Basics of Fourier Analysis

A high level overview and key theorems

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Outline

Outline

Fourier Series

Basics

- Fourier Series apply to functions that are periodic over some period L
- Any square integral, L-periodic function can be expanded into an infinite sum of pure waves of amplitude A_n, discrete frequencies $\frac{2\pi n}{L}$, $n = 1, 2, ...$, and with phase ϕ_n.

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{N} A_n sin\left(\frac{2\pi nx}{L} + \phi_n\right)$$

- The Fourier Series is guaranteed to converge to $f(x)$ at *almost every point*, but for engineering applications periodicity and square integrability are typically considered sufficient conditions for convergence.

Fourier Series

Other ways of writing the series

- Notice that the Fourier expansion can equivalently be written as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n cos\left(\frac{2\pi nx}{L}\right) + b_n sin\left(\frac{2\pi nx}{L}\right)\right)$$

where the coefficients are given as

$$a_n = A_n sin(\phi_n)$$

and

$$b_n = B_n cos(\phi_n)$$

.

Fourier Series

Complex Notation

- An even more convenient form is to use complex exponentials:

$$f(x) = \sum_{n=-N}^{N} c_n e^{i2\pi nx/L}$$

where

$$c_n = \begin{cases} \frac{1}{2}(a_n - ib_n) & \text{for } n > 0 \\ \frac{a_0}{2} & \text{for } n = 0 \\ c^*_{|n|} & \text{for } n < 0 \end{cases}$$

- Notice that the summation using this convention runs form -N to N and that half of the coefficients are redundant (complex conjugates) assuming that f is real-valued

Fourier Series

Solving for coefficients

- How do we solve for the cofficients c_n?
- It is straightforward given orthgonality of each term in the series. That is, note that

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} e^{2\pi inx/L} e^{-2\pi imx/L} = 0 \text{ for } m \neq n$$

- Thus we can compute the coefficients one by one as

$$c_n = \frac{1}{L} \int_{-L/2}^{L/2} f(x) e^{-2\pi nx/L} dx$$

Fourier Series

Solving for coefficients

- The equivalent expressions in terms of sin/cos expansion is

$$a_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x) cos\left(\frac{2\pi nx}{L}\right) dx$$

and

$$b_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x) sin\left(\frac{2\pi nx}{L}\right) dx$$

- Thus, you should think of the complex form as having a real part which is the cosine terms and an imaginary part which is the sin terms.

Fourier Series

Examples

- Seeing a few simple examples will make this more concrete
- Consider the following function

$$f(x) = \frac{x}{\pi} \text{ for } -\pi < x < \pi$$

$$f(x + 2\pi k) = f(x) \text{ for } -\infty < x < \infty \text{ and } k \in Z$$

- Evaluate the integrals

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) cos(nx) dx = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) sin(nx) dx = \frac{2(-1)^{n+1}}{\pi n}, n \geq 1$$

Fourier Series
Examples

- Thus the function f(x) can be expanded as

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}sin(nx)$$

- Turs out that this series, since $f(x)$ is discontinuous at $(2n+1)\pi$, converges to $f(x)$ everywhere except at odd multiples of π, where it converges to zero.
- See Matlab example SawToothFourier.m

Outline

Fourier Transforms

General Concept

- Fourier series are for representing L-periodic functions
- Their distinguishing feature is that only discrete frequencies are required: specifically, $2\pi n/L$. In the case that the period $L = 2\pi$ only integer frequencies are required.
- The Fourier Transform extends this idea to general, non-periodic square integrable functions
- Though we do not derive it here, the derivation is straightforward and involves taking the limit as the period goes to infinity

Fourier Transform

Definition

- The Fourier Trasnform and Inverse Fourier Trasnform Pair are defined as:

$$\hat{f}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-2\pi i \omega t} dt$$

 and

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(\omega) e^{2\pi i \omega t} d\omega$$

- We call $\hat{f}(\omega)$ the Fourier Transform of $f(t)$. Under suitable conditions it is invertible, and we call $f(t)$ the *Inverse Fourier Transform* of $\hat{f}(\omega)$
- Note that different sources define the normalization differently.

Fourier Transform

Examples

- Example Fourier Transform Pairs

$f(t)$	$\hat{f}(\omega)$
1	$\delta(\omega)$
a	$a\delta(\omega)$
1 for $t \in \left[\frac{a}{2}, \frac{a}{2}\right]$	$sinc(a\pi\omega)$
$e^{-\pi t^2}$	$e^{-\pi\omega^2}$
$cos(2\pi st)$	$\frac{1}{2}[\delta(\omega+s)+\delta(\omega-s)]$
$sin(2\pi st)$	$\frac{1}{2}i[\delta(\omega+s)+\delta(\omega-s)]$

Fourier Transform

Some Properties

- Linearity (note $\mathcal{F}(f)$ denotes Fourier Transform of f)

$$\mathcal{F}(f+g) = \mathcal{F}(f) + \mathcal{F}(g)$$

$$\mathcal{F}(af) = a\mathcal{F}(f)$$

- *Energy* Preserving:

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega$$

  where $|\hat{f}(\omega)|^2$ is called the *power spectrum*.

- Power Spectrum is invariant to shifts

$$|\mathcal{F}(f(t))|^2 = |\mathcal{F}(f(t-a))|^2$$

- One of the most important properties of Fourier transforms is the *Convolution Theorem*
- Denote the convolution between two function $f(t)$ and $g(t)$ as

$$f * g \equiv \tilde{f}(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$$

- It can be shown using the definition of the Fourier Transform that

$$\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g) = \hat{f}\hat{g}$$

  i.e. The convolution of two functions is just the product of their Fourier Transforms.
- Thus we have

$$f * g = \mathcal{F}^{-1}(\hat{f}\hat{g})$$

- The convolution theorem is an important property with profound impacts for signal processing
- Convolution is a common operation in signal processing that amounts to em filtering a signal.
- To see this consider our standard weighted running average Define an N point stationary random signal as:

$$\underline{x} = [x_1, x_2, x_3, ..., x_N]$$

and a filter of P non-zero weights

$$\underline{g} = [g_1, g_2, g_3, ..., g_P]$$

Continued ...

- For a running average, e.g. the $\underline{g}$ values would be

$$\underline{g} = [\frac{g_1}{P}, \frac{g_2}{P}, ...]$$

- Consider then a process that filters f using g, using running average as a conceptual model. We would have

$$\begin{aligned}
\tilde{x}_1 &= x_1 g_1 + x_2 g_2 + ... + x_P g_P \\
\tilde{x}_2 &= x_2 g_1 + x_3 g_2 + ... + x_{p+1} g_P \\
&\vdots \\
\tilde{x}_{N-P+1} &= x_{N-P+1} g_1 + x_{N-P+1} g_2 + ... + x_N g_P
\end{aligned}$$

- Using summation notation we have

$$\tilde{f}_i = \sum_{k=1}^{P} x_{k+i-1} g_k, 1 \leq i \leq n - p + 1$$

- While they look plausibly similar, it can be shown that a discrete version of the convolution integral exactly corresponds to our version above. This will be left to homework.
- Note also that we can write the convolution as matrix operation. What are conditions of invertibility?

- Thinking of convolution as an arbitrary weight running average (or difference) operator connects convolution to filtering – i.e. removing and accentuating certain frequencies in a signal.
- The convolution theorem tells us that filtering in the convolution sense is the equivalent of weighting frequency in Fourier space.
- In fact it is often much more intuitive to think of filtering directly in Fourier space – i.e. consider the shape of $\hat{g}(\omega)$ and consider how it weights the Fourier frequencies.

```
n = 500;                %signal length
f=randn(1,n);         %create a signal
m = 2;                   %filter width

% create filter weights, fill in all zeros
g = [ones(1,m)/m zeros(1,n-m)];

%fourier transform of f
fhat = fft(f);
%fourier transform of g
ghat = fft(g);

%apply filter in fourier space
tmp = fhat.*conj(ghat);

%convert back to time domain
ftilde1 = ifft(tmp);

%compare to physical space filter
%note that convol only requires non-zero weights
ftilde2 = conv(f,g(1:m),'valid');
```

Outline

- A third type of Fourier transform deals with discretely sampled, implicitly periodic data, such as the time series we have studied in this class.
- The Discrete Fourier Transform (DFT) can be derived either from the discretized integral or the Fourier Series Formulation.
- In either case the ideas are exactly the same – we represent the points of the times series exactly in this case as a finite sum of discrete frequencies.

$$X_k \equiv \sum_{n=0}^{N-1} x_n e^{2\pi ikn/N}, \text{ for } k = 0, 1, 2, ...N-1$$

Fourier Analysis

A few key points

- The DFT requires only a finite number of discrete frequencies.
- Only a finite number because we are only matching the function at specified points – frequencies higher than $\omega = .5$ are not resolved in the signal. This is called the *Nyquist* frequency.
- Only integer values because the function is implicitly assumed to be periodic of period L.
- A signal with N real values is transformed into N complex Fourier coefficients in general: But there is redundant information – specifically, the first half of the frequences of the complex conjugate of the second half (verify in matlab or R).

Fourier Analysis

A few key points, cont.

- Energy preservation is retained in discrete sense:

$$\sum_{k=1}^{N} |X_k|^2 = \sum_{t=1}^{N} (x_t)^2$$

  assuming x_t has zero mean (otherwise subtract off mean).
- Convolution theorem applies in discrete sense
- Always remember: Fourier transform of a real signal is symmetric: Fourier transform of a symmetric signal is real
- Verify this in matlab – be careful how symmetry is defined. See course example.

- This then connects us to Shumway's discussion
- The *scaled periodigram* is defined by Shumway as:

$$P(j/n) = \left(\frac{2}{n}\sum_{t=1}^{n} x_t cos(2\pi tj/n)\right)^2 + \left(\frac{2}{n}\sum_{t=1}^{n} x_t sin(2\pi tj/n)\right)^2$$

- This is just the square of the discrete Fourier coefficients at each discrete frequency. They can be calculating by projections just as with our Fourier Series. (Shumway mentions that they can be calculated as regressions with perfect fit).

Fourier Analysis

Computing the coefficients

- However, in practice a fast ($O[n \log n]$) algorithm is used called the Fast Fourier Transform. But the result is exactly the same, just saves the cost of n regressions.
- Note also that for a random process x_t the Fourier coefficients are random, and thus the periodigram is as well. We need to take many samples to approximate the true (expected) value.
- This true (expected) value is the *power spectrum*.
- Note also the key point that the Fourier coefficients (a_n and b_n) can be shown to be *pairwise uncorrelated* approximately; This is a key result for deriving many important relations.

Fourier Analysis

Periodogram vs. Sample Spectrum

- Remember that Shumway defines the quantity $P(\omega_j)$, the "scaled periodogram", as

$$P(j/n) = \left(\frac{2}{n}\sum_{t=1}^{n} x_t cos(2\pi tj/n)\right)^2 + \left(\frac{2}{n}\sum_{t=1}^{n} x_t sin(2\pi tj/n)\right)^2$$

- This is actually not exactly the same as, but proportional to the sample power spectrum, which he denotes as $I(\omega_j)$. Using Shumway's notation

$$I(\omega_j) = d_c^2(\omega_j) + d_s^2(\omega_j)$$

  where d_c and d_c are the cosine and sin transforms respectively.
- Thus, it can be shown that $P(\omega_j) = (4/n)I(\omega_j)$;

Fourier Analysis

Convergence of Sample Spectrum to True Spectrum

- Using the property of nearly pairwise uncorrelated normal Fourier coefficients, Shumway shows that confidence intervals can be placed on the sample spectrum as:

$$\frac{2I(\omega_{j:n})}{\chi_2^2(1-\alpha/2)} \leq f(\omega) \leq \frac{2I(\omega_{j:n})}{\chi_2^2(\alpha/2)}$$

- We will use this expression to place confidence internvals on the soi and recruitment data series.

Fourier Analysis

Fourier Transform of autocorrelation function

- Finally, the most important result of Shumway regards the autocorrelation function. We will derive it using the continuous Fourier Transform where the lag, now denoted as τ is continuous.

$$\gamma(\tau) = \int_{-\infty}^{\infty} g(t)g(t+\tau)dt$$

$$\begin{aligned}
\mathcal{F}[\psi(\tau)] &= \int_{-\infty}^{\infty} exp^{-i\omega\tau} \left[\int_{-\infty}^{\infty} g(t)g(t+\tau)dt \right] d\tau \\
&= \int_{-\infty}^{\infty} g(t) \left[\int_{-\infty}^{\infty} g(\tau+t)e^{-i\omega\tau} d\tau \right] dt
\end{aligned}$$

Fourier Analysis

Fourier Transform of autocorrelation function, cont.

Note that the Fourier Transform of $g(\tau + t)$ is $\hat{g}(\omega)e^{i\omega t}$. Thus

$$\mathcal{F}[\psi(\tau)] = \hat{g}(\omega) \int_{-\infty}^{\infty} g(t)e^{i\omega t} dt = |g(\hat{\omega})|^2$$

- This is an extremely important result
- Says that the information contained in the autocorrelation function is the same as that contained in the power spectrum. They just give different perspectives.
- Keep in mind that how to obtain reliable estimates of the power spectrum for a finite signal still needs to be addressed.