 Curvature: the breakdown of local flatness **6**
- 2.1 2D locally Euclidean manifold . . . 7
- 2.2 3D Locally Euclidean manifold . . . 7
- 2.3 General N-dimensional manifold . . . 8

3 Parallel transport and covariant differentiation on a manifold **8**

4 Space-time curvature: the gravitational field **11**
- 4.1 Introduction . . . 11
  - 4.1.1 Homogeneous 2-dimensional spaces . . . 11
  - 4.1.2 Definition of the Riemann curvature tensor . . . 12
- 4.2 The Riemann curvature tensor from parallel transport . . . 13
- 4.3 Riemann tensor as the commutator of second covariant derivatives . . . 15
- 4.4 Symmetries of the Riemann tensor . . . 17
- 4.5 Measuring curvature from geodesic deviation . . . 18
  - 4.5.1 The non-covariant geodesic deviation equation . . . 18
  - 4.5.2 The covariant geodesic deviation equation . . . 19

5 The Einstein field equations **20**
- 5.1 The Ricci tensor and Ricci scalar . . . 21
- 5.2 The differential Bianchi identities . . . 21
- 5.3 The Einstein tensor and the Einstein gravitational field equations . . . 21
- 5.4 Solving the field equations . . . 22
  - 5.4.1 Parallels with Newtonian gravity . . . 22
  - 5.4.2 Non-linearity and ‘no prior geometry’ . . . 23
  - 5.4.3 Interpreting the solution . . . 23
  - 5.4.4 Number of physical degrees of freedom . . . 23
- 5.5 What is ‘relative’ about general relativity? . . . 24
- 5.6 The physical nature of space and space-time in GR . . . 24

6 Problems **28**
- 6.1 Parallel transport on the unit sphere . . . 28
- 6.2 Riemann curvature tensor of the unit sphere . . . 29
- 6.3 Rindler space-time . . . 30
- 6.4 Local flatness . . . 31

GR-M1-5.pdf

ENS M1 General Relativity - Lecture 5 - Weak Field Gravity

Nick Kaiser

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Contents

1 Introduction **2**

2 Geometrized units **3**

3 Weak field gravity **4**
3.1 Nearly Minkowskian coordinate systems 4
3.2 Transformation of the weak-field metric 4
3.2.1 Global 'background' Lorentz transformations 4
3.2.2 Raising, lowering and contracting indices of the metric perturbation 5
3.2.3 Gauge transformations 5
3.2.4 Transformation of the metric under a gauge transformation 6
3.3 The curvature in weak-field gravity 6
3.3.1 The linearised Riemann tensor 6
3.3.2 Transformation of the linearised Riemann tensor 7
3.4 The linearised Einstein field equations in the Lorenz gauge 7
3.4.1 The Ricci tensor and scalar 7
3.4.2 The trace-reversed metric perturbation 8
3.4.3 The Einstein tensor 8
3.4.4 The Lorenz or de Donder gauge 8
3.4.5 The field equations in the Lorenz gauge 9
3.5 Comments on gauge transformations in GR 9

4 The weak-field metric for stationary or nearly-stationary sources **10**
4.1 The source term for non-relativistic matter 10
4.2 The weak-field metric for stationary sources 10
4.3 The Newtonian limit metric 11
4.4 The weak-field metric for nearly stationary sources 12

5 The physical implications of the weak-field metric **12**
5.1 The light-cone structure and the coordinate speed of light 12
5.2 Constant-$\mathbf{r}$ observers 13
5.3 The warping of time 13
5.4 The gravitational redshift 14
5.5 Light deflection from the gravitational redshift 15
5.6 The spatial geometry of $t =$ constant (hyper)surfaces 16
5.7 Uniform density sphere 18
5.8 Embedding diagrams for other spherical and cylindrical models 19
5.9 The spatial geometry of $t \neq$ constant hypersurfaces 19

6 Particle motion in weak field gravity **20**
6.1 Equation of motion for non-relativistic particles 20
6.2 Energy and Hamiltonian of non-relativistic particles 21
6.2.1 Newtonian dynamics 21
6.2.2 Correspondence between Newtonian dynamics and weak-field theory 22

6.3 Relativistic particle dynamics . . . 23
6.3.1 Classical mechanics of relativistic particles . . . 23
6.3.2 Classical mechanics of relativistic particles in weak-fields . . . 25
6.3.3 Classical mechanics of relativistic particles in the Newtonian limit metric . . . 26
6.4 The geodesic equation for massless particles . . . 27
6.5 Light deflection from the geodesic equation for massless particles . . . 27
6.6 Gauge invariance of light deflection . . . 28

7 Matter waves in weak-field gravity **29**
7.1 The Klein-Gordon equation in weak-field gravity . . . 30
7.2 The dispersion relation for scalar waves . . . 31
7.3 The wave- and group-velocities for scalar waves . . . 32
7.4 The stress-energy tensor for scalar waves . . . 33
7.5 The analogy with EM waves in a plasma . . . 33
7.6 Focussing of scalar matter waves . . . 34
7.7 The Fresnel scale for matter waves . . . 35
7.8 Speckly nature of scalar DM in the multi-streaming regime . . . 35
7.9 Evolution of classical scalar fields via the Schrödinger equation . . . 36
7.9.1 From Klein-Gordon equation to the Schrödinger equation . . . 36
7.9.2 The 5th conservation law: conservation of particle number . . . 38
7.9.3 Speckles and phase vortices from the Schrödinger perspective . . . 38

A Curvature of $t =$ constant surfaces for a uniform density sphere **38**

B The Madelung equation for scalar fields **39**

C Problems **40**
C.1 Problem: self-focusing of a beam of light . . . 40

List of Figures

1 Light cones . . . 13
2 Constant-**r** observers . . . 13
3 Warping of time . . . 14
4 The gravitational redshift . . . 14
5 Light deflection from the gravitational redshift . . . 16
6 Embedding diagrams . . . 17
7 Potential for a uniform density sphere . . . 18
8 Isothermal sphere embedding diagram . . . 19
9 Gauge invariance of light deflection . . . 28
10 The Eddington eclipse expedition . . . 29
11 Analogy with EM waves in the ionosphere . . . 34
12 Development of a multi-streaming region . . . 36
13 Speckly nature of wave-DM . . . 37
14 Speckles and phase-vortices of the Schrödinger field . . . 38
15 From Schrödinger to Madelung . . . 39

1 Introduction

Einstein's field equations provide a natural relativistic generalisation of Poisson's equation with the mass density ρ replaced by the stress-energy tensor $\mathbf{T}$ as the 'source'.

The analogue of the Newtonian gravitational potential is the metric, and, if we work in a local inertial frame (LIF), the 2nd derivatives of the metric are the Riemann curvature which, like the 2nd derivatives of the Newtonian potential – i.e. the Newtonian tidal field – are observable through their influence on the trajectories of neighbouring particles or photons. This – as we shall see in more detail below – ties down

ENS M1 General Relativity - Lecture 6 - Gravitational Waves

Nick Kaiser

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Contents

1 Introduction **2**

2 Propagation of GWs **2**
- 2.1 Plane-wave solutions . . . 2
- 2.2 The wave amplitude . . . 3
- 2.3 The Lorenz gauge (review) . . . 3
- 2.4 The transverse-traceless gauge . . . 4
- 2.5 Proof that we can choose a gauge such that $A^\alpha{}_\alpha = 0$ and $A_{\alpha\beta}U^\beta = 0$ . . . 5

3 Response of particles and matter to GWs **6**
- 3.1 Geodesic motion: . . . 6
- 3.2 The reality and observability of gravitational waves . . . 7

4 Properties of GWs **7**
- 4.1 Polarisation of gravitational waves . . . 7
- 4.2 Energy density of gravitational waves . . . 8

5 Generation of Gravitational Waves **9**
- 5.1 Generation of EM waves . . . 9
  - 5.1.1 Dipole radiation . . . 9
  - 5.1.2 Quadrupole radiation . . . 10
- 5.2 Quadrupole gravitational radiation . . . 10
- 5.3 Estimate of the expected strain . . . 10

6 Detection of Gravitational Waves **10**
- 6.1 The Hulse-Taylor binary pulsar . . . 11
- 6.2 Interferometric detection of gravitational waves . . . 11
- 6.3 GW detection by 'pulsar timing arrays' . . . 12
  - 6.3.1 Stability of milli-second pulsars . . . 12
  - 6.3.2 The stochastic GW background from merging supermassive BHs . . . 13
  - 6.3.3 Pulsar timing arrays . . . 14

List of Figures

1 Polarisation of gravitational waves . . . 7
2 Half wave plates . . . 8
3 The Hulse-Taylor binary pulsar . . . 11
4 The LIGO interferometer . . . 11
5 LIGO observational results . . . 12
6 A puzzle regarding LIGO . . . 13
7 The Hellings-Downs curve . . . 15
8 Spectrum of pulsar timing residuals from NANOGrav . . . 15

ENS M1 General Relativity - Lecture 7 - Black-Holes and Stellar Structure

Nick Kaiser

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Contents

1 Static spherical space-times **2**

2 The Schwarzschild metric **3**
- 2.1 Relation to the weak field metric 3
- 2.2 Relation to the conventional spherically symmetric line element 3
- 2.3 The light-cone structure 3
- 2.4 Constant r observers 4
- 2.5 Singularity of the metric at $r = 2M$ 4
  - 2.5.1 The tidal field at $r = 2M$ 5

3 Radial orbits in Schwarzschild geometry **7**
- 3.1 The cycloidal solution for bound orbits 7
  - 3.1.1 Trajectories in $r - t$ space 8
- 3.2 The Oppenheimer-Snyder model for BH formation 9
- 3.3 Radial orbits and particle dynamics interior to $r = 2M$ 11
  - 3.3.1 Energy of outgoing particles as seen by infalling observers 11
  - 3.3.2 'Emission' of an outgoing particle 13
  - 3.3.3 'Absorption' of an outgoing particle 13
  - 3.3.4 Relation between the energy and the 'arrow of proper-time' 13
  - 3.3.5 The orientability of the space-time manifold. 14
  - 3.3.6 The fate of matter falling through the event horizon 14

4 Rindler space-time **15**

5 Kruskal-Szekeres coordinates **18**

6 Non-radial orbits and the precession of the perihelion of Mercury **21**
- 6.1 Newtonian orbits 21
- 6.2 Nearly circular relativistic orbits 22
- 6.3 Precession of orbits 23

7 The equations of stellar structure **24**
- 7.1 The field equations 24
- 7.2 The equation of hydrostatic equilibrium 25
  - 7.2.1 Hydrostatic equilibrium in static spherically symmetric space-times 25
  - 7.2.2 Hydrostatic equilibrium from the equivalence principle 25
  - 7.2.3 The acceleration of constant-r observers 26
- 7.3 The other equations of stellar structure 27
  - 7.3.1 The $G_{rr} = 8\pi T_{rr}$ and $G_{00} = 8\pi T_{00}$ equations 27
  - 7.3.2 Stellar structure of stars undergoing nuclear fusion 28
  - 7.3.3 Stellar structure of white dwarfs 28
  - 7.3.4 The exterior solution 28
- 7.4 The Tolman-Oppenheimer-Volkov equation 28

7.5 The meaning of $m(r)$ . . . 29
7.6 Does pressure really gravitate in GR? . . . 29
7.7 Limits to the masses of stars . . . 30
7.8 Gravity in the core of a star . . . 31

A The Schwarzschild metric **32**

List of Figures

1 Light cones in Schwarzschild geometry . . . 4
2 Geodesic deviation and its relation to curvature . . . 6
3 The cycloidal soution for a bound radial orbit . . . 8
4 Infalling radial orbit in Schwarzschild geometry . . . 9
5 Oppenheimer-Snyder model for stellar collapse . . . 11
6 Radial orbits in Schwarzschild coordinates . . . 12
7 Rindler space-time . . . 16
8 The Kruskal-Szekeres diagram . . . 18
9 Orbital precession . . . 23
10 Why enthalpy appears in the equation of hydrostatic equilibrium . . . 27
11 Bombs in a balloon . . . 30

GR-M1-8.pdf

ENS M1 General Relativity - 8 - The Gravitational Action Principle

Nick Kaiser

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Contents

1 Introduction 1

2 The stress-energy tensor for the matter in flat space-time 2
- 2.1 The Noether currents for matter fields in flat space-time . . . 2
- 2.2 Example: the relativistic scalar field . . . 3

3 The gravitational action 4

4 T as the functional derivative of $\mathcal{L}_m$ with respect to the metric 6

5 The pseudo stress-energy tensor for the gravitational field 7
- 5.1 The pseudo-stress tensor in weak field gravity . . . 8
  - 5.1.1 Static weak fields . . . 8
  - 5.1.2 Newtonian particle dynamics from the gravitational action . . . 9
  - 5.1.3 Gravitational waves . . . 10

A Calculational details 10
- A.1 Evaluation of $\tilde{G}^{\mu\nu}$ . . . 10
- A.2 Stress-energy pseudo-tensor . . . 11
  - A.2.1 The Lagrangian density . . . 11

1 Introduction

The route we have taken in developing GR followed that charted by Einstein. We assert that gravitational phenomena are the influence of curvature of the space-time manifold. Requiring agreement with Newtonian theory we are led – aside from possible ambiguities associate with the cosmological constant – to a unique rank-two contraction $\mathbf{G}$ of the curvature tensor that is 'sourced' by the matter stress-tensor $\mathbf{T}$.

A radically different way of thinking about the field equations is that they are the Euler-Lagrange equations arising from the requirement that the *Einstein-Hilbert action*

$$\boxed{S = \int d^4x \sqrt{g} \left(\frac{R}{16\pi\kappa} + \mathcal{L}_m \right)} \tag{1}$$

where $\sqrt{g}$ is shorthand for $\sqrt{-|g|}$, R is the Ricci scalar – a function of the metric $\mathbf{g}$ and its derivatives – and where $\mathcal{L}_m$ is the Lorentz scalar Lagrangian density for the matter fields, which we shall denote here loosely by $\phi(\vec{x})$, be extremised with respect to variation of the metric.

So just as the usual action principle states that, given some space-time $\mathbf{g}$, the matter field configurations $\phi(\vec{x})$ that actually occur in nature are those that extremise the action for the matter, the *gravitational action principle* states that the space-times that actually occur are those that extremise the complete action given above.

As well as being profound, this is practically useful for several reasons. One is that it is a good starting point for thinking about possible modifications to Einstein's gravity (beyond just adding a cosmological term $\Lambda\mathbf{g}$ to $\mathbf{G}$). Another is that this leads to a stress-energy tensor $T^{\mu\nu}$ for the matter that is guaranteed to be symmetric, whereas, as we saw earlier, in electromagnetism for instance, this was not the case. A third

Astro-L3-7.pdf

L3 Astro - Section 7 - Cosmology 1

Nick Kaiser

Contents

1 The Hubble law **4**

2 The essential features of Einstein's theory of gravity **4**
- 2.1 The fundamental principles . . . 4
  - 2.1.1 The Galilean equivalence principle . . . 4
  - 2.1.2 Space-time is everywhere locally the same as in special relativity . . . 5
  - 2.1.3 Matter controls the curvature of space-time via its stress-energy tensor . . . 5
- 2.2 The machinery of general relativity: differential geometry . . . 6
  - 2.2.1 The metric . . . 6
  - 2.2.2 Parallel transport . . . 6
  - 2.2.3 Covariant differentiation . . . 7
  - 2.2.4 The connection . . . 7
  - 2.2.5 Curvature . . . 7
  - 2.2.6 Geodesic deviation . . . 8
  - 2.2.7 Einstein's equations . . . 8
  - 2.2.8 Raychaudhuri's equation . . . 9
  - 2.2.9 The cosmological constant . . . 9
- 2.3 What is *the* gravitational field? . . . 9

3 Homogeneous Expanding Universe Models **10**
- 3.1 The Friedmann, Lemaitre, Robertson & Walker (FLRW) model . . . 10
  - 3.1.1 The cosmological principle . . . 10
  - 3.1.2 The metric and the line element . . . 10
  - 3.1.3 The Friedmann and continuity equation . . . 12
  - 3.1.4 The acceleration equation . . . 13
  - 3.1.5 The 'equation of state' . . . 13
  - 3.1.6 The expansion rate, critical density and the density parameters . . . 14
  - 3.1.7 Solutions of the Friedmann equations . . . 15
- 3.2 Interpretation of observations in FRW models . . . 16
  - 3.2.1 The cosmological redshift - measurement . . . 16
  - 3.2.2 Conformal distance-redshift relation . . . 18
  - 3.2.3 The angular diameter distance . . . 19
  - 3.2.4 The luminosity distance . . . 20
  - 3.2.5 The deceleration parameter . . . 21
- 3.3 The closed FLRW models . . . 23

4 Cosmic flows as a probe of large-scale structure **25**
- 4.1 Measuring distances to galaxies . . . 25
  - 4.1.1 The 'knee' in the luminosity function . . . 25
  - 4.1.2 Applications: the infall to the Virgo supercluster and the Rubin-Ford effect . . . 26
- 4.2 Distance measurement using internal 'scaling relations' of galaxies . . . 26
  - 4.2.1 The Tully-Fisher relation . . . 26
  - 4.2.2 The fundamental plane for elliptical galazies . . . 27
- 4.3 The Hubble expansion rate and the 'great attractor' . . . 28

5 Mapping, and clustering analysis of, the large-scale structure **29**
5.1 3D structure from redshift surveys . . . 29
5.1.1 The cosmic web . . . 29
5.2 Statistical analysis of redshift surveys . . . 29
5.2.1 The 2-point correlation function - definition . . . 30
5.2.2 Estimating the 2-point function . . . 31
5.2.3 The power spectrum of galaxy clustering . . . 32
5.2.4 Higher order n-point functions . . . 33
5.2.5 Biased galaxy clustering . . . 34
5.3 Peculiar velocities and redshift space distortion . . . 34

6 Clusters of galaxies **36**
6.1 Overview of cluster-cosmology . . . 36
6.2 George Abell's cluster catalog . . . 36
6.3 Cluster Masses from Galaxy Motions . . . 37
6.4 Clusters observed in X-rays . . . 38
6.4.1 Thermal bremsstrahlung . . . 39
6.4.2 Thermal bremsstrahlung from galaxy clusters . . . 41
6.4.3 Cooling flows in clusters . . . 41
6.5 Gravitational lensing by galaxy clusters . . . 42
6.5.1 Cluster masses from giant arcs . . . 42
6.5.2 The Einstein radius and the critical surface density . . . 42
6.5.3 Caustics and critical curves . . . 44
6.5.4 The optical depth for strong lensing . . . 45
6.5.5 Amplification bias and quasar galaxy associations . . . 45
6.5.6 Micro-lensing by point masses . . . 46
6.5.7 Weak lensing and the bullet cluster . . . 47
6.6 A simple model for the formation of galaxy clusters . . . 48
6.7 Evolution of the cluster mass function . . . 50
6.7.1 Self-similar evolution . . . 50
6.8 The Sunyaev-Zel'dovich effect . . . 52
6.8.1 The thermal SZ effect . . . 52
6.8.2 The kinematic SZ effect . . . 53

A The Friedmann equation from the Einstein field equation **54**

List of Figures

1 The Hubble diagram over the ages . . . 4
2 2-dimensional manifold . . . 6
3 The line element in polar coordinates . . . 6
4 Parallel transport . . . 7
5 Curvature defined in terms of parallel transport . . . 8
6 Geodesic deviation . . . 8
7 Parallel transport and curvature . . . 11
8 Homogeneous spaces in 2-dimensions . . . 11
9 Cycloidal and hyper-cycloidal solutions of the Friedmann equation . . . 15
10 The cosmological redshift . . . 17
11 Peebles' argument for the cosmological redshift . . . 17
12 An equatorial slice through our universe . . . 19
13 Angular diameter and luminosity distance vs. redshift . . . 20
14 Baryon acoustic oscillations . . . 21
15 Luminosity distance and conservation of photons . . . 21
16 Hubble diagram for type 1a supernovae . . . 22
17 Mass of the closed FLRW model and embedding diagram . . . 24
18 The multiverse cartoon . . . 24

Astro-L3-8.pdf

L3 Astro - Section 8 - Cosmology 2

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Contents

1 Newtonian Cosmology **2**

1.1 Introduction . . . 2

1.2 Radial orbits in the field of a point mass . . . 3

1.2.1 The equation of motion . . . 3

1.2.2 Parametric (cycloid and hyper-cycloid) solution . . . 3

1.2.3 Behaviour at early times . . . 5

1.3 A uniform density expanding dust sphere . . . 5

1.4 Friedmann, continuity and acceleration equations . . . 6

1.4.1 Re-scaled or 'comoving' coordinates . . . 6

1.4.2 The re-scaled energy equation . . . 6

1.5 Properties of the Newtonian matter dominated cosmological models . . . 7

1.5.1 Redshift: Peebles's argument . . . 7

1.5.2 The redshift as a combination of a Doppler and gravitational effects . . . 7

1.5.3 Comoving-distance vs. redshift relation . . . 8

1.5.4 Angular diameter and luminosity distances in Newtonian cosmology . . . 9

2 Understanding curvature **9**

2.1 Why lensing requires curvature: Hoekstra's paradox . . . 9

2.2 The geodesic deviation equation and light-focussing . . . 10

2.2.1 Raychaudhuri's equation . . . 10

2.2.2 Affine path parameterisation . . . 10

2.2.3 Affine distance in FLRW cosmology . . . 11

2.2.4 Light focussing in FLRW cosmology . . . 11

2.3 Spatial Curvature in the Milne Model . . . 12

2.3.1 Introduction . . . 12

2.3.2 Milne's model . . . 12

2.3.3 Milne's metric . . . 13

2.3.4 Affine distance and light-focussing in Milne's model . . . 14

3 The Hot-Big-Bang model **15**

3.1 The radiation dominated era . . . 15

3.2 The effect of pressure on the dynamical evolution of the universe . . . 15

3.2.1 The continuity equation . . . 15

3.2.2 The energy equation . . . 16

3.2.3 The acceleration equation . . . 16

3.3 The thermal history of the universe and big-bang nucleosynthesis . . . 17

4 Problems with FRW Cosmology **18**

4.1 The fine-tuning problem . . . 18

4.2 The horizon problem . . . 19

4.3 The age problem . . . 20

5 Inflation **20**
5.1 Inflation in the early universe . . . 20
5.2 Late Time Inflation and the 'Concordance Model' . . . 23

List of Figures

1 Cycloid and hypercycloid solutions to Friedmann's equation . . . 4
2 Why the gravitational redshift is 'hidden' in cosmology . . . 8
3 Hoekstra's paradox . . . 10
4 The Milne model . . . 13
5 The metric in Milne's model . . . 14
6 Causal structure of the matter dominated FRW model . . . 19
7 CMB photon paths in conformal coordinates . . . 20
8 Potential for the scalar field in 'chaotic' inflation . . . 21
9 The comoving horizon size vs time in the inflationary scenario . . . 23
10 The concordance model . . . 25

1 Newtonian Cosmology

1.1 Introduction

The FLRW models were developed in the 20's, just before the discovery that the Universe we inhabit is expanding, and just after Einstein had created his theory of gravity (GR) within which they are formulated. In 1934 E.A. Milne, in two papers, one with Bill McCrea, noted the close resemblance of the Friedmann equations to those of Newtonian dynamics and stated that *"All of the phenomena observable at the present could have been predicted by the founders of mathematical hydrodynamics in the 18th century, or even by Newton himself"*.

At the same time, and even up to the 60s, the understanding of GR was in a process of development. The Schwarzschild solution for a black-hole, for example, was found in 1916, but the physical meaning of the gravitational radius $r = 2GM/c^2$ was not fully appreciated for decades. In cosmology, the interpretation of the FLRW models was also problematic, particularly in relation to the reason for the cosmological redshift. One finds, in many text-books and articles, statements about the 'expansion of space' and how this causes light to be redshifted. One reads, in Harrison (2000) for instance, that *"expansion redshifts are produced by expansion of space between bodies that are stationary in space"*. In many other works, the fact that light is redshifted in an expanding universe is held to be self-evident. The FLRW metric contains the expansion factor $a(\tau)$, and the metric plays the role, in GR, of the potential in Newtonian gravity. If one formulates Maxwell's equations in FLRW coordinates, one finds a term (containing the Hubble expansion rate $H = \dot{a}/a$) that is widely interpreted as expressing the *'coupling of electromagnetism to the gravitational field'*. But the gravitational field is the *tide* – or the curvature of space-time – not the expansion. The appearance of the additional 'Hubble damping term' is properly seen not to be a physical effect but simply a 'coordinate artefact'.

In the foundation of GR, there is nothing one can really identify as the 'expansion of space' *per se*; all there are are measurable distances, from which one can distill the metric (and which relates these to coordinate separations – which are arbitrary). Distances we measure between galaxies are increasing, and the amount of space in a volume enclosed by a set of galaxies is increasing, and indeed the total amount of space in a closed FRW model is a well defined and is, in general, changing. But nowhere in GR do we find any way of actually measuring the expansion of space *itself.* Analogies are often drawn with expanding rubber balloons, but the expansion of a balloon is something one can measure; mark some points and then measure their distance with a ruler. In GR there is no way to anchor objects to space-time. Indeed, the principle on which the theory is based is that locally space-time is Minkowskian. Such space-time has an absolute sense of *rotation* – if you are rotating with respect to it then you can feel it – but is has no sense of expansion. In cosmology 'stuff' – including radiation – is expanding, but space, of itself, is not. The idea that the expansion of space – or, as it is often said, the 'fabric of space-time' – is a real phenomenon with local physically measurable effects, is a pernicious one. In 1945, Einstein wrote a paper with Ernst Straus entitled

Astro-L3-9.pdf

Cosmology 3 - Cosmological Structure Formation

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Contents

1 Linear perturbation theory **2**
- 1.1 Perturbation growth in the matter dominated era 2
  - 1.1.1 The spherical 'top-hat' (or 'Swiss-cheese') model 2
  - 1.1.2 General (i.e. non-spherical) perturbations 3
  - 1.1.3 Newtonian pressure-free perturbation theory 4
- 1.2 Perturbations with non-vanishing pressure 5
  - 1.2.1 The sound speed and the Jeans' length 5
  - 1.2.2 The equation of motion 6
  - 1.2.3 Adiabatic damping of sound waves 6
  - 1.2.4 Silk damping 6
  - 1.2.5 Sound waves in the radiation dominated era 6
  - 1.2.6 Isentropic vs isocurvature fluctuations 7

2 Scenarios for structure formation **8**
- 2.1 The adiabatic, baryon dominated universe 8
- 2.2 The 'hot dark matter' (HDM) scenario 9
- 2.3 The 'cold dark matter' (CDM) scenario 10

3 Seeds for structure formation **11**
- 3.1 Spontaneous structure formation 11
- 3.2 Structure from topological defects 11
- 3.3 Primordial fluctuogenesis in inflation 11

4 Non-linear structure via N-body simulations **13**
- 4.1 The 'background-plus-perturbation' picture 13
- 4.2 The Dmitriev & Zel'dovich equations 13

5 Anisotropy of the CMB **15**
- 5.1 Temperature anisotropy 15
- 5.2 Polarisation 15

List of Figures

1 The 'Swiss-cheese' model for a density perturbation 2
2 Isentropic and isocurvature perturbations 7
3 Evolution of 'adiabatic perturbations' in a baryon dominated universe 8
4 Transfer function for 'adiabatic perturbations' in a baryon dominated universe 9
5 Density perturbations in a hot dark matter dominated universes 10
6 Density perturbations in dark matter dominated universes 10
7 Density perturbations generated spontaneously by local effects 12
8 The Dmitriev and Zeldovich equations 14
9 Large-angle anisotropy of the CMB 15
10 Observed temperature anisotropy of the CMB 16