

Classical Control

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Preface

This first course in control systems is aimed at students in electrical, mechanical, and aerospace engineering. Such students will have taken linear algebra, calculus, physics (mechanics), and electric circuits. Electric circuits is ideal background because its tools include differential equations, Laplace transforms, and transfer functions. This control course is normally given in the third or fourth undergraduate year.

The subject of this first course is classical control. That means the systems to be controlled are single-input, single-output and stable except possibly for a single pole at the origin, and design is done in the frequency domain, usually with Bode diagrams. A second undergraduate course, sometimes called modern control although that term is out of date, would normally be a state-space course, meaning multi-input, multi-output and possibly unstable plants, with controllers being observer based or optimal with respect to a linear-quadratic criterion. The consensus of opinion among educators seems to be that every electrical engineering graduate should know the rudiments of control systems, and that the rudiments reside in classical control.

However, state models, normally relegated to the second course, play an important role in the first third of this course because they are very efficient and unifying. From a nonlinear state model, it is easy to linearize at an equilibrium just by computing Jacobians. It is also natural to introduce the important concept of stability in the state equation framework.

On the other hand, classical control is a course that follows electric circuit theory in the standard curriculum. Circuit theory introduces phasors, i.e., sinusoidal steady state, Laplace transforms, transfer functions, poles and zeros, and Bode plots. Classical control should build on that base, and therefore transfer functions and their poles and zeros play a major role. In particular, pole locations are used, not only in the final-value theorem, but also to characterize boundedness of signals. This is possible because only signals having rational Laplace transforms are considered. This is, of course, a severe restriction, but is justified under the belief that undergraduate engineering students should not be burdened with unfamiliar mathematics just for the sake of generality.

The main topics of the course are modeling of physical systems and obtaining state equations, linearization, Laplace transforms, how poles are related to signal behaviour, transfer functions, stability, feedback loops and stability, the Nyquist criterion, stability margins, elementary loopshaping, and limitations to performance. There is also a case study of a DC-DC converter. The book is intended to be just long enough to cover classical control.

To students

This is a first course on the subject of feedback control. For many of you it will be your one and only course on the subject, and therefore the aim is to give both some breadth and some depth. From this course it is hoped that you learn the following:

- The value of block diagrams.
- How to model a system using differential equations.
- What it means for a system to be linear and time-invariant.
- How to put the model in the form of a state equation.
- How to find equilibria and linearize the model at an equilibrium (if there is one).
- The value of graphical simulation tools like Simulink.
- Why we use transfer functions and the frequency domain.
- The correlation between pole locations and time-domain behaviour.
- What stability means.
- What feedback is and why it is important.
- How to determine if a feedback system is stable by checking poles.
- The beautiful Nyquist stability criterion and the meaning of stability margin.
- How to design a simple feedback loop using frequency-domain methods.
- What makes a system easy or hard to control.

Thanks

This book logically precedes the book *Feedback Control Theory* that I co-wrote with John Doyle and Allen Tannenbaum. I'm very grateful to them for that learning experience.

I first taught classical control at the University of Waterloo during 1982 - 84. In fact, because of the co-op program I taught it (EE380) six times in three years. Thanks to the many UW students who struggled while I learned how to teach. In 1984 I came back to the University of Toronto and I got to teach classical control to Engineering Science students. What a great audience!

Chapter 1

Introduction

Without control systems there could be no manufacturing, no vehicles, no computers, no regulated environment—in short, no technology. Control systems are what make machines, in the broadest sense of the term, function as intended. This course is about the modeling of physical systems and the analysis and design of feedback control systems.

1.1 Familiar examples of control systems

We begin with some examples that everyone knows. A familiar example of a control system is the one we all have that regulates our internal body temperature at 37° C. As we move from a warm room to the cold outdoors, our body temperature is maintained, or regulated. Nature has many other interesting examples of control systems: Large numbers of fish move in schools, their formations controlled by local interactions; the population of a species, which varies over time, is a result of many dynamic interactions with predators and food supplies. And human organizations are controlled by regulations and regulatory agencies.

Other familiar examples of control systems: autofocus mechanism in cameras, cruise control and anti-lock brakes in cars, thermostat temperature control systems. The Google car is perhaps the most impressive example of this type: You get in your car, program your destination, and your car drives you there safely under computer control.

1.2 What is in this book

Systems and control is an old subject. That is because when machines were invented, their controllers had to be invented too, for otherwise the machines would not work. The control engineering that is taught today originated with the invention of electric machines—motors and generators.

Today the subject is taught in the following way. First comes electric circuit theory. That is signals and systems for a specific system. Then comes abstract signals and systems theory. And then come the applications of that theory, namely communication systems and control systems. So one must view the subject of feedback control as part of the chain

electric circuit theory \rightarrow signals and systems \rightarrow feedback control.

The most important principle in control is feedback. This doesn't mean just that a control system has sensors and that control action is based on measured signals. It means that the signal to

be controlled is sensed, compared to a desired reference, and action then taken based on the error. This is the structure of a feedback loop. And because the action is based on the error, instability can result¹ If unstable, the machine is not operable. This raises the fundamental problem of feedback control: To design a feedback controller that acts fast enough and provides acceptable performance, including stability, in spite of unavoidable modelling errors. This, then, is what the course is about:

1. Obtaining mathematical models of physical systems.
2. Having a handle on model errors.
3. Understanding the concept of stability.
4. Having a mathematical tool to test for stability of a feedback loop. For us that is the Nyquist criterion.
5. Having a mathematical tool to test for stability margin.
6. Characterizing desired performance.
7. Designing feedback controllers to obtain good performance if it is possible.

1.3 MATLAB and Scilab

MATLAB, which stands for *matrix laboratory*, is a commercial software application for matrix computations. It has related toolboxes, one of which is the Control Systems Toolbox. Scilab is similar but with the noteworthy difference that it is free. These tools, MATLAB especially, is used in industry for control system analysis and design. You must become familiar with it. It is a script language and easy to learn. There are tutorials on the Web, for example at the Mathworks website.

1.4 Notation

Generally, signals are written in lower case, for example, $x(t)$. Their transforms are capitalized: $X(s)$ is the Laplace transform of $x(t)$. Resistance values, masses, and other physical constants are capitals: R , M , etc. In signals and systems the unit step is usually denoted $u(t)$, but in control $u(t)$ denotes a plant input. Following Zemanian², we denote the unit step by $1_+(t)$.

1.5 Problems

1. Find on the Web three examples of control systems not mentioned in this chapter. Make your list as diverse as possible.
2. Imagine yourself flying a model helicopter with a hand-held radio controller. Draw the block diagram with you in the loop. (We will formally do block diagrams in the next chapter. This problem is to get you thinking, although you are free to read ahead.)

¹Try to balance a pencil vertically up on your hand. You won't be able to do it.

²*Distribution Theory and Transform Analysis*, A.H. Zemanian, Dover

3. A fourth-year student who took ECE311 last year has designed an amazing device. It's a wooden box of cube shape that can balance itself on one of its edges; while it is balanced on the edge, if you tapped it lightly it would right itself. How does it work? That is, draw a schematic diagram of your mechatronic system.
4. Historically, control systems go back at least to ancient Greek times. More recently, in 1769 a feedback control system was invented by James Watt: the flyball governor for speed control of a steam engine. Sketch the flyball governor and explain in a few sentences how it works. (Stability of this control system was studied by James Clerk Maxwell, whose equations you know and love.)
5. The topic of vehicle formation is of current research interest. Consider two cars that can move in a horizontal straight line. Let car #1 be the leader and car #2 the follower. The goal is for car #1 to follow a reference speed and for car #2 to maintain a specified distance behind. Discuss how this might be done.

Solution In broad terms, we need to control the speed of car #1 and then the distance from car #2 to car #1. Regulating the speed of a car at a desired constant value is cruise control. So a cruise controller must be put on car #1. With this in place, on car #2 we need a proximity controller to regulate the speed of car #2 so that the distance requirement is achieved.

Let us fill in some details and bring in some ideas that will come later in the course. The best way to work on a problem like this is to try to get a block diagram. Figure 1.1 has two diagrams. The top one is a schematic diagram where the two cars are represented as boxes on wheels. They are moving to the right. The symbols y_1 and y_2 represent their positions with respect to a reference point; any reference point will do. These positions are functions of time. The symbols u_1 and u_2 represent forces applied to the cars, forces produced by engines of some type. Let the velocities of the two cars be denoted v_1 and v_2 . Presumably, the forces are to be used to achieve the two given specifications, v_1 equals some desired constant v_{ref} , and $y_1 - y_2$ equals some desired value d_{ref} .

Back to Figure 1.1, the bottom diagram is a block diagram, where blocks denote subsystems or functional relationships and where arrows denote signals. Let us begin with the part of the diagram containing the symbol u_1 . This is a force, and you know from Newton's second law that force equals mass times acceleration. So if we integrate force and divide by the mass we get the velocity v_1 . This is shown in the diagram by the block with input u_1 and output v_1 . The desired value of v_1 is v_{ref} . The feedback principle, as we will learn in this course, is to define the velocity error $e_v = v_{ref} - v_1$, and then to take corrective action: reduce e_v if it is positive, increase e_v if it is negative. We can increase or decrease e_v only by increasing or decreasing the force u_1 . This is shown in the block diagram by a block, the controller, with input e_v and output the force u_1 . Finally, the lower loop is a feedback loop whose goal is to get d_y , defined as $y_2 - y_1$, equal to d_{ref} . The controller from e_y to u_2 has to be designed to do that.

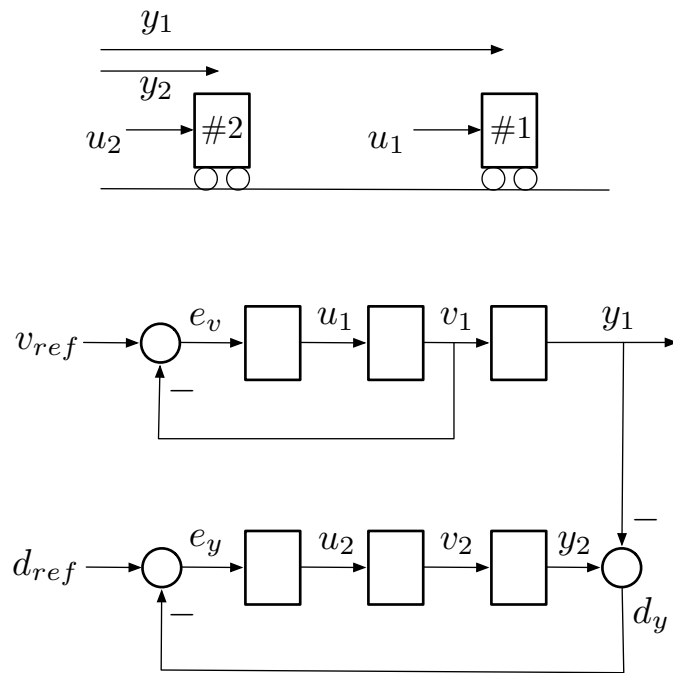


Figure 1.1: The two-cars problem. Top: A schematic diagram of the two cars. Bottom: A block diagram.

Chapter 2

Mathematical Models of Physical Systems

It is part of being human to try to understand the world, natural and human-designed, and our way of understanding something is to formulate a model of it. All of physics—Kepler’s laws of planetary motion, Newton’s laws of motion, his law of gravity, Maxwell’s laws of electromagnetism, the Navier-Stokes equations of fluid flow, and so on—all are models of how things behave. Control engineering is an ideal example of the process of modelling and design of models.

In this chapter we present a common type of model called state equations. The nonlinear form is

$$\begin{aligned}\dot{x} &= f(x, u) \\ y &= h(x, u)\end{aligned}$$

and the linear form is

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du.\end{aligned}$$

Here u, x, y are vectors that are functions of time. The dot, \dot{x} , signifies derivative with respect to time. The components of the vector $u(t)$ are the inputs to the system; an input is an independent signal. The components of $y(t)$ are the outputs, the dependent signals we are interested in. And the vector $x(t)$ is the state; its purpose and meaning will be developed in the chapter. The four symbols A, B, C, D are matrices with real coefficients. These equations are linear, because $Ax + Bu$ and $Cx + Du$ are linear functions of (x, u) , and time-invariant, because the coefficient matrices A, B, C, D do not depend on time.

2.1 Block diagrams

The importance of block diagrams in control engineering cannot be overemphasized. One could easily argue that you don’t understand your system until you have a block diagram of it. This section teaches how to draw block diagrams.

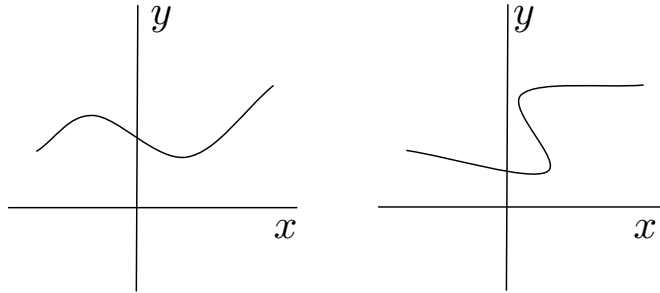


Figure 2.1: The curve on the left is the graph of a function; the curve on the right is not.

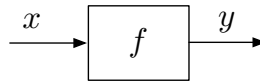


Figure 2.2: The block diagram for the function $y = f(x)$.

1. *Block diagram of a function.* Let us recall what a function is. If X and Y are two sets, a **function** from X to Y is a rule that assigns to each element of X a unique element of Y . The terms function, mapping, and transformation are synonymous. The notation

$$f : X \longrightarrow Y$$

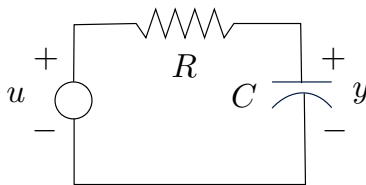
means that X and Y are sets and f is a function from X to Y . We typically write $y = f(x)$ for a function. To repeat, for each x there must be a unique y such that $y = f(x)$; $y_1 = f(x)$ and $y_2 = f(x)$ with $y_1 \neq y_2$ is not allowed.¹ Now let f be a function $\mathbb{R} \longrightarrow \mathbb{R}$. This means that f takes a real variable x and assigns a real variable y , written $y = f(x)$. So f has a graph in the (x, y) plane. For f to be a function, every vertical line must intersect the graph in a unique point, as shown in Figure 2.1. Figure 2.2 shows the block diagram of the function $y = f(x)$. Thus a box represents a function and the arrows represent variables; the input is the independent variable, the output the dependent variable.

2. *Electric circuit example.* Figure 2.3 shows a simple RC circuit with voltage source u . We consider u to be an independent variable and the capacitor voltage y to be a dependent variable. Thinking of the circuit as a system, we view u as the input and y as the output. Let us review how we could compute the output knowing the input. The familiar circuit equation is

$$RC\dot{y} + y = u.$$

Since this is a first-order differential equation, given a time $t > 0$, to compute the voltage $y(t)$ at that time we would need an initial condition, say $y(0)$, together with the input voltage $u(\tau)$

¹The term **multivalued function** is sometimes used if two different values are allowed, but we shall not use that term.

Figure 2.3: An RC circuit.Figure 2.4: Block diagram of RC circuit.

for τ ranging from 0 to t . We could write this symbolically in the form

$$y = f(u, y(0)),$$

which says that the signal y is a function of the signal u and the initial voltage $y(0)$. A block diagram doesn't usually show initial conditions, so for this circuit the block diagram would be Figure 2.4. Inside the box we could write the differential equation or the transfer function, which you might remember is $1/(RCs + 1)$. Here's a subtle point: In the block diagram should the signals be labelled u and y or $u(t)$ and $y(t)$? Both are common, but the second may suggest that y at time t is a function of u only at time t and not earlier. This, of course, is false.

3. *Mechanics example.* The simplest vehicle to control is a cart on wheels. Figure 2.5 depicts the situation where the cart can move only in a straight line on a flat surface. There are two arrows in the diagram. One represents a force applied to the cart; this has the label u , which is a force in Newtons. The direction of the arrow is just a reference direction that we are free to choose. With the arrow as shown, if u is positive, then the force is to the right; if u is negative, the force is to the left. The second arrow, the one with a barb on it, depicts the position of the center of mass of the cart measured from a stationary reference position. The symbol y

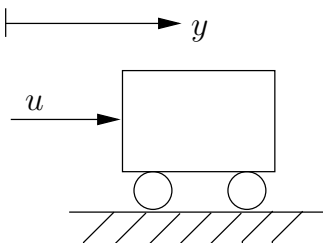
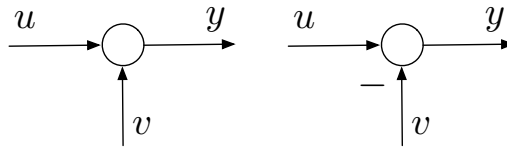


Figure 2.5: A simple vehicle.



Figure 2.6: Block diagram of a cart.

Figure 2.7: The first summing junction stands for $y = u + v$; the second for $y = u - v$.

stands for position in meters. This is a **schematic diagram**, not a block diagram, because it doesn't say which of u , y causes the other. Newton's second law tells us that there's a mathematical relationship between u and y , namely, $u = M\ddot{y}$, that is, force equals mass times acceleration. We take the viewpoint that u is an independent variable, and thus it is viewed as an input. Recall that $M\ddot{y} = u$ is a second-order differential equation. Given the forcing term $u(t)$, we need two initial conditions, say, position $y(0)$ and velocity $\dot{y}(0)$, to be able to solve for $y(t)$. More specifically, for a fixed time $t > 0$, in order to compute $y(t)$ we would need $y(0)$ and $\dot{y}(0)$ and also $u(\tau)$ over the time range from $\tau = 0$ to $\tau = t$. So symbolically we have

$$y = f(u, y(0), \dot{y}(0)).$$

Again, we leave the initial conditions out of the block diagram to get Fig. 2.6. Inside the box we could put the differential equation or the transfer function, which is $1/(Ms^2)$ —the Laplace transform of y divided by the Laplace transform of u . If you don't see this, don't worry—we'll do transfer functions in detail later.

4. *Summing junctions.* Block diagrams also may have summing junctions, as in Figure 2.7. Also, we may need to allow a block to have more than one input, as in Figure 2.8. This means that y is a function of u and v , $y = f(u, v)$.



Figure 2.8: Two inputs.

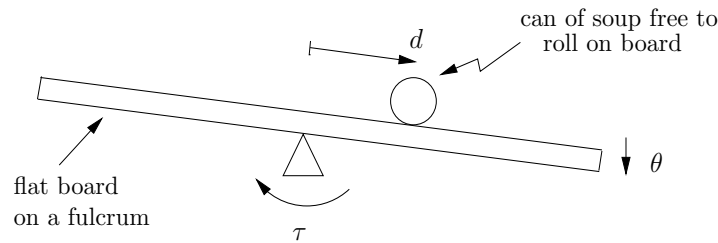


Figure 2.9: A can rolls on a board.

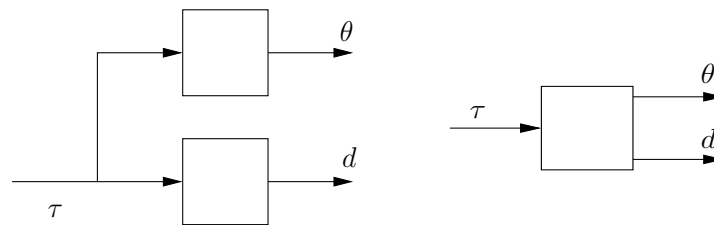


Figure 2.10: One input, two outputs.

5. *Multi-output example.* Figure 2.9 shows a can rolling on a see-saw. Suppose a torque τ is applied to the board. Let θ denote the angle of tilt and d the distance of roll. Then both θ and d are functions of τ . The block diagram could be either of the ones in Figure 2.10.
6. *Water tank.* Figure 2.11 shows a water tank. The arrow indicates water flowing out. Try to draw the block diagram; the answer is in the footnote.²
7. *Cart and motor drive.* This is a harder example. Consider a cart with a motor drive, a DC motor that produces a torque. See Figure 2.12. The input is the voltage u to the motor, the output the cart position y . We want the model from u to y . To model the inclusion of a motor, draw the free body diagram in Figure 2.13. Moving from right to left, we have a force

²There are no input signals. Assuming the geometry of the tank is fixed and considering there is an initial time, say $t = 0$, the flow rate out at any time $t > 0$ depends uniquely on the height of water at time $t = 0$. Therefore, if we let $y(t)$ denote the flowrate out, the block diagram would be a single box, no input, one output labelled y .

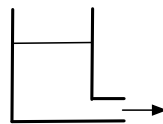


Figure 2.11: Water tank.

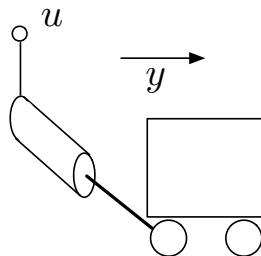


Figure 2.12: A cart with a motor drive.

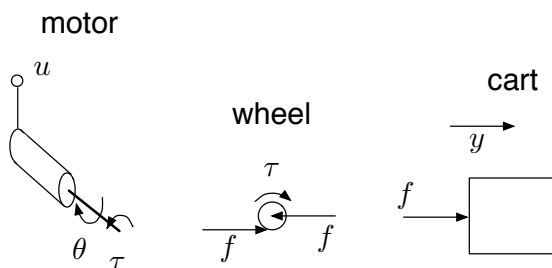


Figure 2.13: Free body diagram.

f on the cart via the wheel through the axle:

$$M\ddot{y} = f.$$

For the wheel, an equal and opposite force f appears at the axle and a horizontal force occurs where the wheel contacts the floor. If the inertia of the wheel is negligible, the two horizontal forces are equal. Finally, there is a torque τ from the motor. Equating moments about the axle gives

$$\tau = fr,$$

where r is the radius of the wheel. Now we turn to the motor. The electric circuit equation is

$$L\frac{di}{dt} + Ri = u - v_b,$$

where v_b is the back emf. The torque produced by the motor satisfies

$$\tau_m = Ki.$$

Newton's second law for the motor shaft gives

$$J\ddot{\theta} = \tau_m - \tau.$$

Then the back emf is

$$v_b = K_b\dot{\theta}.$$

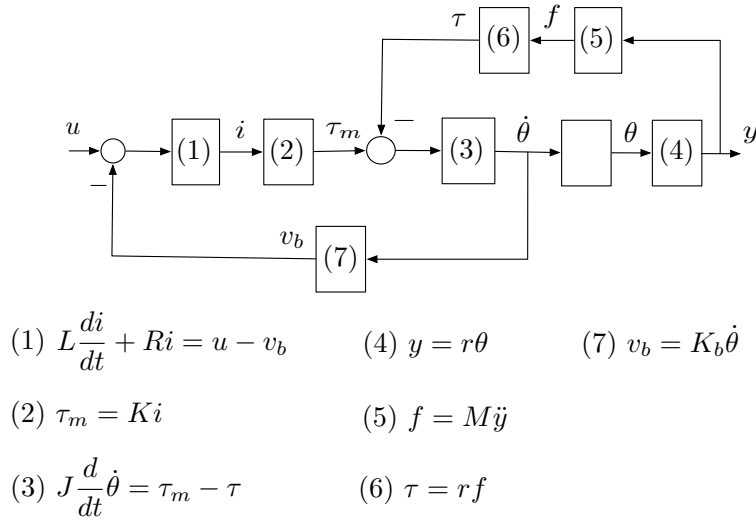
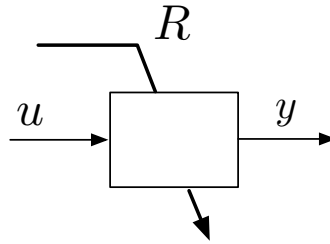


Figure 2.14: The cart and motor drive block diagram.

Figure 2.15: RC circuit where R can be set externally.

Finally, the relationship between shaft angle and cart position is

$$y = r\theta.$$

Combining all this gives the block diagram of Figure 2.14.

8. Occasionally we might want to indicate that a parameter is a variable, or that it can be set at different values. For example, suppose that in the RC circuit in Figure 2.3 several values of R are available for us to select. The the value R ican be selected from an external power (us), and we can indicate this by an arrow going through the box as in Figure 2.15.
9. *Summary.* A block diagram is composed of arrows, boxes, and summing junctions. The arrows represent signals. The boxes represent systems or system components; mathematically they are functions that map one or more signals to one or more other signals. An exogenous input to a block diagram, e.g., u in Figure 2.4, is an independent variable. Other signals are dependent variables.

2.2 State equations

State equations are fundamental to the subject of dynamical systems. In this section we look at a number of examples. We begin with the notion of linearity.

1. *The concept of linearity.* To say that a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is linear means the graph is a straight line through the origin; there's only one straight line that is not allowed—the y -axis. Thus $y = ax$ defines a linear function for any real constant a ; the equation defining the y -axis is $x = 0$. The function $y = 2x + 1$ is not linear—its graph is a straight line, but not through the origin. In your linear algebra course you were taught that a **linear function**, or **linear transformation**, is a function f from a vector space \mathcal{X} to another (or the same) vector space \mathcal{Y} having the property

$$f(a_1x_1 + a_2x_2) = a_1f(x_1) + a_2f(x_2)$$

for all vectors x_1, x_2 in \mathcal{X} and all real numbers³ a_1, a_2 . If the vector spaces are $\mathcal{X} = \mathbb{R}^n$, $\mathcal{Y} = \mathbb{R}^m$, and if f is linear, then it has the form $f(x) = Ax$, where A is an $m \times n$ matrix. Conversely, every function of this form is linear. This is a useful fact, so let us record it as the next item. We emphasize that \mathbb{R}^n denotes the vector space of n -dimensional column vectors, and so the basis is fixed.

2. *Characterization of a linear function.* If A is an $m \times n$ real matrix and f is the function $f(x) = Ax$, then f is linear. Conversely, if f is a linear function from \mathbb{R}^n to \mathbb{R}^m , then there is a unique $m \times n$ matrix A such that $f(x) = Ax$.
3. *Proof.* Suppose A is given and $f(x) = Ax$. Then f is linear because

$$A(a_1x_1 + a_2x_2) = a_1Ax_1 + a_2Ax_2.$$

Conversely, suppose f is linear. We are going to build the matrix A column by column. Let e_1 denote this vector of dimension n :

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Take the first column of A to be $f(e_1)$. Take the second column to be $f(e_2)$, where

$$e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

³This definition assumes that the vector spaces are over the field of real numbers.

And so on. Then certainly $f(x) = Ax$ for x equal to any of the vectors e_1, e_2, \dots . But a general x has the form

$$x = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix} = a_1 e_1 + \dots + a_n e_n.$$

By linearity,

$$f(x) = f(a_1 e_1 + \dots + a_n e_n) = a_1 f(e_1) + \dots + a_n f(e_n).$$

But $a_1 f(e_1) + \dots + a_n f(e_n)$ can be written as a matrix times a vector:

$$\begin{bmatrix} f(e_1) & f(e_2) & \dots & f(e_n) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}.$$

And this product is exactly Ax .

4. *Extension.* This concept of linear function extends beyond vectors to signals. In this book a signal is a function of time. For example, consider a capacitor, whose constitutive law is

$$i = C \frac{dv}{dt}.$$

Here, i and v are not constants, or vectors—they are functions of time. If we think of the current signal i as a function of the voltage signal v , then the function is linear. This is because

$$C \frac{d(a_1 v_1 + a_2 v_2)}{dt} = a_1 C \frac{dv_1}{dt} + a_2 C \frac{dv_2}{dt}.$$

On the other hand, if we try to view v as a function of i , then we have a problem, because we need, in addition, an initial condition $v(0)$ (or at some other initial time) to uniquely define v , not just i . Let us set $v(0) = 0$. Then v can be written as the integral of i like this:

$$v(t) = \frac{1}{C} \int_0^t i(\tau) d\tau.$$

This *does* define a linear function $v = f(i)$.

5. *Terminology.* In control engineering, the system to be controlled is termed the **plant**.
6. *Example.* Figure 2.16 shows a cart on wheels, driven by a force u and subject to air resistance. Typically air resistance creates a force depending on the velocity, \dot{y} ; let us say this force is a possibly nonlinear function $D(\dot{y})$. Assuming M is constant, Newton's second law gives

$$M\ddot{y} = u - D(\dot{y}).$$

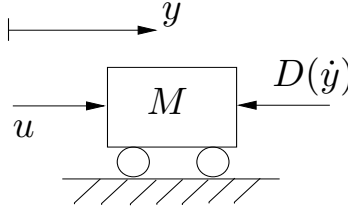


Figure 2.16: Cart on wheels.

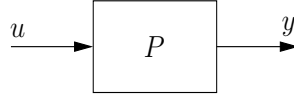


Figure 2.17: Block diagram of the cart.

This is a single second-order differential equation. It will be convenient to put it into two simultaneous first-order equations by defining two so-called state variables, in this example position and velocity:

$$x_1 := y, \quad x_2 := \dot{y}.$$

Then

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{M}u - \frac{1}{M}D(x_2) \\ y &= x_1. \end{aligned}$$

These equations can be combined into

$$\dot{x} = f(x, u) \tag{2.1}$$

$$y = h(x), \tag{2.2}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad f(x, u) = \begin{bmatrix} x_2 \\ \frac{1}{M}u - \frac{1}{M}D(x_2) \end{bmatrix}, \quad h(x) = x_1.$$

The function f is nonlinear if D is, while h is linear in view of Section 2.2, Paragraph 2 and

$$h(x) = \begin{bmatrix} 1 & 0 \end{bmatrix} x.$$

Equations (2.1) and (2.2) constitute a state equation model of the system. The block diagram is shown in Figure 2.17. Here P is a possibly nonlinear system, u (applied force) is the input, y (cart position) is the output, and

$$x = \begin{bmatrix} \text{cart pos'n} \\ \text{cart velocity} \end{bmatrix}$$

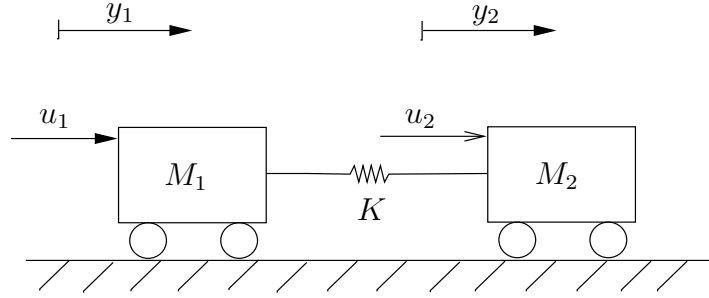


Figure 2.18: A 2-input, 2-output system.

is the state of P . (We'll define state later.) As a special case, suppose the air resistance is a linear function of velocity:

$$D(x_2) = D_0 x_2, \quad D_0 \text{ a constant.}$$

Then f is linear:

$$f(x, u) = Ax + Bu, \quad A := \begin{bmatrix} 0 & 1 \\ 0 & -D_0/M \end{bmatrix}, \quad B := \begin{bmatrix} 0 \\ 1/M \end{bmatrix}.$$

Defining $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$, we get the state model

$$\dot{x} = Ax + Bu, \quad y = Cx. \tag{2.3}$$

This model is linear; it is also time-invariant because the matrices A, B, C do not vary with time. Thus the model is linear, time-invariant (LTI).

7. *Notation.* It is convenient to write vectors sometimes as column vectors and sometimes as n -tuples, i.e., ordered lists. For example

$$x := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad x = (x_1, x_2).$$

We shall use both.

8. *General statement.* An important class of models is

$$\dot{x} = f(x, u), \quad y = h(x, u),$$

where u, x, y are vectors that are functions of time, that is, $u(t), x(t), y(t)$. This model is nonlinear if f and/or h is nonlinear, but it is time-invariant because neither f nor h depends directly on time. Denote the dimensions of u, x, y by, respectively, m, n, p .

9. *Example.* An example where $m = 2, n = 4, p = 2$ is shown in Figure 2.18. For practice you should get the state equation, by taking

$$u = (u_1, u_2), \quad x = (y_1, \dot{y}_1, y_2, \dot{y}_2), \quad y = (y_1, y_2).$$

You will get equations

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du, \end{aligned}$$

where A is 4×4 , B is 4×2 , C is 2×4 , and D is 2×2 .

10. *Example.* A time-varying example is the cart where the mass is decreasing with time because fuel is being used up (or because the cart is on fire). Newton's second law in this case is that force equals the rate of change of momentum, which is mass times velocity. So we have

$$\frac{d}{dt}Mv = u, \quad \dot{y} = v.$$

Expanding the first derivative gives

$$\dot{M}v + M\dot{v} = u, \quad \dot{y} = v.$$

Taking the state variables as usual to be $x_1 = y$, $x_2 = v$, we have

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{\dot{M}}{M}x_2 + \frac{1}{M}u \\ y &= x_1. \end{aligned}$$

In vector form:

$$\dot{x} = A(t)x + B(t)u, \quad y = Cx$$

$$A(t) = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{\dot{M}(t)}{M(t)} \end{bmatrix}, \quad B(t) = \begin{bmatrix} 0 \\ -\frac{1}{M(t)} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The matrices A and B are functions of t because M is a function of t . This model is linear time-varying.

11. *Terminology.* Explanation of the meaning of the **state of a system**: The state x at time t should encapsulate all the system dynamics up to time t , that is, no additional prior information is required. More precisely, the concept for x to be a state is this: For any t_0 and t_1 , with $t_0 < t_1$, knowing $x(t_0)$ and knowing the input $\{u(t) : t_0 \leq t \leq t_1\}$, we can compute $x(t_1)$, and hence the output $y(t_1)$.
12. *Passive circuit.* The customary state variables are inductor currents and capacitor voltages. For a mechanical system the customary state variables are positions and velocities of all masses. The reason for this choice is illustrated as follows. Consider Figure 2.19, a cart with no external applied force. The differential equation model is $M\ddot{y} = 0$, or equivalently, $\ddot{y} = 0$.

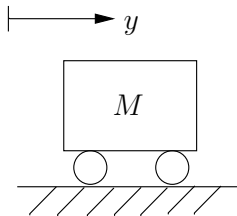


Figure 2.19: A cart with no forces.

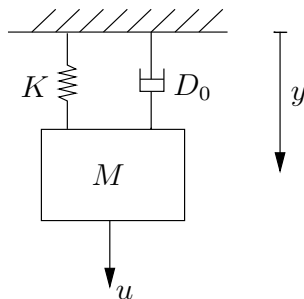


Figure 2.20: Mass-spring-damper.

To solve the equation we need two initial conditions, namely, $y(0)$ and $\dot{y}(0)$. So the state x could not simply be the position y , nor could it simply be the velocity \dot{y} . To determine the position at a future time, we need both the position and velocity at a prior time. Since the equation of motion, $\ddot{y} = 0$, is second order, we need two initial conditions, implying we need a 2-dimensional state vector.

13. *Another mechanical example.* Figure 2.20 shows a mass-spring-damper system. The rest length of the spring is y_0 . Figure 2.21 shows the free-body diagram. The dynamic equation is

$$M\ddot{y} = u + Mg - K(y - y_0) - D_0\dot{y}.$$

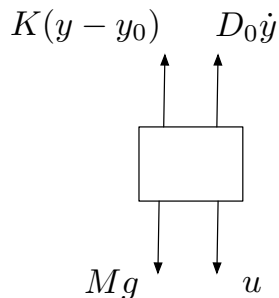


Figure 2.21: Free-body diagram.

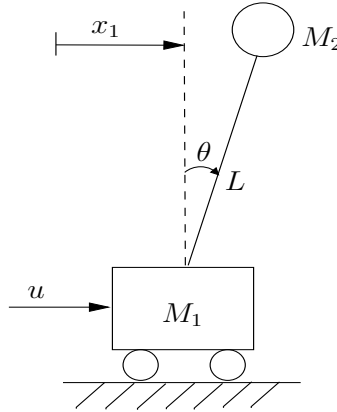


Figure 2.22: Cart-pendulum.

We take position and velocity as state variables,

$$x = (x_1, x_2), \quad x_1 = y, \quad x_2 = \dot{y},$$

and then we get the equations

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{M}u + g - \frac{K}{M}x_1 + \frac{K}{M}y_0 - \frac{D_0}{M}x_2 \\ y &= x_1. \end{aligned}$$

These equations have the form

$$\dot{x} = Ax + Bu + c, \quad y = Cx, \tag{2.4}$$

where the matrices A , B , C , and the vector c are

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{D_0}{M} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix}, \quad C = [1 \quad 0], \quad c = \begin{bmatrix} 0 \\ g + \frac{K}{M}y_0 \end{bmatrix}.$$

The constant vector c is known, and hence is taken as part of the system rather than as a signal. Equations (2.4) have the form

$$\dot{x} = f(x, u), \quad y = h(x),$$

where f is not linear, because of the presence of c , while h is linear.

14. *A favourite toy control problem.* The problem is to get a cart automatically to balance a pendulum, as shown in Figure 2.22. The cart can move in a straight line on a horizontal table. The position of the cart is x_1 referenced to a stationary point. The pendulum, modeled as a point mass on the end of a rigid rod, is attached to a small rotary joint on the cart, so that the pendulum can fall either way but only in the direction that the cart can move. There

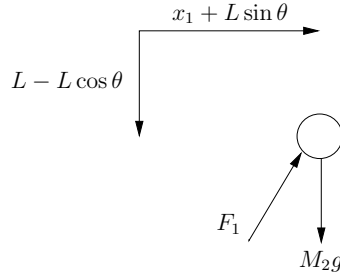


Figure 2.23: Free-body diagram of the pendulum.

is a drive mechanism that produces a force u on the cart. The figure shows the pendulum falling forward. Obviously, the cart has to speed up to keep the pendulum balanced, and the control problem is to design something that will produce a suitable force. That “something” is a controller, and how to design it is one of the topics in this book. The natural state is

$$x = (x_1, x_2, x_3, x_4) = (x_1, \theta, \dot{x}_1, \dot{\theta}).$$

We bring in a free body diagram, Figure 2.23, for the pendulum. The position of the ball is shown in a rectangular coordinate system with two axes: one is horizontally to the right; the other is vertically down. The axes intersect at an origin defined by the conditions $x_1 = 0$ and $\theta = 0$. Newton’s law for the ball in the horizontal direction is

$$M_2 \frac{d^2}{dt^2} (x_1 + L \sin \theta) = F_1 \sin \theta$$

and in the vertical direction (down) is

$$M_2 \frac{d^2}{dt^2} (L - L \cos \theta) = M_2 g - F_1 \cos \theta.$$

The horizontal forces on the cart are u and $-F_1 \sin \theta$. Thus

$$M_1 \ddot{x}_1 = u - F_1 \sin \theta.$$

These are three equations in the four signals x_1, θ, u, F_1 . We have to eliminate F_1 . Use the identities

$$\frac{d^2}{dt^2} \sin \theta = \ddot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta, \quad \frac{d^2}{dt^2} \cos \theta = -\ddot{\theta} \sin \theta - \dot{\theta}^2 \cos \theta$$

to get

$$M_2 \ddot{x}_1 + M_2 L \ddot{\theta} \cos \theta - M_2 L \dot{\theta}^2 \sin \theta = F_1 \sin \theta \tag{2.5}$$

$$M_2 L \ddot{\theta} \sin \theta + M_2 L \dot{\theta}^2 \cos \theta = M_2 g - F_1 \cos \theta \tag{2.6}$$

$$M_1 \ddot{x}_1 = u - F_1 \sin \theta. \tag{2.7}$$

We can eliminate F_1 : Add (2.5) and (2.7) to get

$$(M_1 + M_2)\ddot{x}_1 + M_2L\ddot{\theta}\cos\theta - M_2L\dot{\theta}^2\sin\theta = u;$$

multiply (2.5) by $\cos\theta$, (2.6) by $\sin\theta$, add, and cancel M_2 to get

$$\ddot{x}_1\cos\theta + L\ddot{\theta} - g\sin\theta = 0.$$

Collect the latter two equations as

$$\begin{bmatrix} M_1 + M_2 & M_2L\cos\theta \\ \cos\theta & L \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} u + M_2L\dot{\theta}^2\sin\theta \\ g\sin\theta \end{bmatrix}.$$

Solve:

$$\ddot{x}_1 = \frac{u + M_2L\dot{\theta}^2\sin\theta - M_2g\sin\theta\cos\theta}{M_1 + M_2\sin^2\theta}$$

$$\ddot{\theta} = \frac{-u\cos\theta - M_2L\dot{\theta}^2\sin\theta\cos\theta + (M_1 + M_2)g\sin\theta}{L(M_1 + M_2\sin^2\theta)}.$$

Finally, in terms of the state variables we have

$$\dot{x}_1 = x_3$$

$$\dot{x}_2 = x_4$$

$$\dot{x}_3 = \frac{u + M_2Lx_4^2\sin x_2 - M_2g\sin x_2\cos x_2}{M_1 + M_2\sin^2 x_2}$$

$$\dot{x}_4 = \frac{-u\cos x_2 - M_2Lx_4^2\sin x_2\cos x_2 + (M_1 + M_2)g\sin x_2}{L(M_1 + M_2\sin^2 x_2)}.$$

Again, these have the form

$$\dot{x} = f(x, u).$$

We might take the output to be

$$y = \begin{bmatrix} x_1 \\ \theta \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = h(x).$$

The system is highly nonlinear; as you would expect, it can be approximated by a linear system for $|\theta|$ small enough, say less than 5° .

15. *Water tank.* Figure 2.24 shows water flowing into a tank in an uncontrolled way, and water flowing out at a rate controlled by a valve: The signals are x , the height of the water, u , the area of opening of the valve, and d , the flowrate in. Let A denote the cross-sectional area of the tank, assumed constant. Then conservation of mass gives

$$A\dot{x} = d - (\text{flow rate out}).$$

Also

$$(\text{flow rate out}) = (\text{const}) \times \sqrt{\Delta p} \times (\text{area of valve opening}),$$

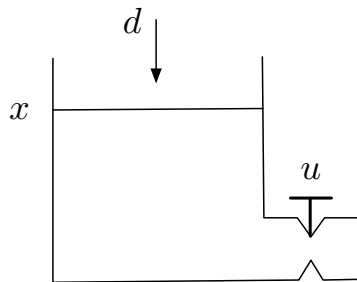
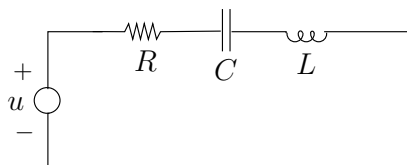


Figure 2.24: Water tank.

Figure 2.25: RLC circuit.

where Δp denotes the pressure drop across the valve, this being proportional to x . Thus

$$(\text{flow rate out}) = c\sqrt{x}u$$

and hence

$$A\dot{x} = d - c\sqrt{x}u.$$

The state equation is therefore

$$\dot{x} = f(x, u, d) = \frac{1}{A}d - \frac{c}{A}\sqrt{x}u.$$

16. *Exclusions.* Not all systems have state models of the form

$$\dot{x} = f(x, u), \quad y = h(x, u).$$

One example is the differentiator: $y = \dot{u}$. A second is a time delay: $y(t) = u(t - 1)$. Finally, there are PDE models, e.g., the vibrating violin string with input the bow force.

17. *Another electric circuit example.* Consider the RLC circuit in Figure 2.25. There are two energy storage elements, the inductor and the capacitor. It is natural to take the state variables to be voltage drop across C and current through L : Figure 2.26. Then KVL gives

$$-u + Rx_2 + x_1 + L\dot{x}_2 = 0$$

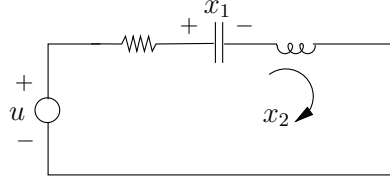


Figure 2.26: State variables.

and the capacitor equation is

$$x_2 = C\dot{x}_1.$$

Thus

$$\dot{x} = Ax + Bu, \quad A = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}.$$

18. *Higher order differential equations.* Finally, consider a system with input $u(t)$ and output $y(t)$ and differential equation

$$\frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_0 y = b_0 u.$$

The coefficients a_i, b_0 are real numbers. We can put this in the state equation form as follows. Define the state

$$x = (x_1, x_2, \dots, x_n) = \left(y, \dot{y}, \frac{d^2 y}{dt^2}, \dots, \frac{d^{n-1} y}{dt^{n-1}} \right).$$

Then

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= -a_{n-1}x_n - \cdots - a_0x_1 + b_0u. \end{aligned}$$

The state model for these equations is

$$\dot{x} = Ax + Bu, \quad y = Cx,$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

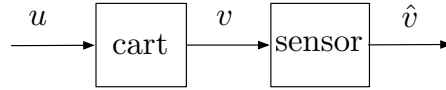


Figure 2.27: A cart with a sensor to measure velocity.

$$C = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

The case

$$\frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_0 y = b_m \frac{d^m u}{dt^m} + b_{m-1} \frac{d^{m-1} u}{dt^{m-1}} + \cdots + b_0 u$$

is somewhat trickier. This has a state model if and only if $n \geq m$. We shall return to this in the next chapter. Briefly, one gets the transfer function from u to y and then gets a state model from the transfer function.

19. *Summary of state equations.* Many systems can be modeled in the form

$$\dot{x} = f(x, u), \quad y = h(x, u),$$

where u, x, y are vectors: u is the input, x the state, and y the output. This model is nonlinear if either f or h is not linear. However, the model is time invariant because neither f nor h has t as an argument. The linear time-invariant (LTI) case is

$$\dot{x} = Ax + Bu, \quad y = Cx + Du.$$

The functions $f(x, u) = Ax + Bu$ and $h(x, u) = Cx + Du$ are linear.

2.3 Sensors and actuators

In this section we briefly discuss sensors and actuators.

1. *Sensors.* Temperatures, positions, currents, forces, and so on can be measured by instruments called **sensors**. A common problem for motion control of mobile robots is to sense the forward velocity of the robot. This can be done by an optical rotary sensor. If placed on a robot's wheel, this gives a fixed number of voltage pulses for each revolution of the wheel, so by counting the number of pulses per second, you have a measurement of speed. If there are two wheels on the ends of an axle and each wheel has a rotary sensor, the two wheel turning rates can be used to determine the heading angle and forward speed.

Suppose the speed of a cart is measured in this way. With v denoting the velocity of the cart, the output of the sensor measuring the speed would usually be denoted \hat{v} . See Figure 2.27.

For a variety of reasons, all measurements have errors. Notice that physically \hat{v} may be a voltage. Although a sensor is a dynamical system itself and therefore could be modelled, it is common to model it simply as a device that adds noise; see Figure 2.28. The noise can never be known exactly, so control engineers assume some generic noise signal using common sense and past experience. For example, one might take random white noise with a certain mean and a certain variance.

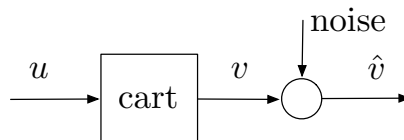
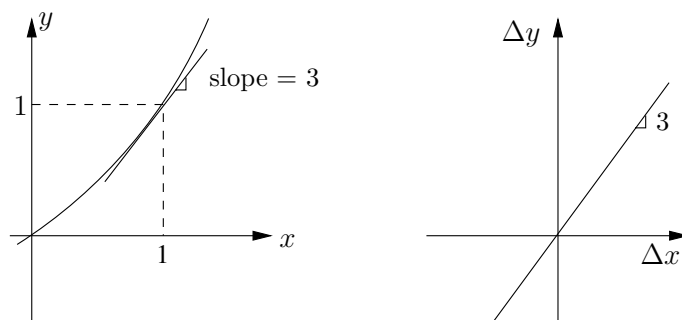


Figure 2.28: Position measurement modelled via additive noise.

Figure 2.29: Graph of the nonlinear function $y = f(x)$ and its linearization at the point x_0 .

2. *Actuators.* A sensor is typically at the output of the plant. At the input may be an actuator. Example: Paragraph 7 in Section 2.1 studies a cart with a motor drive. The force on the cart has to be produced by something, a motor in this case, and this motor is an example of an **actuator**. The actuator may be modelled or not. For example, in the motor-cart system, if the time-constant of the motor is much smaller than the fastest time-constant of the cart, then the dynamics of the motor can be neglected.

2.4 Linearization

Recap: Many systems can be modeled by nonlinear state equations of the form

$$\dot{x} = f(x, u), \quad y = h(x, u),$$

where u, x, y are vectors. There might be disturbance inputs present, but for now we suppose they are lumped into u . There are techniques for controlling nonlinear systems, but that is an advanced subject. Fortunately, many systems can be linearized about an equilibrium point. In this section we see how to do this.

1. *Example.* Linearizing just a function, not a dynamical system model. Let us linearize the function $y = f(x) = x^3$ about the point $x_0 = 1$. Figure 2.29 shows how to do it. At $x = x_0$, the value of y is $y_0 = f(x_0) = 1$. If x varies in a small neighbourhood of x_0 , then y varies in a small neighbourhood of y_0 . The graph of f near the point (x_0, y_0) can be approximated by the tangent to the curve, as shown in the left-hand figure. The slope of the tangent is the

derivative of f at x_0 , $f'(x_0) = 3$. Therefore,

$$\frac{\Delta y}{\Delta x} \approx f'(x_0) = 3.$$

For the linearized function, we merely replace the approximation symbol by an equality:

$$\Delta y = 3\Delta x.$$

Notice that

$$\Delta y = y - y_0, \quad \Delta x = x - x_0.$$

So the linearized function approximates the nonlinear one in the neighbourhood of the point where the derivative is evaluated. Obviously, this approximation gets better and better as $|\Delta x|$ gets smaller and smaller.

2. *Extension.* The method extends to a function $y = f(x)$, where x and y are vectors. Then the derivative is the Jacobian matrix, which we shall denote by $f'(x_0)$.
3. *Example.* Consider the function

$$y = f(x), \quad x = (x_1, x_2, x_3), \quad y = (x_1x_2 - 1, x_3^2 - 2x_1x_3).$$

Let us linearize f at the point $x_0 = (1, -1, 2)$. The linearization is

$$\Delta y = A\Delta x,$$

where A equals the Jacobian of f at the point x_0 . The element in the i^{th} row and j^{th} column is the partial derivative $\partial f_i / \partial x_j$, where f_i is the i^{th} element of f (i.e., y_i). So we have

$$\begin{aligned} f'(x_0) &= \left[\begin{array}{ccc} x_2 & x_1 & 0 \\ -2x_3 & 0 & 2x_3 - 2x_1 \end{array} \right] \bigg|_{x_0} \\ &= \left[\begin{array}{ccc} -1 & 1 & 0 \\ -4 & 0 & 2 \end{array} \right]. \end{aligned}$$

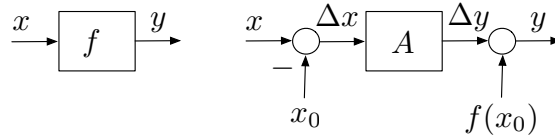
Thus the linearization of $y = f(x)$ at x_0 is $\Delta y = A\Delta x$, where

$$A = f'(x_0) = \left[\begin{array}{ccc} -1 & 1 & 0 \\ -4 & 0 & 2 \end{array} \right].$$

4. *Summary of the preceding two examples.* Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a function and x_0 a vector in \mathbb{R}^n . Assume f is continuously differentiable at the point x_0 . Then the linearization of the equation $y = f(x)$ at the point x_0 is the equation $\Delta y = A\Delta x$, where $A = f'(x_0)$, the Jacobian of f at x_0 . The variables are related by

$$x = x_0 + \Delta x, \quad y = f(x_0) + \Delta y.$$

We will find it useful to relate the block diagram for the equation $y = f(x)$ and the block diagram for the equation $\Delta y = A\Delta x$. These are shown in Figure 2.30. Because x_0 and $f(x_0)$ are known, we should regard the block diagram on the right as having just the input x and just the output y .

Figure 2.30: Block diagram of $y = f(x)$ and its linearization.

5. *Extension.* Now we consider the case $y = f(x, u)$, where u, x, y are all vectors, of dimensions m, n, p , respectively. We want to linearize at the point (x_0, u_0) . We can combine x and u into one vector $v = \begin{bmatrix} x \\ u \end{bmatrix}$ of dimension $n + m$. Then we have the situation in the preceding example, $y = f(v)$. The Jacobian of f is an $n \times (n + m)$ matrix. Define

$$v_0 = \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}, \quad \begin{bmatrix} A & B \end{bmatrix} = f'(v_0).$$

In this way we get the linearization

$$\begin{aligned} \Delta y &= f'(v_0) \Delta v \\ &= \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta u \end{bmatrix} \\ &= A \Delta x + B \Delta u. \end{aligned}$$

6. *Linearization of the differential equation*

$$\dot{x} = f(x, u).$$

First, **assume** there is an equilibrium point, that is, a constant solution $x(t) \equiv x_0, u(t) \equiv u_0$. This is equivalent to saying that $0 = f(x_0, u_0)$. Now consider a nearby solution:

$$x(t) = x_0 + \Delta x(t), \quad u(t) = u_0 + \Delta u(t).$$

We have

$$\begin{aligned} \dot{x}(t) &= f[x(t), u(t)] \\ &= f(x_0, u_0) + A \Delta x(t) + B \Delta u(t) + \text{higher order terms}, \end{aligned}$$

where

$$\begin{bmatrix} A & B \end{bmatrix} = f'(x_0, u_0).$$

Since $\dot{x} = \dot{\Delta x}$ and $f(x_0, u_0) = 0$, we have the linearized equation to be

$$\dot{\Delta x} = A \Delta x + B \Delta u.$$

Similarly, the output equation $y = h(x, u)$ linearizes to

$$\Delta y = C \Delta x + D \Delta u,$$

where

$$\begin{bmatrix} C & D \end{bmatrix} = h'(x_0, u_0).$$

7. *Summary.* To linearize the system $\dot{x} = f(x, u)$, $y = h(x, u)$, select, if one exists, an equilibrium point, that is, constant vectors x_0, u_0 such that $f(x_0, u_0) = 0$. If the functions f and h are continuously differentiable at this equilibrium, compute the Jacobians $\begin{bmatrix} A & B \end{bmatrix}$ and $\begin{bmatrix} C & D \end{bmatrix}$ of f and h at the equilibrium point. Then the linearized system is

$$\dot{\Delta x} = A\Delta x + B\Delta u, \quad \Delta y = C\Delta x + D\Delta u.$$

This linearized system is a valid approximation of the nonlinear one in a sufficiently small neighbourhood of the equilibrium point. How small a neighbourhood? There is no simple answer.

8. *Example of the cart-pendulum.* See page 21. An equilibrium point

$$x_0 = (x_{10}, x_{20}, x_{30}, x_{40}), \quad u_0$$

satisfies $f(x_0, u_0) = 0$, i.e.,

$$x_{30} = 0$$

$$x_{40} = 0$$

$$u_0 + M_2 L x_{40}^2 \sin x_{20} - M_2 g \sin x_{20} \cos x_{20} = 0$$

$$-u_0 \cos x_{20} - M_2 L x_{40}^2 \sin x_{20} \cos x_{20} + (M_1 + M_2)g \sin x_{20} = 0.$$

Multiply the third equation by $\cos x_{20}$ and add to the fourth:

$$-M_2 g \sin x_{20} \cos^2 x_{20} + (M_1 + M_2)g \sin x_{20} = 0.$$

Factor the left-hand side:

$$(\sin x_{20})(M_1 + M_2 \sin^2 x_{20}) = 0.$$

The right-hand factor is positive; it follows that $\sin x_{20} = 0$ and therefore x_{20} equals 0 or π , that is, the pendulum is straight down or straight up. Thus the equilibrium points are described by

$$x_0 = (\text{arbitrary}, 0 \text{ or } \pi, 0, 0), \quad u_0 = 0.$$

We have to choose $x_{20} = 0$ (pendulum up) or $x_{20} = \pi$ (pendulum down). Let us take $x_{20} = 0$. Then the Jacobian computes to

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{M_2}{M_1}g & 0 & 0 \\ 0 & \frac{M_1+M_2}{M_1} \frac{g}{L} & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M_1} \\ -\frac{1}{LM_1} \end{bmatrix}.$$

We just applied the general method of linearizing. For this example, there's actually a faster way, which is to approximate $\sin \theta = \theta$, $\cos \theta = 1$ in the original equations.

2.5 Interconnections of linear subsystems

Frequently, a system is made up of components connected together in some arrangement. This raises the question, if we have state models for components, how can we assemble them into a state model for the overall system?

1. *Review.* This section involves some matrix algebra. Let us summarize what you need to know. If we have two vectors, x_1 and x_2 , of dimensions n_1 and n_2 , we can stack them as a vector of dimension $n_1 + n_2$. Of course, we can stack them either way:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ or } \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}.$$

Then, two state equations

$$\begin{aligned} \dot{x}_1 &= A_1 x_1 + B_1 u_1 \\ \dot{x}_2 &= A_2 x_2 + B_2 u_2 \end{aligned}$$

can be combined into one state equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Be careful: If you stack x_2 above x_1 , the matrices will be different. In the matrix

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

the two zeros are themselves matrices of all zeros. Usually, we don't care how many rows or columns they have, but actually their sizes can be deduced. For example, if x_1 has dimension n_1 and x_2 has dimension n_2 , then the sizes of zero blocks must be as shown here:

$$\begin{bmatrix} A_1 & 0_{n_1 \times n_2} \\ 0_{n_2 \times n_1} & A_2 \end{bmatrix}.$$

This is because the upper-right zero must have the same number of rows as A_1 and the same number of columns as A_2 ; likewise for the lower-left zero.

Finally, multiplication of a block vector by a block matrix, assuming the dimensions are correct, works just as if the blocks were 1×1 . For example, in the product

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

we multiply just as though A_1, A_2, x_1, x_2 were scalars:

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_1 x_1 \\ A_2 x_2 \end{bmatrix}.$$

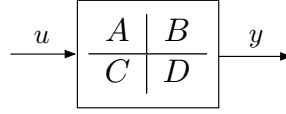
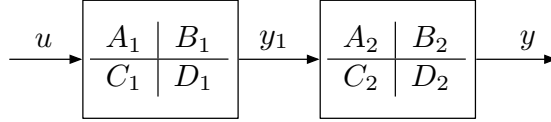
Figure 2.31: Block diagram of the state model $\dot{x} = Ax + Bu$, $y = Cx + Du$.

Figure 2.32: Series connection.

2. *Notation.* There is a very handy way to embed a state model into a block diagram. Suppose we have the state model

$$\dot{x} = Ax + Bu \quad (2.8)$$

$$y = Cx + Du. \quad (2.9)$$

The input is u , the output is y , and x is the state. A block diagram for this component has u on the input arrow and y on the output arrow. The system in the box is modeled by the state equations. It is convenient to encode these equations into the block diagram as in Figure 2.31. The symbol $\frac{A}{C} \mid \frac{B}{D}$ in the block is just an abbreviation for equations (2.8), (2.9).

3. *Example, series connection.* Figure 2.32 shows a series connection of two subsystems. We want to get a state model from u to y . Write the state equations for the two blocks:

$$\dot{x}_1 = A_1x_1 + B_1u$$

$$\dot{x}_2 = A_2x_2 + B_2y_1.$$

Let us take the combined state to be

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Then the combined preceding two equations become

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} y_1.$$

The intermediate signal y_1 , being the output of the left-hand block, equals $C_1x_1 + D_1u$, or, in terms of the combined state and u ,

$$y_1 = \begin{bmatrix} C_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + D_1u. \quad (2.10)$$

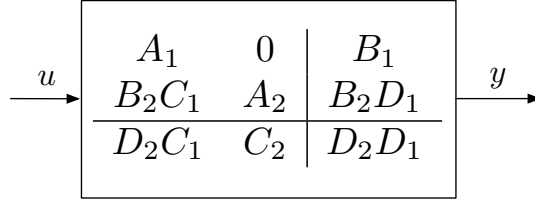


Figure 2.33: Series connection as a combined system.

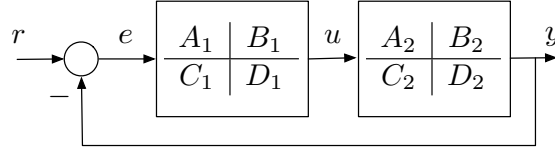


Figure 2.34: Feedback connection.

Substituting this into the preceding equation gives

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ B_2C_1 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2D_1 \end{bmatrix} u.$$

Then, the system output y is

$$\begin{aligned} y &= C_2x_2 + D_2y_1 \\ &= \begin{bmatrix} 0 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + D_2y_1. \end{aligned}$$

Substituting in y_1 from (2.10) gives

$$y = \begin{bmatrix} D_2C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + D_2D_1u.$$

The combined state equations can be written in the standard form

$$\dot{x} = Ax + Bu, \quad y = Cx + Du,$$

where

$$A = \begin{bmatrix} A_1 & 0 \\ B_2C_1 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2D_1 \end{bmatrix}$$

$$C = \begin{bmatrix} D_2C_1 & C_2 \end{bmatrix}, \quad D = D_2D_1.$$

We conclude that the block diagram Figure 2.32 can be simplified to Figure 2.33.

4. *Example, feedback connection.* Figure 2.34 shows a feedback arrangement. Feedback is introduced later in the book, but here we merely want to do some manipulations with state models and block diagrams. Specifically, we want to derive a state model from r to y . The derivation is simpler under the assumption $D_2 = 0$, so we make that assumption. The block diagram has two blocks. Write the state equations for the two blocks:

$$\begin{aligned}\dot{x}_1 &= A_1x_1 + B_1e \\ \dot{x}_2 &= A_2x_2 + B_2u.\end{aligned}$$

Combine:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} e \\ u \end{bmatrix}. \quad (2.11)$$

Write the equations for e and u in terms of r, x_1, x_2 :

$$\begin{aligned}e &= r - C_2x_2 \\ u &= C_1x_1 + D_1e \\ &= C_1x_1 + D_1r - D_1C_2x_2.\end{aligned}$$

Combine:

$$\begin{bmatrix} e \\ u \end{bmatrix} = \begin{bmatrix} 0 & -C_2 \\ C_1 & -D_1C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} I \\ D_1 \end{bmatrix} r.$$

Substitute into (2.11) and simplify:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & -B_1C_2 \\ B_2C_1 & A_2 - B_2D_1C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2D_1 \end{bmatrix} r.$$

Then the output y is

$$y = C_2x_2 = \begin{bmatrix} 0 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

We therefore have the combined model

$$\dot{x} = Ax + Br, \quad y = Cx$$

where

$$A = \begin{bmatrix} A_1 & -B_1C_2 \\ B_2C_1 & A_2 - B_2D_1C_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2D_1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & C_2 \end{bmatrix}.$$

We conclude that the block diagram Figure 2.34 can be simplified to Figure 2.35.

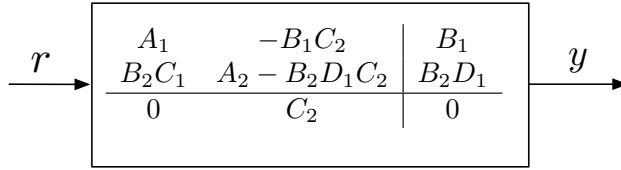


Figure 2.35: Feedback connection as a combined system.

2.6 “Nothing is as abstract as reality”

The title of this section is a quote from the artist Giorgio Morandi.

Mathematical models of physical things are approximations of reality. Consider, for example, a real swinging pendulum and a mathematical model of it. The mathematical model involves a parameter L , the length, whereas the real pendulum does not have a real length at the subatomic scale. The length L is an attribute of an idealized pendulum. Going past this issue of what L is, the mathematical model assumes perfect rigidity of the pendulum. From what we know about reality, nothing is perfectly rigid; that is, rigidity is a concept, an approximation to reality. So if we wanted to make our model “closer to reality,” we could allow some elasticity by adopting a partial differential equation model and we may thereby have a better approximation. But no model is real. There could not be a sequence M_0, M_1, \dots of models that are better and better approximations of reality and such that M_k converges to reality. If M_k does indeed converge, the limit is a model, and no model is real.

The only sensible question is, what do we mean by a “good model,” or, if we have two models, how can we say which is better? We can test our model against the real thing. That is, we can do several tests on the real thing, perform the same test on the model, and compare the resulting measured data with the simulated data. If the two sets of data are close, and if the measuring instruments are reasonably accurate, then we can say that the model is quite good.

For more along these lines, see the article “What’s bad about this habit,” N. D. Mermin, *Physics Today*, May 2009, pages 8, 9.

2.7 Problems

1. Consider an electric circuit consisting of, in series, a voltage source supplying $u(t)$ volts, a resistor, an inductor, and a battery of 10 V. Take the state to be the current. Find the state equation $\dot{x} = f(x, u)$. Find all equilibria and linearize about one of them. Hint: The circuit would be linear were it not for the battery.
2. Let x and y be vectors and A a matrix. Consider the block diagram in Figure 2.36. According to the diagram, the matrix A must partition into two blocks: $A = \begin{bmatrix} A_1 & A_2 \end{bmatrix}$. Then $y = A_1x + A_2y$. When does this equation define a function from x to y ?

Solution The equation $y = A_1x + A_2y$ defines a function from x to y if and only if for every x there exists a unique y satisfying $(I - A_2)y = A_1x$. For the vector y to be unique, the matrix $I - A_2$ must be invertible. In this case, $y = (I - A_2)^{-1}A_1x$. This equation defines a

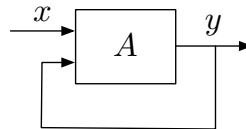


Figure 2.36: Feedback in a matrix system.

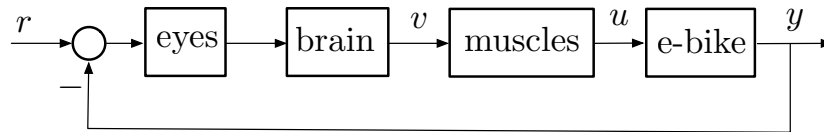


Figure 2.37: E-bike problem.

function from x to y . Thus the answer is, Figure 2.36 defines a function if and only if $I - A_2$ is invertible (equivalently, 1 is not an eigenvalue of A_2).

- Imagine you are riding an e-bike at constant speed along a street. You are steering the e-bike to control the direction you are going. Draw the block diagram.

Solution The e-bike, with you sitting on it, is a system component with input u , your force turning the handlebars, and output y , the direction of motion of the e-bike. The force u comes from your arm muscles, with input v an electrical voltage generated by your brain. The input to your brain is the error as seen by your eyes, the error between the desired direction r you want to go and the actual direction. The resulting block diagram is shown in Figure 2.37.

- Modeling a bicycle is harder than you might think. Imagine a rider on a bike and the bike on the road. Take the overall output to be the bicycle position (in an (x, y) -plane). What's the overall input? The rider can apply forces to the pedals, so they are inputs; so is the torque applied to the steering wheel; and so is a leaning torque applied by the rider's muscles. Try getting a block diagram where the bicycle is one block and the person another, and there may be more.
- There are two cars and a road. One car is to be driven by a person along the road continuously in one direction. The second car is required to follow the first at a fixed distance, but without a human driver. Suppose the second car has a camera that can see the first car, some mechanism to steer and speed up and slow down, and a computer with a program in it. The program computes a rule to steer and to speed up or slow down accordingly. The system may work well or not—we're not interested in that aspect. Draw a block diagram of this system.
- Suppose you have a car with a GPS navigation system. The system has a screen showing a map and there's an arrow on the map showing where your car is. As the car moves, the map evolves so that the arrow stays in the middle of the screen. Draw a block diagram of this setup.
- Consider a force-feedback joystick connected to a laptop, with a person applying a force to the joystick. Suppose the laptop is connected to another laptop through the Internet. This

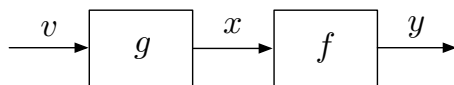


Figure 2.38: Composition of two functions.

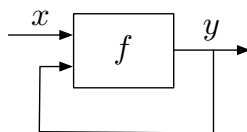


Figure 2.39: Feedback around a function.

second laptop is in a loop controlling a cart. Finally, the cart may bump into an obstacle. This is a telerobotic architecture: The first laptop and the joystick are the master manipulator, the second laptop and the cart are the slave. Let's say the system should have the following capabilities: When the person applies a force to the joystick, the remote cart should move appropriately; when the cart hits the obstacle, the force should be reflected back to the person.

Let us first model the joystick. It is a DC motor and the relevant variables are the voltage to the field windings, the torque that is generated by the magnetic field and applied to the shaft, the torque applied to the shaft by the person, and the shaft angle. Continue modeling in this way and get a block diagram.

8. If the composition $f \circ g$ is defined, i.e., the co-domain of g is contained in the domain of f , then the block diagram is as shown in Figure 2.38. Here v is the overall input and y the overall output of the system composed of f and g combined in the order shown. So far we haven't said what v, x, y represent. For example, they could be real numbers, in which case f and g are functions $\mathbb{R} \rightarrow \mathbb{R}$. Give examples of nonlinear f, g and find the system from v to y .

Solution Let f be defined by $y = x^2 + 1$ and g be defined by $x = \sin(v)$. Then

$$y = x^2 + 1 = \sin^2 v + 1.$$

9. Figure 2.39 is more interesting system. Obviously this represents the equation $y = f(x, y)$. That is, f is a function of two variables, and we've attempted to define a new function by the equation $y = f(x, y)$. For the block diagram to be well-defined, that is, to represent a function, it must be true that for every x there exists a unique y such that $y = f(x, y)$. Give an example of f where the block diagram defines a function and one where it does not.

Solution Let $f(x, y) = 2x - 3y$. Then we can solve $y = f(x, y)$ to get $y = x/2$. On the other hand, if we take f to be $f(x, y) = x + y$, then the equation $y = f(x, y)$ becomes $y = x + y$, which is not solvable for y .

10. Why does the block diagram in Figure 2.40 not define a function?

Solution There is no input, i.e., independent variable.

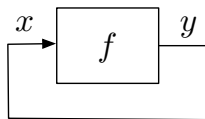


Figure 2.40: Is this a function?

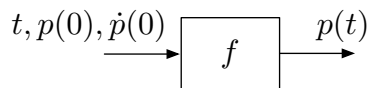


Figure 2.41: Problem about throwing a baseball.

11. A baseball is thrown and we want to model the ball's motion while in flight. Drawing a block diagram means determining what the variables are and how they depend on each other. Let us neglect the ball's rotation and think of it as a point mass. One variable is therefore the position of the ball, with respect to, say, a coordinate system fixed to the earth. Let us denote position at time t by $p(t)$, a three-dimensional vector. What does $p(t)$ depend on? The force of gravity, but that is fixed, not a variable, and therefore not an input. The position of the ball depends obviously on the force with which it was thrown, where it was thrown from, and what direction it was thrown. Equivalently, $p(t)$ depends on the initial position and velocity, $p(0), \dot{p}(0)$, and the time t . Thus the block diagram is Figure 2.41. The input is a vector of 7 real numbers and the output is a vector of 3 real numbers. The equation is

$$p(t) = f(t, p(0), \dot{p}(0)),$$

What parameters does f depend on?

Solution The function f depends only on the mass of the ball and the gravity constant g .

12. Consider two carts connected by a spring. A force u is applied to one of the carts. Let the positions of the carts be y_1, y_2 and suppose we designate y_2 to be the overall output. A common way to model this is via free-body diagrams and Newton's second law. Letting v denote the force applied to cart 1 via the spring, we get equations

$$\begin{aligned} M_1 \ddot{y}_1 &= u - v \\ M_2 \ddot{y}_2 &= v \\ v &= K(y_1 - y_2). \end{aligned}$$

Let us assume zero initial conditions: $y_1(0), \dot{y}_1(0), y_2(0), \dot{y}_2(0)$ all zero. Then the equations have the general form

$$\begin{aligned} y_1 &= f_1(u, v) \\ y_2 &= f_2(v) \\ v &= f_3(y_1, y_2). \end{aligned}$$

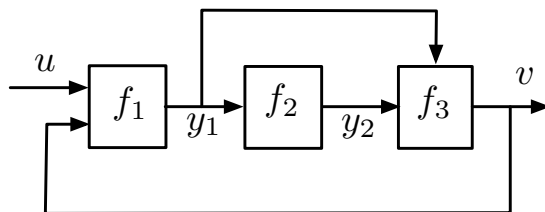


Figure 2.42: Solution.

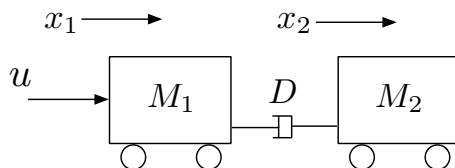


Figure 2.43: Two carts and a damper.

Draw the resulting block diagram.

Solution See Figure 2.42.

13. Suppose f is a linear function from \mathbb{R}^2 to \mathbb{R}^3 , that is, 2 inputs and 3 outputs. Given f , how could you get the matrix A such that $f(x) = Ax$? Hint: Apply an input so that the output equals the first column of A .

Solution Apply the input $(1, 0)$ Suppose the output equals (a_{11}, a_{21}, a_{31}) . Next, apply the input $(0, 1)$ and suppose the output equals (a_{12}, a_{22}, a_{32}) . Then

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}.$$

14. Consider Figure 2.43 showing two carts and a dashpot. (Recall that a dashpot is like a spring except the force is proportional to the *derivative* of the change in length; D is the proportionality constant.) The input is the force u and the positions of the carts are x_1, x_2 . The other state variables are $x_3 = \dot{x}_1, x_4 = \dot{x}_2$. Take $M_1 = 1, M_2 = 1/2, D = 1$. Derive the matrices A, B in the state model $\dot{x} = Ax + Bu$,

Solution

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 2 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

15. This problem concerns a beam balanced on a fulcrum. The angle of tilt of the beam is denoted $\alpha(t)$; a torque, denoted $\tau(t)$, is applied to the beam; finally, a ball rolls on the beam at distance $d(t)$ from the fulcrum. Introduce the parameters:

J moment of inertia of the beam
 J_b moment of inertia of the ball
 R radius of the ball
 M mass of the ball.

The equations of motion are given to you as

$$\left(\frac{J_b}{R^2} + M \right) \ddot{d} + Mg \sin \alpha - Md\dot{\alpha}^2 = 0$$

$$(Md^2 + J + J_b)\ddot{\alpha} + 2Md\dot{d}\dot{\alpha} + Mgd \cos \alpha = \tau.$$

Put this into the form of a nonlinear state model with input τ .

Solution We can take the state to be

$$x = (x_1, x_2, x_3, x_4) = (\alpha, \dot{\alpha}, d, \dot{d}).$$

The input is $u = \tau$. The state equation is $\dot{x} = f(x, u)$, where

$$f(x, u) = \begin{bmatrix} x_2 \\ \frac{1}{Mx_3^2 + J + J_b} (-2Mx_2x_3x_4 - Mgx_3 \cos x_1 + u) \\ x_4 \\ \frac{1}{\frac{J_b}{R^2} + M} (-Mg \sin x_1 + Mx_2^2x_3) \end{bmatrix}.$$

16. Continue with the same ball-and-beam problem. Find all equilibrium points. Linearize the state equation about the equilibrium point where $\alpha = d = 0$.

Solution The equilibrium equation is $f(x, u) = 0$, that is,

$$\begin{aligned} x_2 &= 0 \\ Mgx_3 \cos x_1 &= u \\ x_4 &= 0 \\ \sin x_1 &= 0. \end{aligned}$$

Thus the equilibrium points are

$$x = (0, 0, d, 0), \quad u = Mgd, \quad d \text{ arbitrary}$$

and

$$x = (\pi, 0, d, 0), \quad u = -Mgd, \quad d \text{ arbitrary.}$$

Now we are asked to linearize at

$$x = (0, 0, 0, 0), \quad u = 0.$$

The Jacobians are computed to be

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{Mg}{J+J_b} & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{Mg}{\frac{J_b}{R^2}+M} & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{J+J_b} \\ 0 \\ 0 \end{bmatrix}.$$

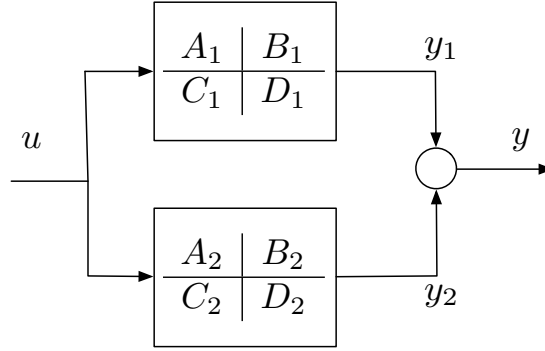


Figure 2.44: Parallel connection.

17. Let A be an $n \times n$ real matrix and $b \in \mathbb{R}^n$. Define the function

$$f(x) = x^T A x + b^T x \quad f : \mathbb{R}^n \longrightarrow \mathbb{R},$$

where T denotes transpose. Linearize the equation $y = f(x)$ at the point x_0 .

Solution The linearized equation is $\Delta y = f'(x_0)\Delta x$, that is,

$$\Delta y = (2x_0^T A + b^T)\Delta x.$$

18. Linearize the water-tank example.

Solution The state equation is

$$\dot{x} = f(x, u, d) = \frac{1}{A} - \frac{c}{A}\sqrt{x}u.$$

The equilibria are those values (x_0, u_0, d_0) satisfying $d_0 = c\sqrt{x_0}u_0$. The linearized equation is

$$\dot{\Delta x} = -\frac{cu_0}{2A\sqrt{x_0}}\Delta x - \frac{c\sqrt{x_0}}{A}\Delta u.$$

This is valid for $x_0 > 0$.

19. Connect two subsystems in parallel as in Figure 2.44. Find the state model of the combined system.

Solution If we take the state vector to be $x = (x_1, x_2)$, then

$$\left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & D_1 + D_2 \end{array} \right].$$

20. A propos of Section 2.6, comment on the following quote and statements:

(a) “I took a tenth-order model, but the real flexible beam is infinite dimensional.”

- (b) From Maxwell's equations, the model of a transmission line is a partial differential equation.
- (c) Every real system is actually nonlinear.

Solution

- (a) The statement was once made with the justification that a real beam has infinitely many molecules. But the statement is illogical, since dimension is a mathematical concept and a real system does not have a dimension.
- (b) It is correct that Maxwell's equations lead to a PDE model of a transmission line. However there is no unique model of a transmission line, since a PDE can be reduced to any number of lumped models.
- (c) Strictly speaking, this statement is illogical, since linearity is a mathematical concept. However, it is frequently used loosely, for example, to reflect the fact that a real system will saturate if an input is applied of too high amplitude.

Chapter 3

The Laplace Transform

We have seen that the systems of interest for us are modelled by linear constant-coefficient differential equations. We have also seen how to combine several coupled differential equations into the form

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du,\end{aligned}$$

where u, x, y are vectors. The next step is to transform this from the time domain to the frequency domain. The one-sided Laplace transform is the fundamental tool used in control engineering to do this transformation. We will arrive at the model

$$Y(s) = G(s)U(s),$$

where $U(s)$ and $Y(s)$ are the Laplace transforms of, respectively, the input $u(t)$ and the output $y(t)$, and $G(s)$ is the so-called transfer function, or transfer matrix if the dimension of $U(s)$ or $Y(s)$ is greater than 1. The function $G(s)$ is determined by the matrices A, B, C, D .

It is interesting that the subject of communication systems uses the Fourier transform and the subject of control systems uses the one-sided Laplace transform. The systems in the two subjects are quite similar—they are linear time-invariant. However communication systems are stable while the plants in control systems are frequently not. By definition, velocity is the derivative of position, and this leads to an integrator in the model. As a dynamical system, an integrator is unstable. Regarding signals, in communication theory signals are modeled as being stationary. By contrast, in control systems commands are abrupt.

In this chapter we go over the definition of Laplace transform, the conditions for existence, how pole locations reflect qualitative behaviour in the time domain, the final-value theorem, transfer functions, and stability.

3.1 Definition of the Laplace transform

1. *Setup.* Let $f(t)$ be a continuous-time function. The time variable t can range over all time, $-\infty < t < \infty$, or perhaps only over all non-negative time, $0 \leq t < \infty$. We assume $f(t)$ is real-valued, which is typically true in applications, where $f(t)$ is a voltage, velocity, or some other physical variable. We make two additional assumptions about f .

2. *Piecewise continuous.* The first assumption is **piecewise continuous for $t \geq 0$** . This means that f is continuous except possibly at a countable number of times $0 = t_0 < t_1 < \dots$. The widths of the intervals $[t_k, t_{k+1}]$ must be such that $t_{k+1} - t_k \geq b$ for some positive b and all k ; that is, we don't allow $f(t)$ to have infinitely many jumps during a finite time interval. For example, every sinusoid is continuous and therefore piecewise continuous, a square wave is not continuous but is piecewise continuous, and the blowing-up exponential e^t is continuous and therefore piecewise continuous. On the other hand, this function is not, because it has infinitely many jumps in finite time:

$$f(t) = \begin{cases} 0, & \text{if } t \text{ is a rational number} \\ 1, & \text{if } t \text{ is not a rational number.} \end{cases}$$

3. *Exponentially bounded.* The second assumption is exponentially bounded for $t \geq 0$. This means that if it blows up, there is some exponential that blows up faster. That is,

$$|f(t)| \leq Me^{at} \tag{3.1}$$

for some $M \geq 0$ and some $a \in \mathbb{R}$ and all $t > 0$. For example $f(t) = e^t$ is exponentially bounded but $f(t) = e^{(t^2)}$ is not. The signal

$$f(t) = \begin{cases} \frac{1}{t-1}, & t \neq 1 \\ 0, & t = 1 \end{cases},$$

which blows up in finite time, is not exponentially bounded.

4. *The Laplace integral.* For a function satisfying these two conditions, its Laplace transform is then defined to be

$$F(s) = \int_0^\infty f(t)e^{-st} dt,$$

where s is a complex variable. The function $f(t)$ may or may not be zero for negative time—it doesn't matter, because the Laplace transform is one-sided and ignores the values of $f(t)$ for $t < 0$. The piecewise continuity assumption implies that the Riemann integral

$$\int_0^T f(t)e^{-st} dt$$

exists for every finite T , and then the exponential boundedness assumption implies that the limit

$$\lim_{T \rightarrow \infty} \int_0^T f(t)e^{-st} dt$$

exists if the real part of s is sufficiently large. To see this latter point, letting $\operatorname{Re} s$ denote “real part of s ” and assuming (3.1), we have

$$\begin{aligned} \int_0^T |f(t)e^{-st}| dt &\leq M \int_0^T e^{at} |e^{-st}| dt \\ &= M \int_0^T e^{at} e^{-(\operatorname{Re} s)t} dt \\ &= M \int_0^T e^{-((\operatorname{Re} s)-a)t} dt. \end{aligned}$$

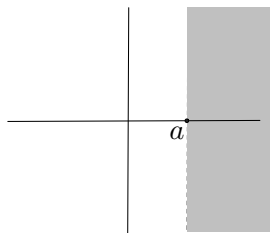


Figure 3.1: An example of a right half-plane—the shaded region.

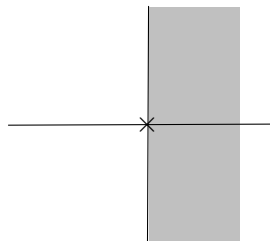


Figure 3.2: The ROC for the Laplace transform of the unit step.

Define $\alpha = ((\operatorname{Re} s) - a)$ and assume α is positive. Then continuing we have

$$\begin{aligned} \int_0^T |f(t)e^{-st}| dt &\leq M \int_0^T e^{-\alpha t} dt \\ &= \frac{M}{\alpha} (1 - e^{-\alpha T}) \\ &\leq \frac{M}{\alpha}. \end{aligned}$$

Thus the limit

$$\lim_{T \rightarrow \infty} \int_0^T |f(t)e^{-st}| dt$$

exists, and hence so does

$$\lim_{T \rightarrow \infty} \int_0^T f(t)e^{-st} dt.$$

We conclude that $F(s)$ exists provided $(\operatorname{Re} s) - a > 0$, that is, $\operatorname{Re} s > a$. Thus the region of convergence (ROC) of the Laplace integral is an open right half-plane, as shown in Figure 3.1.

5. *Example.* The unit step $f(t) = \mathbf{1}_+(t)$. The smallest value of a such that (3.1) holds for some M is $a = 0$. The ROC, shown in Figure 3.2, is the open right half-plane $\operatorname{Re} s > 0$. For s

within the ROC, we can compute the Laplace transform as follows:

$$\begin{aligned}
 F(s) &= \int_0^{\infty} f(t)e^{-st} dt \\
 &= \int_0^{\infty} e^{-st} dt \\
 &= \left(-\frac{1}{s} e^{-st} \right)_{t=\infty} - \left(-\frac{1}{s} e^{-st} \right)_{t=0} \\
 &= \frac{1}{s}.
 \end{aligned}$$

Thus the Laplace transform of the unit step is $1/s$ and the ROC is $\text{Re } s > 0$. The ROC is an open right half-plane and the Laplace transform has a pole on the boundary of this half-plane, at $s = 0$. This pole is marked by an \times in Figure 3.2. Observe that the constant function $f(t) = 1$, for all t , has the same Laplace transform as the unit step.

6. *General remarks about the ROC.* The ROC is an open right half-plane. Open means that it does not contain its boundary. There are no poles inside the ROC. If the ROC is not equal to the entire complex plane, there is a pole on the boundary of the ROC. For example, if we know that

$$F(s) = \frac{1}{(s^2 + 1)(2s - 1)},$$

then we know that the ROC is $\text{Re } s > \frac{1}{2}$, because the poles are at $\pm j, \frac{1}{2}$. If we had thought that the ROC was to the right of the imaginary poles, we would have been wrong because then the pole at $\frac{1}{2}$ would have been inside the ROC. So, given $F(s)$, to find the ROC simply locate all the poles, draw a vertical line through the pole (or poles) that is farthest to the right; then the ROC is to the right of this vertical line. In this way the ROC of a Laplace integral is uniquely determined by the Laplace transform $F(s)$ itself, and therefore we don't need to record the ROC along with the function $F(s)$.

7. *Additional point.* For the unit step, the Laplace integral converges if and only if s lies in the ROC, the open right half-plane $\text{Re } s > 0$. But the function $1/s$ itself is well defined for every s except $s = 0$. Therefore, once we have the Laplace transform $F(s)$ we can, if we need to, permit s to take on values outside the ROC (except at poles). It's just that the Laplace integral won't converge for such s . This remark will later give justification for drawing the Bode plot of $1/(s - 1)$: The Bode plot is determined by taking $s = j\omega$, whereas this s , on the imaginary axis, lies outside the ROC.
8. *Example.* A blowing-up exponential. Let $f(t) = e^{2t}$. Clearly $f(t)$ satisfies (3.1) for $M = 1$, $a = 2$. The ROC is thus $\text{Re } s > 2$. For such s we compute that

$$\begin{aligned}
 F(s) &= \int_0^{\infty} e^{2t} e^{-st} dt \\
 &= \int_0^{\infty} e^{-(s-2)t} dt \\
 &= \frac{1}{s-2}.
 \end{aligned}$$

9. *Example.* A sinusoid. Let $f(t) = \sin(\omega t)$, where ω is a positive constant frequency. Obviously, $|f(t)| \leq 1$ for all $t \geq 0$, and therefore $f(t)$ satisfies (3.1) for $M = 1$, $a = 0$. The ROC is thus $\text{Re } s > 0$. To compute $F(s)$ it is convenient to use Euler's formula:

$$e^{j\theta} = \cos \theta + j \sin \theta.$$

This gives

$$f(t) = \sin(\omega t) = \frac{1}{2j} (e^{j\omega t} - e^{-j\omega t}).$$

Then

$$\begin{aligned} F(s) &= \int_0^\infty \frac{1}{2j} (e^{j\omega t} - e^{-j\omega t}) e^{-st} dt \\ &= \frac{1}{2j} \int_0^\infty (e^{(j\omega-s)t} - e^{-(j\omega+s)t}) dt \\ &= -\frac{1}{2j} \left(\frac{1}{j\omega-s} + \frac{1}{j\omega+s} \right) \\ &= \frac{\omega}{s^2 + \omega^2}. \end{aligned}$$

10. *Example.* A derivative. Regarding the Laplace transform of $\dot{f}(t)$, it is convenient to assume $f(t)$ does not have a jump at $t = 0$, for otherwise $f(t)$ is not differentiable at $t = 0$. Integrating by parts, we have

$$\begin{aligned} \int_0^\infty \dot{f}(t) e^{-st} dt &= f(t) e^{-st} \Big|_{t=0}^{t=\infty} + \int_0^\infty f(t) s e^{-st} dt \\ &= -f(0) + sF(s). \end{aligned}$$

In particular, if the initial value of $f(t)$ equals 0, i.e., $f(0) = 0$, then differentiating in the time domain corresponds to multiplying by s in the s -domain.

11. *Table.* On the next page is a short table of Laplace transforms. Longer ones can be found on the Web. You should derive some of the entries. Euler's formula is very helpful. For example

$$e^{at} \sin \omega t = \frac{1}{2j} (e^{(a+j\omega)t} - e^{(a-j\omega)t}).$$

Table of Laplace Transforms

$f(t)$	$F(s)$	
$1_+(t)$	$\frac{1}{s}$	unit step
e^{at}	$\frac{1}{s-a}$	
$\dot{f}(t)$	$sF(s) - f(0)$	valid if $f(t)$ is differentiable at $t = 0$
$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$	linearity
$f(t) * g(t)$	$F(s)G(s)$	convolution
t^n	$\frac{n!}{s^{n+1}}$	
$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$	
$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$	
$e^{at} \sin \omega t$	$\frac{\omega}{(s-a)^2 + \omega^2}$	
$e^{at} \cos \omega t$	$\frac{s-a}{(s-a)^2 + \omega^2}$	
$t \sin \omega t$	$\frac{2\omega s}{(s^2 + \omega^2)^2}$	
$t \cos \omega t$	$\frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$	

12. *Inversion.* The inversion problem is to find $f(t)$ given $F(s)$. This is normally done using a table. For example, given $F(s) = \frac{3s+17}{s^2-4}$, let us find $f(t)$. We use partial fraction expansion to get terms that are in the table:

$$F(s) = \frac{c_1}{s-2} + \frac{c_2}{s+2}, \quad c_1 = \frac{23}{4}, \quad c_2 = -\frac{11}{4}.$$

Then we invert $F(s)$ term by term using linearity of the Laplace transform (Section 3.2):

$$f(t) = c_1 e^{2t} + c_2 e^{-2t} = \frac{23}{4} e^{2t} - \frac{11}{4} e^{-2t}$$

We do not know the value of $f(t)$ for $t < 0$.

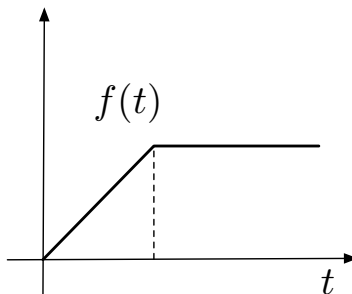


Figure 3.3: The signal ramps up to a constant value.

13. *Impulse.* You may have noticed that the table does not contain the unit impulse $\delta(t)$. We have elected not to include it for two reasons. First, it requires special handling. The impulse is neither piecewise continuous nor exponentially bounded. In fact, it is not even a real function because ∞ is not a number. There is a rigorous way to handle $\delta(t)$, but it is beyond the scope of this book. And without the rigorous framework it is not possible to answer, for example, if $\delta(t)1_+(t)$ is defined, if $\delta(t)^2$ is defined, if $\dot{\delta}(t)$ is defined, if the derivative of the step equals the impulse, and so on. The second reason for not including $\delta(t)$ is that it is not important in control engineering. The impulse is not a physical signal.

3.2 Linearity and convolution

1. *Linearity.* The Laplace transform maps a class of time-domain functions $f(t)$ into a class of complex-valued functions $F(s)$. The mapping $f(t) \mapsto F(s)$ is linear, that is, the Laplace transform of $a_1f_1(t) + a_2f_2(t)$ equals

$$a_1F_1(s) + a_2F_2(s).$$

2. *Example.* Let $f(t)$ denote the signal in Figure 3.3 that ramps up to the constant 1 at time $t = 1$. We shall use linearity to find the Laplace transform of this signal. We have $f = f_1 + f_2$, where $f_1(t) = t$, the unit ramp starting at time 0, and $f_2(t)$ is the ramp of slope -1 starting at time 1:

$$f_2(t) = \begin{cases} 0, & 0 \leq t \leq 1 \\ -f_1(t-1), & t > 1. \end{cases}$$

By linearity, $F(s) = F_1(s) + F_2(s)$. From the table or direct computation $F_1(s) = 1/s^2$. Also

$$\begin{aligned}
 F_2(s) &= \int_0^\infty f_2(t)e^{-st} dt \\
 &= - \int_0^\infty f_1(t-1)e^{-st} dt \\
 &= - \int_1^\infty f_1(t-1)e^{-st} dt \\
 &= - \int_0^\infty f_1(\tau)e^{-s(1+\tau)} d\tau \\
 &= -e^{-s}F_1(s) \\
 &= -e^{-s} \frac{1}{s^2}.
 \end{aligned}$$

Thus

$$F(s) = \frac{1 - e^{-s}}{s^2}.$$

3. *Superposition.* A special case of linearity is that the Laplace transform of a sum equals the sum of the Laplace transforms: If

$$f(t) = g(t) + h(t),$$

then

$$F(s) = G(s) + H(s).$$

4. *Multiplication.* What about multiplication? If

$$f(t) = g(t)h(t),$$

does it follow that

$$F(s) = G(s)H(s)?$$

Certainly not. Rather, multiplication in the s -domain corresponds to convolution in the time domain. For this treatment of convolution we shall assume that $g(t)$ and $h(t)$ equal zero for $t < 0$. Such signals are said to be **causal**. The definition of the **convolution** of $g(t)$ and $h(t)$ is

$$f(t) = \int_{-\infty}^\infty g(t-\tau)h(\tau)d\tau.$$

We see that $f(t)$ is causal too. The conventional notation for convolution is

$$f(t) = g(t) * h(t).$$

Actually, this notation is incorrect. It suggests that f at time t is obtained from g and h at time t . This is obviously false: f at time t depends on g and h over their whole domains of definition. The correct way to write convolution is $f = g * h$, or, if you want to show t , $f(t) = (g * h)(t)$. However, we shall stick to common practice and write $f(t) = g(t) * h(t)$.

5. *Theorem.* If $g(t)$ and $h(t)$ are causal and $f(t) = \int_{-\infty}^{\infty} g(t-\tau)h(\tau)d\tau$, then $F(s) = G(s)H(s)$.
6. *Proof.* All the integrals in this proof range from $-\infty$ to ∞ and hence we shall not write the limits. We have

$$F(s) = \int f(t)e^{-st}dt.$$

Substitute in the convolution integral for $f(t)$:

$$F(s) = \int \int g(t-\tau)h(\tau)d\tau e^{-st}dt.$$

Change the order of integration:

$$F(s) = \int \int g(t-\tau)h(\tau)e^{-st}dtd\tau.$$

Change $t - \tau$ to r :

$$F(s) = \int \int g(r)h(\tau)e^{-s(\tau+r)}drd\tau.$$

Bring $h(\tau)e^{-s\tau}$ outside the inner integral:

$$F(s) = \int \int g(r)e^{-sr}drh(\tau)e^{-s\tau}d\tau.$$

The inner integral equals $G(s)$, which can come outside the outer integral:

$$F(s) = G(s) \int h(\tau)e^{-s\tau}d\tau.$$

Thus

$$F(s) = G(s)H(s).$$

3.3 Pole locations

In this section we study how the poles of $F(s)$ affect the qualitative behaviour of $f(t)$. We will do this only for a special class of $F(s)$, namely, rational functions. We begin by defining what those are.

1. *Example.* We saw that the Laplace transform of the unit step $f(t) = \mathbf{1}_+(t)$ is $F(s) = 1/s$. This function of s is an example of a **rational function**, meaning it has a numerator and a denominator, both of which are polynomials. Other examples of rational functions are

$$\frac{1}{s}, \quad \frac{1}{s^2}, \quad \frac{s}{s^2 + 1}.$$

These are the Laplace transforms of, respectively,

$$1, \quad t, \quad \cos t.$$

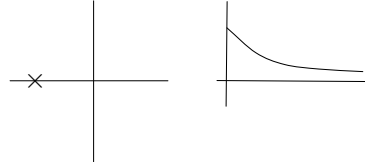
An example of a non-rational Laplace transform is e^{-s} .¹

¹Rational Laplace transforms where the numerator degree is not less than the denominator degree, such as 1 and s , have impulses in the time domain. For example, the inverse Laplace transform of s is $\delta(t)$.

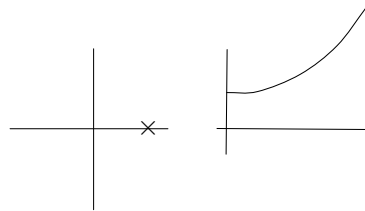
2. *Poles.* Let $F(s)$ be rational. Its poles are the values of s that make the denominator equal to zero. The locations of the poles give us information about the behaviour of $f(t)$. Consider for example the Laplace transform pair

$$f(t) = e^{-t}, \quad F(s) = \frac{1}{s+1}, \quad \text{pole} = -1.$$

We observe that a single negative pole corresponds to a decaying exponential in the time domain. This can be illustrated by the sketch



which depicts on the left the pole location in the complex plane and on the right the graph of $f(t)$. Likewise, this sketch



depicts that a single positive pole corresponds to a blowing-up exponential in the time domain.

3. *Correlation.* How pole locations are related to time-domain behaviour:
- (a) A single real negative pole corresponds to a decaying exponential. The farther left the pole is, the faster $f(t)$ decays.
 - (b) A single pole at $s = 0$ corresponds to a constant in the time domain.
 - (c) A single real positive pole corresponds to an exponential that blows up. The farther right the pole is, the faster $f(t)$ blows up.
 - (d) A complex conjugate pair of poles with $\text{Re } s < 0$ corresponds to a sinusoid with decaying amplitude.
 - (e) A complex conjugate pair of poles on the imaginary axis corresponds to a sinusoid with constant amplitude.
 - (f) A complex conjugate pair of poles with $\text{Re } s > 0$ corresponds to a sinusoid with amplitude that blows up.

Then there are the cases where the poles have higher multiplicity than one. We do three cases for illustration:

- (g) A double real negative pole corresponds to an amplitude-modified decaying exponential.

Example:

$$F(s) = \frac{1}{(s+1)^2} \implies f(t) = te^{-t}.$$

- (h) A double real positive pole corresponds to an amplitude-modified blowing-up exponential.

Example:

$$F(s) = \frac{1}{(s-1)^2} \implies f(t) = te^t.$$

- (i) A complex conjugate pair of poles on the imaginary axis of multiplicity two corresponds to a sinusoid with ramp-like amplitude. Example:

$$F(s) = \frac{1}{(s^2 + 1)^2} \implies f(t) = \frac{1}{2}(\sin t - t \cos t).$$

4. *Goodness.* Because signals that blow up are (usually) unwanted, to a control engineer the left half-plane is “good” and the right half-plane is “bad.”
5. *Caution.* As a final remark, it may have occurred to you that pole locations alone are a good indicator of a signal’s behaviour. Caution is advised. For example, suppose $F(s)$ has the poles $\{-2 \pm 10j, -10\}$. They may suggest severe oscillations in $f(t)$. But the inverse Laplace transform could actually be

$$f(t) = 50e^{-10t} + 0.001e^{-2t} \cos(10t),$$

in which case the “severe” oscillations will never be observed.

3.4 Bounded signals and the final-value theorem

In this section we continue our study of how pole locations affect qualitative behaviour in the time domain. The questions in this section are, when is $f(t)$ bounded and when does it have a final value, that is, when does $\lim_{t \rightarrow \infty} f(t)$ exist, and, if it does exist, what is the value of this limit? We shall answer these questions for the case that $F(s)$ is a rational function.

1. *Bounded.* A signal $f(t)$ is bounded if $|f(t)| \leq M$ for some M and all $t \geq 0$. For example, $\mathbf{1}_+(t)$ and $\cos(t)$ are bounded but e^t is not. Unbounded signals are potentially harmful. For example, an unbounded voltage in a circuit would eventually fry the electronics. On the other hand, if a radio signal is sent from earth toward a remote star, its distance from earth is (virtually) unbounded, but no harm is done. The final value of a signal $f(t)$ may be of importance in a control system. For example, $f(t)$ may be an error signal, in which case its final value is the steady-state error.
2. *Terminology.* Since $F(s)$ is rational, its numerator, $N(s)$, and its denominator, $D(s)$, have well-defined degrees. We say $F(s)$ is

strictly proper if $\text{degree } N(s) < \text{degree } D(s)$

proper if $\text{degree } N(s) \leq \text{degree } D(s)$

and

improper if $\text{degree } N(s) > \text{degree } D(s)$.

We shall assume $F(s)$ is strictly proper, for otherwise $f(t)$ contains an impulse and/or its derivative.

3. *Expansion.* By partial fraction expansion, we can uniquely write $F(s)$ as

$$F(s) = G_1(s) + G_2(s) + G_3(s),$$

where $G_1(s)$ has all the poles with negative real part, $G_2(s)$ has all the poles only at $s = 0$, and $G_3(s)$ has all the other poles.

4. *Example.*

$$\begin{aligned} F(s) &= G_1(s) + G_2(s) + G_3(s) \\ G_1(s) &= \frac{1}{s+1} + \frac{2}{(s+2)^2 + 20} \\ G_2(s) &= \frac{2}{s} + \frac{4}{s^2} \\ G_3(s) &= \frac{1}{s^2 + 1} + \frac{2}{s - 10}. \end{aligned}$$

From the table, the inverse Laplace transform is

$$\begin{aligned} f(t) &= g_1(t) + g_2(t) + g_3(t) \\ g_1(t) &= e^{-t} + \frac{2}{\sqrt{20}} e^{-2t} \sin \sqrt{20}t \\ g_2(t) &= 2 + 4t \\ g_3(t) &= \sin t + 2e^{10t}. \end{aligned}$$

Notice that $g_1(t)$ is bounded and converges to 0 and therefore it has a final value; $g_2(t)$ blows up because there's a repeated pole at $s = 0$; $g_3(t)$ blows up. So for this example, $f(t)$ is unbounded and does not have a final value.

5. *Example.* Modify the example to

$$\begin{aligned} F(s) &= G_1(s) + G_2(s) + G_3(s) \\ G_1(s) &= \frac{1}{s+1} + \frac{2}{(s+2)^2 + 20} \\ G_2(s) &= \frac{2}{s} \\ G_3(s) &= 0. \end{aligned}$$

Then $f(t)$ is bounded and does have a final value, namely, 2. Notice that the pole of $G_2(s)$ is simple, i.e., the multiplicity is 1. This example surely leads you to believe the following two facts.

6. *Fact. Characterization of boundedness.* Suppose $F(s)$ is rational and strictly proper.
- (a) If $F(s)$ has no poles in $\operatorname{Re} s \geq 0$, then $f(t)$ is bounded.
 - (b) If $F(s)$ has no poles in $\operatorname{Re} s \geq 0$ except a simple pole at $s = 0$ and/or some simple complex-conjugate pairs of poles on the imaginary axis, then $F(s)$ is bounded.
 - (c) In all other cases $f(t)$ is unbounded.
7. *Fact. Characterization of having a final value.* Suppose $F(s)$ is rational and strictly proper.
- (a) If $F(s)$ has no poles in $\operatorname{Re} s \geq 0$, then $f(t)$ has a final value, namely 0.
 - (b) If $F(s)$ has no poles in $\operatorname{Re} s \geq 0$ except a simple pole at $s = 0$, then $f(t)$ has a final value, which equals $\lim_{s \rightarrow 0} sF(s)$.
 - (c) In all other cases $f(t)$ does not have a final value.
8. *Be careful.* Remember that you have to *know* that $f(t)$ has a final value, by examining the poles of $F(s)$, before you can calculate $\lim_{s \rightarrow 0} sF(s)$ and claim it is the final value. Many students have been tricked by Problem 4.

3.5 Transfer functions

We return to a system with a state model:

$$\dot{x} = Ax + Bu, \quad y = Cx + Du. \quad (3.2)$$

The system is linear and, because A, B, C, D are constant matrices, time invariant. The input is u and the output is y . We are going to take Laplace transforms of these equations.

1. *Extension to vectors.* The signals u, x, y may be vectors. We define the Laplace transform of a vector $u(t)$ to be the vector of Laplace transforms. That is, if

$$u(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{bmatrix},$$

then we define

$$U(s) = \begin{bmatrix} U_1(s) \\ \vdots \\ U_m(s) \end{bmatrix},$$

Likewise for $X(s)$ and $Y(s)$. From the Laplace transform table, the Laplace transform of the derivative $\dot{x}(t)$ equals $sX(s) - x(0)$; this assumes sufficient smoothness. The **transfer function** of the system (3.2) is the function $G(s)$ satisfying $Y(s) = G(s)U(s)$ when we take Laplace transforms of (3.2) with $x(0) = 0$.

2. *Example.* Suppose the four state matrices are

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0.$$

Take Laplace transforms with zero initial conditions on x . We get

$$sX(s) = AX(s) + BU(s) \tag{3.3}$$

$$Y(s) = CX(s). \tag{3.4}$$

On the left-hand side of (3.3), $X(s)$ is multiplied by the variable s , while on the right-hand side $X(s)$ is multiplied by the matrix A . To be able to get a common coefficient of $X(s)$ we have to turn the coefficient s into a matrix. We do that via the 2×2 matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then $sX(s) = sIX(s)$. Using this in (3.3) gives

$$sIX(s) = AX(s) + BU(s).$$

Take $AX(s)$ to the left:

$$(sI - A)X(s) = BU(s). \tag{3.5}$$

Let us look at the matrix $sI - A$:

$$\begin{bmatrix} s & -1 \\ -1 & s \end{bmatrix}.$$

Recall that the inverse of a matrix equals the adjoint matrix divided by the determinant:

$$(sI - A)^{-1} = \frac{1}{\det(sI - A)} \operatorname{adj} (sI - A). \tag{3.6}$$

For this matrix, the adjoint equals

$$\begin{bmatrix} s & 1 \\ 1 & s \end{bmatrix}$$

and the determinant equals $s^2 - 1$. Thus we have found that $sI - A$ has an inverse:

$$(sI - A)^{-1} = \frac{1}{s^2 - 1} \begin{bmatrix} s & 1 \\ 1 & s \end{bmatrix}.$$

Therefore from (3.5)

$$X(s) = \frac{1}{\det(sI - A)} \operatorname{adj} (sI - A) BU(s).$$

To get $Y(s)$ we need to multiply $X(s)$ by the matrix C . Since $\frac{1}{\det(sI - A)}$ is a scalar, it commutes with C and we arrive at

$$Y(s) = \frac{1}{\det(sI - A)} C \operatorname{adj} (sI - A) BU(s)$$

and

$$\begin{aligned} \frac{1}{\det(sI - A)} C \operatorname{adj}(sI - A) B &= \frac{1}{s^2 - 1} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & 1 \\ 1 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{1}{s^2 - 1}. \end{aligned}$$

We have thus derived the transfer function from u to y :

$$Y(s) = G(s)U(s), \quad G(s) = \frac{1}{s^2 - 1}.$$

3. *Example.* An example use of this transfer function: Suppose the input is the unit step: $u(t) = \mathbf{1}_+(t)$. Then

$$\begin{aligned} Y(s) &= \frac{1}{s(s^2 - 1)} \\ &= \frac{1}{s(s+1)(s-1)} \\ &= -\frac{1}{s} + \frac{1/2}{s-1} + \frac{1/2}{s+1} \end{aligned}$$

and so

$$y(t) = -1 + \frac{1}{2}e^t + \frac{1}{2}e^{-t}, \quad t \geq 0.$$

To recap, in this example we took a state model and, for a step input, we found the output using Laplace transforms. Notice that the transfer function has a pole in the right half-plane; hence $Y(s)$ does too; and hence $y(t)$ does not have a final value.

4. *Extension.* Here the input and output are vectors. Suppose the four state matrices are

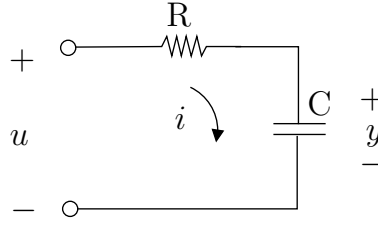
$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad D = 0.$$

You may derive that

$$G(s) = C(sI - A)^{-1}B = \frac{1}{s^2(s^2 + 2)} \begin{bmatrix} s^2 + 1 & 1 \\ 1 & s^2 + 1 \end{bmatrix}.$$

Since $Y(s) = G(s)U(s)$, we call $G(s)$ the transfer matrix from u to y . The dimension of both u and y is 2 and consequently $G(s)$ is a 2×2 matrix. Writing out the components of $Y(s)$, $G(s)$, and $U(s)$ gives

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}.$$

Figure 3.4: An RC filter.

5. *Summary.* For the system modeled by

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

there exists a transfer matrix $G(s)$ associated with (3.2), namely,

$$G(s) = C(sI - A)^{-1}B + D. \quad (3.7)$$

6. *State models.* Consider (3.7) in the single input, single output case ($G(s)$ is 1×1) and with $D = 0$. Then $G(s)$ has a numerator and a denominator and these are both polynomials. In view of (3.6) we have

$$\begin{aligned} G(s) &= C(sI - A)^{-1}B \\ &= \frac{1}{\det(sI - A)} C \operatorname{adj}(sI - A) B \\ &= \frac{N(s)}{D(s)}, \end{aligned}$$

where

$$N(s) = C \operatorname{adj}(sI - A) B, \quad D(s) = \det(sI - A). \quad (3.8)$$

If the high-frequency gain D is nonzero, then

$$N(s) = C \operatorname{adj}(sI - A) B + D \det(sI - A), \quad D(s) = \det(sI - A).$$

In rare pathological examples the two polynomials in (3.8) can have a common factor, which rightly should be cancelled. This pathological case won't occur in this course.

7. *Generality.* The method summarized in Paragraph 5 is general. However, some systems are sufficiently simple that we can get the transfer function directly, without first getting a state model.
8. *Example.* Figure 3.4 shows an RC filter. The circuit equations are

$$-u + Ri + y = 0, \quad i = C \frac{dy}{dt}.$$

Therefore

$$RC\dot{y} + y = u.$$

Apply Laplace transforms with zero initial conditions:

$$RCsY(s) + Y(s) = U(s).$$

Therefore the transfer function is

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{RCs + 1}.$$

Or, by the voltage-divider rule using impedances:

$$G(s) = \frac{\frac{1}{Cs}}{R + \frac{1}{Cs}} = \frac{1}{RCs + 1}.$$

The product RC has units of seconds and is called the **time constant** of the circuit or of the transfer function. The pole is at $s = -1/RC$ and therefore the smaller is the time constant, the farther left is the pole. The **DC gain** of the circuit is $G(0) = 1$. That is, if $u(t)$ is a constant voltage, then in steady state so is $y(t)$, and $y(t) = u(t)$. This is a lowpass circuit. If we had taken the output to be the voltage drop across the resistor, then the transfer function would have been

$$G(s) = \frac{RCs}{RCs + 1}.$$

This is a highpass circuit.

9. Some other transfer functions.

- (a) $G(s) = 2$ represents a pure gain. The output equals twice the input, for every input.
- (b) $G(s) = 1/s$ is the ideal integrator. The input and output are related by $y(t) = \int_{-\infty}^t u(\tau) d\tau$.
- (c) $G(s) = 1/s^2$ is the double integrator.
- (d) $G(s) = s$ is the differentiator. The input and output are related by $y(t) = \dot{u}(t)$. A differentiator can be at best an approximation to a real system. For example, if the input is $\sin(\omega t)$, then the output is $\omega \cos(\omega t)$. So as ω becomes larger and larger, the output amplitude becomes larger and larger too. This cannot happen in a real circuit.
- (e) $G(s) = e^{-\tau s}$ with $\tau > 0$ is a time delay system—note that the transfer function is not rational.
- (f)

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

is the standard second-order transfer function, where $\omega_n > 0$ is the natural frequency and $\zeta \geq 0$ is the damping constant. An *RLC* circuit has this transfer function.

(g)

$$G(s) = K_1 + \frac{K_2}{s} + K_3 s$$

is the transfer function of the proportional-integral-derivative (PID) controller. Notice that $G(s)$ is improper, since, if we put it in the form

$$G(s) = \frac{K_3 s^2 + K_1 s + K_2}{s},$$

we see that the numerator has higher degree than the denominator. Again, $G(s)$ can at best be only an approximation to a real system. A better approximation might be

$$G(s) = K_1 + \frac{K_2}{s} + \frac{K_3 s}{\varepsilon s + 1},$$

where ε is a small positive number.

10. *Summary.* General procedure for getting the transfer function of a system:

- (a) Apply the laws of physics etc. to get differential equations governing the behaviour of the system. Put these equations in state form. In general these are nonlinear.
- (b) Find an equilibrium, if there is one. If there are several equilibria, you have to select one. If there isn't even one, this method doesn't apply.
- (c) Linearize about the equilibrium point.
- (d) If the linearized system is time invariant, take Laplace transforms with zero initial state.
- (e) Solve for the output $Y(s)$ in terms of the input $U(s)$.

The transfer function from input to output satisfies

$$Y(s) = G(s)U(s).$$

In general $G(s)$ is a matrix: If $\dim u = m$ and $\dim y = p$ (m inputs, p outputs), then $G(s)$ is $p \times m$. In the single-input, single-output case, $G(s)$ is a scalar-valued transfer function.

11. *Realization.* There is a converse problem to getting the transfer function and that is, given a transfer function, to find a corresponding state model. That is, given $G(s)$, find A, B, C, D such that

$$G(s) = C(sI - A)^{-1}B + D..$$

The state model is called a **realization** of $G(s)$. The state matrices are never unique: Each $G(s)$ has an infinite number of state models. But it is a fact that every proper, rational $G(s)$ has a state realization. Let us see how to do this in the single-input/ single-output case, where $G(s)$ is 1×1 .

12. *Example.* First, a $G(s)$ with a constant numerator:

$$G(s) = \frac{1}{2s^2 - s + 3}.$$

Therefore the output and input are related in the s -domain by

$$\frac{Y(s)}{U(s)} = \frac{1}{2s^2 - s + 3},$$

or equivalently,

$$(2s^2 - s + 3)Y(s) = U(s).$$

Now go back to the time domain. We know from the table that $sY(s)$ maps to $\dot{y}(t)$. Likewise $s^2Y(s)$ maps to $\ddot{y}(t)$. Thus the corresponding differential equation model is

$$2\ddot{y} - \dot{y} + 3y = u.$$

Taking $x_1 = y, x_2 = \dot{y}$, we get

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{2}x_2 - \frac{3}{2}x_1 + \frac{1}{2}u \\ y &= x_1\end{aligned}$$

and thus

$$A = \begin{bmatrix} 0 & 1 \\ -3/2 & 1/2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}, \quad C = [1 \quad 0], \quad D = 0.$$

This technique extends to

$$G(s) = \frac{\text{constant}}{\text{polynomial of degree } n}.$$

13. *Example.* A nonconstant numerator:

$$G(s) = \frac{s - 2}{2s^2 - s + 3}.$$

Introduce an auxiliary signal $V(s)$:

$$Y(s) = (s - 2)V(s), \quad V(s) = \frac{1}{2s^2 - s + 3}U(s).$$

Then the transfer function from U to V has a constant numerator. We have

$$\begin{aligned}2\ddot{v} - \dot{v} + 3v &= u \\ y &= \dot{v} - 2v.\end{aligned}$$

Defining

$$x_1 = v, \quad x_2 = \dot{v},$$

we get

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{2}x_2 - \frac{3}{2}x_1 + \frac{1}{2}u \\ y &= x_2 - 2x_1\end{aligned}$$

and so

$$A = \begin{bmatrix} 0 & 1 \\ -3/2 & 1/2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} -2 & 1 \end{bmatrix}, \quad D = 0.$$

This extends to any strictly proper rational function.

14. *Proper but not strictly proper.* If the numerator and denominator of $G(s)$ have the same degree, then we can divide the numerator by the denominator and get

$$G(s) = c + G_1(s),$$

where c is a constant and $G_1(s)$ is strictly proper. In this case we get A, B, C to realize $G_1(s)$, and then just set $D = c$.

3.6 Stability

Stability is one of the most important concepts in this course. Everyone has an intuitive understanding of what stability and instability mean. In recent years we have seen the governments of many countries overthrown; if the constitution of a country is not widely accepted or the government does not adhere to the rule of law, instability can result. If you have cancer, your body is subject to rampant growth of cancer cells—this is a form of instability. If drivers on the highway drive too closely to the cars ahead, the system is unstable, for if one car decelerates quickly, a pileup can easily result. On the other hand, if you play a game of basketball, your heartrate will increase, but when the game is over, assuming you are healthy, your heartrate will return to normal.

1. *Intuitive definition.* Very roughly, a system is stable if it has these two properties: It returns to equilibrium after a perturbation of its state; the system can accommodate a persistent disturbance.
2. *Example.* Figure 3.5 shows a cart attached to the wall by a spring and damper and possibly subjected to wind gusts. If you do a free-body diagram of the cart, and account for the forces through the spring and the damper, from Newton's second law you will get

$$M\ddot{y} = d - Ky - D\dot{y}.$$

Let us simplify by taking the numerical values $M = K = D = 1$. Then

$$\ddot{y} = d - y - \dot{y}.$$

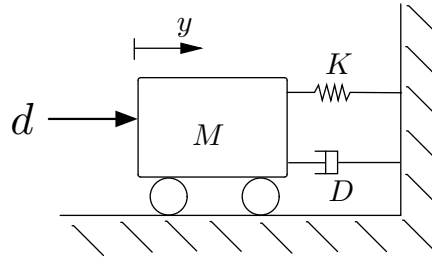


Figure 3.5: A cart attached to a wall and subject to wind gusts.

Taking, as usual, the state variables $x_1 = y$, $x_2 = \dot{y}$ and the state vector $x = (x_1, x_2)$, and taking the output to be y , lead to the state model

$$\begin{aligned}\dot{x} &= Ax + Ed \\ y &= Cx \\ A &= \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad E = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.\end{aligned}$$

There are two questions to be asked concerning stability for this system, which we treat one at a time.

3. *First question.* In the absence of an input d , will $x(t)$ converge to 0 as t goes to infinity for every $x(0)$? From our study of pole locations we know how to answer this question. Take Laplace transforms of $\dot{x} = Ax$. We get

$$sX(s) - x(0) = AX(s).$$

Solve for $X(s)$:

$$X(s) = (sI - A)^{-1}x(0).$$

Recall from (3.6) that the inverse of $sI - A$ equals the adjoint of the matrix divided by the determinant of the matrix. Thus for A as above

$$\begin{aligned}\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} &= \frac{1}{\det(sI - A)} \operatorname{adj}(sI - A) x(0) \\ &= \frac{1}{s^2 + s + 1} \begin{bmatrix} s + 1 & 1 \\ -1 & s \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}\end{aligned}$$

and so

$$X_1(s) = \frac{(s + 1)x_1(0) + x_2(0)}{s^2 + s + 1}, \quad X_2(s) = \frac{-x_1(0) + sx_2(0)}{s^2 + s + 1}.$$

We now apply Section 3.4, Paragraph 7. The zeros of $s^2 + s + 1$ have negative real part. Thus for every $x(0)$, the final value of $x(t)$ is zero.

4. *Second question.* If $d(t)$ is bounded, does it follow that $y(t)$ is bounded too? That is, can a bounded wind gust make the cart's position become unbounded? It seems unlikely. For this second question we may as well assume $x(0) = 0$ because we've addressed the initial condition response in the first question. Again, we take Laplace transforms:

$$\begin{aligned} sX(s) &= AX(s) + ED(s) \\ Y(s) &= CX(s). \end{aligned}$$

Solve for $Y(s)$:

$$\begin{aligned} Y(s) &= \frac{1}{\det(sI - A)} C \operatorname{adj}(sI - A) ED(s) \\ &= \frac{1}{s^2 + s + 1} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s+1 & 1 \\ -1 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} D(s) \\ &= \frac{1}{s^2 + s + 1} D(s). \end{aligned}$$

So the question reduces to, given

$$Y(s) = \frac{1}{s^2 + s + 1} D(s),$$

if $d(t)$ is bounded, does it follow that $y(t)$ is bounded? The answer is yes, from Section 3.4, Paragraph 6.

5. *General definition of stability.* For the system modeled by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du. \end{aligned}$$

This system is defined to be **stable** provided,

$$\text{with } u = 0 \text{ and } x(0) \text{ arbitrary, the final value of } x(t) \text{ equals } 0 \quad (3.9)$$

and

$$\text{with } x(0) = 0, y(t) \text{ is bounded for every bounded } u(t). \quad (3.10)$$

Notice, first, that (3.9) holds if and only if the zeros of the polynomial $\det(sI - A)$ all have negative real part, and, secondly, if (3.9) holds, then automatically so does (3.10).

6. *Terminology.* The preceding definition of stability is a little different from that in other books. Our definition combines two properties, namely, (3.9) and (3.10). In other books, these two properties are kept separate. Condition (3.9) is called asymptotic stability in the sense of Lyapunov and condition (3.10) is called bounded-input, bounded-output stability. If you go on to more advanced courses in control, in particular, a course in nonlinear control, you will have to separate the two conditions.
7. *More terminology.* You may recall from linear algebra that the polynomial $\det(sI - A)$ is called the **characteristic polynomial** of the matrix A . Also, the zeros of this polynomial equal the **eigenvalues** of A . Using the term eigenvalues, we have the following simple, concise theorem on stability.

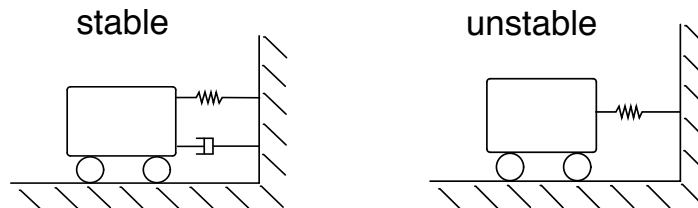


Figure 3.6: An undamped mass/spring system is unstable.

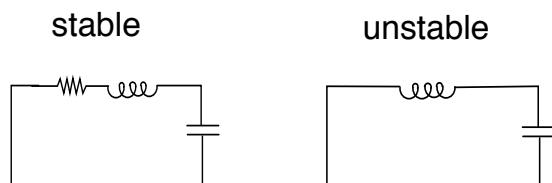


Figure 3.7: A lossless circuit is unstable.

8. *Theorem.* The system

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

is stable if and only if the eigenvalues of A all have negative real part.

9. *Examples without an input.* The system $\dot{x} = -x$ is stable. The system $\dot{x} = 0$ is unstable because $A = 0$ and its eigenvalue is not negative. The system $\dot{x} = x$ is unstable.

10. *Examples by pictures.* A cart in Figure 3.6 and a circuit in Figure 3.7.

11. *The Routh-Hurwitz criterion.* In practice, one checks feedback stability using MATLAB or Scilab to calculate the eigenvalues of A or, equivalently, the roots of the characteristic polynomial. However, it is sometimes useful, and also of historical interest, to have an easy test for simple cases. Consider a general characteristic polynomial with real coefficients:

$$p(s) = s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0.$$

Let us say that $p(s)$ is **stable** if all its zeros have $\text{Re } s < 0$. The Routh-Hurwitz criterion is an algebraic test for $p(s)$ to be stable, without having to calculate the zeros. Instead of studying the complete criterion, here are the results for $n = 1, 2, 3$:

(a) $p(s) = s + a_0$: $p(s)$ is stable, obviously, iff a_0 is positive.

(b) $p(s) = s^2 + a_1s + a_0$: $p(s)$ is stable iff a_0, a_1 are both positive.

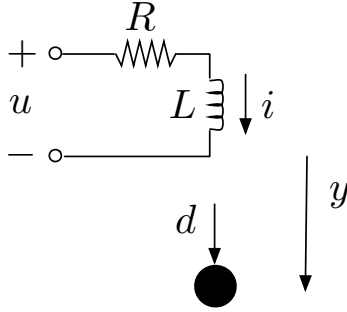


Figure 3.8: Magnetic levitation.

- (c) $p(s) = s^3 + a_2s^2 + a_1s + a_0$: $p(s)$ is stable iff a_0, a_1, a_2 are all positive and in addition $a_1a_2 > a_0$.
12. *Magnetic levitation.* Imagine an electromagnet suspending an iron ball—Figure 3.8. The ball is subject to gravity. A current passes through a coil, a magnetic field is produced, and the resulting magnetic force competes with the gravitational force on the ball. To balance the ball, the magnetic force must be exactly right, neither too strong nor too weak. As you can imagine, you could never adjust the voltage u by hand to balance the ball—feedback control is required. The signals are these: u is the voltage applied to the electromagnet, i is the current in the coil, y is the position down of the ball, and d is a possible disturbance force—for example, we may like to tap the ball while it's balanced. Let the input be the voltage u and the output the position y of the ball below the magnet. Then

$$L \frac{di}{dt} + Ri = u.$$

Also, it can be derived that the magnetic force on the ball has the form Ki^2/y^2 , K a constant. Thus

$$M\ddot{y} = Mg + d - K \frac{i^2}{y^2}.$$

Realistic numerical values are $M = 0.1$ Kg, $R = 15$ ohms, $L = 0.5$ H, $K = 0.0001$ Nm²/A², $g = 9.8$ m/s². Substituting in these numbers gives the equations

$$0.5 \frac{di}{dt} + 15i = u$$

$$0.1 \frac{d^2y}{dt^2} = 0.98 + d - 0.0001 \frac{i^2}{y^2}.$$

Define state variables $x = (x_1, x_2, x_3) = (i, y, \dot{y})$. Then the nonlinear state model is $\dot{x} = f(x, u, d)$, where

$$f(x, u, d) = (-30x_1 + 2u, x_3, 9.8 + 10d - 0.001x_1^2/x_2^2).$$

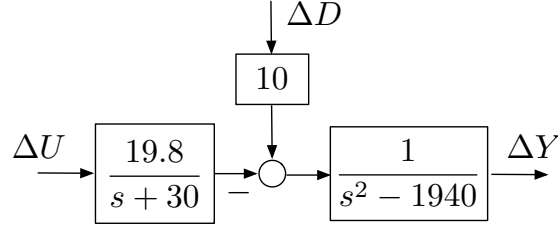


Figure 3.9: Block diagram of magnetic levitation system.

Suppose we want to stabilize the ball at $y = 1$ cm, or 0.01 m. We need a linear model valid in the neighbourhood of that value. Solve for the equilibrium point (x_0, u_0, d_0) where $x_{2_0} = 0.01$ and $d_0 = 0$:

$$-30x_{1_0} + 2u_0 = 0, \quad x_{3_0} = 0, \quad 9.8 - 0.001x_{1_0}^2/0.01^2 = 0.$$

Thus

$$x_0 = (0.99, 0.01, 0), \quad u_0 = 14.85.$$

The linearized model is

$$\dot{\Delta x} = A\Delta x + B\Delta u + E\Delta d, \quad \Delta y = C\Delta x,$$

where A equals the Jacobian of f with respect to x , evaluated at (x_0, u_0, d_0) , B equals the same except with respect to u , and E equals the same except with respect to d :

$$A = \begin{bmatrix} -30 & 0 & 0 \\ 0 & 0 & 1 \\ -0.002x_1/x_2^2 & 0.002x_1^2/x_2^3 & 0 \end{bmatrix}_{(\bar{x}, \bar{u})} = \begin{bmatrix} -30 & 0 & 0 \\ 0 & 0 & 1 \\ -19.8 & 1940 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \quad C = [0 \quad 1 \quad 0], \quad E = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}.$$

The eigenvalues of A are $-30, \pm 44.05$, the units being s^{-1} . The corresponding time constants are $1/30 = 0.033, 1/44.05 = 0.023$ s. The first is the time constant of the electric circuit; the second, the time constant of the magnetics. There are no complex conjugate eigenvalues because there is no oscillatory motion in the open-loop system. Take Laplace transforms with $\Delta x(0) = 0$:

$$\begin{aligned} \Delta Y(s) &= C(sI - A)^{-1}B\Delta U(s) + C(sI - A)^{-1}E\Delta D(s) \\ &= -\frac{19.8}{(s+30)(s^2-1940)}\Delta U(s) + \frac{10}{s^2-1940}\Delta D(s). \end{aligned}$$

The block diagram is shown in Figure 3.9. The polynomial $\det(sI - A)$ is $(s+30)(s^2-1940)$ and therefore, since there is a zero (of the polynomial, and hence a pole of the transfer function)

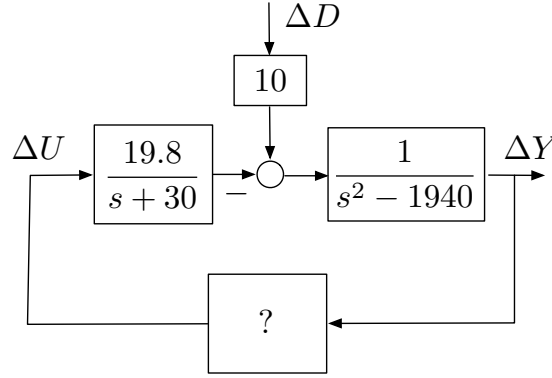


Figure 3.10: Block diagram of magnetic levitation system with a feedback controller.

at $\sqrt{1940}$ in the right half-plane, the system is unstable. To stabilize, we could contemplate using feedback, as in Figure 3.10. Can we design a controller (unknown box) so that the resulting system is stable? Let us see if we can stabilize using a pure gain, that is, by making the voltage variation $\Delta u(t)$ directly proportional to the ball's position error $\Delta y(t)$. Take $\Delta u = F\Delta y$. Then

$$\begin{aligned}\dot{\Delta x} &= A\Delta x + B\Delta u + E\Delta d \\ &= A\Delta x + BF\Delta y + E\Delta d \\ &= A\Delta x + BFC\Delta x + E\Delta d \\ &= (A + BFC)\Delta x + E\Delta d\end{aligned}$$

and thus the original matrix A has changed to the matrix $A + BFC$. Substituting in for A, B, C we get

$$\begin{aligned}A + BFC &= \begin{bmatrix} -30 & 0 & 0 \\ 0 & 0 & 1 \\ -19.8 & 1940 & 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} F \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -30 & 2F & 0 \\ 0 & 0 & 1 \\ -19.8 & 1940 & 0 \end{bmatrix}.\end{aligned}$$

The new characteristic polynomial is therefore

$$\det(sI - A - BFC) = s^3 + 30s^2 - 1940s + 30 \times 1940 - 39.6F.$$

It follows from the Routh-Hurwitz test that this can never be a stable polynomial because the coefficient of s is negative. So, unfortunately, by Paragraph 8 the magnetic levitation system cannot be stabilized by a pure gain controller; a more complex controller is required. We shall leave this example now and study feedback control in more depth.

3.7 Problems

1. Sketch the function

$$f(t) = \begin{cases} t+1, & 0 \leq t \leq 10 \\ -2e^t, & t > 10 \end{cases}$$

and find its Laplace transform, including the region of convergence.

Solution Let

$$f_1(t) = \begin{cases} t+1, & 0 \leq t \leq 10 \\ 0, & t > 10 \end{cases}, \quad f_2(t) = \begin{cases} 0, & 0 \leq t \leq 10 \\ -2e^t, & t > 10 \end{cases}.$$

so that $f = f_1 + f_2$, and therefore $F = F_1 + F_2$. Since both $F_1(s)$ and $F_2(s)$ must converge, the ROC of F equals the intersection of the ROCs of F_1 and F_2 . We have

$$\begin{aligned} F_1(s) &= \int_0^{10} (t+1)e^{-st} dt \\ &= \int_0^{10} te^{-st} dt + \int_0^{10} e^{-st} dt. \end{aligned}$$

Integrate the first term by parts:

$$\begin{aligned} F_1(s) &= \left(-\frac{1}{s}te^{-st} \Big|_0^{10} + \int_0^{10} \frac{1}{s}e^{-st} dt \right) + \int_0^{10} e^{-st} dt \\ &= -\frac{10}{s}e^{-10s} + \left(\frac{1}{s} + 1 \right) \int_0^{10} e^{-st} dt \\ &= -\frac{10}{s}e^{-10s} + \left(\frac{1}{s} + 1 \right) \frac{1}{s}(1 - e^{-10s}). \end{aligned}$$

The function f_2 is easier:

$$\begin{aligned} F_2(s) &= - \int_{10}^{\infty} 2e^{(1-s)t} dt \\ &= 2e^{10(1-s)} \frac{1}{1-s}. \end{aligned}$$

Thus

$$F(s) = -\frac{10}{s}e^{-10s} + \left(\frac{1}{s} + 1 \right) \frac{1}{s}(1 - e^{-10s}) + 2e^{10(1-s)} \frac{1}{1-s}.$$

The ROC is $\{s : \operatorname{Re} s > 1\}$.

2. (a) Find the inverse Laplace transform of $G(s) = \frac{1}{2s^2 + 1}$.
 (b) Repeat for $G(s) = \frac{1}{s^2}$.

(c) Repeat for $G(s) = \frac{s}{2s^2 + 1}$.

Solution These can be looked up from the table. For example, the inverse Laplace transform of

$$G(s) = \frac{1}{2s^2 + 1} = \frac{0.5}{s^2 + 0.5} = \frac{0.5}{s^2 + \sqrt{0.5}^2}$$

is a sinusoid of frequency $\sqrt{0.5}$: $g(t) = \sqrt{0.5} \sin \sqrt{0.5}t$. The inverse Laplace transform of $G(s) = \frac{1}{s^2}$ is a ramp: $g(t) = t$.

3. Explain why we don't use Laplace transforms for the system

$$\ddot{y}(t) + 2t\dot{y}(t) - y(t) = u(t).$$

Solution The Laplace transform of the product $t\dot{y}(t)$ is not equal to the product of the Laplace transforms of t and $\dot{y}(t)$, and therefore the Laplace transform technique does not convert the differential equation to an algebraic equation.

4. Find the final value of $f(t)$ when

$$F(s) = \frac{s^2 + 1}{s(s^3 + s + 1)}.$$

5. What is the ROC of the Laplace integral if

$$F(s) = \frac{s^2 + 1}{(s - 1)^2(2s + 3)}?$$

Solution The rightmost pole is at $s = 1$. Thus the ROC is the right half-plane to the right of the point 1.

6. Consider a mass-spring system where $M(t)$ is a known function of time. The equation of motion in terms of force input u and position output y is

$$\frac{d}{dt}M\dot{y} = u - Ky$$

(i.e., rate of change of momentum equals sum of forces), or equivalently

$$M\ddot{y} + \dot{M}\dot{y} + Ky = u.$$

This equation has time-dependent coefficients. So there's no transfer function $G(s)$, hence no impulse-response function $g(t)$, hence no convolution equation $y = g * u$. Find a linear state model.

Solution For a state model, take

$$x = (y, \dot{y}).$$

The state equation is

$$\dot{x} = Ax + Bu, \quad A = \begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{\dot{M}}{M} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix}.$$

7. Consider Problem 14 of Chapter 2. Find the transfer function from u to x_1 . Do it both by hand (from the state model) and by Scilab or MATLAB.

Solution

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 2 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad C = [1 \quad 0 \quad 0 \quad 0]$$

The transfer function is

$$C(sI - A)^{-1}B.$$

Computing by hand gives the transfer function

$$\frac{s^2 + 2s}{s^4 + 3s^3}.$$

The numerator and denominator have a common factor. When the common factor is cancelled, the transfer function reduces to

$$\frac{s + 2}{s^3 + 3s^2} = \frac{s + 2}{s^2(s + 3)}.$$

The following MATLAB code computes the numerator and denominator:

```
A=[0 0 1 0;0 0 0 1;0 0 -1 1;0 0 2 -2];
B=[0 0 1 0]';
C=[1 0 0 0];
[num,den]=ss2tf(A,B,C,0,1);
```

8. Find a state model (A, B, C, D) for the system with transfer function

$$G(s) = \frac{-2s^2 + s + 1}{s^2 - s - 4}.$$

9. Consider the parallel connection of G_1 and G_2 , the LTI systems with transfer functions

$$G_1(s) = \frac{10}{s^2 + s + 1}, \quad G_2(s) = \frac{1}{0.1s + 1}.$$

- (a) Find state models for G_1 and G_2 .
 - (b) Find a state model for the overall system.
10. The impedance of a 1-Farad capacitor is $G(s) = 1/s$. What is the inverse Laplace transform, $g(t)$? What is the Fourier transform of $g(t)$? Is the Fourier transform equal to $G(j\omega)$? Hint: No.

Solution From the Laplace transform table, the inverse Laplace transform of $1/s$ is the unit step, $g(t) = 1_+(t)$. The Fourier transform of $g(t)$ happens to be

$$\pi\delta(j\omega) + \frac{1}{j\omega}.$$

(You may have seen this in your Signals and Systems text.) Observe that merely setting $s = j\omega$ in the Laplace transform does not produce the Fourier transform. The following is true, but beyond the scope of the course: If $G_{LT}(s)$ is the Laplace transform of a signal $g(t)$ that is zero for $t < 0$, if the ROC of $G_{LT}(s)$ includes the imaginary axis, and if $G_{FT}(j\omega)$ is the Fourier transform of $g(t)$, then $G_{LT}(j\omega) = G_{FT}(j\omega)$.

11. Consider a system modeled by

$$\dot{x} = Ax + Bu, \quad y = Cx + Du,$$

where $\dim u = \dim y = 1$, $\dim x = 2$.

(a) Suppose that

$$u = 0, \quad x(0) = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \implies y(t) = e^{-t} - 0.5e^{-2t}$$

and

$$u = 0, \quad x(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \implies y(t) = -0.5e^{-t} - e^{-2t}.$$

Find $y(t)$ for $u = 0$, $x(0) = \begin{bmatrix} 2 \\ 0.5 \end{bmatrix}$.

(b) Now suppose

$$u \text{ a unit step, } x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \implies y(t) = 0.5 - 0.5e^{-t} + e^{-2t}$$

$$u \text{ a unit step, } x(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \implies y(t) = 0.5 - e^{-t} + 1.5e^{-2t}.$$

Find $y(t)$ when u is a unit step and $x(0) = 0$.

Solution This is an exercise in using linearity. In the first part, the three initial conditions are related by

$$\begin{bmatrix} 2 \\ 0.5 \end{bmatrix} = \frac{5}{3} \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Thus

$$y(t) = \frac{5}{3}(e^{-t} - 0.5e^{-2t}) - \frac{1}{3}(-0.5e^{-t} - e^{-2t}).$$

In the second part

$$y(t) = 2(e^{-t} - 0.5e^{-2t}) - (-0.5e^{-t} - e^{-2t}).$$

12. Consider $\dot{x} = Ax$ with

$$A = \begin{bmatrix} 0 & -2 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Is the system stable?

Solution The eigenvalues of A are $-1, \pm j$. Thus, no.

13. Consider $\dot{x} = Ax$ with

$$A = \begin{bmatrix} 0 & 1 \\ a & 0 \end{bmatrix}.$$

Is there a constant a for which the system is stable?

Solution The characteristic polynomial is $s^2 - a$. By Routh-Hurwitz, the polynomial has a zero in $\text{Re } s \geq 0$ for every a . Thus, no.

14. Consider the cart-spring system. Its equation of motion is

$$M\ddot{y} + Ky = 0,$$

where $M > 0$, $K > 0$. Take the natural state, $x = (y, \dot{y})$, and prove that the system is unstable.

Solution The characteristic polynomial is $s^2 + (K/M) = 0$. The roots are purely imaginary.

15. The transfer function of an LC circuit is $G(s) = 1/(LCs^2 + 1)$. Show that the circuit is unstable.

Solution In a state model the matrix A will have imaginary eigenvalues.

16. A rubber ball is tossed straight into the air, rises, then falls and bounces from the floor, rises, falls, and bounces again, and so on. Let c denote the coefficient of restitution, that is, the ratio of the velocity just after a bounce to the velocity just before the bounce. Thus $0 < c < 1$. Neglecting air resistance, show that there are an infinite number of bounces in a finite time interval.

Hint: Assume the ball is a point mass. Let $x(t)$ denote the height of the ball above the floor at time t . Then $x(0) = 0$, $\dot{x}(0) > 0$. Model the system before the first bounce and calculate the time of the first bounce. Then specify the values of x, \dot{x} just after the first bounce. And so on.

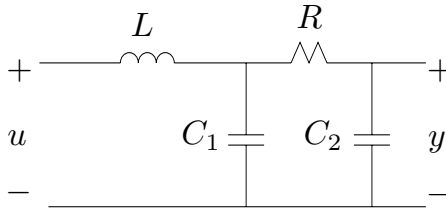
17. The linear system $\dot{x} = Ax$ can have more than one equilibrium point. Characterize the set of equilibrium points. Give an example A for which there's more than one.

Solution The equilibria are the vectors x satisfying $Ax = 0$. The set of such vectors is called the nullspace, or kernel, of A . For the matrix

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

every vector is an equilibrium.

18. Derive a state model for the circuit



Is the circuit stable?

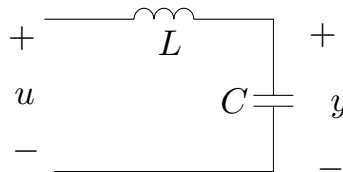
19. Consider a system modeled by the nonlinear state equation $\dot{x} = f(x, u)$, where the state x has dimension 2, the input u has dimension 1, and where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad f(x, u) = \begin{bmatrix} x_1^2 + x_1 x_2 + 2u \\ -x_1^2 x_2 + u \end{bmatrix}.$$

The solution $\bar{x}_1 = 0$, $\bar{x}_2 = 1$, $\bar{u} = 0$ is an equilibrium point. Linearize the system about this equilibrium point, ending up with an equation of the form

$$\Delta \dot{x} = A \Delta x + B \Delta u.$$

20. Consider the LC circuit



with input voltage $u(t)$ and output voltage $y(t)$.

- Find the transfer function from u to y .
 - Is y bounded when u is the unit step?
 - Give a bounded input u for which y is not bounded.
21. Refer to the standard feedback system. Find the final value of $e(t)$ (if it exists) for this data:

$$r(t) = 1_+(t), \quad d(t) = 0, \quad P(s) = \frac{s+2}{s+1}, \quad C(s) = 1.$$

Repeat for

$$r(t) = e^{-t} 1_+(t), \quad d(t) = 0, \quad P(s) = \frac{s-2}{s+1}, \quad C(s) = 1.$$

Repeat for

$$r(t) = 0, \quad d(t) = t1_+(t), \quad P(s) = \frac{1}{s(s+3)}, \quad C(s) = 10.$$

22. Consider the system with transfer function $1/(s+1)$. Let the input be a periodic square wave of minimum value 0 and peak value 1 (i.e., its DC value is $1/2$), and of period 5 seconds. Find the Fourier series of this input in the form

$$\sum_k a_k e^{j\omega_k t}.$$

That is, find the frequencies ω_k and the coefficients a_k . By linearity and time-invariance, the output has the form

$$\sum_k b_k e^{j\omega_k t}.$$

What are the constants b_k ?

Chapter 4

The Feedback Loop and its Stability

Control systems are most often based on the principle of feedback, whereby the signal to be controlled is compared to a desired reference signal and the discrepancy used to compute corrective control action. When you go from your home to the university you use this principle continuously without even thinking about it. To emphasize how effective feedback is, imagine you have to program a mobile robot, with no vision capability and no GPS, and therefore no feedback, to go open loop from your home to your university classroom; the program has to include all motion instructions to the motors that drive the robot: Turn the right wheel 2.7285 radians, then turn both wheels 6.522 radians, and so on. The program would be unthinkableably long, and in the end the robot would be way off target, as depicted in Figure 4.1, because the initial heading would be at least slightly off.

In this chapter and the next we develop the basic theory and tools for feedback control analysis and design in the frequency domain. “Analysis” means you already have a controller and you want to study how good (or bad) it is; “design” of course means you want to design a controller to meet certain specifications. The most fundamental specification is stability. Typically, good performance requires high-gain controllers, yet typically the feedback loop will become unstable if the gain is too high. The stability criterion we will study is the Nyquist criterion, dating from 1932. The Nyquist criterion is a graphical technique involving the open-loop frequency response function, magnitude and phase.

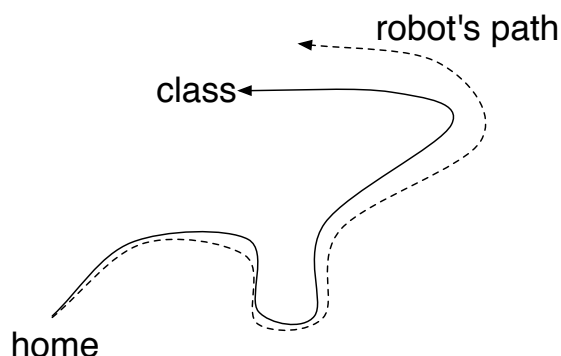


Figure 4.1: How feedback works.

There are two main approaches to control analysis and design. The first, the one we are doing in this course, is the older, so-called “classical” approach in the frequency domain. Specifications are based on closed-loop gain, bandwidth, and stability margin. Design is done using Bode plots.¹ The second approach, which is the subject of a second course on control, is in the time domain and uses state-space models instead of transfer functions. Specifications may be based on closed-loop poles. This second approach is known as the “state-space approach” or “modern control”, although it dates from the 1960s and 1970s.

These two approaches are complementary. Classical control is appropriate for a single-input, single-output plant, especially if it is open-loop stable. The state-space approach is appropriate for multi-input, multi-output plants; it is especially powerful in providing a methodical procedure to stabilize an unstable plant. Stability margin is very transparent in classical control and less so in the state-space approach. Of course, simulation must accompany any design approach. For example, in classical control you typically design for a desired bandwidth and stability margin; you test your design by simulation; you evaluate, and then perhaps modify the stability margin, redesign, and test again.

Beyond these two approaches is optimal control, where the controller is designed by minimizing a mathematical function. In this context classical control extends to \mathcal{H}^∞ optimization and state-space control extends to Linear-quadratic Gaussian (LQG) control.

For all these techniques of analysis and design there are computer tools, the most popular being the Control System Toolbox of MATLAB.

4.1 The cart-pendulum example

In this section we use the cart-pendulum example to introduce the important concept of stability of a feedback system.

1. *The setup.* Figure 4.2 shows the linearized cart-pendulum. The figure defines x_1, x_2 . Now define $x_3 = \dot{x}_1$, $x_4 = \dot{x}_2$. Take $M_1 = 1$ kg, $M_2 = 2$ kg, $L = 1$ m, and $g = 9.8$ m/s². Then the state model is

$$\dot{x} = A_p x + B_p u, \quad A_p = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -19.6 & 0 & 0 \\ 0 & 29.4 & 0 & 0 \end{bmatrix}, \quad B_p = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

Subscript p stands for “plant.” Let us designate the cart position as the only output: $y = x_1$. Then

$$C_p = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}.$$

¹One classical technique is called “root locus.” A root locus is a graph of the closed-loop poles as a function of a single real parameter, for example a controller gain. In root locus you try to get the closed-loop poles into a desired region (with good damping, say) by manipulating controller poles and zeros. You could easily pick this up if you wish using MATLAB’s *rltool*.

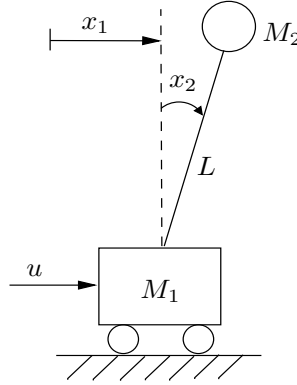
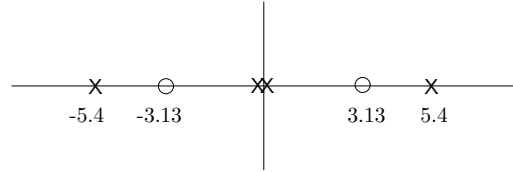


Figure 4.2: Cart-pendulum.

Figure 4.3: Poles and zeros of $P(s)$.

The transfer function from u to y is

$$\begin{aligned}
 P(s) &= C_p(sI - A_p)^{-1}B_p \\
 &= \frac{1}{\det(sI - A_p)} C_p \operatorname{adj}(sI - A_p) B_p \\
 &= \frac{s^2 - 9.8}{s^2(s^2 - 29.4)}.
 \end{aligned}$$

The poles and zeros of $P(s)$ are shown in Figure 4.3. Having three poles in $\operatorname{Re} s \geq 0$, the plant is quite unstable. The right half-plane zero doesn't contribute to the degree of instability, but, as we shall see, it does make the plant quite difficult to control. We can understand intuitively why that is so: To go forward, the cart must initially go backward in order to tilt the pendulum forward. The block diagram of the plant by itself is shown in Figure 4.4.

2. *Wrong notion.* It might occur to you to stabilize the cart-pendulum merely by canceling the unstable poles, that is, by multiplying $P(s)$ by a controller transfer function $C(s)$ so that the product $P(s)C(s)$ has no right half-plane poles. **This will not work; the standard feedback loop will not be stable.** Instead of canceling the unstable poles, we must move them into the left half-plane by means of feedback. We will explain this very important point during the remainder of this chapter.

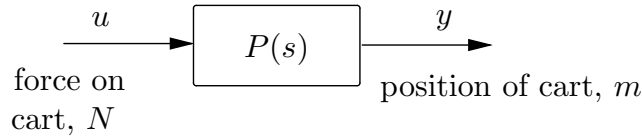


Figure 4.4: Block diagram of the plant.

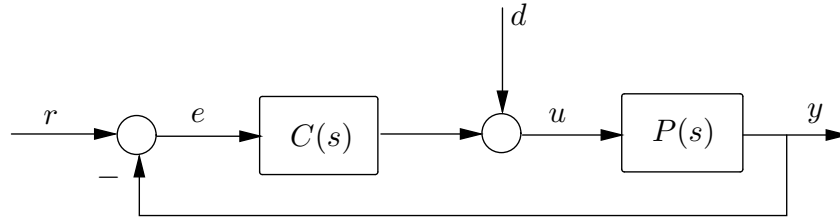


Figure 4.5: Feedback control of the cart-pendulum.

3. *Correct way.* Let us consider stabilizing the plant by feeding back the cart position, y , comparing it to a reference r , and setting the error $r - y$ as the controller input, as shown in Figure 4.5. Here $C(s)$ is the transfer function of the controller to be designed. The signal d is a possible disturbance force on the cart.

4. *A solution.* One controller that does in fact stabilize is

$$C(s) = \frac{10395s^3 + 54126s^2 - 13375s - 6687}{s^4 + 32s^3 + 477s^2 - 5870s - 22170}.$$

This controller was obtained by a more advanced method than is considered in this course. A state realization of $C(s)$ is

$$C(s) = C_c(sI - A_c)^{-1}B_c$$

$$A_c = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 22170 & 5870 & -477 & -32 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C_c = \begin{bmatrix} -6687 & -13375 & 54126 & 10395 \end{bmatrix}.$$

The controller itself, $C(s)$, is unstable, as is $P(s)$. But when the controller and plant are connected in feedback, then, as long as the loop remains unbroken, the standard feedback loop is stable. If the pendulum starts to fall, the controller causes the cart to move, in

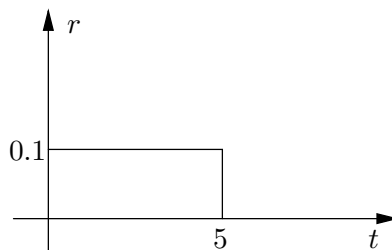
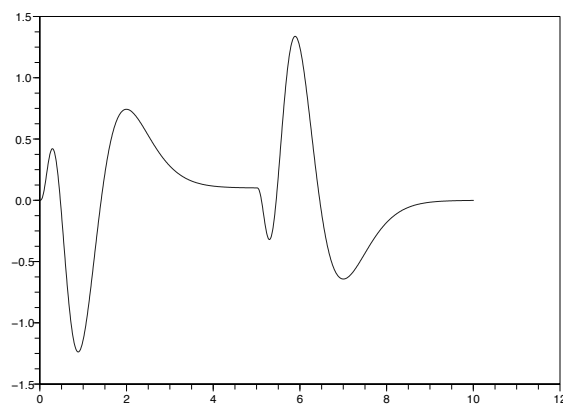
Figure 4.6: Test input r .

Figure 4.7: Plot of the cart position vs. time.

the right direction, to make the pendulum tend to come vertical again. You're invited to simulate the standard feedback loop; for example, let r be the signal shown in Figure 4.6. This corresponds to a command that the cart move right 0.1 m for 5 seconds, then return to its original position. Figure 4.7 is a plot of the cart position x_1 versus t . The cart moves rather wildly as it tries to balance the pendulum—it is not a **good** controller design—but it does stabilize.

5. *Unstable controller.* We mentioned that our controller $C(s)$ is open-loop unstable. It can be proved (it is beyond the scope of this course) that **every** controller that stabilizes this $P(s)$ is itself unstable. The general result² is this: There exists a stable $C(s)$ that stabilizes an unstable $P(s)$ if and only if this **parity test** holds: between every pair of real zeros of $P(s)$ in $\text{Re } s \geq 0$ there exists an even number of poles. For the cart-pendulum,

$$P(s) = \frac{s^2 - 9.8}{s^2(s^2 - 29.4)}.$$

²A proof can be found in *Feedback Control Theory*, by J. Doyle, B. Francis, and A. Tannenbaum.

We count the point $s = +\infty$ as a real zero for this result, so the real zeros in $\text{Re } s \geq 0$ are $\sqrt{9.8}, +\infty$. Between these two zeros is just one pole $\sqrt{29.4}$, not an even number of poles. Thus $P(s)$ fails the parity test.

6. *Feedback stability.* Let us define what it means for the standard feedback loop in Figure 4.5 to be stable. For this, we choose to view the system as having inputs (r, d) and outputs (e, u) , although, as we shall see, these choices don't affect the definition of stability. Take the states of $P(s)$ and $C(s)$ to be, respectively, x_p and x_c , and then take the state of the combined standard feedback loop to be $x_{cl} = (x_p, x_c)$. You can derive this model:

$$\dot{x}_{cl} = A_{cl}x_{cl} + B_{cl} \begin{bmatrix} r \\ d \end{bmatrix} \quad (4.1)$$

$$\begin{bmatrix} e \\ u \end{bmatrix} = C_{cl}x_{cl} + \begin{bmatrix} r \\ d \end{bmatrix} \quad (4.2)$$

$$A_{cl} = \begin{bmatrix} A_p & B_p C_c \\ -B_c C_p & A_c \end{bmatrix}, \quad B_{cl} = \begin{bmatrix} 0 & B_p \\ -B_c & 0 \end{bmatrix}$$

$$C_{cl} = \begin{bmatrix} -C_p & 0 \\ 0 & C_c \end{bmatrix}.$$

According to our definition in Section 3.6, Paragraph 8, this system is stable if and only if the eigenvalues of A_{cl} all lie in the open left half-plane. You can check this is true using this MATLAB code:

```
A_c=[0 1 0 0;0 0 1 0;0 0 0 1;22170 5870 -477 -32];
B_c=[0;0;0;1];
C_c=[-6687 -13375 54126 10395];
A_p=[0 0 1 0;0 0 0 1;0 -19.6 0 0;0 29.4 0 0];
B_p=[0;0;1;-1];
C_p=[1 0 0 0];

A_cl=[A_p B_p*C_c ; -B_c*C_p A_c];
eig(A_cl)
```

7. *Summary.* The stability of the standard feedback loop is characterized by the eigenvalues of the closed-loop A -matrix, A_{cl} . Since stability depends only on the A_{cl} matrix, this confirms that the choice of inputs and outputs in Paragraph 6 was not important. In fact we could have set r, d to zero and derived the equation $\dot{x}_{cl} = A_{cl}x_{cl}$.

4.2 The standard feedback loop

We continue with the block diagram in Figure 4.8.

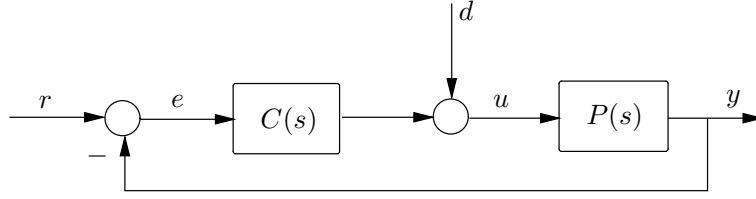


Figure 4.8: Standard feedback loop.

1. *Notation.* The notation is this: $P(s)$ is the plant transfer function, $C(s)$ is the controller transfer function, $r(t)$ is a reference (or command) input, $e(t)$ is the tracking error, $d(t)$ is a disturbance, $u(t)$ is the plant input, and $y(t)$ is the plant output. From now on the plant and controller are single-input, single output. That is, $P(s)$ and $C(s)$ are not matrices. We shall **assume throughout** that $P(s)$, $C(s)$ are rational, $C(s)$ is proper, and $P(s)$ is strictly proper.
2. *From now on.* We are going to emphasize transfer function models instead of state models. But it is important to understand that the two models are more-or-less equivalent.
3. *Closed-loop transfer functions.* We now derive the transfer function model for the standard feedback loop. As in the preceding section, we choose to view the system as having inputs (r, d) and outputs (e, u) . Write the equations in the Laplace domain for the outputs of the summing junctions:

$$\begin{aligned} E &= R - PU \\ U &= D + CE. \end{aligned}$$

Assemble into a vector equation:

$$\begin{bmatrix} 1 & P \\ -C & 1 \end{bmatrix} \begin{bmatrix} E \\ U \end{bmatrix} = \begin{bmatrix} R \\ D \end{bmatrix}. \quad (4.3)$$

Now bring in numerators and denominators:

$$P = \frac{N_p}{D_p}, \quad C = \frac{N_c}{D_c}. \quad (4.4)$$

There is no reason why N_p and D_p would have common factors; likewise for N_c and D_c . But it could happen that N_p and D_c have a common factor, or that N_c and D_p have a common factor. Substitute (4.4) into (4.3) and clear fractions:

$$\begin{bmatrix} D_p & N_p \\ -N_c & D_c \end{bmatrix} \begin{bmatrix} E \\ U \end{bmatrix} = \begin{bmatrix} D_p & 0 \\ 0 & D_c \end{bmatrix} \begin{bmatrix} R \\ D \end{bmatrix}.$$

Invert the matrix on the left:

$$\begin{aligned} \begin{bmatrix} E \\ U \end{bmatrix} &= \frac{1}{D_p D_c + N_p N_c} \begin{bmatrix} D_c & -N_p \\ N_c & D_p \end{bmatrix} \begin{bmatrix} D_p & 0 \\ 0 & D_c \end{bmatrix} \begin{bmatrix} R \\ D \end{bmatrix} \\ &= \frac{1}{D_p D_c + N_p N_c} \begin{bmatrix} D_p D_c & -D_c N_p \\ D_p N_c & D_p D_c \end{bmatrix} \begin{bmatrix} R \\ D \end{bmatrix} \end{aligned} \quad (4.5)$$

On the other hand, the state model for the same system has the form (see (4.1) and (4.2))

$$\begin{aligned} \dot{x}_{cl} &= A_{cl} x_{cl} + B_{cl} \begin{bmatrix} r \\ d \end{bmatrix} \\ \begin{bmatrix} e \\ u \end{bmatrix} &= C_{cl} x_{cl} + D_{cl} \begin{bmatrix} r \\ d \end{bmatrix} \end{aligned}$$

from which

$$\begin{aligned} \begin{bmatrix} E \\ U \end{bmatrix} &= \left(\frac{1}{\det(sI - A_{cl})} C_{cl} \operatorname{adj}(sI - A_{cl}) B_{cl} + D_{cl} \right) \begin{bmatrix} R \\ D \end{bmatrix} \\ &= \frac{1}{\det(sI - A_{cl})} [C_{cl} \operatorname{adj}(sI - A_{cl}) B_{cl} + \det(sI - A_{cl}) D_{cl}] \begin{bmatrix} R \\ D \end{bmatrix} \end{aligned}$$

Comparing this and (4.5) we conclude that the two polynomials

$$D_p D_c + N_p N_c, \quad \det(sI - A_{cl})$$

are equivalent, in that they have the same zeros. Therefore we are justified in calling $D_p D_c + N_p N_c$, product of denominators plus product of numerators, the **closed-loop characteristic polynomial**.

4. *Theorem.* The closed-loop feedback system is stable if and only if the zeros of $D_p D_c + N_p N_c$ are all in the open left half-plane.
5. *Examples.* Consider

$$P(s) = \frac{1}{s-1}, \quad C(s) = K.$$

The plant is open-loop unstable, having a pole at $s = 1$. The characteristic polynomial of the feedback system is $s - 1 + K = s + K - 1$. Thus the feedback system is stable if and only if $K > 1$. As another example take

$$P(s) = \frac{1}{s^2 - 1}, \quad C(s) = \frac{s-1}{s+1}.$$

The plant has an unstable pole at $s = 1$, which the controller cancels. But the closed-loop characteristic polynomial is

$$(s^2 - 1)(s + 1) + (s - 1),$$

which has a zero at $s = 1$. Thus the feedback system is unstable.

6. *Robustness.* Consider Figure 4.8 and assume the closed loop is stable. If we slightly perturb the coefficients in the numerator or denominator, or both, of $P(s)$, the closed loop will still be stable. This follows from the mathematical fact that the zeros of a polynomial are continuous functions of the coefficients of the polynomial. So if all the zeros satisfy $\text{Re } s < 0$, they continue to do so when the coefficients are perturbed. We say that feedback stability is a *robust* property of a feedback loop.
7. *Important point.* An unstable plant cannot be stabilized by cancelling unstable poles. If $P(s)$ has a pole at $s = p$ with non-negative real part, and if $C(s)$ cancels that pole, the characteristic polynomial will still have a zero at $s = p$. Unstable plants can be stabilized only by moving unstable poles into the left half-plane.
8. *Closed-loop transfer functions* for Figure 4.8: Solve (4.3):

$$\begin{bmatrix} 1 & P \\ -C & 1 \end{bmatrix} \begin{bmatrix} E \\ U \end{bmatrix} = \begin{bmatrix} R \\ D \end{bmatrix}$$

becomes

$$\begin{bmatrix} E \\ U \end{bmatrix} = \begin{bmatrix} 1 & P \\ -C & 1 \end{bmatrix}^{-1} \begin{bmatrix} R \\ D \end{bmatrix}.$$

The solution is

$$\begin{bmatrix} E \\ U \end{bmatrix} = \begin{bmatrix} \frac{1}{1+PC} & -\frac{P}{1+PC} \\ \frac{C}{1+PC} & \frac{1}{1+PC} \end{bmatrix} \begin{bmatrix} R \\ D \end{bmatrix}.$$

So, for example, the transfer function from D to E is $-P/(1+PC)$.

9. *General case.* How to find closed-loop transfer functions in general: Figure 4.9 shows a somewhat complex block diagram. Suppose we want to find the transfer function from r to y . In the Laplace domain the block diagram is a graphical representation of algebraic equations. We merely write and then solve the equations. First, label using the symbol x_i (in the time domain) the outputs of the summing junctions. Then write the equations at the summing junctions:

$$\begin{aligned} X_1 &= R - P_2 P_5 X_2 \\ X_2 &= P_1 X_1 - P_2 P_4 X_2 \\ Y &= P_3 X_1 + P_2 X_2. \end{aligned}$$

Assemble as

$$\begin{bmatrix} 1 & P_2 P_5 & 0 \\ -P_1 & 1 + P_2 P_4 & 0 \\ -P_3 & -P_2 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ Y \end{bmatrix} = \begin{bmatrix} R \\ 0 \\ 0 \end{bmatrix}.$$

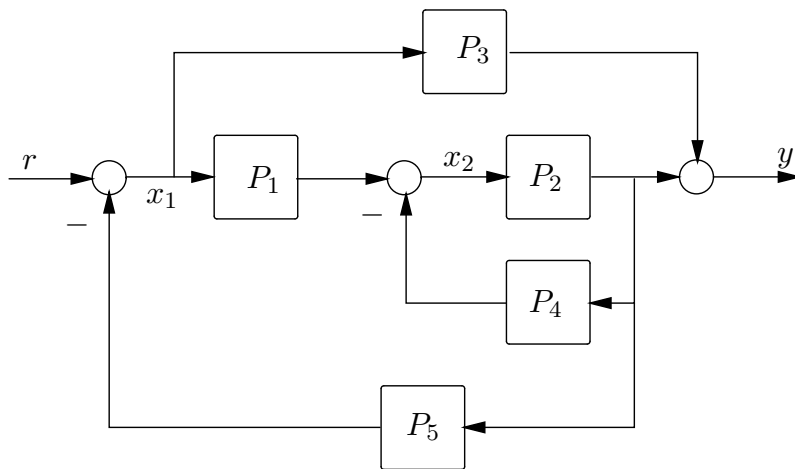


Figure 4.9: Complex block diagram.

Solve for Y by Cramer's rule:

$$Y = \frac{\det \begin{bmatrix} 1 & P_2 P_5 & R \\ -P_1 & 1 + P_2 P_4 & 0 \\ -P_3 & -P_2 & 0 \end{bmatrix}}{\det \begin{bmatrix} 1 & P_2 P_5 & 0 \\ -P_1 & 1 + P_2 P_4 & 0 \\ -P_3 & -P_2 & 1 \end{bmatrix}}.$$

Simplify:

$$Y = GR, \quad G = \frac{P_1 P_2 + P_3(1 + P_2 P_4)}{1 + P_2 P_4 + P_1 P_2 P_5}.$$

4.3 Tracking a reference signal

In this section we study Figure 4.10 and the performance requirement that the plant output $y(t)$ should follow a specified reference signal $r(t)$. Of course there is the additional requirement of stability of the feedback loop—that is always a requirement.

1. *Cruise control.* In cruise control in a car, you set the reference speed, say 100 km/h, and a controller regulates the speed to a prescribed setpoint. How does this work? The answer lies in the final-value theorem. As a simple example, suppose

$$P(s) = \frac{1}{s+1}, \quad C(s) = 1, \quad R(s) = \frac{100}{s}.$$

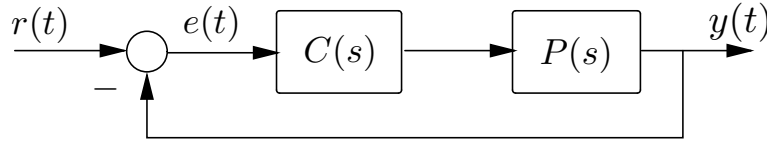


Figure 4.10: Setup for reference tracking.

The closed-loop characteristic polynomial is $s + 2$ and hence the feedback loop is stable. The transfer function from R to E is

$$\frac{1}{1 + P(s)C(s)} = \frac{s + 1}{s + 2}.$$

Thus

$$E(s) = \frac{100(s + 1)}{s(s + 2)}.$$

Hence the steady-state tracking error is $\lim_{t \rightarrow \infty} e(t) = 50$. If we change the controller to $C(s) = 50$, then $\lim_{t \rightarrow \infty} e(t) = 100/51 \approx 2$. So increasing the controller gain from 1 to 50 reduces the tracking error from 50 to 2. High controller gain seems to be a good thing as far as the tracking error is concerned. However high gain over a wide bandwidth is expensive. In this example we need high gain only at DC since the input is a constant (i.e., a DC signal). Let us therefore try the integral controller $C(s) = 1/s$, which has infinite gain at DC. This should give zero steady-state error if only the feedback loop is stable. The closed-loop characteristic polynomial is now

$$s(s + 1) + 1 = s^2 + s + 1.$$

The closed-loop system is stable. The steady-state tracking error is zero:

$$\begin{aligned} \lim_{t \rightarrow \infty} e(t) &= \lim_{s \rightarrow 0} sE(s) \\ &= \lim_{s \rightarrow 0} s \frac{1}{1 + P(s)C(s)} \frac{100}{s} \\ &= \lim_{s \rightarrow 0} \frac{1}{1 + \frac{1}{s(s+1)}} 100 \\ &= \lim_{s \rightarrow 0} \frac{s(s+1)}{s(s+1) + 1} 100 \\ &= 0. \end{aligned}$$

Notice that $R(s)$ has a pole at $s = 0$ and so we can think of $R(s)$ as being generated by an integrator. We are endowing $C(s)$ with an integrator too. This latter integrator is called an **internal model** of the former.

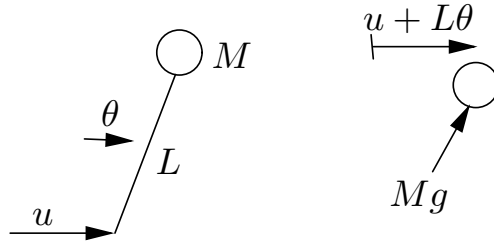


Figure 4.11: Balancing a pendulum.

2. *Sharing.* This time take

$$C(s) = \frac{1}{s}, \quad P(s) = \frac{2s + 1}{s(s + 1)}$$

and take r to be a ramp, $r(t) = r_0 t$. Then $R(s) = r_0/s^2$ and so

$$E(s) = \frac{s + 1}{s^3 + s^2 + 2s + 1} r_0.$$

Again $e(t) \rightarrow 0$; perfect tracking of a ramp. Here $C(s)$ and $P(s)$ together provide the internal model, a double integrator.

3. *Generalization.* Assume $P(s)$ is strictly proper, $C(s)$ is proper, and the feedback system is stable. If $P(s)C(s)$ contains an internal model of the unstable part of $R(s)$, then perfect asymptotic tracking occurs, i.e., $e(t) \rightarrow 0$.
4. *Example.* Consider this problem:

$$R(s) = \frac{r_0}{s^2 + 1}, \quad P(s) = \frac{1}{s + 1}.$$

Design $C(s)$ to achieve perfect asymptotic tracking of the sinusoid $r(t)$, as follows. From the theorem, we should try something of the form

$$C(s) = \frac{1}{s^2 + 1} C_1(s),$$

that is, we embed an internal model in $C(s)$, and allow an additional factor to achieve feedback stability. You can check that $C_1(s) = s$ works. Notice that we have effectively created a notch filter from R to E , a notch filter with zeros at $s = \pm j$. \square

5. *Example.* An inverted pendulum balanced on your hand—see Figure 4.11. The equation is

$$\ddot{u} + L\ddot{\theta} = Mg\theta.$$

Thus

$$s^2 U + s^2 L \theta = Mg \theta.$$

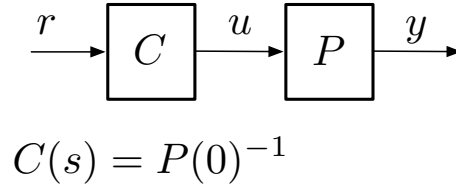


Figure 4.12: Perfect open-loop tracking.

So the transfer function from u to θ equals

$$\frac{-s^2}{Ls^2 - Mg}.$$

This plant transfer function has a zero at $s = 0$, that is, its DC gain is zero. Explain why this is true: It is impossible to balance the pendulum at $\theta = 10$ degrees.

6. *The internal model principle.* Consider a plant with transfer function $P(s)$ and having the two properties that, first, all poles have negative real part and, second, that the DC gain is nonzero. Let the input to $P(s)$ and the output from $P(s)$ be denoted $u(t)$ and $y(t)$. Suppose we want that $y(t)$ should asymptotically track a constant reference signal $r(t)$. If $r(t)$ can be directly measured, that is, we can put a sensor on it, then the open-loop controller $C(s) = P(0)^{-1}$ with input $r(t)$ and output $u(t)$ works perfectly. See Figure 4.12. This is open-loop control. So it seems that an integrator (the internal model) is not necessary after all. But wait: The solution in Figure 4.12 requires perfect knowledge of the DC gain of the plant. This is difficult if not impossible to get. A controller $C(s)$ is said to be non-robust if it doesn't work as intended when the plant model is perturbed slightly. By contrast, the feedback solution in Figure 4.10 with $C(s)$ having an integrator—an internal model—is robust, because $P(s)$ can be perturbed slightly, or need not be modelled perfectly accurately. Indeed, as long as the feedback loop is stable, it will remain so under sufficiently small perturbation of the coefficients of $P(s)$. The internal model principle is that only a feedback controller with an internal model is a robust solution to the tracking problem.

4.4 Principle of the argument

The Nyquist criterion is a test for a feedback loop to be stable. The criterion is the greatest thing since sliced bread and is based on the principle of the argument from complex function theory. The word “argument” refers to the angle of a complex number.

1. *The Principle of the argument.* The principle of the argument involves two things: a curve in the complex plane and a transfer function. Consider, as in Figure 4.13, a closed path (or curve or contour) in the s -plane, with no self-intersections and with negative, i.e., clockwise (CW) orientation. We name the path \mathcal{D} (because later it is going to have the shape of that letter): Now let $G(s)$ be a rational function that does not have a zero or a pole on the curve \mathcal{D} . For every point s in the complex plane, $G(s)$ is a point in the complex plane. We draw two copies of the complex plane to avoid clutter, s in one called the s -plane, $G(s)$ in the other

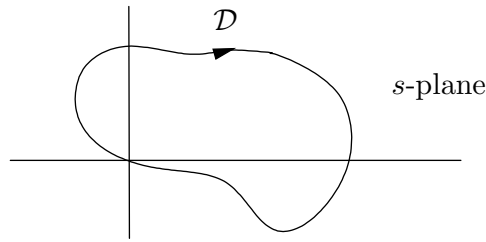

 Figure 4.13: The path \mathcal{D} .

 Figure 4.14: Since \mathcal{D} encircles the zero of $G(s)$ so \mathcal{G} encircles the origin once CW.

called the G -plane. As s goes once around \mathcal{D} from any starting point, the point $G(s)$ traces out a closed curve denoted \mathcal{G} , the image of \mathcal{D} under $G(s)$.

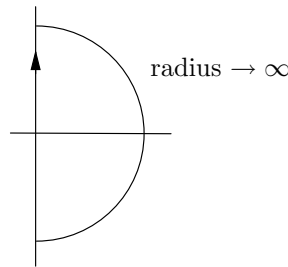
2. *Example.* Begin with $G(s) = s - 1$. We could have as in Figure 4.14. Notice that \mathcal{G} is just \mathcal{D} shifted to the left one unit. Since \mathcal{D} encircles one zero of $G(s)$, \mathcal{G} encircles the origin once CW. Let us keep $G(s) = s - 1$ but change \mathcal{D} as in Figure 4.15. Now \mathcal{D} encircles no zero of $G(s)$ and \mathcal{G} has no encirclements of the origin. Now consider instead

$$G(s) = \frac{1}{s - 1}.$$

The angle of $G(s)$ equals the negative of the angle of $s - 1$:

$$\angle G(s) = \angle 1 - \angle(s - 1) = -\angle(s - 1).$$

From this we get that if \mathcal{D} encircles the pole CW, then \mathcal{G} encircles the origin once counter-clockwise (CCW). Based on these examples, we now see the relationship between the number of poles and zeros of $G(s)$ in \mathcal{D} and the number of times \mathcal{G} encircles the origin.

Figure 4.15: \mathcal{D} does not encircle the zero of $G(s)$.Figure 4.16: The curve \mathcal{D} used in the Nyquist criterion.

3. *Principle of the Argument.* Suppose $G(s)$ has no poles or zeros on \mathcal{D} , but \mathcal{D} encloses n poles and m zeros of $G(s)$. Then \mathcal{G} encircles the origin exactly $n - m$ times CCW.
4. *Proof.* Write $G(s)$ in this way

$$G(s) = K \frac{\prod_i (s - z_i)}{\prod_i (s - p_i)}$$

with K a real gain, $\{z_i\}$ the zeros, and $\{p_i\}$ the poles. Then for every s on \mathcal{D}

$$\angle G(s) = \angle K + \sum \angle(s - z_i) - \sum \angle(s - p_i).$$

If z_i is enclosed by \mathcal{D} , the net change in $\angle(s - z_i)$ is -2π ; otherwise the net change is 0. Hence the net change in $\angle G(s)$ equals $m(-2\pi) - n(-2\pi)$, which equals $(n - m)2\pi$.

5. *Nyquist contour.* The special \mathcal{D} we use for the Nyquist contour is shown in Figure 4.16. Then \mathcal{G} is called the **Nyquist plot** of $G(s)$.
6. *Leading up.* If $G(s)$ has no poles or zeros on \mathcal{D} , then the Nyquist plot encircles the origin exactly $n - m$ times CCW, where n equals the number of poles of $G(s)$ in $\text{Re } s > 0$ and m

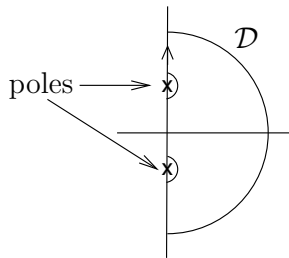


Figure 4.17: Indenting around poles.

equals the number of zeros of $G(s)$ in $\text{Re } s > 0$. From this follows this important observation: Suppose $G(s)$ has no poles on \mathcal{D} . Then $G(s)$ has no zeros in $\text{Re } s \geq 0$ iff \mathcal{G} doesn't pass through the origin and encircles it exactly n times CCW, where n equals the number of poles in $\text{Re } s > 0$. Thus we have a graphical test for a rational function not to have any zeros in the right half-plane.

7. *Clarifying idea.* Note that $G(s)$ has no poles on \mathcal{D} iff $G(s)$ is proper and $G(s)$ has no poles on the imaginary axis; and $G(s)$ has no zeros on \mathcal{D} iff $G(s)$ is not strictly proper and $G(s)$ has no zeros on the imaginary axis.
8. *Indenting.* In our application, if $G(s)$ actually does have poles on the imaginary axis, we have to indent around them. You can indent either to the left or to the right; we shall always indent to the right. See Figure 4.17. Note that we are indenting around **poles** of $G(s)$ on the imaginary axis. Zeros of $G(s)$ are irrelevant at this point.

4.5 Nyquist stability criterion

Now we apply the principle of the argument to derive the Nyquist criterion.

1. *Setup.* The setup is the standard feedback loop with plant $P(s)$ and controller $KC(s)$, where K is a real gain and $C(s)$ and $P(s)$ are rational transfer functions. We're after a graphical test for stability involving the Nyquist plot of $P(s)C(s)$. We could also have a transfer function $F(s)$ in the feedback path, but we'll take $F(s) = 1$ for simplicity. We allow a variable gain K to make the criterion a little more useful; with this assumption we need to draw the Nyquist plot only once even though K is not necessarily fixed. The **assumptions** are these:
 - (a) $P(s)$, $C(s)$ are proper, with at least one of them strictly proper.
 - (b) The product $P(s)C(s)$ has no pole-zero cancellations in $\text{Re } s \geq 0$. We have to assume this because the Nyquist criterion doesn't test for it, and such cancellations would make the feedback system not stable, as we saw before.
 - (c) The gain K is nonzero. This is made only because we are going to divide by K at some point.

2. *The Nyquist criterion.* Let n denote the number of poles of $P(s)C(s)$ in $\text{Re } s > 0$. Construct the Nyquist plot of $P(s)C(s)$, indenting to the right around poles on the imaginary axis. Then the feedback system is stable iff the Nyquist plot doesn't pass through $-\frac{1}{K}$ and encircles it exactly n times CCW.
3. *Proof.* The closed-loop characteristic polynomial is

$$D_p D_c + K N_p N_c.$$

and therefore the feedback loop is stable iff this polynomial has no zeros in $\text{Re } s \geq 0$. Since we have assumed no unstable pole-zero cancellations, you can show that the following two functions have the same right half-plane zeros:

$$D_p D_c + K N_p N_c, \quad 1 + K P(s)C(s).$$

Define $G(s) = 1 + K P(s)C(s)$. Therefore, feedback stability is equivalent to the condition that $G(s)$ has no zeros in $\text{Re } s \geq 0$. So we are going to apply the principle of the argument to get a test for $G(s)$ to have no RHP zeros. Note that $G(s)$ and $P(s)C(s)$ have the same poles in $\text{Re } s \geq 0$, so $G(s)$ has precisely n there. Since \mathcal{D} indents around poles of $G(s)$ on the imaginary axis and since $G(s)$ is proper, $G(s)$ has no poles on \mathcal{D} . Thus by Paragraph 6 in the preceding section, the feedback system is stable iff the Nyquist plot of $G(s)$ doesn't pass through 0 and encircles it exactly n times CCW. Since $P(s)C(s) = \frac{1}{K}G(s) - \frac{1}{K}$, this latter condition is equivalent to: the Nyquist plot of $P(s)C(s)$ doesn't pass through $-\frac{1}{K}$ and encircles it exactly n times CCW.

4. *Review of logic.* Let us review the logic of the proof:

- The feedback loop is stable iff the characteristic polynomial has no right half-plane (RHP) zeros.
- Because of the assumption about no RHP pole-zero cancellations, this is equivalent to the condition that $G = 1 + KPC$ has no RHP zeros.
- The principle of the argument says \mathcal{G} encircles the origin $n - m$ times CCW.
- Therefore $m = 0$, i.e., G has no RHP zeros, iff \mathcal{G} encircles the origin n times CCW.
- Therefore feedback stability holds iff the Nyquist plot of PC encircles the point $-1/K$ a total of n times CCW.

Go over that logic a few times.

5. *Aside.* There is a subtle point that may have occurred to you concerning the Nyquist criterion. Consider the plant transfer function $P(s)$. If it has an unstable pole, then the ROC of the Laplace transform does not include the imaginary axis. And yet the Nyquist plot involves the function $P(j\omega)$, which is a Laplace transform evaluated on the imaginary axis, which is outside the ROC. This may seem to be a contradiction, but it is not. In mathematical terms we have employed analytic continuation to extend the function $P(s)$ to be defined outside the region of convergence of the Laplace integral.

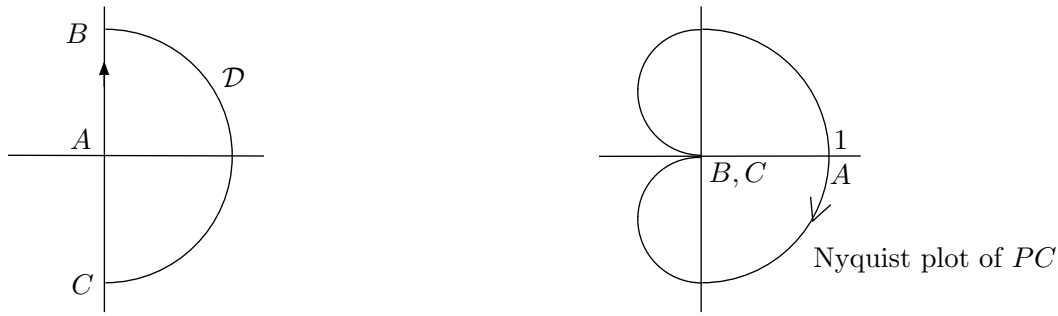


Figure 4.18: First Nyquist plot example.

4.6 Examples of drawing Nyquist plots

In this section you'll learn how to draw Nyquist plots and how to apply the Nyquist criterion.

1. *First example.* The first example is this:

$$PC(s) = \frac{1}{(s+1)^2}.$$

Figure 4.18 shows the curve \mathcal{D} and the Nyquist diagram. The curve \mathcal{D} is divided into segments whose ends are the points A, B, C . We map \mathcal{D} one segment at a time. The points in the right-hand plot are also labelled A, B, C , so don't be confused: the left-hand A is mapped by PC into the right-hand A . The first segment is from A to B , that is, the positive imaginary axis. On this segment, $s = j\omega$ and you can derive from

$$PC(j\omega) = \frac{1}{(j\omega + 1)^2}$$

that

$$\operatorname{Re} PC(j\omega) = \frac{1 - \omega^2}{(1 - \omega^2)^2 + (2\omega)^2}, \quad \operatorname{Im} PC(j\omega) = \frac{-2\omega}{(1 - \omega^2)^2 + (2\omega)^2}.$$

As s goes from A to B , $PC(s)$ traces out the curve in the lower half-plane. You can see this by noting that the imaginary part of $PC(j\omega)$ remains negative, while the real part changes sign once, from positive to negative. This segment of the Nyquist plot starts at the point 1, since $PC(0) = 1$. As s approaches B , that is, as ω goes to $+\infty$, $PC(j\omega)$ becomes approximately

$$\frac{1}{(j\omega)^2} = -\frac{1}{\omega^2},$$

which is a negative real number going to 0. This is also consistent with the fact that PC is strictly proper. Next, the semicircle from B to C has infinite radius and hence is mapped by PC to the origin. Finally, the line segment from C to A in the left-hand graph is the

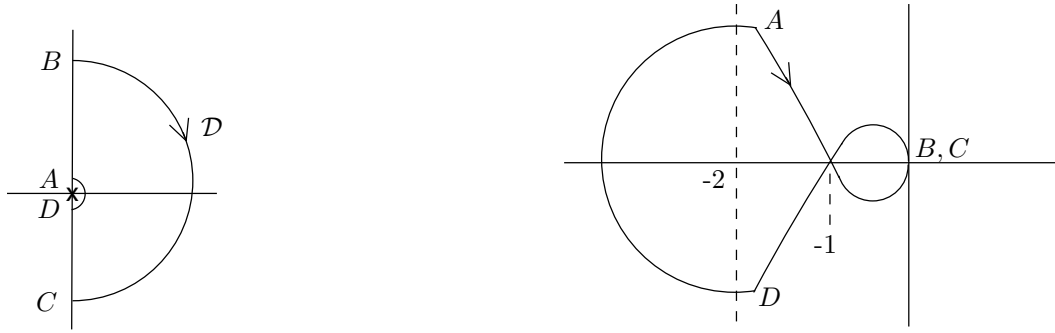


Figure 4.19: Second Nyquist plot example.

complex conjugate of the already-drawn segment from A to B . So the same is true in the right-hand graph. (Why? Because PC has real coefficients, and therefore $PC(\bar{s}) = \overline{PC(s)}$.) In conclusion, we've arrived at the closed path in the right-hand graph—the Nyquist plot. In this example, the curve has no self-intersections.

Now we are ready to apply the Nyquist criterion and determine for what range of K the feedback system is stable. The transfer function PC has no poles inside \mathcal{D} and therefore $n = 0$. So the feedback system is stable iff the Nyquist plot encircles $-1/K$ exactly 0 times CCW. This means, does not encircle it. Thus the conditions for stability are $-1/K < 0$ or $-1/K > 1$; that is, $K > 0$ or $-1 < K < 0$; that is, $K > -1, K \neq 0$. The condition $K \neq 0$ is ruled out by our initial assumption (which we made only because we were going to divide by K). But now, at the end of the analysis, we can check directly that the feedback system actually is stable for $K = 0$. So finally the condition for stability is $K > -1$. You can readily confirm this by applying Routh-Hurwitz to the closed-loop characteristic polynomial, $(s + 1)^2 + K$.

2. *Second example.* Next, we look at

$$PC(s) = \frac{s + 1}{s(s - 1)}$$

for which

$$\operatorname{Re} PC(j\omega) = -\frac{2}{\omega^2 + 1}, \quad \operatorname{Im} PC(j\omega) = \frac{1 - \omega^2}{\omega(\omega^2 + 1)}.$$

The plots are shown in Figure 4.19. Since there is a pole at $s = 0$ we have to indent around it. As stated before, we will always indent to the right. Let us look at A , the point $j\varepsilon$, ε small and positive. We have

$$\operatorname{Re} PC(j\varepsilon) = -\frac{2}{\varepsilon^2 + 1} \approx -2, \quad \operatorname{Im} PC(j\varepsilon) = \frac{1 - \varepsilon^2}{\varepsilon(\varepsilon^2 + 1)} \approx +\infty.$$

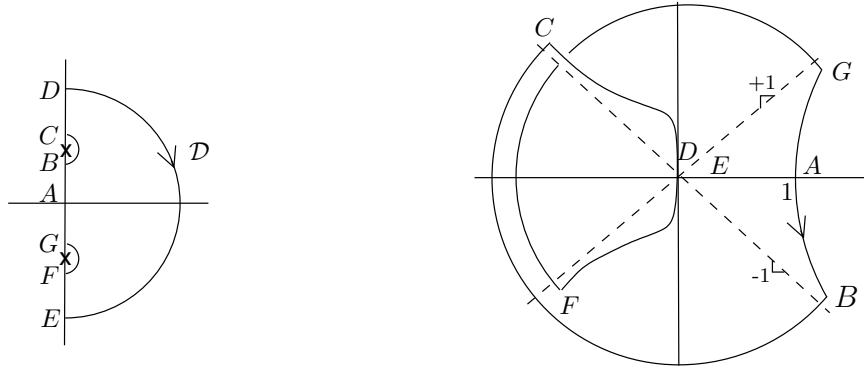


Figure 4.20: Third Nyquist plot example.

This point is shown on the right-hand graph. The mapping of the curve segment $ABCD$ follows routinely. Finally, the segment DA . On this semicircle, $s = \varepsilon e^{j\theta}$ and θ goes from $\pi/2$ to $+\pi/2$; that is, the direction is CCW. We have

$$PC \approx -\frac{1}{\varepsilon e^{j\theta}} = -\varepsilon e^{-j\theta} = \varepsilon e^{j(\pi-\theta)}.$$

Since θ is increasing from D to A , the angle of PC , namely $\pi - \theta$ decreases, and in fact undergoes a net change of $-\pi$. Thus on the right-hand graph, the curve from D to A is a semicircle of infinite radius and the direction is CW. There is another, quite nifty, way to see this. Imagine a particle making one round trip on the \mathcal{D} contour in the left-hand graph. As the particle goes from C to D , at D it makes a right turn. Exactly the same thing happens in the right-hand graph: Moving from C to D , the particle makes a right turn at D . Again, on the left-hand plot, the segment DA is a half-revolution turn about a pole of multiplicity one, and consequently, the right-hand segment DA is a half-revolution too, though opposite direction.

Now to apply the Nyquist criterion, $n = 1$. Therefore we need exactly 1 CCW encirclement of the point $-1/K$. Thus feedback stability holds iff $-1 < -1/K < 0$; equivalently, $K > 1$.

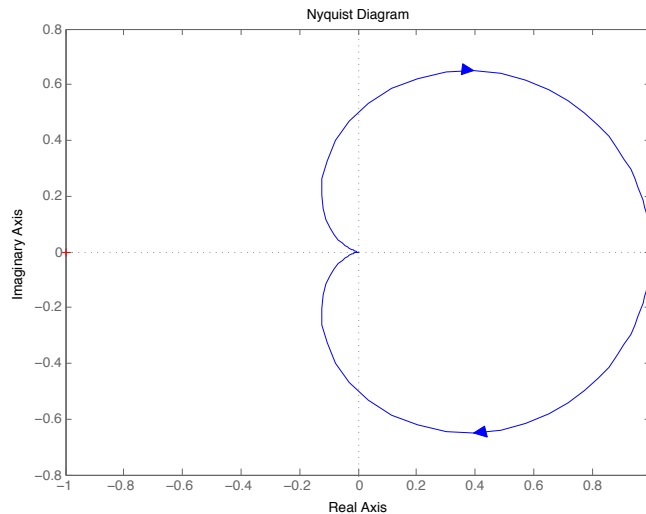
3. *Third example.* The third example is

$$PC(s) = \frac{1}{(s+1)(s^2+1)}$$

for which

$$\operatorname{Re} PC(j\omega) = \frac{1}{1-\omega^4}, \quad \operatorname{Im} PC(j\omega) = \frac{-\omega}{1-\omega^4}.$$

You should be able to draw the graphs based on the preceding two examples. See Figure 4.20. To apply the criterion, we have $n = 0$. Feedback stability holds iff $-1/K > 1$; equivalently, $-1 < K < 0$.

Figure 4.21: Nyquist plot of $1/(s+1)^2$.

4. *Rationale.* You might wonder why you have to learn to draw Nyquist plots instead of letting MATLAB draw them. The reason is simple: MATLAB cannot draw them. More specifically, it cannot draw them near an open-loop pole, and there frequently is an open-loop pole on the imaginary axis, for example with integral control.

4.7 Using the Nyquist criterion

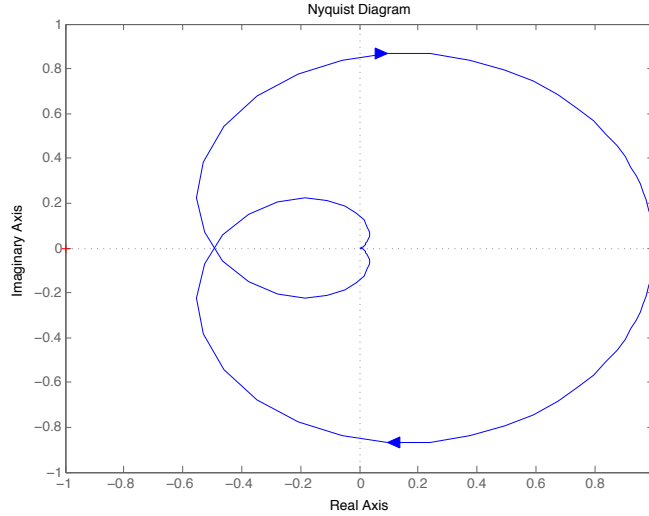
In this section we see how insightful Nyquist plots can be.

1. *Accuracy.* First we clarify that the Nyquist plots in the preceding section are not accurate—they were drawn for visual clarity. For example, Figure 4.21 is the accurate Nyquist plot of $1/(s+1)^2$ shown inaccurately in Figure 4.18. Figure 4.21 was generated using MATLAB and the code

```
s=tf('s');
G=1/(s+1)^2;
nyquist(G)
```

Looking at Figure 4.21 alone, can we tell that the feedback system with unity gain ($K = 1$) is stable? No. By the Nyquist criterion, we must know how many encirclements there must be of the critical point -1 , that is, we must know the number of open-loop unstable poles there are. In Figure 4.21 there are no encirclements of -1 ; therefore there must be no unstable open-loop poles for us to conclude feedback stability.

2. *MATLAB's limitations.* We also clarify that, if there are imaginary axis open-loop poles, MATLAB will not do the indentation around them, and therefore the Nyquist plot computed

Figure 4.22: Nyquist plot of $(1 - s)/(s + 1)^3$.

by MATLAB will not be a closed contour. Consequently, you yourself must close the contour to be able to count encirclements.

3. *Continued.* Let us continue with Figure 4.21, the Nyquist plot of

$$P(s)C(s) = \frac{1}{(s + 1)^2}.$$

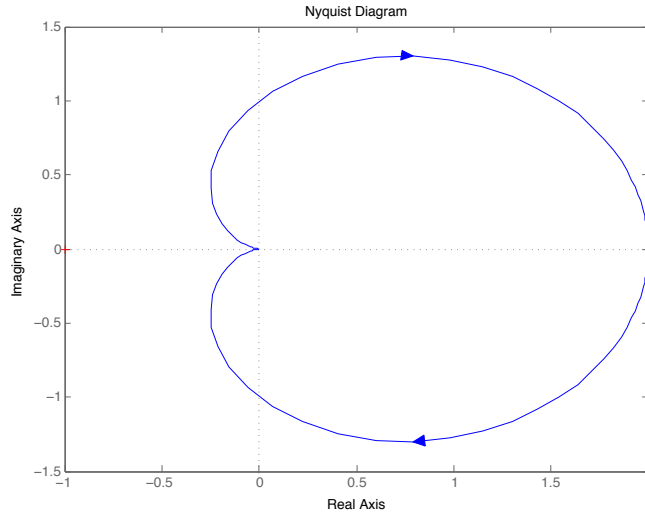
The feedback loop is stable for all positive K . This $P(s)C(s)$ has no right half-plane zeros and we say it is therefore **minimum phase**, for a reason to be given now. Suppose that we multiply $P(s)C(s)$ by the factor $(1 - s)/(1 + s)$. Then

$$P(s)C(s) = \frac{1}{(s + 1)^2} \frac{1 - s}{1 + s}.$$

The Nyquist plot of this $P(s)C(s)$ is Figure 4.22. Look at the effect of the factor $(1 - s)/(1 + s)$. The Nyquist plot now intersects the negative real axis at $-1/2$, and therefore, starting at $K = 1$, if K is increased the feedback loop will become unstable at $K = 2$. Thus the extra factor $(1 - s)/(1 + s)$ in $P(s)C(s)$ has reduced the amount of beneficial high gain we may employ. The factor

$$\frac{1 - s}{1 + s}$$

is called an **all-pass** transfer function: Its magnitude Bode plot equals 1 at all frequencies, meaning it will pass a sinusoid of any frequency without attenuating its amplitude. An all-pass transfer function has pole-zero symmetry with respect to the imaginary axis: If there is a zero at $s = z$, then there is a pole at its reflection across the imaginary axis. A transfer function with at least one right half-plane zero is said to be **non-minimum phase**, because the right half-plane zero has made the phase more negative than it would be without the right half-plane zero. The conclusion is that the achievable performance for a non-minimum-phase plant is generally less than for the analogous minimum-phase plant.

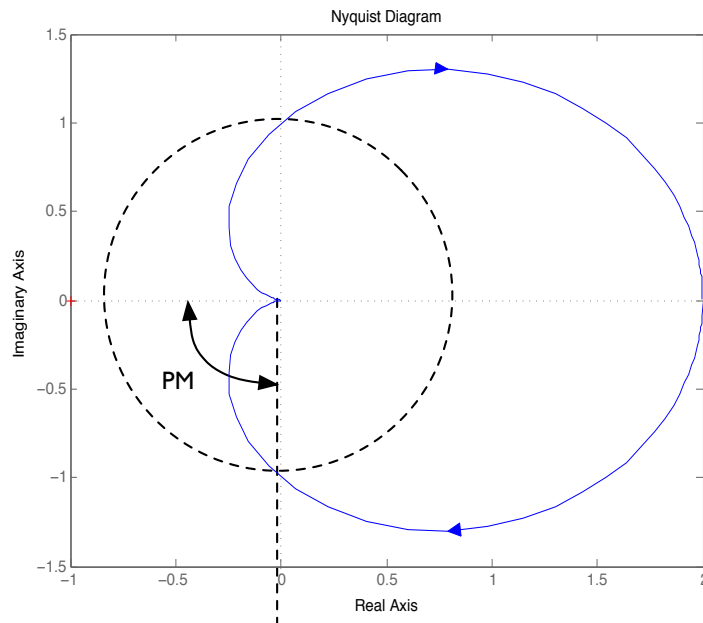
Figure 4.23: Nyquist plot of $2/(s+1)^2$.

4. *Time delay.* A more familiar illustration of the preceding point is offered by the effect of sensor-actuation non-collocation. Imagine a shower stall where the valve to adjust the ratio of hot-to-cold water is some distance before the shower head. The valve is the actuator and your skin contains the temperature sensors. Suppose you are standing in the shower with the water running, and the temperature of the water at your skin is too cold. You adjust the valve, but it takes some time to feel the change in water temperature at your skin. It is clearly harder to control the water temperature with this delay. Mathematically, it is harder to control the plant $P(s)e^{-sT}$ than it is to control the plant $P(s)$. The time-delay term e^{-sT} is an example of a non-rational all-pass transfer function.
5. *Distance from instability.* If a feedback loop is stable, how stable is it? In other words, how far is it from being unstable? This depends entirely on our plant model, how we got it, and what uncertainty there is about the model. In the frequency domain context, uncertainty is naturally measured in terms of magnitude and phase as functions of frequency. There are three measures of stability margin.
6. *Phase margin.* The first measure is called the phase margin. Consider

$$C(s) = 2, \quad P(s) = \frac{1}{(s+1)^2}.$$

The Nyquist plot of $P(s)C(s)$ is shown in Figure 4.23. Since the gain of 2 is incorporated into the Nyquist plot, the critical point is -1 . There are no encirclements of the critical point, so the feedback system is stable. The phase margin is related to the distance from the critical point -1 to the point where the Nyquist plot crosses the unit circle. Precisely, draw the unit circle and draw the straight radial line from the origin through the point where the circle intersects the Nyquist plot. See Figure 4.24. The **phase margin** is the angle shown as PM. In the present example, this equals 90 degrees. So mathematically,

$$\text{PM} = -180 - (\text{the phase of } P(j\omega)C(j\omega) \text{ at } \omega \text{ where } |P(j\omega)C(j\omega)| = 1.)$$

Figure 4.24: Phase margin of $2/(s+1)^2$.

The value of ω where $|P(j\omega)C(j\omega)| = 1$ is called the **gain crossover frequency**, denoted ω_{gc} . MATLAB has a function to compute PM. Phase margin is widely used to predict “ringing”, meaning, oscillatory response. But it must be used with caution. It does not measure how close the feedback loop is to being unstable. More precisely, if the phase margin is small (say 5 or 10 degrees), then you are close to instability and there may be ringing, but if the phase margin is large (say 60 degrees), then you can’t conclude anything—the Nyquist plot could still be dangerously close to the critical point.

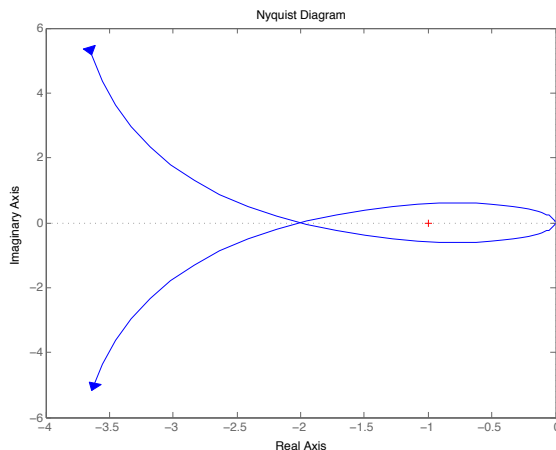
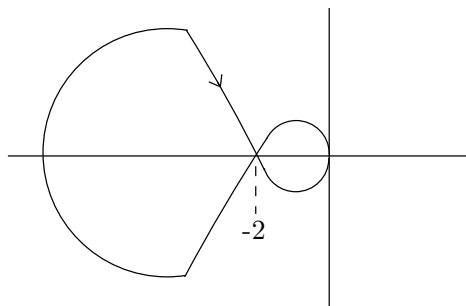
7. *Gain margin.* The second measure of stability margin is called the gain margin. It is a measure of the distance from the critical point to where the Nyquist plot intersects the negative real axis. Consider Figure 4.22. The critical point is -1 and therefore the nominal gain is $K = 1$. The gain can be increased until $-\frac{1}{K} = -\frac{1}{2}$, that is, $K = 2$. Gain margin is measured in dB, so

$$\text{GM} = 20 \log_{10} 2 = 6 \text{ dB}.$$

Mathematically,

$$\text{GM} = -20 \log_{10} |P(j\omega)C(j\omega)| \text{ at } \omega \text{ where the phase equals } -180 \text{ degrees.}$$

8. *Stability margin.* The third measure of stability margin is called **the stability margin**. It equals the actual distance from the critical point to the closest point on the Nyquist plot. This is the best of the three. One can construct a pernicious example where the phase and gain margins are acceptable and yet the Nyquist plot comes dangerously close to the critical point.

Figure 4.25: Nyquist plot of $2(s+1)/(s(s-1))$ via MATLAB.Figure 4.26: Nyquist plot of $2(s+1)/(s(s-1))$, approximate sketch.

9. *Be careful.* We emphasize that the stability margins are defined only when the feedback loop is stable—the system has to be stable before it has a margin of stability.
10. *A final example.* Take

$$C(s) = 2, \quad P(s) = \frac{s+1}{s(s-1)}.$$

The Nyquist plot of $P(s)C(s)$ as computed by MATLAB is shown in Figure 4.25. MATLAB cannot close the curve, so we must. See Figure 4.26. The critical point is -1 and we need 1 CCW encirclement, so the feedback system is stable. The phase margin computed by MATLAB is 37 degrees. We see from Figure 4.26 that, starting from $K = 1$, we can increase K without limit without instability occurring, but we can decrease K only to 0.5 at which point the feedback loop is unstable. MATLAB computes the gain margin to be -6 dB, meaning the gain can be reduced by a factor of 2 until instability occurs. The stability margin, as read from the Nyquist plot, is about $1/2$. The precise value is 0.57 and is computed by plotting the Bode plot of $1/(1+PC)$, reading off the peak magnitude, and taking the reciprocal.

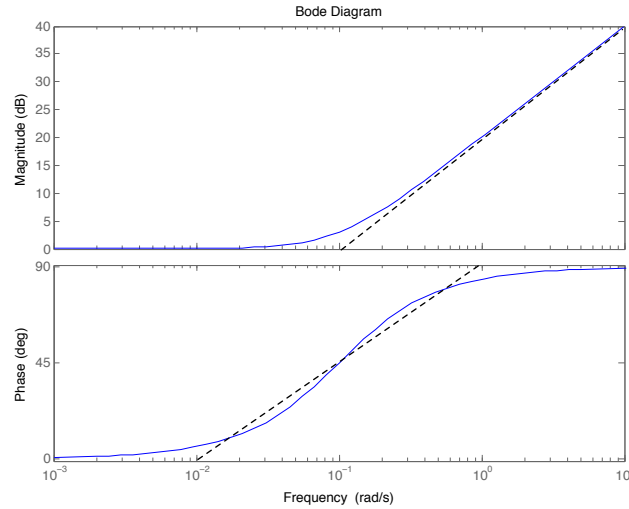


Figure 4.27: Bode plot of $G(s) = Ts + 1$, $T = 10$, piecewise-linear approximation in dashed line.

4.8 Bode plots

1. *Limitation of Nyquist.* Suppose you have tentatively selected a controller but the performance under simulation is inadequate and you therefore want to improve the controller by altering it. The Nyquist plot tells all we need to know about stability of a feedback loop. Its limitation is that it is not very convenient for designing a controller. The Bode plot, based on the fact that the logarithm of a product equals the sum of the logarithms, was invented for the purpose of design, where a plant $P(s)$ is multiplied by a controller $C(s)$. Therefore we need to translate the Nyquist criterion into a condition on Bode plots.
2. *Review of Bode plots.* We now review Bode plots. The Bode plot of a transfer function $G(s)$ consists of two diagrams: the magnitude of $G(j\omega)$ versus ω and the angle of $G(j\omega)$ versus ω . The axes are as follows: ω is in radians/second on a logarithmic scale; $|G(j\omega)|$ is converted to decibels and then the scale is linear in dB; $\angle G(j\omega)$ is in degrees on a linear scale. You are encouraged to be able to draw simple Bode plots by hand—we will see why this is a useful skill. For example, $G(s) = s^n$, n a positive integer. The magnitude is $|G(j\omega)| = \omega^n$. On a log-log scale the graph of ω^n versus ω is a straight line of slope n . With the vertical axis in dB, the graph is a straight line of slope $20n$ dB per decade. That is, if ω is multiplied by 10, $20n$ dB is added to the magnitude. On the other hand, the graph of the phase of $G(s)$ is a horizontal line of $90n$ degrees.
3. *Continued.* Terms like $G(s) = Ts + 1$, T positive, come up frequently. For $T = 10$ the Bode plot and its piecewise-linear approximation are shown in Figure 4.27. At low frequency the magnitude is approximately 1, or 0 dB. At high frequency the magnitude is approximately $T\omega$, the graph of which is a straight line of slope 20 dB/dec. These two straight lines meet at the point where $\omega = 1/T$ and the magnitude is 0 dB. So the piecewise-linear approximation of the magnitude plot is as shown. The piecewise-linear approximation of the phase is the dashed line shown. The two corners are at frequencies $1/10T$ and $10/T$.

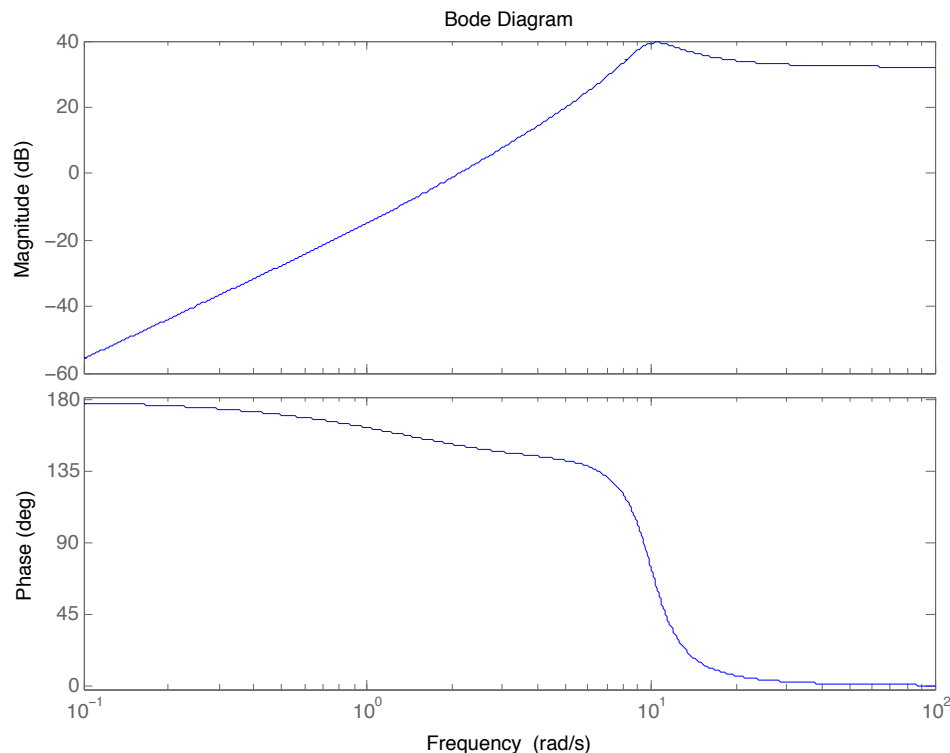


Figure 4.28: Bode plot example.

4. *Example.* Consider the example

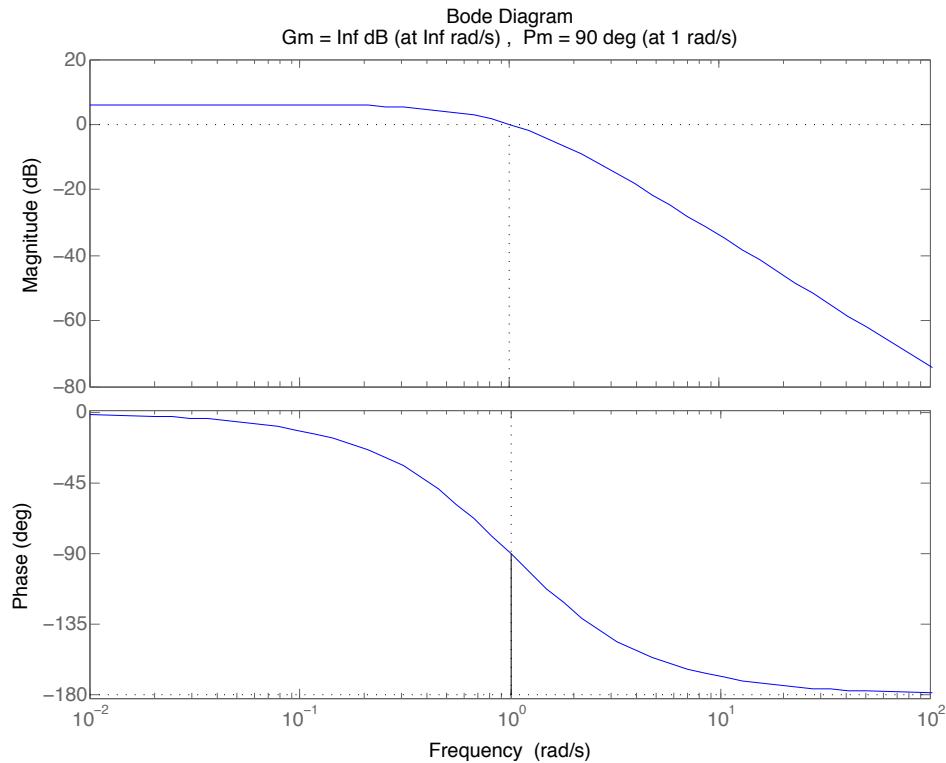
$$G(s) = \frac{40s^2(s-2)}{(s-5)(s^2+4s+100)}.$$

Its Bode plot is drawn in Figure 4.28. This was produced by MATLAB. Notice that the magnitude axis is in decibels, the phase axis in degrees, and the frequency axis in radians/s. This $G(s)$ has a right half-plane zero and a right half-plane pole; also two zeros at the origin; and lightly damped resonant complex poles. For $s = j\omega$ and small ω ,

$$G(j\omega) \approx (\text{negative constant}) \omega^2,$$

which has magnitude equal to zero at DC and that increases with slope 40 dB/decade, and that has 180 degrees phase. At high frequency $G(j\omega) \approx 40$, which has zero phase.

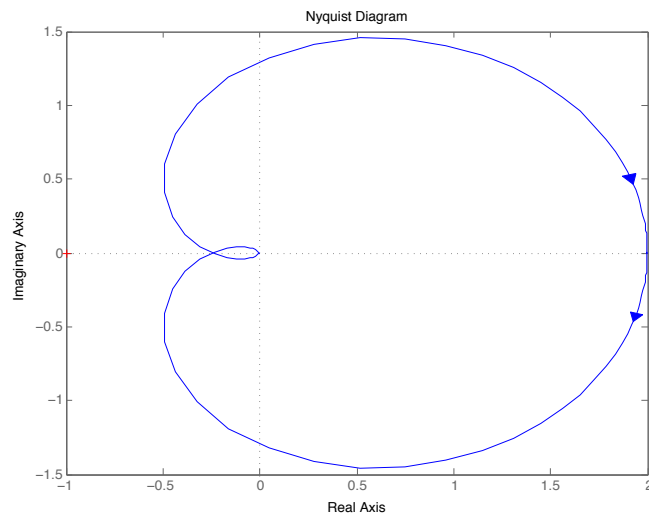
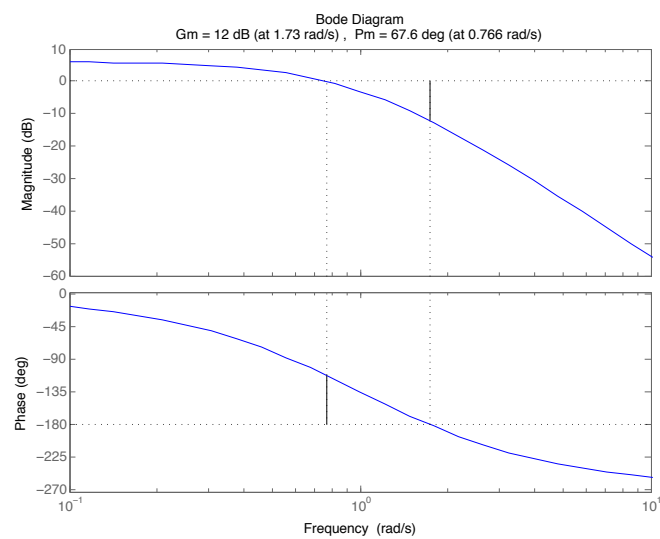
5. *Stability margins.* Let us now look at stability margins in terms of Bode plots. Consider $P(s)C(s) = 2/(s+1)^2$ and assume there are no right half-plane pole-zero cancellations. The Nyquist plot is in Figure 4.24. Since there are no unstable open-loop poles, and since there is no encirclement of the critical point, the feedback loop is stable. As we see from the Nyquist plot, the gain margin is infinite since the Nyquist plot does not cross the negative real axis, while the phase margin is 90 degrees. The MATLAB commands

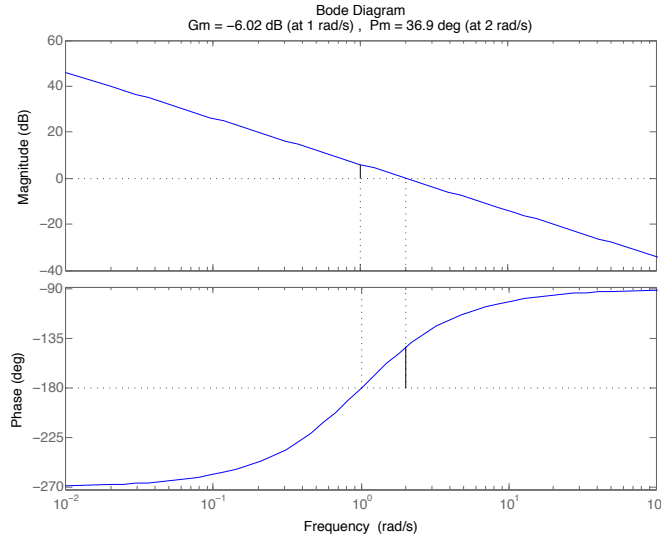
Figure 4.29: Phase and gain margins of $2/(s+1)^2$.

```
s=tf('s');
PC=2/(s+1)^2;
margin(PC)
```

produce Figure 4.29, which is the Bode plot together with the calculation of the gain and phase margins. The phase margin is shown on the vertical line on the phase plot. In detail, at the frequency where the magnitude of PC equals 1 (the gain crossover frequency), the phase is 90 degrees above where it needs to be for stability. That is, at the gain crossover frequency, 90 degrees of additional phase lag can be accommodated before the feedback loop becomes unstable.

6. *Example.* As a second example, let us take $P(s)C(s) = 2/(s+1)^3$. The Nyquist plot is shown in Figure 4.30. The closed loop satisfies the Nyquist criterion and therefore the feedback loop is stable. We see that the gain margin is finite this time. The phase and gain margins are shown explicitly in Figure 4.31. The gain margin is shown as the vertical line on the magnitude plot, and the phase margin is shown as the vertical line on the phase plot. At the phase crossover frequency, the gain can be increased by 12 dB before instability results.
7. *Example.* As a third example let us take $P(s)C(s) = 2(s+1)/(s(s-1))$. The Nyquist plot

Figure 4.30: Nyquist plot of $2/(s+1)^3$.Figure 4.31: Phase and gain margins of $2/(s+1)^3$.

Figure 4.32: Phase and gain margins of $2(s+1)/(s(s-1))$.

is in Figure 4.26. We see that the Nyquist test holds—one CCW encirclement of the critical point. The Bode plot is in Figure 4.32. The gain margin equals -6 dB, meaning the gain can be reduced by half before instability results, and the phase margin equals 37 degrees.

8. *Stability margin.* Finally, the stability margin, SM. It is defined to be the distance from the critical point -1 to the closest point on the Nyquist plot:

$$\begin{aligned} \text{SM} &= \min_{\omega} | -1 - P(j\omega)C(j\omega) | \\ &= \min_{\omega} | 1 + P(j\omega)C(j\omega) |. \end{aligned}$$

Thus

$$\frac{1}{\text{SM}} = \max_{\omega} \frac{1}{|1 + P(j\omega)C(j\omega)|}.$$

The transfer function

$$S(s) = \frac{1}{1 + P(s)C(s)}$$

is the transfer function from r to e in the standard feedback loop. We conclude that $1/\text{SM}$ equals the maximum height of the Bode magnitude plot of the transfer function $S = 1/(1 + PC)$.

4.9 Problems

1. Consider the block diagram in Figure 4.8. Find the transfer function from d to y .

Solution Write the equations at the summing junctions:

$$\begin{bmatrix} 1 & P \\ -C & 1 \end{bmatrix} \begin{bmatrix} E \\ U \end{bmatrix} = \begin{bmatrix} R \\ D \end{bmatrix}.$$

Set $R = 0$, invert the matrix, and solve for U :

$$U = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & P \\ -C & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ D \end{bmatrix}.$$

This gives

$$U = \frac{1}{1 + PC} D.$$

Since $Y = PU$, so

$$Y = \frac{P}{1 + PC} D$$

and we conclude that the transfer function from D to Y equals $P/(1 + PC)$.

2. Consider the cart-pendulum system of Figure 4.2 but take the output y to be the pendulum angle x_2 instead of the cart position. Carry through the development in Section 4.1, Paragraph 1 and see what happens. For example, try to see if you can stabilize the feedback loop.

Solution Now the transfer function from u to y equals

$$\frac{-s^2}{s^2(s^2 - 29.4)}.$$

The factor s^2 cancels and the transfer function becomes

$$\frac{-1}{s^2 - 29.4}.$$

This is the apparent plant from u to y , the pendulum angle. The feedback loop cannot be stabilized because, for any feedback controller, the closed-loop matrix A_{cl} will have two eigenvalues at 0.

3. Consider the standard feedback loop where $P(s) = 1/(s + 1)$ and $C(s) = K$.
 - (a) Find the minimum $K > 0$ such that the steady-state absolute error $|e(t)|$ is less than or equal to 0.01 when r is the unit step and $d = 0$.
 - (b) Find the minimum $K > 0$ such that the steady-state absolute error $|e(t)|$ is less than or equal to 0.01 for all inputs of the form

$$r(t) = \cos(\omega t), \quad 0 \leq \omega \leq 4$$

with $d = 0$.

Solution For the first part,

$$\begin{aligned} E(s) &= \frac{1}{1 + P(s)C(s)} R(s) \\ &= \frac{1}{1 + \frac{K}{s+1}} \frac{1}{s} \\ &= \frac{s+1}{s(s+1+K)}. \end{aligned}$$

The final value of $e(t)$ equals $1/(1+K)$. Thus the minimum K satisfies

$$\frac{1}{1+K} = 0.01,$$

i.e., $K = 99$.

For the second part, the steady-state $e(t)$ is a sinusoid of amplitude

$$\left| \frac{j\omega + 1}{j\omega + 1 + K} \right|.$$

The minimum K satisfies

$$\left| \frac{4j + 1}{4j + 1 + K} \right| = 0.01,$$

i.e.,

$$K = \sqrt{17 \times 10^4 - 16} - 1 = 411.3.$$

4. Same block diagram but now with

$$P(s) = \frac{1}{s^2 - 1}, \quad C(s) = \frac{s-1}{s+1}.$$

Is the feedback loop stable?

Solution No. There's a right-half plane pole-zero cancellation.

5. Consider the standard feedback control system with

$$P(s) = \frac{5}{s+1}, \quad C(s) = K_1 + \frac{K_2}{s}.$$

It is desired to find constants K_1 and K_2 so that (i) the closed-loop poles (i.e., roots of the characteristic polynomial) lie in the half-plane $\text{Re } s < -4$ (this is for a desired speed of transient response), and (ii) when $r(t)$ is the ramp of slope 1 and $d = 0$, the final value of the absolute error $|e(t)|$ is less than or equal to 0.05. Draw the region in the (K_1, K_2) -plane for which these two specs are satisfied.

Solution The characteristic polynomial is

$$s^2 + (1 + 5K_1)s + 5K_2.$$

Define $r = s + 4$ so that the real part of s is less than 4 if and only if the real part of r is negative. The polynomial in terms of r is

$$(r - 4)^2 + (1 + 5K_1)(r - 4) + 5K_2 = r^2 + (5K_1 - 7)r + (12 - 20K_1 + 5K_2).$$

By Routh-Hurwitz, the zeros of the r -polynomial are in the open left half plane if and only if the coefficients are positive:

$$5K_1 - 7 > 0, \quad 12 - 20K_1 + 5K_2 > 0.$$

For the second spec, with the unit-ramp input,

$$E(s) = \frac{s(s+1)}{s(s+1) + 5(K_1s + K_2)} \frac{1}{s^2}.$$

The final value of $e(t)$ equals $1/5K_2$, and thus the spec is

$$\frac{1}{5K_2} \leq 0.05.$$

The two specs together are therefore defined by the inequalities

$$K_1 > 1.4, \quad K_2 \geq 4, \quad 12 - 20K_1 + 5K_2 > 0.$$

6. Standard feedback loop. Suppose $P(s) = 5/(s+1)$, $r(t) = 0$, and $d(t) = \sin(10t)$. Design a proper $C(s)$ so that the feedback system is stable and $e(t)$ converges to 0.

Solution The transfer function from d to e equals

$$-\frac{P}{1+PC}.$$

The Laplace transform $D(s)$ has poles at $s = \pm 10j$. So for $e(t)$ to converge to 0, the transfer function $P/(1+PC)$ must have zeros at $s = \pm 10j$. For this to happen, $C(s)$ must have poles at $s = \pm 10j$. So let's try the controller

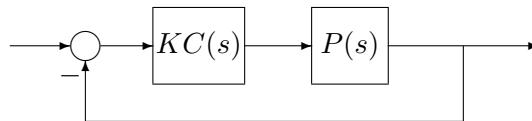
$$C(s) = \frac{as+b}{s^2+100}.$$

We need to choose a, b , if possible, to stabilize the feedback loop. The characteristic polynomial is

$$5(as+b) + (s+1)(s^2+100) = s^3 + s^2 + (100+5a)s + (100+5b).$$

According to Routh-Hurwitz, $a = 1, b = 0$ works.

7. Consider the feedback system



with

$$P(s) = \frac{s+2}{s^2+2}, \quad C(s) = \frac{1}{s}.$$

Sketch the Nyquist plot of PC . How many encirclements are required of the critical point for feedback stability? Determine the range of real gains K for stability of the feedback system.

8. Repeat with

$$P(s) = \frac{4s^2+1}{s(s-1)^2}, \quad C(s) = 1.$$

9. Repeat with

$$P(s) = \frac{s^2+1}{(s+1)(s^2+s+1)}, \quad C(s) = 1.$$

10. Compute the Nyquist plot of

$$G(s) = \frac{s(4s^2+5s+4)}{(s^2+1)^2}.$$

How many times does it encircle the point $(1,0)$? What does this say about the transfer function $G(s) - 1$?

11. Consider the standard feedback system with

$$P(s) = \frac{10}{(5s+1)(s+2)}, \quad C(s) = \frac{K}{s}.$$

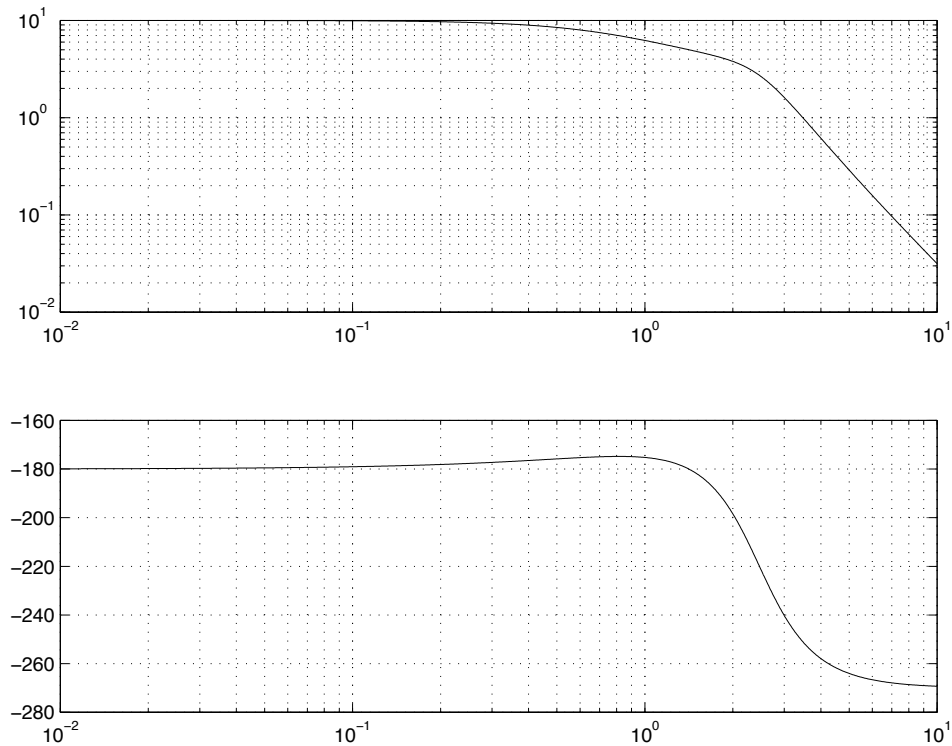
Find the range of gains K for which the feedback loop is stable.

12. Consider the transfer function

$$P(s) = \frac{-0.1s+1}{s(s-1)(s^2-1)}.$$

Draw the piecewise-linear approximation of the Bode plot of P .

13. The Bode plot of $P(s)C(s)$ is given below (magnitude in absolute units, not dB; phase in degrees). The phase starts at -180° and ends at -270° . Sketch the Nyquist plot of $P(s)C(s)$. Assuming the feedback system is stable, what are the gain and phase margins?



14. Consider the standard feedback system with

$$C(s) = K, \quad P(s) = \frac{s^2 + 1}{(s^2 + 2)(s - 2)}.$$

- Sketch the Nyquist plot of $P(s)$. Include correct asymptotes to infinity and to the origin.
- Indicate the number of encirclements (and their orientations) of all possible critical points.
- For what range of K is the feedback system stable?

15. Consider the transfer function

$$P(s) = \frac{-0.1s + 1}{(s + 1)(s - 1)(s^2 + s + 4)}.$$

What are the phases of $P(j\omega)$ near DC and at high frequency. Using MATLAB, compute the Bode plot of P , thus checking your answer.

Solution For s small, $P(s) \approx -1/4$, whose phase is π or $-\pi$, that is, ± 180 degrees. When s is large, $P(s) \approx -0.1/s^3$ and so

$$P(j\omega) \approx -\frac{0.1}{(j\omega)^3} = \frac{0.1}{j\omega^3}.$$

The angle of $1/j$ is -90 degrees, so this is the high-frequency phase. Thus the Nyquist plot approaches the origin tangent to the negative imaginary axis.

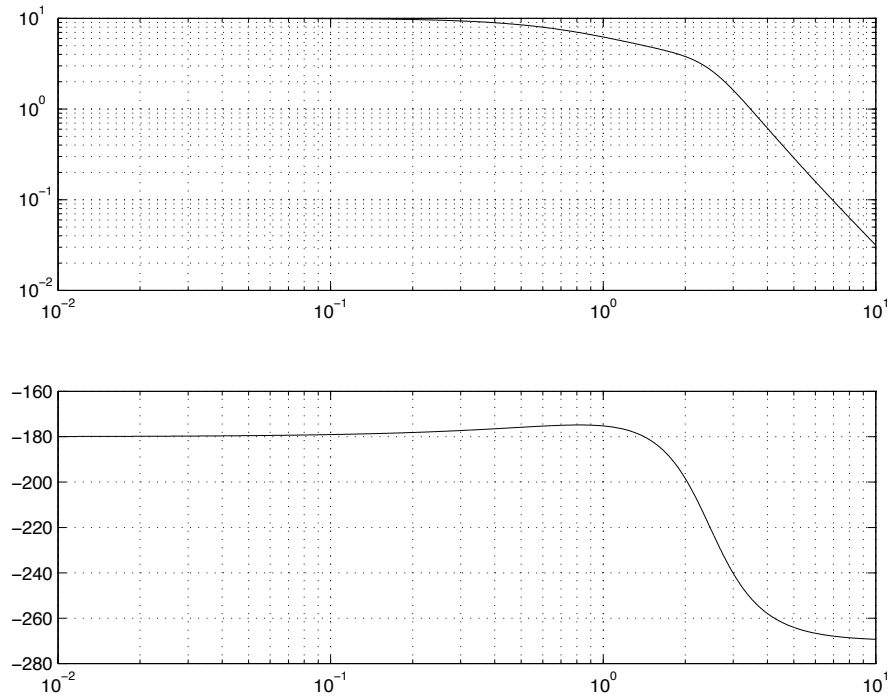


Figure 4.33: Bode plot.

16. Consider the two plants

$$P_1(s) = \frac{1}{s-1}, \quad P_2(s) = \frac{1}{10s-1}.$$

Do you think one is harder to stabilize than the other?

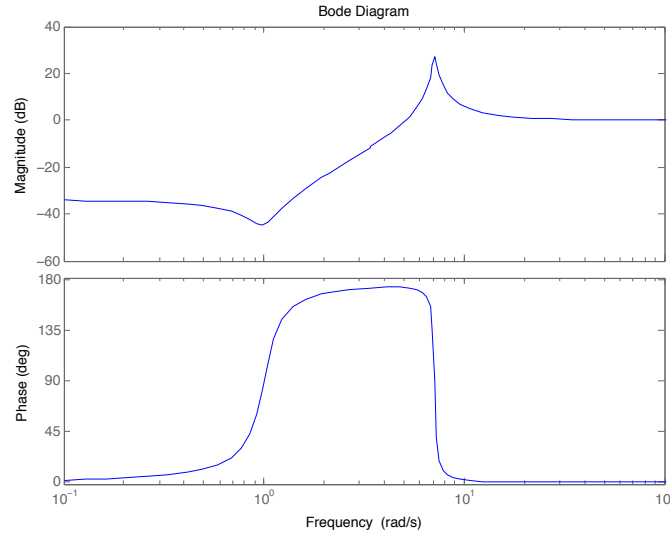
Solution The transfer functions have equal DC gain and are unstable. One has a pole at $s = 1$ and the other at $s = 0.1$. One way to answer the question is to see how much gain is required to stabilize. For $P(s) = \frac{1}{s-1}$ and $C(s) = K$, $K > 1$ is required, while for $P(s) = \frac{1}{10s-1}$ and $C(s) = K$, $K > 1$ is still required. From this viewpoint, there's no difference.

17. Consider the standard feedback system with $C(s) = K$. The Bode plot of $P(s)$ is given in Figure 4.33 (magnitude in absolute units, not dB; phase in degrees). The phase starts at -180° and ends at -270° . You are also given that $P(s)$ has exactly one pole in the right half-plane. For what range of gains K is the feedback system stable?

18. Consider the standard feedback system with

$$C(s) = 16, \quad P(s) = \frac{1}{(s+1)(30s+1)(s^2/9 + s/3 + 1)} e^{-s}.$$

This plant has a time delay, making the transfer function irrational. It is common to use a

Figure 4.34: Bode plot of S .

Padé approximation of the time delay. The second order Padé approximation is

$$e^{-s} = \frac{s^2 - 6s + 12}{s^2 + 6s + 12},$$

which is a rational allpass function. Using this approximation in $P(s)$, graph using MATLAB the Bode plot of $P(s)C(s)$. Can you tell from the Bode plot that the feedback system is stable? Use MATLAB to get the gain and phase margins. Finally, what is the stability margin (distance from the critical point to the Nyquist plot)?

19. Consider

$$P(s) = \frac{10}{s^2 + 0.3s + 1}, \quad C(s) = 5.$$

The Bode plot of the transfer function S is shown in Figure 4.34.

- Show that the feedback system is stable by looking at the closed-loop characteristic polynomial.
- What is the distance from the critical point to the Nyquist plot of PC ?
- If $r(t) = \cos(t)$, what is the steady-state amplitude of the tracking error $e(t)$?

Solution

- The characteristic polynomial is

$$s^2 + 0.3s + 1 + 5 \times 10 = s^2 + 0.3s + 51.$$

All the coefficients are positive; by Routh-Hurwitz, the feedback loop is stable.

- (b) From the Bode plot, the peak magnitude of S equals about 25 dB, or $10^{(25/20)} = 17.78$. Thus the stability margin equals $1/17.78 = 0.0562$.
- (c) The sinusoid r has magnitude 1 and is a sinusoid of frequency 1 rad/s. The magnitude of $S(j\omega)$ at $\omega = 1$ equals 0.006. This equals the steady-state amplitude of $e(t)$.
20. Consider the complex block diagram in Figure 4.9. What would be a good definition for the characteristic polynomial of this system?

Solution The idea is to generalize what we did in Section 2.5, namely, build up a state model for the overall system from state models for the components, and then take the characteristic polynomial of the overall system to be the characteristic polynomial of the overall A -matrix.

For simplicity we will assume each transfer function is strictly proper. In the block diagram erase x_1 and x_2 (because we need to use them for states) and set $r = 0$ (because we need only the closed-loop A -matrix and not B or C). For each $i = 1, \dots, 5$, label u_i as the input and y_i as the output of $P_i(s)$; also, let x_i denote the state. Then we have state models of the following form:

$$\begin{aligned} \dot{x}_1 &= A_1 x_1 + B_1 u_1 \\ &\vdots \\ \dot{x}_5 &= A_5 x_5 + B_5 u_5. \end{aligned}$$

Combine these equations by defining

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_5 \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ \vdots \\ u_5 \end{bmatrix}$$

$$A = \begin{bmatrix} A_1 & 0 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 & 0 \\ 0 & 0 & A_3 & 0 & 0 \\ 0 & 0 & 0 & A_4 & 0 \\ 0 & 0 & 0 & 0 & A_5 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & 0 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 & 0 \\ 0 & 0 & B_3 & 0 & 0 \\ 0 & 0 & 0 & B_4 & 0 \\ 0 & 0 & 0 & 0 & B_5 \end{bmatrix}$$

so that we have $\dot{x} = Ax + Bu$. Now write the output equation for each block,

$$\begin{aligned} y_1 &= C_1 x_1 \\ &\vdots \\ y_5 &= C_5 x_5, \end{aligned}$$

and assemble them as $y = Cx$. Finally, write the input equations taking into account the summing junctions

$$\begin{aligned} u_1 &= -y_5 \\ u_2 &= y_1 - y_4 \\ u_3 &= -y_5 \\ u_4 &= y_2 \\ u_5 &= y_2 \end{aligned}$$

and assemble them as $u = Ky$. Then we have

$$\dot{x} = Ax + Bu = Ax + BKy = Ax + BKCx = (A + BKC)x.$$

The characteristic polynomial of the system equals the characteristic polynomial of the matrix $A + BKC$. In particular, the system is stable iff all the eigenvalues of $A + BKC$ have negative real parts.

Chapter 5

Introduction to Classical Control Design

This chapter is an introduction to classical control design, which is for stable plants, or, at worst, plants with poles at $s = 0$. We introduce phase lag and phase lead controllers, also called compensators. These very simple, first-order controllers are used traditionally to increase the phase margin, with the goal of reducing oscillations to a step change in reference input. The phase lag controller does have a negative phase Bode plot, which would not be useful in itself because it would typically reduce the phase margin and therefore move the feedback loop closer to instability. The phase lag controller is actually used to move the gain crossover frequency to a lower frequency where the phase margin is adequate. The phase lead controller, on the other hand, has positive phase and can therefore be used to increase the phase margin.

PID (proportional-integral-derivative controllers) are limiting cases of lag and lead controllers. In particular, the integral controller has a phase lag of 90 degrees at every frequency, and a proportional-derivative controller has a phase lead at every frequency.

Design using phase lag and phase lead controllers is an example of loopshaping, meaning the Bode plot or Nyquist plot of the open-loop transfer function $P(s)C(s)$ is altered or shaped to have desirable properties. Loopshaping is the basic design technique in classical control. Classical control, having been developed for electric machines, cannot handle open-loop unstable plants. For example, it cannot be used for the magnetic levitation system.

In the final section of the chapter we look at limitations to achievable performance. These observations are due to Bode in his profound 1945 book *Network Analysis and Feedback Amplifier Design*.

5.1 Loopshaping

1. *Design problem.* Consider the unity feedback system in Figure 5.1. The design problem is this: Given $P(s)$, the nominal plant transfer function, maybe some uncertainty bounds, and some performance specs, design an implementable $C(s)$. The performance specs would include, as a bare minimum, stability of the feedback system. The simplest situation is where the performance can be specified in terms of the transfer function

$$S := \frac{1}{1 + PC},$$

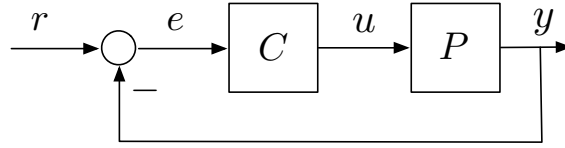


Figure 5.1: Unity feedback loop.

which is called the **sensitivity function**.

2. *Terminology.* Here's the reason for this name. Denote by T the transfer function from r to y , namely,

$$T = \frac{PC}{1 + PC}.$$

Of relevance is the relative perturbation in T due to a relative perturbation in P :

$$\begin{aligned} \lim_{\Delta P \rightarrow 0} \frac{\Delta T/T}{\Delta P/P} &= \lim_{\Delta P \rightarrow 0} \frac{\Delta T}{\Delta P} \frac{P}{T} \\ &= \frac{dT}{dP} \cdot \frac{P}{T} \\ &= \frac{d}{dP} \left(\frac{PC}{1 + PC} \right) \cdot P \cdot \frac{1 + PC}{PC} \\ &= S. \end{aligned}$$

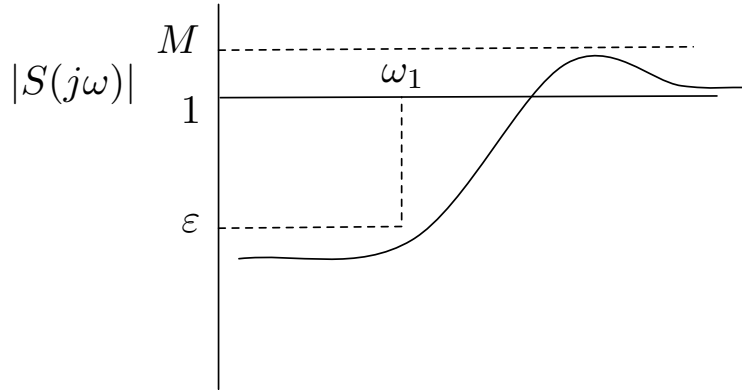
So S is a measure of the sensitivity of the closed-loop transfer function to variations in the plant transfer function.

3. *Specs in terms of S .* For us, S is important for two reasons: First, S is the transfer function from r to e . Thus we want $|S(j\omega)|$ to be small over the range of frequencies of r . Secondly, the peak magnitude of S is the reciprocal of the stability margin. Thus a typical desired magnitude plot of S is as in Figure 5.2. Here ω_1 is the maximum frequency of r , ε is the maximum permitted relative tracking error, $\varepsilon < 1$, and M is the maximum value of $|S|$, $M > 1$. If $|S|$ has this shape and the feedback system is stable, then for the input $r(t) = \cos \omega t$, $\omega \leq \omega_1$ we have $|e(t)| \leq \varepsilon$ in steady state, and the stability margin $1/M$. A typical value for M is 2 or 3. In these terms, the design problem can be stated as follows: Given P , M , ε , ω_1 ; design C so that the feedback system is stable and $|S|$ satisfies $|S(j\omega)| \leq \varepsilon$ for $\omega \leq \omega_1$ and $|S(j\omega)| \leq M$ for all ω .
4. *Example.* Take

$$P(s) = \frac{10}{0.2s + 1}.$$

This is a typical transfer function of a DC motor. Let's take a PI controller:

$$C(s) = K_1 + \frac{K_2}{s}.$$

Figure 5.2: Design specs in terms of S .

Then any M , ε , ω_1 are achievable by suitable K_1 , K_2 . To see this, begin with

$$\begin{aligned} S(s) &= \frac{1}{1 + \frac{10(K_1s + K_2)}{s(0.2s + 1)}} \\ &= \frac{s(0.2s + 1)}{0.2s^2 + (1 + 10K_1)s + 10K_2}. \end{aligned}$$

For suitable K_1 , K_2 defined in terms of K_3 , we can write the preceding expression as

$$\frac{5s(0.2s + 1)}{(s + K_3)^2}.$$

But K_3 is still freely designable. Sketch the Bode plot S and confirm that any $M > 1$, $\varepsilon < 1$, ω_1 can be achieved.

5. In practice it is common to combine interactively the shaping of S with a time-domain simulation.
6. *From S to L .* Now, S is a nonlinear function of C . So in fact it is easier to design the loop transfer function $L := PC$ instead of $S = \frac{1}{1 + L}$. Notice that if $|L|$ is much greater than 1, then $|S| \approx \frac{1}{|L|}$. A typical desired plot is shown in Figure 5.3. In shaping $|L|$, we don't have a direct handle on the stability margin, unfortunately. However, we do have control over the gain and phase margins, as we'll see. In the next two sections we present two simple loopshaping controllers.

5.2 Phase lag controller

1. *Cruise control.* We tackle the example problem of cruise control of a hypothetical toy cart. To get a plant model, we argue as follows: If we were to apply a constant force to the cart, it would go at a constant speed. If we were to increase the force, the cart would gradually speed

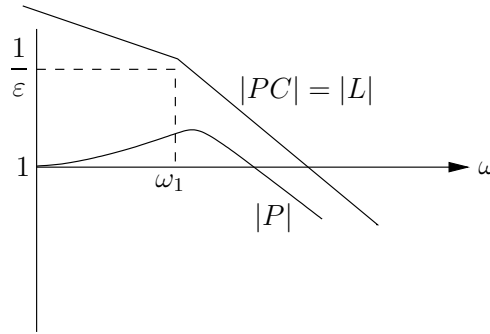
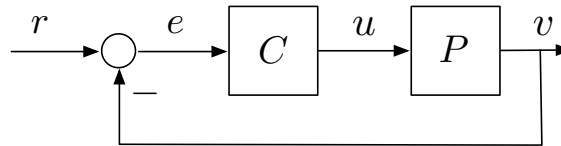
Figure 5.3: Design spec for the loop gain L .

Figure 5.4: Block diagram of cruise control problem.

up, but its speed would not oscillate. Thus a plausible model is first order:

$$\dot{v} = u - \frac{1}{2}v.$$

Here v is the velocity of the cart, u the force, and $-v/2$ a drag term. We would like the cart velocity to follow our command, r . In particular, if we select an increase in speed, the cart should respond briskly and accelerate.

2. *Specs.* In control engineering it is rarely appropriate to write down precise specifications that we may or may not be able to achieve. In industry it is common to write the specs after the design is finished and tested (when successful achievement of specs is guaranteed). It is more sensible to rely on previous designs and work up incrementally, starting with the simplest controller that is likely to work well.
3. *Block diagram.* The block diagram is as shown in Figure 5.4: u is the force, v the velocity, and r the velocity command. The plant transfer function is

$$P(s) = \frac{2}{2s + 1}.$$

The step response of $P(s)$ is shown in Figure 5.5. The plot confirms that the DC gain (i.e., $P(0)$) equals 2 and the time constant (the value of T when $P(s)$ is written as $c/(Ts + 1)$) is about 2 seconds. We want the DC gain from r to v to be 1; that is, we want the cart to go at exactly the commanded speed in steady state. That requires integral control. So we take

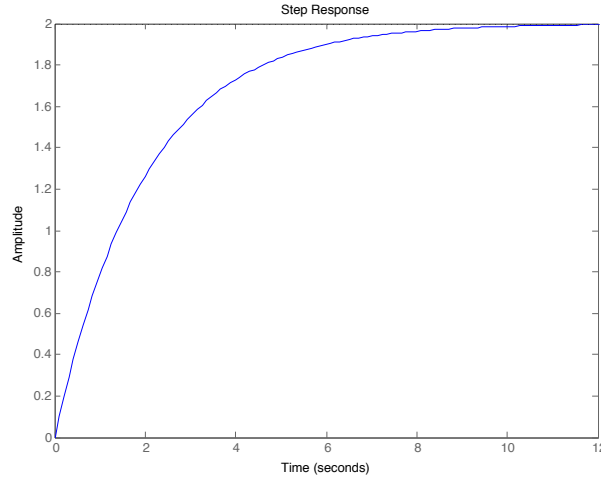


Figure 5.5: Open-loop step response.

$C(s) = K/s$. What value of K to take? Let us simply simulate for a few values of K to get a sense of what response is feasible, recognizing that the larger the value of K , the more the controller costs. The transfer function from r to v is

$$\frac{P(s)C(s)}{1 + P(s)C(s)} = \frac{2K}{2s^2 + s + 2K}.$$

Figure 5.6 shows closed-loop step responses for two values of K . The final values are correct in both cases because of the integrator in $C(s)$, but for $K = 0.1$ the response is too slow, although there's no overshoot, and for $K = 0.8$ the overshoot is too large.

4. *Bode plot with first controller.* Let us look at the controller $C(s) = K/s$, $K = 0.8$, in the frequency domain. Figure 5.7 shows the Bode plot of $P(s)C(s)$. The phase margin is 31 degrees. In general, if the PM is very small, then the stability margin SM is small too (the converse is not necessarily true). A small PM indicates that there are closed-loop poles near the imaginary axis. This is the cause of an oscillatory step response. If we could increase the PM, we might reasonably expect to reduce the oscillations in the step response. Now 31 degrees is not a very small PM, but nevertheless let us see the effect of increasing it. So far we have the controller $0.8/s$, which shall be denoted by $C_1(s)$. Let us adjoin this to $P(s)$ and define a modified plant

$$P_1(s) = C_1(s)P(s) = \frac{0.8 \times 2}{s(2s + 1)}$$

At the end, we will put $C_1(s)$ back into the controller.

5. *Phase lag methodology.* Look again at the Bode plot of $P_1(s)$, Figure 5.7. Our goal is to increase the phase margin, to, say, 60 degrees, and we observe that there is 60 degrees of phase lag, but it is at 0.313 radians/second, which is not the gain crossover frequency—the gain equals 12.5 dB at this frequency. This suggests looking for a controller with a Bode plot like that in Figure 5.8. At the frequency 0.313 and higher, the gain is -12.5 dB, while the phase is nearly zero.

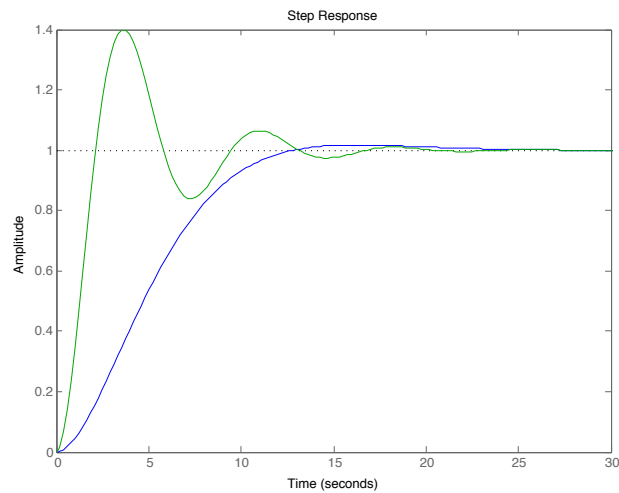


Figure 5.6: Closed-loop step responses $v(t)$ vs. t for the controller $C(s) = K/s$; $K = 0.1$ (blue), $K = 0.8$ (green).

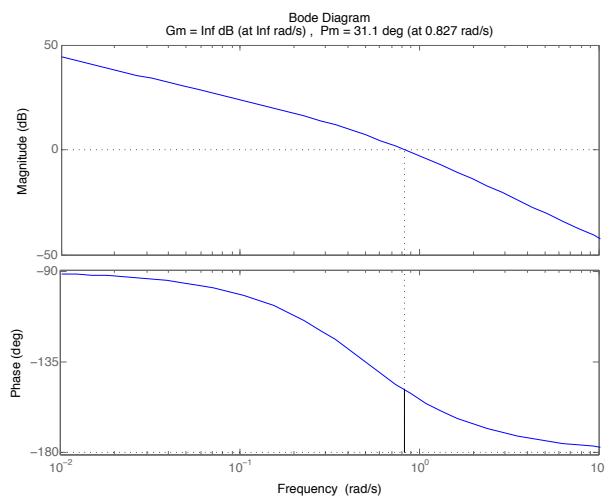


Figure 5.7: Bode plot of $P(s)C_1(s)$, $C_1(s) = 0.8/s$. The phase margin is 31 degrees.

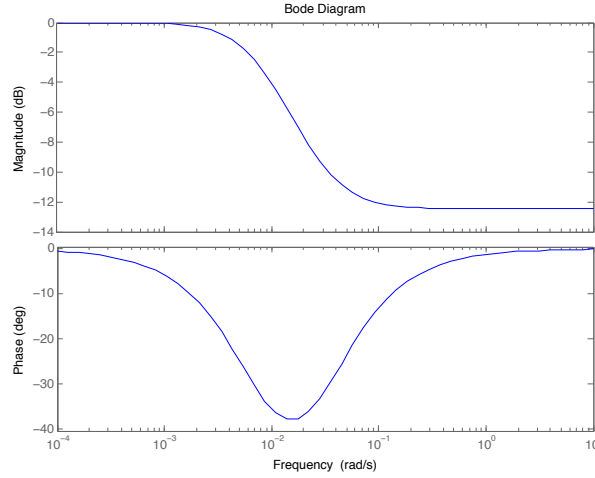


Figure 5.8: Bode plot of desired controller; phase lag.

6. *Controller.* The transfer function having a Bode plot of the form in Figure 5.8 is

$$C_2(s) = \frac{\alpha T s + 1}{T s + 1},$$

where α, T are positive real numbers, with $\alpha < 1$. This is the **phase lag controller**. Its characteristics are these:

- (a) DC gain of 1.
- (b) High-frequency gain of α .
- (c) Phase lag concentrated in the frequency range from $0.1/T$ to $10/\alpha T$.

This last point is based on our analysis of the Bode plot of the term $Ts + 1$ (and likewise $\alpha Ts + 1$); see Section 4.8, Paragraph 3.

7. *Procedure.* Here is the procedure for choosing α and T :

- (a) Start with the Bode plot of $P_1(s)$.
- (b) Go to the frequency where the phase of $P_1(s)$ gives the desired phase margin. In this example it is 0.313 rad/s.
- (c) Read off the magnitude of $P_1(s)$. It is 12.5 dB in this example.
- (d) Set α to be the reciprocal of this magnitude: $\alpha = -12.5$ dB. That is,

$$\alpha = 10^{-12.5/20} = 0.2371$$

- (e) Equate the corner frequency $10/\alpha T$ and the new gain crossover frequency, and solve for T :

$$\frac{10}{\alpha T} = 0.313 \implies T = 134.2$$

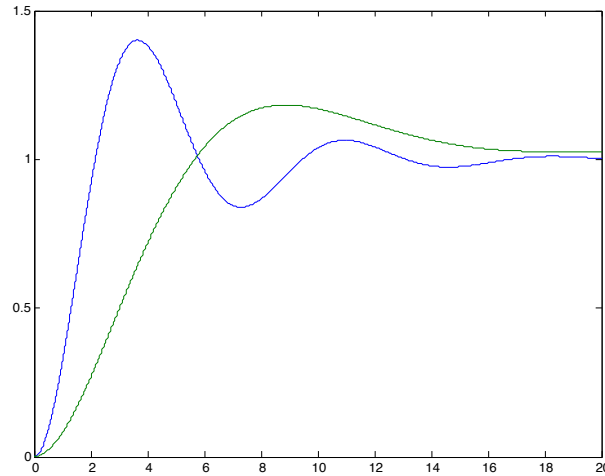


Figure 5.9: Closed-loop step responses: controller $0.8/s$ (blue); lag controller (green).

Altogether, the controller is

$$C(s) = C_1(s)C_2(s) = \frac{0.8}{s} \times \frac{31.95s + 1}{134.2s + 1}.$$

Figure 5.9 shows the step responses for the controller $C_1(s)$ and the controller $C(s)$. The stability margin (recall that this is the distance from -1 to the Nyquist plot of $P(s)C(s)$) is 0.70 , a large value.

8. *Summary.* Let us summarize this example. The plant is first-order, stable, and has non-unity DC gain. We wanted the output to be able to track a step with zero steady-state error, to respond as quickly as possible, and preferably not have any overshoot. To get zero steady-state error we first chose an integral controller. We adjusted the gain K by simulating and compromising between speed of response and amount of overshoot. The overshoot was, however appreciable. We tried a phase lag controller to increase the phase margin and thereby decrease the overshoot. The overshoot did reduce, but at the cost of slower response. The resulting stability margin was large, a consequence of the fact that this is a very easy plant to control.

5.3 Phase lead controller

1. *The phase lead transfer function.* The phase lead controller has exactly the same form as the phase lag controller,

$$C(s) = \frac{\alpha Ts + 1}{Ts + 1}, \tag{5.1}$$

except that $\alpha > 1$ instead of $\alpha < 1$. Look again at the Bode plot of the phase lag controller, Figure 5.8. The Bode plot of a phase lead controller is shown in Figure 5.10. The piecewise-linear approximation of the magnitude is shown too. There are two corner frequencies: the

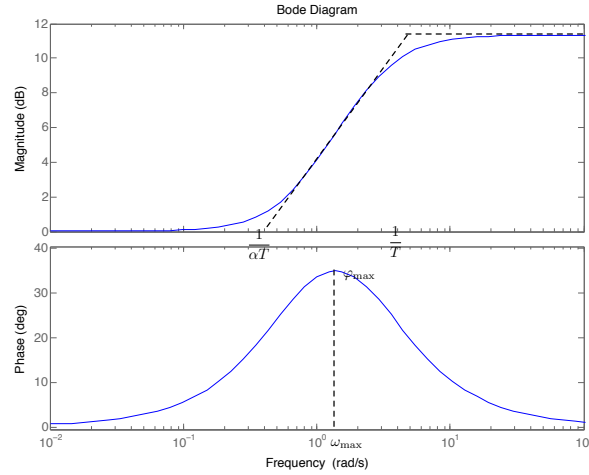


Figure 5.10: A phase lead controller; piecewise-linear approximation of magnitude also shown.

corner $1/\alpha T$ frequency at which the numerator magnitude $|\alpha T j\omega + 1|$ starts to increase, and $1/T$ at which the denominator magnitude $|T j\omega + 1|$ starts to increase. The high-frequency gain is α . Also labelled on the figure are ω_{\max} , the frequency where the phase angle is maximum, and φ_{\max} , the maximum phase angle. The phase lead controller is used to increase the phase angle at the frequency ω_{\max} . However, this is complicated by the fact that the magnitude is not 0 dB at the frequency ω_{\max} . So when we try to increase the phase margin, the gain crossover frequency will increase too.

2. *Formulas.* We'll need three formulas valid for the transfer function (5.1):

$$\omega_{\max} = \frac{1}{T\sqrt{\alpha}} \quad (5.2)$$

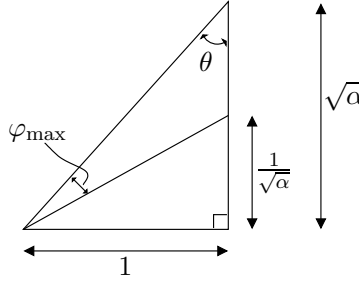
$$|C(j\omega_{\max})| = \sqrt{\alpha} \quad (5.3)$$

$$\varphi_{\max} = \sin^{-1} \frac{\alpha - 1}{\alpha + 1} \quad (5.4)$$

3. *Proofs.* The frequency ω_{\max} is the midpoint between the corners $\frac{1}{\alpha T}$ and $\frac{1}{T}$ on the logarithmically scaled frequency axis. Thus

$$\begin{aligned} \log_{10} \omega_{\max} &= \frac{1}{2} \left(\log_{10} \frac{1}{\alpha T} + \log_{10} \frac{1}{T} \right) \\ &= \frac{1}{2} \log_{10} \frac{1}{\alpha T^2} \\ &= \log_{10} \frac{1}{T\sqrt{\alpha}}. \end{aligned}$$

This proves (5.2). The magnitude of $C(j\omega)$ at ω_{\max} is the midpoint between 1 and α on the

Figure 5.11: Calculating φ_{\max} by the law of sines.

logarithmically scaled magnitude axis. Thus

$$\begin{aligned}\log_{10} |C(j\omega_{\max})| &= \frac{1}{2}(\log_{10} 1 + \log_{10} \alpha) \\ &= \log_{10} \sqrt{\alpha}.\end{aligned}$$

This proves (5.3). Finally, the angle φ_{\max} is the angle of $C(j\omega_{\max})$, which equals the angle of the complex number

$$\left. \frac{\alpha Tj\omega + 1}{Tj\omega + 1} \right|_{\omega=\omega_{\max}} = \frac{1 + \sqrt{\alpha}j}{1 + \frac{1}{\sqrt{\alpha}}j}.$$

By Figure 5.11 and the law of sines,

$$\frac{\sin \varphi_{\max}}{\sqrt{\alpha} - \frac{1}{\sqrt{\alpha}}} = \frac{\sin \theta}{\sqrt{1 + \frac{1}{\alpha}}}.$$

But $\sin \theta = \frac{1}{\sqrt{1+\alpha}}$. Thus

$$\sin \varphi_{\max} = \left(\sqrt{\alpha} - \frac{1}{\sqrt{\alpha}} \right) \frac{1}{\sqrt{1+\alpha}\sqrt{1+\frac{1}{\alpha}}} = \frac{\alpha - 1}{\alpha + 1}.$$

This proves (5.4).

4. *Phase lead properties.* Let us summarize the properties of the phase lead controller:

- (a) DC gain of 1.
- (b) High-frequency gain of α .
- (c) Phase lead of φ_{\max} at the frequency ω_{\max} . Gain of $\sqrt{\alpha}$ at this frequency.

5. *Design procedure.* Now we return to the design example of Section 5.2. The plant is

$$P(s) = \frac{2}{2s + 1}.$$

For the reasons given in that section, we first apply the controller $C_1(s) = 0.8/s$. We shall design a lead controller $C_2(s)$ for

$$P_1(s) = P(s)C_1(s) = \frac{1.6}{s(2s+1)}.$$

The steps are these:

- (a) Start with the Bode plot of $P_1(s)$, Figure 5.7.
- (b) Decide how much phase lead is needed. The phase margin for $P_1(s)$ is 31 degrees. To get to 60 degrees we would need to add 29 degrees. But the gain crossover frequency is going to increase somewhat, to a frequency where the existing phase margin is less than 31 degrees. Let's guess that adding 35 degrees will be adequate.
- (c) Set $\varphi_{\max} = 35$ degrees and solve (5.4) for α : $\alpha = 3.69$.
- (d) From formula (5.3) we will be adding a gain of $\sqrt{\alpha} = 1.92$, or 5.67 dB, at the new gain crossover frequency. The magnitude of P_1 equals -5.67 dB at $\omega = 1.37$ rad/s. Therefore 1.37 will be the new gain crossover frequency, and thus 1.37 is the value of ω_{\max} .
- (e) Solve (5.2) for T : $T = 0.38$.

The resulting phase lead controller is

$$C_2(s) = \frac{1.4s+1}{0.38s+1},$$

whose Bode plot has already been drawn: Figure 5.10. The final controller is

$$C(s) = \frac{0.8}{s} \times \frac{1.4s+1}{0.38s+1}.$$

The resulting stability margin is 0.76, slightly larger than for the lag controller. Figure 5.12 shows the closed-loop step response and compares it with that of the phase lag controller and also just the integral controller $0.8/s$. To test the robustness, i.e., tolerance to parameter variations, of this controller, the drag term in the plant model was changed from $-v/2$ to $-v/3$. For the same controller the step response is shown in Figure 5.13. The response degrades (the overshoot is higher), but is still probably acceptable.

6. Finally, Figure 5.14 shows Bode plots of the sensitivity function S for the lag and lead controllers. The lead controller is better from this point of view too, because $|S|$ is smaller over a wider low frequency range (better tracking capability) and its peak value is smaller (better stability margin).
7. *Summary.* With the phase lead controller the overshoot did reduce very nicely and the response speeded up. If no other factors are relevant, the phase lead controller is the best of the three: It is the fastest, gives the least overshoot, and has a good stability margin.

5.4 Limitations

In this section we look at some limitations to classical control design, and also some limitations to achievable performance. These will help us decide if a plant is easy or hard to control.

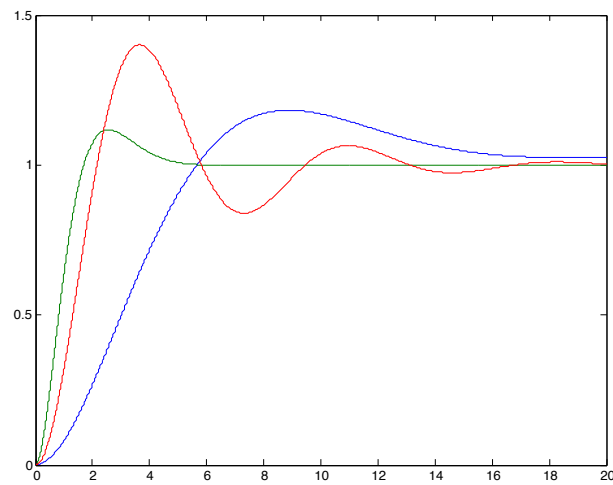


Figure 5.12: Closed-loop step responses: controller $0.8/s$ (red), lag controller (blue), and lead controller (green).

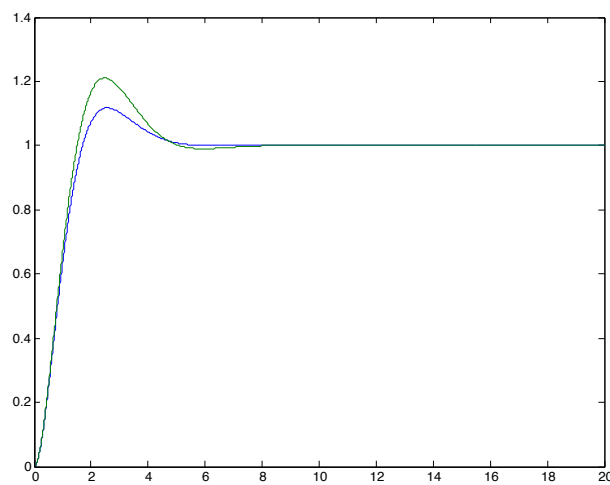


Figure 5.13: Closed-loop step responses of the lead controller: original plant (blue) and drag term perturbed to $-v/3$ (green).

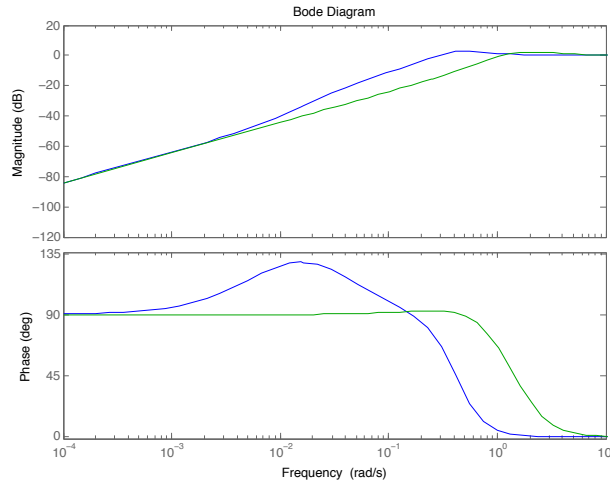
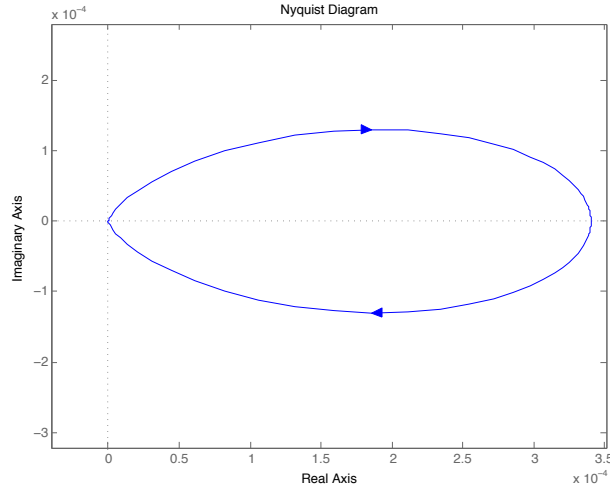


Figure 5.14: Bode plots of S : lag controller (blue); lead controller (green).

1. *High gain is desirable but there's a limit.* As we saw, good tracking in a feedback loop requires high gain at low frequency (in order that $|S|$ be small at low frequency). However, one might argue that instability will result if the loop gain is high enough. The reasoning here is that “everything is infinite dimensional at high enough frequency.” That is, lumped models (finite dimensional models of circuits and mechanical systems) become distributed. Therefore you might argue that all physical things roll off with higher and higher phase lag as frequency increases, and therefore the Nyquist diagram crosses the negative real axis at some frequency. And therefore, the logic would demand, there is a finite gain margin. The problem with this argument is that the linear model becomes less and less accurate as frequency increases. So it becomes more and more of a stretch to talk about magnitude, phase, and Nyquist plots.
2. *Unstable plants.* Phase lag and phase lead controllers are sometimes useful, as shown in the preceding two sections. However, for them to be applicable, the plant has to be quite easy to control. The most they can do is improve a feedback loop that already works quite well. Let us look closely at a more challenging plant and see if phase lag or phase lead can help us. The maglev system of Section 3.6, Paragraph 12 is challenging to control (although not as challenging as the cart-pendulum). The plant transfer function from voltage input to ball-position output is

$$P(s) = -\frac{19.8}{(s + 30)(s^2 - 1940)}.$$

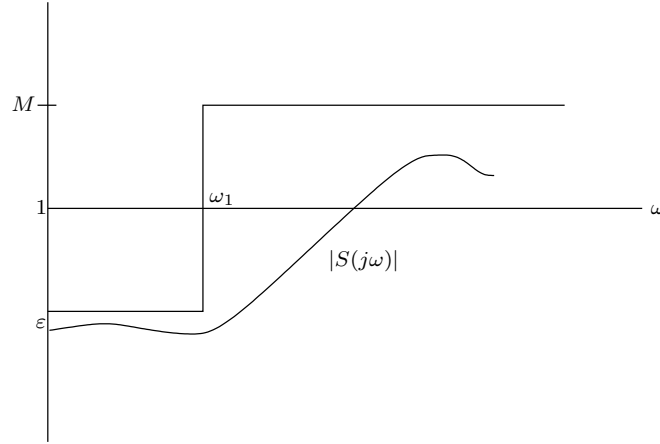
Recall that this is the transfer function of the nonlinear system linearized at a desired ball position. The control objective is to levitate the ball, that is, to stabilize the linear feedback loop. Can we stabilize $P(s)$ using phase lag or phase lead controllers? Let us see. Let $C(s)$ be any phase lag or phase lead controller. The plant $P(s)$ has a right half-plane pole, namely, at $s = \sqrt{1940}$. Thus for feedback stability the Nyquist plot of $P(s)C(s)$ must have exactly one CCW encirclement of the point -1 . The Nyquist plot of $P(s)$ itself is shown in Figure 5.15. Notice the scaling on the axes: The plot lives in the small rectangle with corners

Figure 5.15: Nyquist plot of $P(s)$ for the maglev example.

at $(0, 0)$, $(0.00035, 0)$, $(0.00035, 0.0002)$, $(0, 0.0002)$. Being small is not the problem—we could easily use a controller to scale the Nyquist diagram up in size. The problem is that the Nyquist plot has a CW orientation and this cannot be changed by a phase lag or phase lead controller. Indeed, the loopshaping of classical control cannot help us in the stabilization problem.

3. *Bode's phase formula.* We don't need Bode's precise formula. What is more instructive is an approximate relationship that results from it. And this relationship actually requires a rather strong assumption. For a plant $P(s)$ and controller $C(s)$, assume the loop gain $L(s) = P(s)C(s)$ is stable, minimum phase, and of positive DC gain. If the slope of $|L(j\omega)|$ near gain crossover (that is, near the frequency where the gain equals 1) is $-n$, then $\arg(L(j\omega))$ at gain crossover is approximately $-n\frac{\pi}{2}$. What we learn from this observation is that in transforming $|P|$ to $|PC|$ via, say, lag or lead compensation, we should not attempt to roll off $|PC|$ too sharply near gain crossover. If we do, $\arg(PC)$ will be too large near crossover, resulting in a negative phase margin and hence instability.
4. *Example.*
5. *The waterbed effect.* Turn to Figure 5.16. This concerns the ability to achieve the following spec on the sensitivity function S : Let us suppose $M > 1$ and $\omega_1 > 0$ are fixed. Can we make ε arbitrarily small? That is, can we get arbitrarily good tracking over a finite frequency range, while maintaining a given stability margin $(1/M)$? Or is there a positive lower bound for ε ? The answer is that arbitrarily good performance in this sense is achievable if and only if $P(s)$ is minimum phase. Thus, non-minimum phase plants have bounds on achievable performance: As $|S(j\omega)|$ is pushed down on one frequency range, it pops up somewhere else, like a waterbed. The precise result is this: Suppose $P(s)$ has a zero at $s = z$ with $\text{Re } z > 0$. Let $A(s)$ denote the allpass factor of $S(s)$. Then there are positive constants c_1, c_2 , depending only on ω_1 and z , such that

$$c_1 \log \varepsilon + c_2 \log M \geq \log |A(z)^{-1}| \geq 0.$$

Figure 5.16: Nyquist plot of $P(s)$ for the maglev example.

6. *Example.* Consider the plant

$$P(s) = \frac{1-s}{(s+1)(s-p)}, \quad p > 0, \quad p \neq 1$$

and assume $p > 0$ and $p \neq 1$, that is, the pole p is unstable and is not cancelled by a zero. Let $C(s)$ be a stabilizing controller. Then

$$S = \frac{1}{1+PC} \Rightarrow S(p) = 0.$$

Thus $\frac{s-p}{s+p}$ is an allpass factor of S . There may be other allpass factors, so what we can say is that $A(s)$ has the form

$$A(s) = \frac{s-p}{s+p} A_1(s),$$

where $A_1(s)$ is some allpass TF (may be 1). In this example, the RHP zero of $P(s)$ is $s = 1$. Thus

$$|A(1)| = \left| \frac{1-p}{1+p} \right| |A_1(1)|.$$

Now $|A_1(1)| \leq 1$ (why?), so

$$|A(1)| \leq \left| \frac{1-p}{1+p} \right|$$

and hence

$$|A(1)^{-1}| \geq \left| \frac{1+p}{1-p} \right|.$$

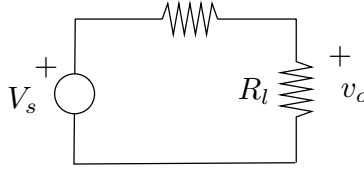


Figure 5.17: A voltage divider.

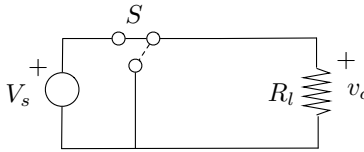


Figure 5.18: A switching circuit.

The theorem gives

$$c_1 \log \varepsilon + c_2 \log M \geq \log \left| \frac{1+p}{1-p} \right|.$$

Thus, if $M > 1$ is fixed, $\log \varepsilon$ cannot be arbitrarily negative, and hence ε cannot be arbitrarily small. In fact the situation is much worse if $p \approx 1$, that is, if the RHP plant pole and zero are close.

5.5 Case Study: Buck Converter

Many consumer electronic appliances, such as laptops, have electronic components that require regulated voltages. The power may come from a battery. For example, a regulated value of 2 V may be needed from a battery rated at 12 V. This raises the problem of stepping down a source voltage V_s to a regulated value at a load. In this section we study the simplest such voltage regulator.

1. Let's first model the load as a fixed resistor, R_l (subscript "l" for "load"). Then of course a voltage dividing resistive circuit suggests itself¹: See Figure 5.17. This solution is unsatisfactory for several obvious reasons. First, it is inefficient because some current is uselessly heating up the non-load resistor. A second reason is that the battery will drain and therefore v_o will decrease too.
2. *Switched mode power supply.* As a second attempt at a solution, let us try a switching circuit as in Figure 5.18. The switch S , typically a MOSFET, opens and closes periodically according to a duty cycle, as follows. Let T be the period in seconds. The time axis is divided into intervals of width T :

$$\dots, [0, T), [T, 2T), [2T, 3T), \dots$$

¹The output voltage, v_o , is written lower case because eventually it's not going to be perfectly constant.

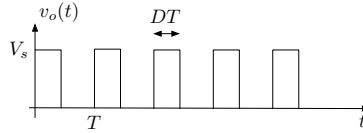


Figure 5.19: Square-wave voltage.

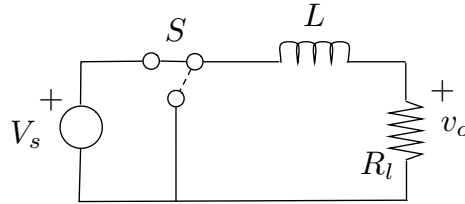


Figure 5.20: Switched regulator with a filter.

Over the interval $[0, T)$, the switch is closed for the subinterval $[0, DT)$ and then open for $[DT, T)$, where D , the duty cycle, is a number between 0 and 1. Likewise, for every other interval. The duty cycle will have to be adjusted for each interval, but for now let's suppose it is constant. The idea in the circuit is to choose D to get the desired regulated value of v_o . For example, if we want v_o to be half the value of V_s , we choose $D = 1/2$. In this case $v_o(t)$ would look as shown in Figure 5.19. Clearly, the average value of $v_o(t)$ is correct, but $v_o(t)$ is far from being constant. How about efficiency? Over the interval $[0, DT)$, S is closed and the current flowing is V_s/R_l . The input and output powers are equal. Over the interval $[DT, T)$, S is open and the current flowing is 0. The input and output powers are again equal. Therefore the efficiency is 100%. However we have not accounted for the power required to activate the switches.

3. *Inclusion of a filter.* Having such large variations in v_o is of course unacceptable. And this suggests we need a circuit to filter out the variations. Let us try adding an inductor as in Figure 5.20. This circuit is equivalent to the one in Figure 5.21 where the input is as shown in Figure 5.22. A square wave into a circuit can be studied using Fourier series, which we proceed to do.

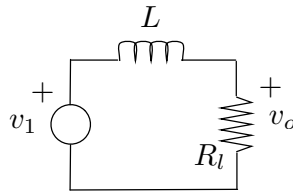


Figure 5.21: Square-wave input voltage.

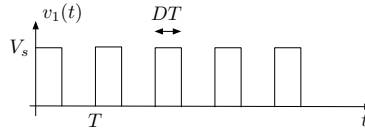


Figure 5.22: Graph of input.

4. Our goal now is to see quantitatively how the filter affects the ripple in the output voltage. Since $v_1(t)$ is periodic, so is $v_o(t)$ in steady state. The transfer function from v_1 to v_o , by the voltage divider rule for impedances, equals

$$H(s) = \frac{R_l}{Ls + R_l}.$$

The DC gain equals 1, i.e., $H(0) = 1$. Thus the average of $v_o(t)$ equals the average of $v_1(t)$, namely, DV_s . The sinusoidal basis functions of period T are

$$w_k(t) = \frac{1}{\sqrt{T}} e^{j2\pi kt/T}, \quad k = 0, \pm 1, \pm 2, \dots$$

The scalar factor $1/\sqrt{T}$ is to normalize the sinusoid. Then the Fourier series of the input is

$$v_1(t) = \sum_k a_k w_k(t),$$

where the Fourier coefficients are given by

$$a_k = \frac{1}{\sqrt{T}} \int_0^T v_1(t) e^{-j2\pi kt/T} dt = \frac{1}{\sqrt{T}} V_s \int_0^{DT} e^{-j2\pi kt/T} dt.$$

For $k = 0$ integration yields

$$a_0 = \sqrt{T} V_s D.$$

Thus the $k = 0$ term in the Fourier series of $v_1(t)$ is the DC term

$$a_0 w_0(t) = V_s D,$$

as expected. For $k = 1$ we have

$$a_1 = \sqrt{T} V_s \frac{1 - e^{-j2\pi D}}{j2\pi}$$

and so the $k = 1$ term is

$$a_1 w_1(t) = V_s \frac{1 - e^{-j2\pi D}}{j2\pi} e^{j2\pi t/T},$$

whose magnitude is

$$V_s \left| \frac{1 - e^{-j2\pi D}}{j2\pi} e^{j2\pi t/T} \right| = \frac{V_s}{\pi} \sin(\pi D).$$

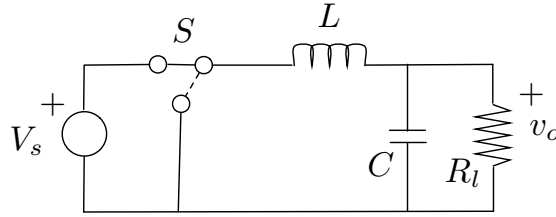


Figure 5.23: Buck converter.

This is maximum when $D = 1/2$, for which the magnitude is V_s/π . Passing this Fourier series through the circuit gives

$$v_o(t) = \sum_k b_k w_k(t), \quad b_k = H(j2\pi k/T) a_k.$$

Thus $b_0 = a_0$ and the $k = 0$ term in the Fourier series of $v_o(t)$ is

$$b_0 w_0(t) = V_s D,$$

again as expected. The $k = 1$ Fourier coefficient is

$$b_1 = H(j2\pi/T) a_1 = \frac{R_l T}{j2\pi L + R_l T} \sqrt{T} V_s \frac{1 - e^{-j2\pi D}}{j2\pi}$$

and then it turns out that

$$|b_1 w_1(t)| = \left| \frac{R_l T}{j2\pi L + R_l T} \right| \frac{V_s}{\pi} \sin(\pi D) \leq \frac{V_s R_l T}{2\pi^2 L}.$$

Clearly, this upper bound can be made arbitrarily small either by making T small enough or by making the time constant L/R_l large enough.

5. In conclusion, the average output voltage $v_o(t)$ equals DV_s , duty cycle \times input voltage, and there's a ripple whose first harmonic can be made arbitrarily small by suitable design of the switching frequency or the circuit time constant.
6. In practice, the circuit also has a capacitor, as in Figure 5.23. This is called a DC-DC buck converter.
7. Left out of the discussion so far is that in reality V_s is not constant—the battery drains—and R_l is not a fully realistic model for a load. In practice a controller is designed in a feedback loop from v_o to switch S . A battery drains fairly slowly, so it is reasonable to assume V_s is constant and let the controller make adjustments accordingly. As for the load, a more realistic model includes a current source, reflecting the fact that the load draws current.
8. In practice, the controller can be either analog or digital. Consider analog control. The block diagram is Figure 5.24. The controller's job is to regulate the voltage v_o , ideally making $v_o = V_{ref}$, by adjusting the duty cycle; the load current i_l is a disturbance.

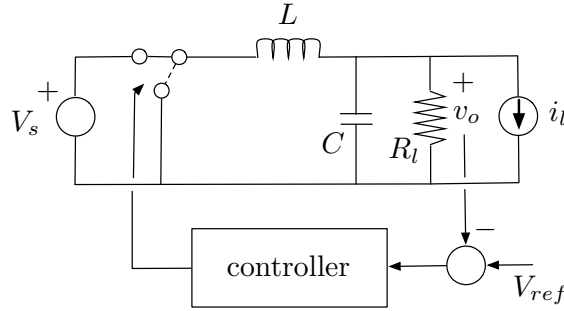


Figure 5.24: Analog controller for voltage regulation.

9. Let v_1 denote the square-wave input voltage to the circuit. Also, let i_L denote the inductor current. Then Kirchhoff's current law at the upper node gives

$$i_L = i_l + \frac{v_o}{R_l} + C \frac{dv_o}{dt}$$

and Kirchhoff's voltage law gives

$$L \frac{di_L}{dt} + v_o = v_1.$$

Define the state $x = (v_o, i_L)$. Then the preceding two equations can be written in state-space form as

$$\dot{x} = Ax + B_1 i_l + B_2 v_1 \quad (5.5)$$

$$v_o = Cx, \quad (5.6)$$

where

$$A = \begin{bmatrix} -\frac{1}{R_l C} & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -\frac{1}{C} \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Thus the transfer function from i_l to v_o is

$$C(sI - A)^{-1}B_1 = -\frac{\frac{1}{C}s}{s^2 + \frac{1}{R_l C}s + \frac{1}{LC}}$$

and from v_1 to v_o is

$$C(sI - A)^{-1}B_2 = \frac{\frac{1}{C}}{s^2 + \frac{1}{R_l C}s + \frac{1}{LC}}.$$

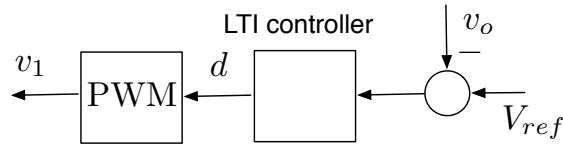


Figure 5.25: Pulse width modulator.

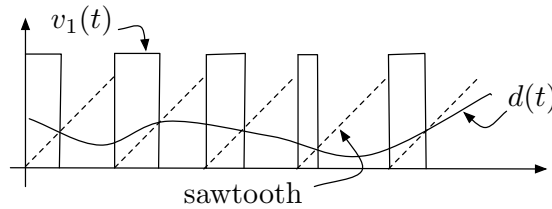


Figure 5.26: Analog form of PWM.

10. Let us emphasize that the system with inputs (v_1, i_l) and output v_o is linear time-invariant. But v_1 itself depends on the duty cycle. We now allow the duty cycle to vary from period to period. The controller is made of two parts: a linear time-invariant (LTI) circuit and a nonlinear pulse-width modulator (PWM); see Figure 5.25. The PWM works like this: The continuous-time signal $d(t)$, which is the output of the LTI controller, is compared to a sawtooth waveform of period T , producing $v_1(t)$ as shown in Figure 5.26. It is assumed that $d(t)$ stays within the range $[0, 1]$, and the slope of the sawtooth is $1/T$. We have in this way arrived at the block diagram of the control system: The plant block and notation refer to the state-space equations (5.5), (5.6). Every block here is linear time-invariant except for the PWM.
11. *Linearization.* The task now is to linearize the plant. But the system does not have an equilibrium, that is, a state where all signals are constant. This is because by its very nature $v_1(t)$ cannot be constant because it is generated by a switch. What we should do is linearize

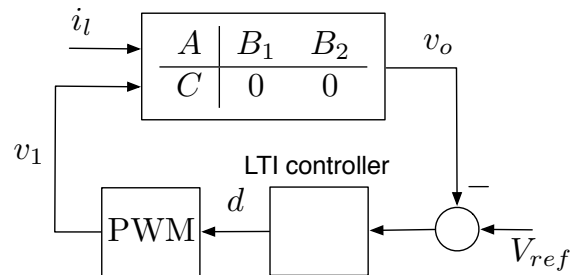


Figure 5.27: Block diagram of control system.

about a periodic solution.

12. Current practice is not to linearize exactly, but to approximate and linearize in an ad hoc manner. Consider the figure titled “Block diagram of analog control system.” To develop a small-signal model, assume the controller signal $d(t)$ equals a constant, $d(t) = D$, the nominal duty cycle about which we shall linearize, assume the load current $i_l(t)$ equals zero, and assume the feedback system is in steady state; of course, for this to be legitimate the feedback system has to be stable. Then $v_1(t)$ is periodic, and its average value equals DV_s . The DC gain from v_1 to v_o equals 1, and so the average value of $v_o(t)$ equals DV_s too. Assume D is chosen so that the average regulation error equals 0, i.e., $DV_s = V_{ref}$. Now let $d(t)$ be perturbed away from the value D :

$$d(t) = D + \Delta d(t).$$

Assume the switching frequency is sufficiently high that $\Delta d(t)$ can be approximated by a constant for any switching period $[kT, (k+1)T)$. Looking at the figure titled “Analog form of PWM,” we see that, over the period $[kT, (k+1)T)$, the average of $v_1(t)$ equals $V_s d(t)$. Thus, without a very strong justification we approximate $v_1(t)$ via

$$v_1(t) = V_s d(t) = DV_s + V_s \Delta d(t).$$

Define $\bar{v}_1 = DV_s$ and $\Delta v_1(t) = V_s \Delta d(t)$. Likewise we write x , i_l , and v_o as constants plus perturbations:

$$\begin{aligned} x &= \bar{x} + \Delta x \\ i_l &= 0 + \Delta i_l \\ v_o &= \bar{v}_o + \Delta v_o. \end{aligned}$$

With these approximations, the steady state equations of the plant become

$$\begin{aligned} 0 &= A\bar{x} + B_2\bar{v}_1 = AV_s D + B_2\bar{v}_1 \\ \bar{v}_o &= DV_s \end{aligned}$$

and the small-signal model is

$$\begin{aligned} \dot{\Delta x} &= A\Delta x + B_1\Delta i_l + B_2V_s\Delta d \\ \Delta v_o &= C\Delta x. \end{aligned}$$

The block diagram of this small-signal linear system is shown in Figure 5.28.

13. *Design.* The design of a buck converter is quite involved, but engineering firms have worked out recipes in equation form for the sizing of circuit components. We are concerned only with the feedback controller design, so we start with given parameter values. The following are typical:

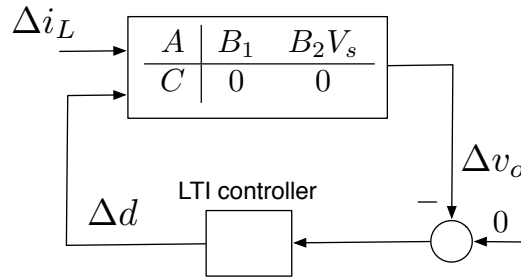
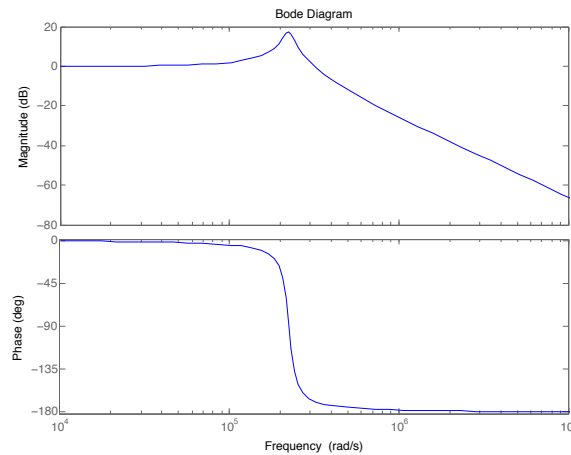


Figure 5.28: Block diagram of linearized system.

Figure 5.29: Bode plot from Δd to Δv_o .

V_s	8 V
L	$2 \mu\text{H}$
C	$10 \mu\text{F}$
R_l	3.3Ω
D	0.45
$1/T$	1 mHz

The transfer function from Δd to Δv_o is second order with the Bode plot in Figure 5.29. If the loop were closed with the controller transfer function of 1, the phase margin would be only 11° .

14. Typical closed-loop specifications involve the recovery to a step demand in load current. For a step disturbance in Δi_l , the output voltage Δv_o will droop. The droop should be limited in amplitude and duration, while the duty cycle remains within the range from 0 to 1. The nominal value of load voltage is $V_s D = 3.6$ V. Therefore the nominal load current is $V_s D / R_l = 1.09$ A. For our tests, we used a step of magnitude 20% of this nominal value. Shown in Figure 5.30 is the test response of $\Delta v_o(t)$ for the unity controller. The response is very

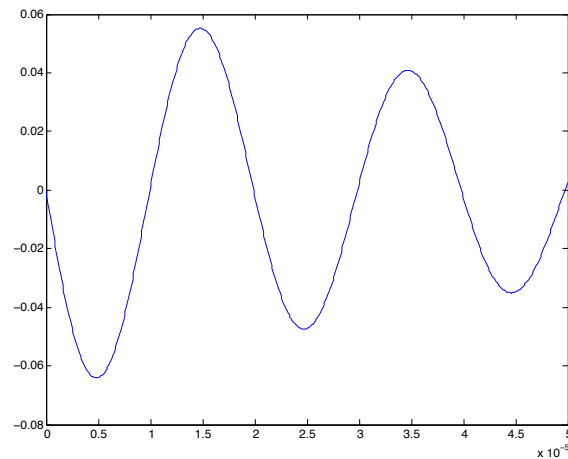


Figure 5.30: Step response from Δi_L to Δv_o for unity controller.

oscillatory, with a long settling time. The problem with this plant is it is very lightly damped. The damping ratio is only 0.07. The sensible and simple solution is to increase the damping by using a derivative controller. The controller $K(s) = 7 \times 10^{-6}s$ increases the damping to 0.8 without changing the natural frequency. Such a controller causes difficulty in computer-aided design because it doesn't have a state-space model. The obvious fix is to include a fast pole:

$$K(s) = \frac{7 \times 10^{-6}s}{10^{-10}s + 1}.$$

The resulting test response in output voltage is in Figure 5.31. Moreover, the duty cycle stayed within the required range. Finally, we emphasize that the design must be tested by a full-scale Simulink model.

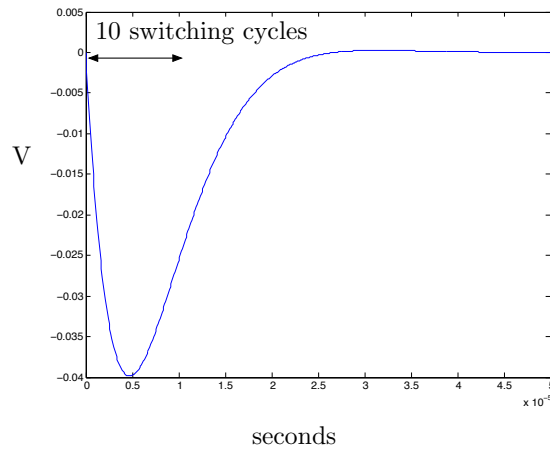
The MATLAB program for this example is given below.

```
% Circuit values

clear
V_s=8;
L=2*10^(-6);
C=10^(-5);
R_l=3.3;
D=.45;
T=10^(-6);

% State space matrices

A=[-1/(R_l*C) 1/C; -1/L 0];
B1=[-1/C ;0];
```

Figure 5.31: Step response from Δi_L to Δv_o for designed controller.

```

B2=[0;1/L];
CC=[1 0];

% Transfer functions: P is from d to v_o; P1 is from i_l to v_o.

s=tf('s');

[num,den]=ss2tf(A,B2,CC,0);
P=tf(num,den);

[num,den]=ss2tf(A,B1,CC,0);
P1=tf(num,den);

clf
% bode(P)

% Specs:
% 1. Nominal v_o is V_sD. Thus nominal current is v_sD/R_l.
% Say the step in \Delta {i}_l is 20% of this.
% 2: Number of cycles to recover.
% 3. d must remain between 0 and 1.

% Design of controller K. The symbol K is used because
% C is used for the capacitor.
% The problem is P is underdamped: zeta = 0.07.
% Therefore add derivative control to make zeta 0.8.

```

```

K=7*10^(-6)*s/(10^(-10)*s+1);

PK=P*K;
Q1=P1/(1+PK); % transfer function from load current to output voltage
Q1=minreal(Q1); % pole-zero cancellation

Q2=P1*K/(1+PK); % transfer function from load current to duty cycle
Q2=minreal(Q2);

clf

% return
t=0:10^(-7):5*10^(-5);
v=0.2*(V_s*D/R_l)*step(Q1,t); % output voltage
plot(t,v)
return
clf
d=D+0.2*(V_s*D/R_l)*step(Q2,t); % duty cycle
plot(t,d)

```

5.6 Problems

1. Take $P(s) = 0.1/(s^2 + 0.7s + 1)$. Design a lag compensator $C(s)$ to reduce the DC gain from r to e and to increase the phase margin. Include all Bode plots, together with closed-loop step responses.
2. For the plant $P(s) = 1/s^2$ design a lead compensator to get a gain crossover frequency of 10 rad/s. Make the phase margin as large as possible. Include all Bode plots, together with closed-loop step responses.
3. For the plant $P(s) = 1/(s + 1)$ design a PID compensator to get a gain crossover frequency of 2 rad/s and a phase margin as large as possible. Include all Bode plots, together with closed-loop step responses.
4. The plant transfer function is $P(s) = \frac{1}{s(s + 1)}$ and the controller transfer function $C(s)$ is the product $KC_1(s)$. Design a gain K and a lag compensator $C_1(s)$ to achieve a phase margin of 40° and a steady-state tracking error of 5% for a unit ramp input r .
5. Consider the plant $P(s) = 1/(\tau s + 1)$, a simple first order lag with time constant τ and unity DC gain. Let us say that $\tau = 10$ and that this represents a large time constant, i.e., this plant is sluggish. Mathematically, there is no obstacle in speeding this plant up without limit. For example, take

$$C(s) = K \frac{\tau s + 1}{s}.$$

Then the closed-loop transfer function is

$$\frac{P(s)C(s)}{1 + P(s)C(s)} = \frac{1}{\frac{1}{K}s + 1}.$$

This also has unity DC gain but its time constant is $1/K$, which can be arbitrarily small. What's the catch, or can we really make a Boeing 747 fly like a hummingbird?