

# Renormalization Group and Central Limit Theorem

November 8, 2025

Define the random variable  $X_n$ , i.i.d., with probability density function  $p(x)$  (without loss of generality, we assume  $\langle X_n \rangle = 0$ ). Then the distribution of the sum  $X_1 + X_2$  is given by convolution.

**Theorem 1.** *The probability density function  $p_{X_1+X_2}(x)$  is the convolution of  $p$ , i.e.*

$$p_{X_1+X_2}(x) = C[p](x) = p * p(x) = \int dy p(x-y)p(y). \quad (1)$$

*Proof.* The theorem can be seen intuitively: the probability for  $x_1 + x_2 = x$  is the sum of all possibilities where  $x_1 = x_1$  and  $x_2 = x - x_1$ . A mathematically rigorous proof would be to consider the cumulative distribution function  $F(X_1 + X_2 \leq x)$  and then take the derivative.  $\square$

**Corollary 2.** *We know that the Fourier transform of a convolution is a product, so*

$$\mathcal{F}[p_{X_1+X_2}] = (\mathcal{F}[p])^2. \quad (2)$$

**Theorem 3.** *There is a relation between variances:*

$$\sigma_{X_1+X_2} = \sqrt{2}\sigma_X. \quad (3)$$

*Proof.* Since  $X_1$  and  $X_2$  are independent,  $\langle X_1 X_2 \rangle = \langle X_1 \rangle \langle X_2 \rangle = 0$ . Then

$$\sigma_{X_1+X_2} = \sqrt{\langle (X_1 + X_2)^2 \rangle} = \sqrt{\langle X_1^2 \rangle + \langle X_2^2 \rangle} = \sqrt{2}\sigma_X. \quad (4)$$

$\square$

To apply the "coarse-graining" idea of the renormalization group, we must renormalize the parameters (shrink the distribution) so that the variance remains invariant. Define the coarse-graining as

$$T[p](x) = \sqrt{2}C[p](\sqrt{2}x). \quad (5)$$

The  $\sqrt{2}$  factor ensures normalization of the probability distribution.

**Theorem 4.** *Denote  $\hat{f}$  as  $\mathcal{F}[f]$ . After Fourier transformation, we have*

$$\widehat{T[p]}(k) = \hat{p}^2(k/\sqrt{2}). \quad (6)$$

*Proof.*

$$\widehat{T[p]}(k) = \int dx e^{-ikx} \sqrt{2} \int dy p(\sqrt{2}x - y)p(y). \quad (7)$$

Decompose  $e^{-ikx} = e^{-i\frac{k}{\sqrt{2}}(\sqrt{2}x-y+y)}$  and rewrite equation (7):

$$\widehat{T[p]}(k) = \int d(\sqrt{2}x) e^{ik'(\sqrt{2}x-y)} p(\sqrt{2}x - y) \int dy e^{ik'y} p(y). \quad (8)$$

The first integral is simply a translation, so the result is independent of  $y$ . Finally, we obtain

$$\widehat{T[p]}(k) = \hat{p}^2(k') = \hat{p}^2(k/\sqrt{2}). \quad (9)$$

$\square$

**Example 1.** We can apply the above result to compute the Gaussian distribution. We know that

$$\hat{p}^*(k) = \int dx e^{-ikx} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} = e^{-\frac{\sigma^2 k^2}{2}}. \quad (10)$$

Then

$$\widehat{T[p^*]}(k) = e^{-\sigma^2(k/\sqrt{2})^2} = e^{-\frac{\sigma^2 k^2}{2}} = \hat{p}^*(k). \quad (11)$$

This shows that the Gaussian  $p^*$  is indeed a fixed point.

We can consider the linearized theory around the fixed point. Assume  $p = p^* + \epsilon f$ , and that the coarse-graining operator is locally linear (either in position or momentum space), i.e.

$$T[p^* + \epsilon f] = p^* + \epsilon Af + O(\epsilon^2), \quad (12)$$

where  $A$  is a linear operator. We can then try to solve the eigenvalue problem  $Af_n = \lambda_n f_n$ .

**Theorem 5.** The eigenvector  $A\hat{f}_n = \lambda_n \hat{f}_n$  satisfies

$$\hat{f}_n(k) = \frac{2}{\lambda_n} \hat{p}^*(k/\sqrt{2}) \hat{f}_n(k/\sqrt{2}). \quad (13)$$

*Proof.* By Theorem 4 we know

$$\widehat{T[p^* + \epsilon f_n]}(k) = (\hat{p}^* + \epsilon \hat{f}_n)^2(k/\sqrt{2}) = \hat{p}^*(k) + 2\epsilon \hat{p}^* \hat{f}_n(k/\sqrt{2}) + O(\epsilon^2). \quad (14)$$

Comparing the first-order term with the eigenvalue equation immediately gives the desired result.  $\square$

By Theorem 5 we know that  $\hat{f}_n$  must satisfy

$$\hat{f}_n(k) = \frac{2}{\lambda_n} e^{-\frac{\sigma^2 k^2}{4}} \hat{f}_n(k/\sqrt{2}). \quad (15)$$

Let  $\hat{f}_n = e^{-\frac{\sigma^2 k^2}{2}} g_n$ , then

$$g_n(k) = \frac{2}{\lambda_n} g_n(k/\sqrt{2}). \quad (16)$$

The only possible solutions are homogeneous functions. If we also require  $\hat{f}_n$  to be smooth and well-defined for all  $k$ , then we must have

$$g_n(k) = Ck^n, \quad \lambda_n = 2^{1-n/2}, \quad n \in \mathbb{N}. \quad (17)$$

For  $n = 0$  the result is meaningless, since we only rescale the same probability distribution. There are still 2 eigenvectors with  $\lambda \geq 1$ . For  $n = 1$ ,  $\hat{f}_1 = ke^{-\frac{\sigma^2 k^2}{2}} \Rightarrow f_1 \propto \partial_x p^*$ . So

$$p^* + \epsilon f_1 \sim p^* + \epsilon \partial_x p^* \sim p^*(x + \delta), \quad (18)$$

which basically implies a translation of the mean value. For  $n = 2$ ,  $\hat{f}_2 = k^2 e^{-\frac{\sigma^2 k^2}{2}} \sim \partial_\sigma p^*$ , thus

$$p^* + \epsilon f_2 \sim p^* + \epsilon \partial_\sigma p^*, \quad (19)$$

which essentially means a first-order shift in  $\sigma$ . These cases are not interesting, so we will ignore them for now. The first nontrivial perturbation appears at  $n = 3 \Rightarrow \hat{f}_3 = k^3 e^{-\frac{\sigma^2 k^2}{2}}$ , and in real space

$$f_3 \sim \partial_x^3 e^{-\frac{x^2}{2\sigma^2}} = e^{-\frac{x^2}{2\sigma^2}} \left( \frac{2x}{\sigma^4} - \frac{x^3}{\sigma^6} \right). \quad (20)$$

When coarse-graining the system  $l$  times (that means adding together  $2^l X$ ), the amplitude of the  $n$ th eigenvector scales as  $\lambda_n^l$ , so the leading correction should come from  $f_3$ , which has the largest eigenvalue.

**Example 2.** Take coin flips as an example. The following figure shows the difference between the binomial distribution of  $m$  heads in  $N$  flips and the Gaussian distribution with the same mean and variance.

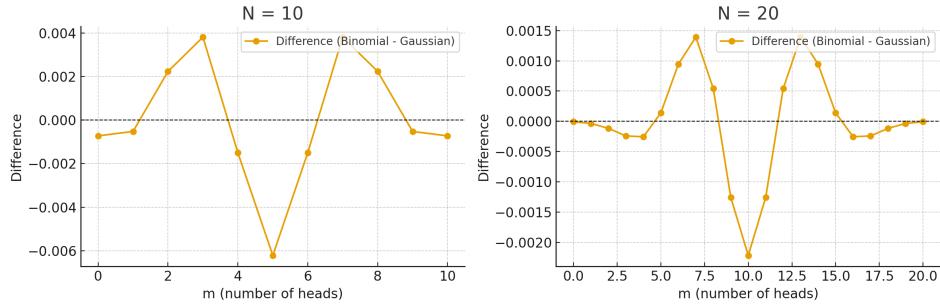


Figure 1: Difference between binomial distribution and Gaussian distribution

We see the deviation is an even function (with respect to the mean) because there is a  $\mathbb{Z}_2$  symmetry head $\rightarrow$ tail in the problem. Therefore, we should compare the deviation with the first-order even-parity perturbation, which is  $f_4 \sim \partial_x^4 e^{-\frac{x^2}{2\sigma^2}}$ .

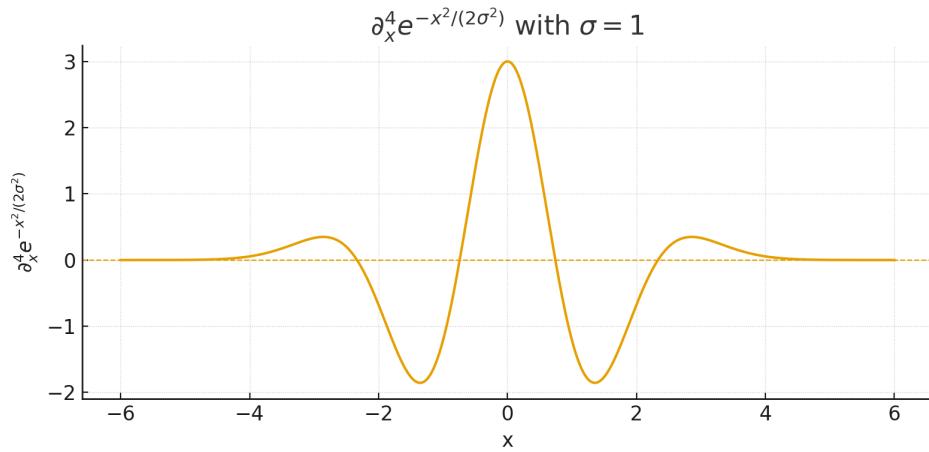


Figure 2: Plot of  $f_4$  with  $\sigma = 1$

Thus, the renormalization group method provides the first-order correction to the problem.