MPCS 58020 2017: Homework 3 Solutions

April 30, 2017

1. **Shumway 3.3** Using eq 3.11 from Shumway:

$$
\begin{aligned}
x_t &= -\sum_{j=1}^{\infty} \phi^{-j} w_{t+j} \\
\mathrm{E}(x_t) &= \mathrm{E}\left(-\sum_{j=1}^{\infty} \phi^{-j} w_{t+j}\right) \\
&= -\sum_{j=1}^{\infty} \phi^{-j}\,\mathrm{E}(w_{t+j}) \\
&= -\sum_{j=1}^{\infty} \phi^{-j} 0 \\
&= 0
\end{aligned}
$$

so the process is stationary with zero mean.

For the autocovariance:

$$
\begin{aligned}
\gamma(h) = \mathrm{cov}(x_{t+h}, x_t) &= \mathrm{E}[(x_{t+h} - \bar{x})(x_t - \bar{x})] = \mathrm{E}[x_{t+h} x_t] \\
&= \mathrm{E}\left[\left(-\sum_{j=1}^{\infty} \phi^{-j} w_{t+h+j}\right)\left(-\sum_{k=1}^{\infty} \phi^{-k} w_{t+k}\right)\right] \\
&= \mathrm{E}\left[(\phi^{-1} w_{t+h+1} + \phi^{-2} w_{t+h+2} + \dots)(\phi^{-1} w_{t+1} + \dots + \phi^{-h-1} w_{t+h+1} + \phi^{-h-2} w_{t+h+2} + \dots)\right] \\
&= \mathrm{E}\left[\phi^{-2} w_{t+h+1} w_{t+1} + \dots + \phi^{-h-2} w_{t+h+1}^2 + \phi^{-h-4} w_{t+h+2}^2 + \dots\right] \\
&= \mathrm{E}\left[\phi^{-2} w_{t+h+1} w_{t+1} + \dots + \sum_{j=1}^{\infty} \phi^{-h-2j} w_{t+h+j}^2\right]
\end{aligned}
$$

when taking the expectation, all the terms with $w_i w_j$, $i \neq j$ will become zero, since the w_i are iid. This leaves just the sum:

$$
\begin{aligned}
\gamma(h) &= \mathrm{E}\left[\sum_{j=1}^{\infty} \phi^{-h-2j} w_{t+h+j}^2\right] \\
&= \phi^{-h}\,\mathrm{E}\left[\sum_{j=1}^{\infty} \phi^{-2j} w_{t+h+j}^2\right] \\
&= \phi^{-h} \sum_{j=1}^{\infty} \phi^{-2j}\,\mathrm{E}(w_{t+h+j}^2) \\
&= \sigma_w^2 \phi^{-h} \sum_{j=1}^{\infty} \phi^{-2j} \\
&= \sigma_w^2 \phi^{-h} \phi^{-2} \sum_{j=0}^{\infty} \phi^{-2j} \\
&= \frac{\sigma_w^2 \phi^{-h} \phi^{-2}}{1 - \phi^{-2}}
\end{aligned}
$$

Consider the series $y_t = \phi^{-1} y_{t-1} + v_t$ with $v_t \sim \text{iid}\,\mathrm{N}(0, \sigma_w^2 \phi^{-2})$. Since $|\phi| > 1$, $|\phi^{-1}| < 1$, so y_t is actually a causal AR(1) process with parameters $\phi' = \phi^{-1}$ and is stationary with zero mean. Note that the standard deviation of v_t is $\sigma_w \phi^{-1}$, so by eq 3.7, the autocovariance is:

$$\begin{aligned}\gamma(h) &= \frac{(\sigma_w \phi^{-1})^2 (\phi^{-1})^h}{1-(\phi^{-1})^2} \\ &= \frac{\sigma_w^2 \phi^{-2} \phi^{-h}}{1-\phi^{-2}} \\ &= \frac{\sigma_w^2 \phi^{-h} \phi^{-2}}{1-\phi^{-2}}\end{aligned}$$

2. **Shumway 3.4**

   (a)

$$\begin{aligned}x_t &= 0.80x_{t-1} - 0.15x_{t-2} + w_t - 0.30w_{t-1} \\ (1-0.80B+0.15B^2)x_t &= (1-0.30B)w_t \\ (1-0.30B)(1-0.5B)x_t &= (1-0.30B)w_t \\ (1-0.5B)x_t &= w_t\end{aligned}$$

   This is actually an AR(1) process with $\phi = 0.5$, because of parameter redundancy. It is causal because $|\phi| < 1$.

   (b)

$$\begin{aligned}x_t &= x_{t-1} - 0.50x_{t-2} + w_t - w_{t-1} \\ (1-B+0.50B^2)x_t &= (1-B)w_t\end{aligned}$$

   This is an ARMA(2,1) process. Because the root of $\Theta(z) = (1-z)$ is 1, this is not an invertible process. The roots of $\Phi(z) = 1 - z + 0.5z^2$ are $1 \pm i$, which are both outside the unit circle, so the process is causal.

3. **Shumway 3.6** For the AR(2) model $x_t = -0.9x_{t-2} + w_t$, the autoregressive polynomial is $\Phi(z) = 1 + 0.9z^2$. The roots are $\pm\frac{\sqrt{10}}{3}i$, both of which are outside the unit circle, so the model is causal. By example 3.9 (iii), the ACF has the form $\rho(h) = a|z_1|^{-h}\cos(h\theta + b)$. z_1 is one of the roots so $|z_1| = \frac{\sqrt{10}}{3}$, and θ is defined as the angle of the polar coordinate for $z_1 = |z_1|e^{i\theta}$. In this case, there is no real part, so $\theta = \pi/2$. a and b are determined based on $\rho(0) = 1$ and $\rho(1) = \phi_1/(1-\phi_2) = 0$, since $\phi_1 = 0$. This yields $\rho(0) = 1 = a\cos(b)$ and $\rho(1) = 0 = \frac{3a}{\sqrt{10}}\cos(\pi/2 + b)$. The second equation means $b = 0$, since $\cos(\pi/2) = 0$. Plugging $b = 0$ into the first equation we get $a = 1$, since $\cos(0) = 1$. Therefore the ACF is:

$$\gamma(h) = \left(\frac{\sqrt{10}}{3}\right)^{-h} \cos\left(\frac{\pi}{2}h\right)$$