# MPCS 58020 2017: Solutions for Homework 1a

April 4, 2017

1. An airplane can safely fly if at least half of its engines are functional. If each engine independently functions with probability $p$, for what values of $p$ is a three-engine plane safer than a five-engine plane?

   **Solution** The three engine plane needs either two or three engines to fly safely. Let $E_3$ be the random variable representing the number of working engines.

$$\begin{aligned} P(E_3 = 3) &= p^3 \\ P(E_3 = 2) &= \binom{3}{2} p^2(1-p) = -3p^3 + 3p^2 \\ P(E_3 = 2, 3) &= -2p^3 + 3p^2 \end{aligned}$$

   For $P(E_3 = 2)$, there are $\binom{3}{2} = 3$ ways of choosing which of the two engines are functional. Given one of those choices, the probability that those two are functional and the third is not is $p^2(1-p)$, since each engine being functional is independent. Said another way $P(E_3)$ is the probability that (engines 1 and 2 are functional and 3 is not OR engines 1 and 3 are functional and 2 is not OR engines 2 and 3 are functional and 1 is not). Since each of the probabilities in the OR are independent, we can add the probabilities, and since they are equal, we can just multiply by 3.

   Similarly the five engine plane needs three, four, or five engines:

$$\begin{aligned} P(E_5 = 5) &= p^5 \\ P(E_5 = 4) &= \binom{5}{4} p^4(1-p) = -5p^5 + 5p^4 \\ P(E_5 = 3) &= \binom{5}{3} p^3(1-p)^2 = 10p^5 - 20p^4 + 10p^3 \\ P(E_5 = 3, 4, 5) &= 6p^5 - 15p^4 + 10p^3 \end{aligned}$$

   The three engine plane is safer when:

$$\begin{aligned} P(E_3 = 2, 3) &> P(E_5 = 3, 4, 5) \\ -2p^3 + 3p^2 &> 6p^5 - 15p^4 + 10p^3 \\ 3 - 2p &> 6p^3 - 15p^2 + 10p \\ 0 &> 6p^3 - 15p^2 + 12p - 3 \\ 0 &> 2p^3 - 5p^2 + 4p - 1 \\ 0 &> (p-1)^2(2p-1) \\ 0 &> 2p - 1 \\ \frac{1}{2} &> p \end{aligned}$$

2. Show that De Morgan's laws are true for sets (using either a formal proof or an informal explanation with diagrams):
   (a) $(A \cup B)^c = A^c \cap B^c$
   (b) $(A \cap B)^c = A^c \cup B^c$

   **Solution** For basic proofs like this, the proof is often just a careful application of definitions. For this reason, it's useful to review definitions before starting (but generally not necessary to include in answers). In this case:

$$\begin{aligned} A \cup B &= \{a | a \in A \text{ or } a \in B\} \\ A \cap B &= \{a | a \in A \text{ and } a \in B\} \\ A^c &= \{a | a \notin A\} \end{aligned}$$

   Some other things to keep in mind:

- In math "A or B" always means "A or B or both". The other informal meaning of "or" must be explicitly stated as "A or B but not both".
- To prove set equality, one must prove that each side is a subset of the other side, which is done by picking any element in the set and showing that it is in the other set.
- Writing good proofs is an art, and what can be assumed without explanation depends on context and intended audience. Lots of textbook proofs are too concise, but it's also possible to bog things down in unnecessary detail. For homework assignment, students are really the intended audience, not the grader or instructor.
- While it's useful to write detailed and clear proofs, it can also be fun to try to prove things in a more concise or elegant way, even if it's less intuitively clear. Just don't do this if you decide to write a textbook or course material.

(a) Let $a \in (A \cup B)^c$. Then $a \notin A \cup B$, by the definition of set complement. Suppose $a \in A$. Then $a \in A \cup B$, which is a contradiction, so $a \in A^c$. Similarly it must be that $a \in B^c$, so $a \in A^c \cap B^c$.

Let $a \in A^c \cap B^c$. Then $a \in A^c$ and $a \in B^c$. By definition, this means that $a \notin A$ and $a \notin B$. Suppose $a \in A \cup B$. Then $a \in A$ or $a \in B$, which is a contradiction, so $a \notin (A \cup B)$. Therefore $a \in (A \cup B)^c$.

Since membership in $(A \cup B)^c$ implies membership in $A^c \cap B^c$ and membership in $A^c \cap B^c$ implies membership in $(A \cup B)^c$, $(A \cup B)^c = A^c \cap B^c$.

(b) If $a \in (A \cap B)^c$, then $a \notin A \cap B$. Suppose $a \in A$. If $a$ is also in $B$, then $a \in A \cap B$, a contradiction, so $a \notin B$ and $a \in B^c$. Similarly if $a \in B$ then $a \in A^c$. If $a$ is in neither, then $a \in A^c$ and $a \in B^c$. Since $a \in A^c$ or $a \in B^c$, $a \in A^c \cup B^c$.

If $a \in A^c \cup B^c$, then $a \in A^c$ or $a \in B^c$, and $a \notin A$ or $a \notin B$. Suppose $a \notin A$. Then $a \notin A \cap B$ and $a \in (A \cap B)^c$. Similarly $a \notin B$ implies $a \in (A \cap B)^c$, which covers both cases.

Since membership in $(A \cup B)^c$ implies membership in $A^c \cap B^c$ and membership in $A^c \cap B^c$ implies membership in $(A \cup B)^c$, $(A \cup B)^c = A^c \cap B^c$.

3. An insurance company writes a policy to the effect that an amount of money $A$ must be paid if some event $E$ occurs within a year. If the company estimates that $E$ will occur within a year with probability $p$, what should it charge the customer so that its expected profit will be 10 percent of $A$?

**Solution** The expected cost for a single customer for a year is $Ap + 0(1 - p) = Ap$. We want profit $P = .1A$, so need to set customer cost per year $C$ such that $P = .1A = C - Ap$, or $C = A(.1 + p)$.

4. Two design teams, one named C and the other N, are asked to separately design a new product within a month. From experience we know that:

(a) The probability that team C is successful is 2/3.
(b) The probability that team N is successful is 1/2.
(c) The probability that at least one team is successful is 3/4.

Assuming that exactly one successful design is produced, what is the probability that it was designed by team N?

**Solution** First define the sample space: each team is either successful or unsuccessful, so there are four possible outcomes. We'll use the notation that a team's letter appears if it was successful, and an $O$ appears if not: $\{OO, CO, ON, CN\}$. Re-writing the information given in formal notation:

$$\begin{aligned} P(CO, CN) &= \frac{2}{3} \\ P(ON, CN) &= \frac{1}{2} \\ P(CO, ON, CN) &= \frac{3}{4} \end{aligned}$$

we are asked to find the conditional probability:

$$P(CO|CO, ON) = \frac{P(CO)}{P(CO, ON)}$$

Using $P(A \cup B) = P(A) + P(B) - P(A \cap B)$:

$$\begin{aligned} P(ON, CO, CN) &= P(ON, CN) + P(CO, CN) - P(CN) \\ \frac{3}{4} &= \frac{2}{3} + \frac{1}{2} - P(CN) \\ P(CN) &= \frac{5}{12} \end{aligned}$$

then

$$P(CO) = P(CO, CN) - P(CN) = \frac{1}{4}$$
$$P(ON) = P(ON, CN) - P(CN) = \frac{1}{12}$$

and

$$\begin{aligned} P(ON|ON, CO) &= \frac{P(ON)}{P(ON, CO)} \\ &= \frac{\frac{1}{4}}{\frac{1}{4} + \frac{1}{12}} = \frac{1}{4} \end{aligned}$$

5. Show that
$$A \cup B \cup C = A \cup (A^c \cap B) \cup (A^c \cap B^c \cap C)$$

   **Solution** Let $x \in A \cup B \cup C$. Then $x \in A$ or $x \in B$ or $x \in C$. If $x \in A$, then $x \in A \cup (A^c \cap B) \cup (A^c \cap B^c \cap C)$. This is still true if $x \in A$ and $x \in B$, if $x \in A$ and $x \in C$ or if $x$ is in all three sets. If $x \notin A$ and $x \in B$, then $x \in (A^c \cap B)$ and therefore in the union on the right hand side. If $x \notin A$ and $x \notin B$ and $x \in C$, then $x \in (A^c \cap B^c \cap C)$, and therefore in the union on the right hand side.

   Let $x \in A \cup (A^c \cap B) \cup (A^c \cap B^c \cap C)$. Then $x \in A$ or $x \in (A^c \cap B)$ or $x \in (A^c \cap B^c \cap C)$. If $x \in A$, then $x \in A \cup B \cup C$. If $x \in (A^c \cap B)$, then $x \in B$ and $x \in A \cup B \cup C$. If $x \in (A^c \cap B^c \cap C)$, then $x \in C$ and $x \in A \cup B \cup C$.

   Since membership in each side implies membership in the other side, the sets must be equal.

6. Pat and Taylor have a date at a given time, and each will arrive at the meeting place with a delay between 0 and 1 hour, with all pairs of delays being equally likely. The first to arrive will wait for 15 minutes and will leave if the other has not yet arrived. What is the probability that they will meet?

   **Solution** In this case it's actually easier to calculate the probability that they do not meet. Imagine a line from 0 to 60, with two labeled points along the line randomly selected. Extend each point into a segment of length 15 to represent the window each person is willing to wait. For each segment, $\frac{45}{60} = \frac{3}{4}$ of the line is outside the segment. In order for them to not meet, Pat must be in that $\frac{3}{4}$ section outside Taylor's line, *and* Taylor must be in that $\frac{3}{4}$ section outside Pat's line. Since these events are independent, the probability of a miss is $\left(\frac{3}{4}\right)^2$. The probability that they do meet is therefore $1 - \left(\frac{3}{4}\right)^2 = \frac{7}{16}$.