Path Integral

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1 Quantum Mechanics case

Consider 1d propagator in non-relativistic quantum mechanics e^{-iHt} (we have set $\hbar = 1$),we want to compute the matrix element

$$U(q', t'; q, t) = \langle q'| e^{-iH(t'-t)} |q\rangle \tag{1}$$

Suppose H has the form $\frac{p^2}{2m} + V(q)$,we firstly slice the time interval $[t, t']$ into N slices (and let $\Delta t = \frac{t'-t}{N}$),i.e.

$$U(q', t'; q, t) = \langle q'| \prod_{n=1}^{N} e^{-i(\frac{p^2}{2m}+V(q))\frac{t'-t}{N}} |q\rangle \tag{2}$$

The insert the complete relation $\int dq\, |q\rangle \langle q| = \int dp\, |p\rangle \langle p| = 1$

$$U(q', t'; q, t) = \langle q'| \prod_{n=1}^{N} (\int dp_n dq_n e^{-i\frac{p_n^2}{2m}\Delta t} |p_n\rangle \langle p_n|q_n\rangle e^{-iV(q_n)\Delta t} \langle q_n| + O(\Delta t^2)) |q\rangle \tag{3}$$

When $N \to \infty$,the $O(\Delta t^2)$ term can be ignored (i.e. we are essentially consider the Lie algebra,which is commutative w.r.t. addition).Plugging in $\langle p_n|q_n\rangle = (2\pi)^{-1/2} e^{ip_n q_n}$,and notice that $\langle q_n|$ pairs with $|p_{n-1}\rangle$,$\langle q_1|$ pairs with $|q\rangle$,So

$$U(q', t'; q, t) = (2\pi)^{-N} \int dp_N dq_N e^{ip_N(q_N - q')} \prod_{n=1}^{N-1} [\int dp_n dq_n e^{i(p_n q_n - p_n q_{n+1} - V(q_n)\Delta t - \frac{p_n^2}{2m}\Delta t)}] \delta(q_1 - q) \tag{4}$$

Integrate out p_n for each $n \neq N$ gives

$$\int dp_n e^{i(p_n(q_n - q_{n+1}) - \frac{p_n^2}{2m}\Delta t)} = \int dp_n e^{-\frac{i\Delta t}{2m}(p_n - \frac{m}{\Delta t}(q_n - q_{n+1}))^2} e^{\frac{im}{2\Delta t}(q_n - q_{n+1})^2} = (\frac{2m\pi}{i\Delta t})^{1/2} e^{\frac{im}{2\Delta t}(q_n - q_{n+1})^2} \tag{5}$$

Plugging in equation (4)

$$U(q', t'; q, t) = (\frac{m}{2\pi i\Delta t})^{-(N-1)/2} \prod_{n=1}^{N} [\int dq_n e^{i(\frac{m(q_n - q_{n+1})^2}{2\Delta t} - V(q_n)\Delta t)}] \delta(q' - q_N) \delta(q - q_1) \tag{6}$$

The delta functions are essentially fixing end points.Notice that $\frac{m(q_n - q_{n+1})^2}{2\Delta t} \sim \frac{m\dot{q}_n^2}{2}\Delta t$,so the exponential is essentially

$$\frac{m(q_n - q_{n+1})^2}{2\Delta t} - V(q_n)\Delta t = L(q_n, \dot{q}_n)\Delta t \tag{7}$$

Define the path integral measure as

$$\int D[q] = \lim_{N\to\infty} \prod_{n=1}^{N} (\frac{2\pi i(t'-t)}{Nm})^{1/2} \int dq_n \tag{8}$$

Then the propagator is

$$U(q',t';q,t) = \int_{q(t)=q}^{q(t')=q'} D[q] e^{i\int_t^{t'} L[q]dt} = \int_{q(t)=q}^{q(t')=q'} D[q] e^{iS[q]} \tag{9}$$