PART IV

NONLINEAR FILTERS

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CHAPTER 13

Nonlinear Kalman filtering

> It appears that no particular approximate [nonlinear] filter is consistently better than any other, though ... any nonlinear filter is better than a strictly linear one.
> —Lawrence Schwartz and Edwin Stear [Sch68]

All of our discussion to this point has considered linear filters for linear systems. Unfortunately, linear systems do not exist. All systems are ultimately nonlinear. Even the simple $I = V/R$ relationship of Ohm's Law is only an approximation over a limited range. If the voltage across a resistor exceeds a certain threshold, then the linear approximation breaks down. Figure 13.1 shows a typical relationship between the current through a resistor and the voltage across the resistor. At small input voltages the relationship is approximately linear, but if the power dissipated by the resistor exceeds some threshold then the relationship becomes highly nonlinear. Even a device as simple as a resistor is only approximately linear, and even then only in a limited range of operation.

So we see that linear systems do not really exist. However, many systems are close enough to linear that linear estimation approaches give satisfactory results. But "close enough" can only be carried so far. Eventually, we run across a system that does not behave linearly even over a small range of operation, and our linear approaches for estimation no longer give good results. In this case, we need to explore nonlinear estimators.

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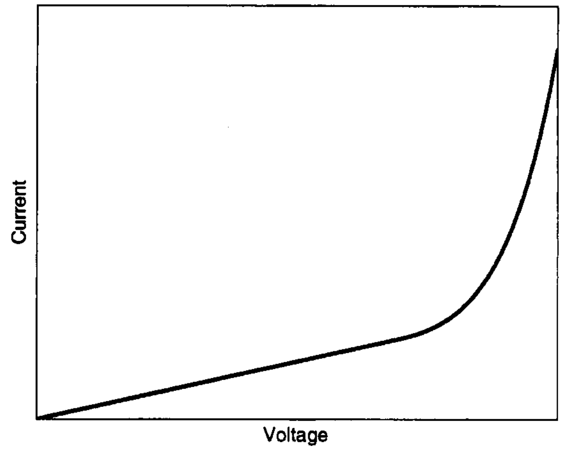

Figure 13.1 Typical current/voltage relationship for a resistor. The relationship is linear for a limited range of operation, but becomes highly nonlinear beyond that range.

Nonlinear filtering can be a difficult and complex subject. It is certainly not as mature, cohesive, or well understood as linear filtering. There is still a lot of room for advances and improvement in nonlinear estimation techniques. However, some nonlinear estimation methods have become (or are becoming) widespread. These techniques include nonlinear extensions of the Kalman filter, unscented filtering, and particle filtering.

In this chapter, we will discuss some nonlinear extensions of the Kalman filter. The Kalman filter that we discussed earlier in this book directly applies only to linear systems. However, a nonlinear system can be linearized as discussed in Section 1.3, and then linear estimation techniques (such as the Kalman or H_∞ filter) can be applied. This chapter discusses those types of approaches to nonlinear Kalman filtering.

In Section 13.1, we will discuss the linearized Kalman filter. This will involve finding a linear system whose states represent the deviations from a nominal trajectory of a nonlinear system. We can then use the Kalman filter to estimate the deviations from the nominal trajectory, and hence obtain an estimate of the states of the nonlinear system. In Section 13.2, we will extend the linearized Kalman filter to directly estimate the states of a nonlinear system. This filter, called the extended Kalman filter (EKF), is undoubtedly the most widely used nonlinear state estimation technique that has been applied in the past few decades. In Section 13.3, we will discuss "higher-order" approaches to nonlinear Kalman filtering. These approaches involve more than a direct linearization of the nonlinear system, hence the expression "higher order." Such methods include second-order Kalman filtering, iterated Kalman filtering, sum-based Kalman filtering, and grid-based Kalman filtering. These filters provide ways to reduce the linearization errors that are inherent in the EKF. They typically provide estimation performance that is better than the EKF, but they do so at the price of higher complexity and computational expense.

Section 13.4 covers parameter estimation using Kalman filtering. Sometimes, an engineer wants to estimate the parameters of a system but does not care about estimating the states. This becomes a system identification problem. The system equations are generally nonlinear functions of the system parameters. System parameters are usually considered to be constant, or slowly time-varying, and a nonlinear Kalman filter (or any other nonlinear state estimator) can be adapted to estimate system parameters.

13.1 THE LINEARIZED KALMAN FILTER

In this section, we will show how to linearize a nonlinear system, and then use Kalman filtering theory to estimate the deviations of the state from a nominal state value. This will then give us an estimate of the state of the nonlinear system. We will derive the linearized Kalman filter from the continuous-time viewpoint, but the analogous derivation for discrete-time or hybrid systems are straightforward.

Consider the following general nonlinear system model:

$$\begin{aligned}
\dot{x} &= f(x, u, w, t) \\
y &= h(x, v, t) \\
w &\sim (0, Q) \\
v &\sim (0, R)
\end{aligned} \tag{13.1}$$

The system equation $f(\cdot)$ and the measurement equation $h(\cdot)$ are nonlinear functions. We will use Taylor series to expand these equations around a nominal control u_0, nominal state x_0, nominal output y_0, and nominal noise values w_0 and v_0. These nominal values (all of which are functions of time) are based on *a priori* guesses of what the system trajectory might look like. For example, if the system equations represent the dynamics of an airplane, then the nominal control, state, and output might be the planned flight trajectory. The *actual* flight trajectory will differ from this nominal trajectory due to mismodeling, disturbances, and other unforeseen effects. But the actual trajectory should be close to the nominal trajectory, in which case the Taylor series linearization should be approximately correct. The Taylor series linearization of Equation (13.1) gives

$$\begin{aligned}
\dot{x} &\approx f(x_0, u_0, w_0, t) + \left.\frac{\partial f}{\partial x}\right|_0 (x - x_0) + \left.\frac{\partial f}{\partial u}\right|_0 (u - u_0) + \\
&\quad \left.\frac{\partial f}{\partial w}\right|_0 (w - w_0) \\
&= f(x_0, u_0, w_0, t) + A\Delta x + B\Delta u + L\Delta w \\
y &\approx h(x_0, v_0, t) + \left.\frac{\partial h}{\partial x}\right|_0 (x - x_0) + \left.\frac{\partial h}{\partial v}\right|_0 (v - v_0) \\
&= h(x_0, v_0, t) + C\Delta x + M\Delta v
\end{aligned} \tag{13.2}$$

The definitions of the partial derivative matrices A, B, C, L, and M are apparent from the above equations. The 0 subscript on the partial derivatives means that they are evaluated at the nominal control, state, output, and noise values. The definitions of the deviations Δx, Δu, Δw, and Δv are also apparent from the above equations.

Let us assume that the nominal noise values $w_0(t)$ and $v_0(t)$ are both equal to 0 for all time. [If they are not equal to 0 then we should be able to write them as the sum of a known deterministic part and a zero-mean part, redefine the noise quantities, and rewrite Equation (13.1) so that the nominal noise values are equal to 0. See Problem 13.1]. Since $w_0(t)$ and $v_0(t)$ are both equal to 0, we see that $\Delta w(t) = w(t)$ and $\Delta v(t) = v(t)$. Further assume that the control $u(t)$ is perfectly known. In general, this is a reasonable assumption. After all, the control input $u(t)$ is determined by our control system, so there should not be any uncertainty in its value. This means that $u_0(t) = u(t)$ and $\Delta u(t) = 0$. However, in reality there may be uncertainties in the outputs of our control system because they are connected to actuators that have biases and noise. If this is the case then we can express the control as $u_0(t) + \Delta u(t)$, where $u_0(t)$ is known and $\Delta u(t)$ is a zero-mean random variable, rewrite the system equations with a perfectly known control signal, and include $\Delta u(t)$ as part of the process noise (see Problem 13.2). Now we define the nominal system trajectory as

$$\begin{aligned} \dot{x}_0 &= f(x_0, u_0, w_0, t) \\ y_0 &= h(x_0, v_0, t) \end{aligned} \tag{13.3}$$

We define the deviation of the true state derivative from the nominal state derivative, and the deviation of the true measurement from the nominal measurement, as follows:

$$\begin{aligned} \Delta\dot{x} &= \dot{x} - \dot{x}_0 \\ \Delta y &= y - y_0 \end{aligned} \tag{13.4}$$

With these definitions Equation (13.2) becomes

$$\begin{aligned} \Delta\dot{x} &= A\Delta x + Lw \\ &= A\Delta x + \tilde{w} \\ \tilde{w} &\sim (0, \tilde{Q}), \qquad \tilde{Q} = LQL^T \\ \Delta y &= C\Delta x + Mv \\ &= C\Delta x + \tilde{v} \\ \tilde{v} &\sim (0, \tilde{R}), \qquad \tilde{R} = MRM^T \end{aligned} \tag{13.5}$$

The above equation is a linear system with state Δx and measurement Δy, so we can use a Kalman filter to estimate Δx. The inputs to the filter consist of Δy, which is the difference between the actual measurement y and the nominal measurement y_0. The Δx that is output from the Kalman filter is an estimate of the difference between the actual state x and the nominal state x_0. The Kalman filter equations for the linearized Kalman filter are

$$\begin{aligned} \Delta\hat{x}(0) &= 0 \\ P(0) &= E\left[(\Delta x(0) - \Delta\hat{x}(0))(\Delta x(0) - \Delta\hat{x}(0))^T\right] \\ \Delta\dot{\hat{x}} &= A\Delta\hat{x} + K(\Delta y - C\Delta\hat{x}) \\ K &= PC^T\tilde{R}^{-1} \\ \dot{P} &= AP + PA^T + \tilde{Q} - PC^T\tilde{R}^{-1}CP \\ \hat{x} &= x_0 + \Delta\hat{x} \end{aligned} \tag{13.6}$$

For the Kalman filter, P is equal to the covariance of the estimation error. In the linearized Kalman filter this is no longer true because of errors that creep into the linearization of Equation (13.2). However, if the linearization errors are small then P should be approximately equal to the covariance of the estimation error. The linearized Kalman filter can be summarized as follows.

The continuous-time linearized Kalman filter

1. The system equations are given as

$$\begin{aligned} \dot{x} &= f(x,u,w,t) \\ y &= h(x,v,t) \\ w &\sim (0,Q) \\ v &\sim (0,R) \end{aligned} \tag{13.7}$$

   The nominal trajectory is known ahead of time:

$$\begin{aligned} \dot{x}_0 &= f(x_0,u_0,0,t) \\ y_0 &= h(x_0,0,t) \end{aligned} \tag{13.8}$$

2. Compute the following partial derivative matrices evaluated at the nominal trajectory values:

$$\begin{aligned} A &= \left.\frac{\partial f}{\partial x}\right|_0 \\ L &= \left.\frac{\partial f}{\partial w}\right|_0 \\ C &= \left.\frac{\partial h}{\partial x}\right|_0 \\ M &= \left.\frac{\partial h}{\partial v}\right|_0 \end{aligned} \tag{13.9}$$

3. Compute the following matrices:

$$\begin{aligned} \tilde{Q} &= LQL^T \\ \tilde{R} &= MRM^T \end{aligned} \tag{13.10}$$

4. Define Δy as the difference between the actual measurement y and the nominal measurement y_0:

$$\Delta y = y - y_0 \tag{13.11}$$

5. Execute the following Kalman filter equations:

$$\begin{aligned} \Delta\hat{x}(0) &= 0 \\ P(0) &= E\left[(\Delta x(0) - \Delta\hat{x}(0))(\Delta x(0) - \Delta\hat{x}(0))^T\right] \\ \Delta\dot{\hat{x}} &= A\Delta\hat{x} + K(\Delta y - C\Delta\hat{x}) \\ K &= PC^T\tilde{R}^{-1} \\ \dot{P} &= AP + PA^T + \tilde{Q} - PC^T\tilde{R}^{-1}CP \end{aligned} \tag{13.12}$$

6. Estimate the state as follows:

$$\hat{x} = x_0 + \Delta\hat{x} \tag{13.13}$$

The hybrid linearized Kalman filter and the discrete-time linearized Kalman filter are not presented here, but if the development above is understood then their derivations should be straightforward.

13.2 THE EXTENDED KALMAN FILTER

The previous section obtained a linearized Kalman filter for estimating the states of a nonlinear system. The derivation was based on linearizing the nonlinear system around a nominal state trajectory. The question that arises is, How do we know the nominal state trajectory? In some cases it may not be straightforward to find the nominal trajectory. However, since the Kalman filter estimates the state of the system, we can use the Kalman filter estimate as the nominal state trajectory. This is sort of a bootstrap method. We linearize the nonlinear system around the Kalman filter estimate, and the Kalman filter estimate is based on the linearized system. This is the idea of the extended Kalman filter (EKF), which was originally proposed by Stanley Schmidt so that the Kalman filter could be applied to nonlinear spacecraft navigation problems [Bel67].

In Section 13.2.1, we will present the EKF for continuous-time systems with continuous-time measurements. In Section 13.2.2, we will present the hybrid EKF, which is the EKF for continuous-time systems with discrete-time measurements. In Section 13.2.3, we will present the EKF for discrete-time systems with discrete-time measurements.

13.2.1 The continuous-time extended Kalman filter

Combine the $\dot{x}_0$ expression in Equation (13.3) with the $\Delta\dot{\hat{x}}$ expression in Equation (13.6) to obtain

$$\dot{x}_0 + \Delta\dot{\hat{x}} = f(x_0, u_0, w_0, t) + A\Delta\hat{x} + K[y - y_0 - C(\hat{x} - x_0)] \tag{13.14}$$

Now choose $x_0(t) = \hat{x}(t)$ so that $\Delta\hat{x}(t) = 0$ and $\Delta\dot{\hat{x}}(t) = 0$. In other words, our linearization trajectory $x_0(t)$ is equal to our linearized Kalman filter estimate $\hat{x}(t)$. Then the nominal measurement expression in Equation (13.3) becomes

$$\begin{aligned} y_0 &= h(x_0, v_0, t) \\ &= h(\hat{x}, v_0, t) \end{aligned} \tag{13.15}$$

and Equation (13.14) becomes

$$\dot{\hat{x}} = f(\hat{x}, u, w_0, t) + K\left[y - h(\hat{x}, v_0, t)\right] \tag{13.16}$$

This is equivalent to the linearized Kalman filter except that we have chosen $x_0 = \hat{x}$, and we have rearranged the equations to obtain $\hat{x}$ directly. The Kalman gain K is the same as that presented in Equation (13.6). But this formulation inputs the measurement y directly, and outputs the state estimate $\hat{x}$ directly. This is often referred to as the extended Kalman–Bucy filter because Richard Bucy collaborated with Rudolph Kalman in the first publication of the continuous-time Kalman filter [Kal61]. The continuous-time EKF can be summarized as follows.

The continuous-time extended Kalman filter

1. The system equations are given as

$$\begin{aligned}\dot{x} &= f(x,u,w,t)\\ y &= h(x,v,t)\\ w &\sim (0,Q)\\ v &\sim (0,R)\end{aligned} \tag{13.17}$$

2. Compute the following partial derivative matrices evaluated at the current state estimate:

$$\begin{aligned}A &= \left.\frac{\partial f}{\partial x}\right|_{\hat{x}}\\ L &= \left.\frac{\partial f}{\partial w}\right|_{\hat{x}}\\ C &= \left.\frac{\partial h}{\partial x}\right|_{\hat{x}}\\ M &= \left.\frac{\partial h}{\partial v}\right|_{\hat{x}}\end{aligned} \tag{13.18}$$

3. Compute the following matrices:

$$\begin{aligned}\tilde{Q} &= LQL^T\\ \tilde{R} &= MRM^T\end{aligned} \tag{13.19}$$

4. Execute the following Kalman filter equations:

$$\begin{aligned}\hat{x}(0) &= E[x(0)]\\ P(0) &= E\left[(x(0)-\hat{x}(0))(x(0)-\hat{x}(0))^T\right]\\ \dot{\hat{x}} &= f(\hat{x},u,w_0,t)+K\left[y-h(\hat{x},v_0,t)\right]\\ K &= PC^T\tilde{R}^{-1}\\ \dot{P} &= AP+PA^T+\tilde{Q}-PC^T\tilde{R}^{-1}CP\end{aligned} \tag{13.20}$$

where the nominal noise values are given as $w_0 = 0$ and $v_0 = 0$.

■ EXAMPLE 13.1

In this example, we will use the continuous-time EKF to estimate the state of a two-phase permanent magnet synchronous motor. The system equations are given in Example 1.4 and are repeated here:

$$\begin{aligned}\dot{i}_a &= \frac{-R}{L}i_a+\frac{\omega\lambda}{L}\sin\theta+\frac{u_a+q_1}{L}\\ \dot{i}_b &= \frac{-R}{L}i_b-\frac{\omega\lambda}{L}\cos\theta+\frac{u_b+q_2}{L}\\ \dot{\omega} &= \frac{-3\lambda}{2J}i_a\sin\theta+\frac{3\lambda}{2J}i_b\cos\theta-\frac{F\omega}{J}+q_3\\ \dot{\theta} &= \omega\end{aligned} \tag{13.21}$$

where i_a and i_b are the currents in the two windings, θ and ω are the angular position and velocity of the rotor, R and L are the winding resistance and inductance, λ is the flux constant, and F is the coefficient of viscous friction. The control inputs u_a and u_b consist of the applied voltages across the two windings, and J is the moment of inertia of the motor shaft and load. The state is defined as

$$x = \begin{bmatrix} i_a & i_b & \omega & \theta \end{bmatrix}^T \tag{13.22}$$

The q_i terms are process noise due to uncertainty in the control inputs (q_1 and q_2) and the load torque (q_3). The partial derivative A matrix is obtained as

$$\begin{aligned} A &= \frac{\partial f}{\partial x} \\ &= \begin{bmatrix} -R/L & 0 & \lambda s/L & x_3\lambda c/L \\ 0 & -R/L & -\lambda c/L & x_3\lambda s/L \\ -3\lambda s/2/J & 3\lambda c/2/J & -F/J & -3\lambda(x_1 c + x_2 s)/2/J \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{aligned} \tag{13.23}$$

where we have used the notation $s = \sin x_4$ and $c = \cos x_4$. Suppose that we can measure the winding currents with sense resistors so our measurement equations are

$$\begin{aligned} y(1) &= i_a + v(1) \\ y(2) &= i_b + v(2) \end{aligned} \tag{13.24}$$

where $v(1)$ and $v(2)$ are independent zero-mean white noise processes with standard deviations equal to 0.1 amps. The nominal control inputs are set to

$$\begin{aligned} u_a(t) &= \sin(2\pi t) \\ u_b(t) &= \cos(2\pi t) \end{aligned} \tag{13.25}$$

The actual control inputs are equal to the nominal values plus q_1 and q_2 (electrical noise terms), which are independent zero-mean white noise processes with standard deviations equal to 0.01 amps. The noise due to load torque disturbances (q_3) has a standard deviation of 0.5 rad/sec^2. Measurements are obtained continuously. Even though our measurements consist only of the winding currents and the system is nonlinear, we can use a continuous-time EKF (implemented in analog circuitry or very fast digital logic) to estimate the rotor position and velocity. The simulation results are shown in Figure 13.2. The four states are estimated quite well. In particular, the rotor position estimate is so good that the true and estimated rotor position traces are not distinguishable in Figure 13.2.

The P matrix quantifies the uncertainty in the state estimates. If the nonlinearities in the system and measurement are not too severe, then the P matrix should give us an idea of how accurate our estimates are. In this example, the standard deviations of the state estimation errors were obtained from the simulation and then compared with the diagonal elements of the steady-state P matrix that came out of the Kalman filter. Table 13.1 shows a comparison of the estimation errors that were determined by simulation and

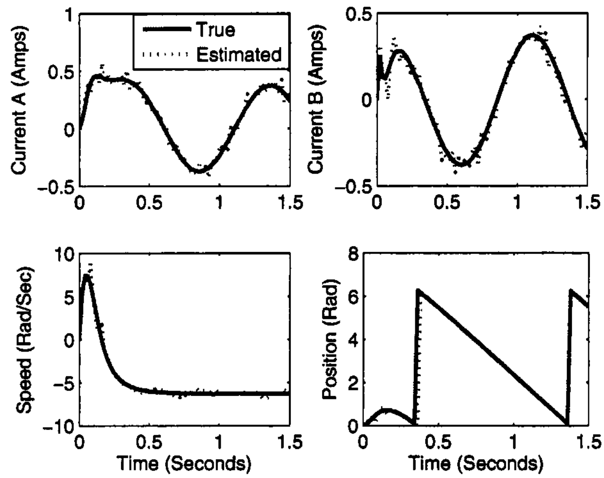

Figure 13.2 Continuous extended Kalman filter simulation results for the two-phase permanent magnet synchronous motor of Example 13.1.

Table 13.1 Example 13.1 results showing one standard deviation state estimation errors determined from simulation results and determined from the P matrix of the EKF. These results are for the two-phase permanent magnet motor simulation. This table shows that the P matrix gives a good indication of the magnitude of the EKF state estimation errors.

	Simulation	P Matrix
Winding A Current	0.054 amps	0.094 Amps
Winding B Current	0.052 amps	0.094 Amps
Speed	0.26 rad/sec	0.44 rad/sec
Position	0.013 rad	0.025 rad

the theoretical estimation errors based on the P matrix. We see that the P matrix gives a good indication of the magnitude of the estimation errors.

$\nabla\nabla\nabla$

13.2.2 The hybrid extended Kalman filter

Many real engineering systems are governed by continuous-time dynamics whereas the measurements are obtained at discrete instants of time. In this section, we will derive the hybrid EKF, which considers systems with continuous-time dynamics and discrete-time measurements. This is the most common situation encountered in practice.

Suppose we have a continuous-time system with discrete-time measurements as follows:

$$\begin{aligned} \dot{x} &= f(x,u,w,t) \\ y_k &= h_k(x_k,v_k) \\ w(t) &\sim (0,Q) \\ v_k &\sim (0,R_k) \end{aligned} \tag{13.26}$$

The process noise $w(t)$ is continuous-time white noise with covariance Q, and the measurement noise v_k is discrete-time white noise with covariance R_k. Between measurements we propagate the state estimate according to the known nonlinear dynamics, and we propagate the covariance as derived in the continuous-time EKF of Section 13.2.1 using Equation (13.20). Recall that the $\dot{P}$ expression from Equation (13.20) is given as

$$\dot{P} = AP + PA^T + LQL^T - PC^T(MRM^T)^{-1}CP \tag{13.27}$$

In the hybrid EKF, we should not include the R term in the $\dot{P}$ equation because we are integrating P between measurement times, during which we do not have any measurements. Another way of looking at it is that in between measurement times we have measurements with infinite covariance ($R = \infty$), so the last term on the right side of the $\dot{P}$ equation goes to zero. This gives us the following for the time-update equations of the hybrid EKF:

$$\begin{aligned} \dot{\hat{x}} &= f(\hat{x},u,w_0,t) \\ \dot{P} &= AP + PA^T + LQL^T \end{aligned} \tag{13.28}$$

where A and L are given in Equation (13.18). The above equations propagate $\hat{x}$ from $\hat{x}_{k-1}^+$ to $\hat{x}_k^-$, and P from P_{k-1}^+ to P_k^-. Note that w_0 is the nominal process noise in the above equation; that is, $w_0(t) = 0$.

At each measurement time, we update the state estimate and the covariance as derived in the discrete-time Kalman filter (Chapter 5):

$$\begin{aligned} K_k &= P_k^- H_k^T (H_k P_k^- H_k^T + M_k R_k M_k^T)^{-1} \\ \hat{x}_k^+ &= \hat{x}_k^- + K_k[y_k - h_k(\hat{x}_k^-, v_0, t_k)] \\ P_k^+ &= (I - K_k H_k) P_k^- (I - K_k H_k)^T + K_k M_k R_k M_k^T K_k^T \end{aligned} \tag{13.29}$$

where v_0 is the nominal measurement noise; that is, $v_0 = 0$. H_k is the partial derivative of $h_k(x_k, v_k)$ with respect to x_k, and M_k is the partial derivative of $h_k(x_k, v_k)$ with respect to v_k. H_k and M_k are evaluated at $\hat{x}_k^-$.

Note that P_k and K_k cannot be computed offline because they depend on H_k and M_k, which depend on $\hat{x}_k^-$, which in turn depends on the noisy measurements. Therefore, a steady-state solution does not (in general) exist to the extended Kalman filter. However, some efforts at obtaining steady-state approximations to the extended Kalman filter have been reported in [Saf78].

The hybrid EKF can be summarized as follows.

The hybrid extended Kalman filter

1. The system equations with continuous-time dynamics and discrete-time measurements are given as follows:

$$\begin{aligned} \dot{x} &= f(x, u, w, t) \\ y_k &= h_k(x_k, v_k) \\ w(t) &\sim (0, Q) \\ v_k &\sim (0, R_k) \end{aligned} \tag{13.30}$$

2. Initialize the filter as follows:

$$\begin{aligned} \hat{x}_0^+ &= E[x_0] \\ P_0^+ &= E\left[(x_0 - \hat{x}_0^+)(x_0 - \hat{x}_0^+)^T\right] \end{aligned} \tag{13.31}$$

3. For $k = 1, 2, \cdots$, perform the following.

   (a) Integrate the state estimate and its covariance from time $(k-1)^+$ to time k^- as follows:

$$\begin{aligned} \dot{\hat{x}} &= f(\hat{x}, u, 0, t) \\ \dot{P} &= AP + PA^T + LQL^T \end{aligned} \tag{13.32}$$

   where F and L are given in Equation (13.18). We begin this integration process with $\hat{x} = \hat{x}_{k-1}^+$ and $P = P_{k-1}^+$. At the end of this integration we have $\hat{x} = \hat{x}_k^-$ and $P = P_k^-$.

   (b) At time k, incorporate the measurement y_k into the state estimate and estimation covariance as follows:

$$\begin{aligned} K_k &= P_k^- H_k^T (H_k P_k^- H_k^T + M_k R_k M_k^T)^{-1} \\ \hat{x}_k^+ &= \hat{x}_k^- + K_k(y_k - h_k(\hat{x}_k^-, 0, t_k)) \\ P_k^+ &= (I - K_k H_k) P_k^- (I - K_k H_k)^T + K_k M_k R_k M_k^T K_k^T \end{aligned} \tag{13.33}$$

   H_k and M_k are the partial derivatives of $h_k(x_k, v_k)$ with respect to x_k and v_k, and are both evaluated at $\hat{x}_k^-$. Note that other equivalent expressions can be used for K_k and P_k^+, as is apparent from Equation (5.19).

■ **EXAMPLE 13.2**

In this example, we will use the continuous-time EKF and the hybrid EKF to estimate the altitude x_1, velocity x_2, and constant ballistic coefficient $1/x_3$ of a body as it falls toward earth. A range-measuring device measures the altitude of the falling body. This example (or a variant thereof) is given in several places, for example [Ath68, Ste94, Jul00]. The equations for this system are

$$\begin{aligned} \dot{x}_1 &= x_2 + w_1 \\ \dot{x}_2 &= \rho_0 \exp(-x_1/k) x_2^2 / 2x_3 - g + w_2 \\ \dot{x}_3 &= w_3 \\ y &= x_1 + v \end{aligned} \tag{13.34}$$

As usual, w_i is the noise that affects the ith process equation, and v is the measurement noise. ρ_0 is the air density at sea level, k is a constant that defines the relationship between air density and altitude, and g is the acceleration due to gravity. The partial derivative matrices for this system are given as follows:

$$
\begin{aligned}
A &= \frac{\partial f}{\partial x} \\
&= \begin{bmatrix} 0 & 1 & 0 \\ A_{21} & A_{22} & A_{23} \\ 0 & 0 & 0 \end{bmatrix} \\
A_{21} &= -\rho_0 \exp(-x_1/k) x_2^2/2kx_3 \\
A_{22} &= \rho_0 \exp(-x_1/k) x_2/x_3 \\
A_{23} &= -\rho_0 \exp(-x_1/k) x_2^2/2x_3^2 \\
C = H &= \frac{\partial h}{\partial x} \\
&= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}
\end{aligned} \tag{13.35}
$$

We will use the continuous-time system equations to simulate the system. For the hybrid system we suppose that we obtain range measurements every 0.5 seconds. The constants that we will use are given as

$$
\begin{aligned}
\rho_0 &= 0.0034 \text{ lb-sec}^2/\text{ft}^4 \\
g &= 32.2 \text{ ft/sec}^2 \\
k &= 22000 \text{ ft} \\
E[v^2(t)] &= 100 \text{ ft}^2 \\
E[w_i^2(t)] &= 0 \quad (i = 1, 2, 3)
\end{aligned} \tag{13.36}
$$

The initial conditions of the system and the estimator are given as

$$
\begin{aligned}
x_0 &= \begin{bmatrix} 100,000 & -6,000 & 1/2,000 \end{bmatrix}^T \\
\hat{x}_0^+ &= \begin{bmatrix} 100,010 & -6,100 & 1/2,500 \end{bmatrix}^T \\
P_0^+ &= \begin{bmatrix} 500 & 0 & 0 \\ 0 & 20,000 & 0 \\ 0 & 0 & 1/250,000 \end{bmatrix}
\end{aligned} \tag{13.37}
$$

We use rectangular integration with a step size of 0.4 msec to simulate the system, the continuous-time EKF, and the hybrid EKF (with a measurement time of 0.5 sec). Figure 13.3 shows estimation-error magnitudes averaged over 100 simulations for the altitude, velocity, and ballistic coefficient reciprocal of the falling body. We see that the continuous-time EKF appears to perform better in general than the hybrid EKF. This is to be expected since more measurements are incorporated in the continuous-time EKF. The RMS estimation errors averaged over 100 simulations was 2.8 feet for the continuous-time EKF and 5.1 feet for the hybrid EKF for altitude estimation, 1.2 feet/s for the continuous-time EKF and 2.0 feet/s for the hybrid EKF for velocity estimation, and 213 for the continuous-time EKF and 246 for the hybrid EKF for the reciprocal of ballistic coefficient estimation. Of course, a continuous-time EKF (in analog hardware) would be more difficult to implement, tune, and modify than a hybrid EKF (in digital hardware).

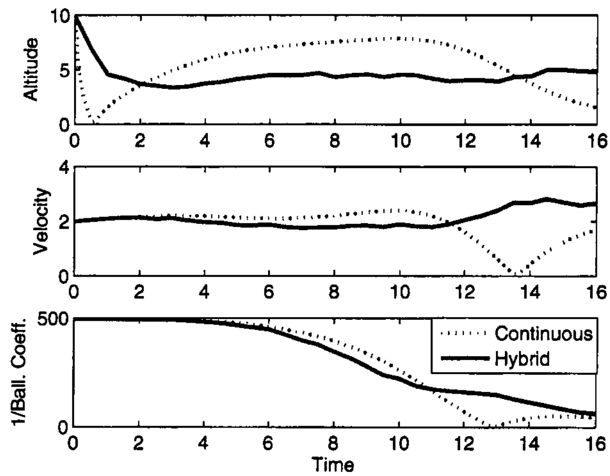

Figure 13.3 Example 13.2 altitude, velocity, and ballistic coefficient reciprocal estimation-error magnitudes of a falling body averaged over 100 simulations. The continuous-time EKF generally performs better than the hybrid EKF.

∇∇∇

13.2.3 The discrete-time extended Kalman filter

In this section, we will derive the discrete-time EKF, which considers discrete-time dynamics and discrete-time measurements. This situation is often encountered in practice. Even if the underlying system dynamics are continuous time, the EKF usually needs to be implemented in a digital computer. This means that there might not be enough computational power to integrate the system dynamics as required in a continuous-time EKF or a hybrid EKF. So the dynamics are often discretized (see Section 1.4) and then a discrete-time EKF can be used.

Suppose we have the system model

$$\begin{aligned} x_k &= f_{k-1}(x_{k-1}, u_{k-1}, w_{k-1}) \\ y_k &= h_k(x_k, v_k) \\ w_k &\sim (0, Q_k) \\ v_k &\sim (0, R_k) \end{aligned} \tag{13.38}$$

We perform a Taylor series expansion of the state equation around $x_{k-1} = \hat{x}_{k-1}^+$ and $w_{k-1} = 0$ to obtain the following:

$$
\begin{aligned}
x_k &= f_{k-1}(\hat{x}_{k-1}^+, u_{k-1}, 0) + \left.\frac{\partial f_{k-1}}{\partial x}\right|_{\hat{x}_{k-1}^+} (x_{k-1} - \hat{x}_{k-1}^+) + \left.\frac{\partial f_{k-1}}{\partial w}\right|_{\hat{x}_{k-1}^+} w_{k-1} \\
&= f_{k-1}(\hat{x}_{k-1}^+, u_{k-1}, 0) + F_{k-1}(x_{k-1} - \hat{x}_{k-1}^+) + L_{k-1}w_{k-1} \\
&= F_{k-1}x_{k-1} + \left[f_{k-1}(\hat{x}_{k-1}^+, u_{k-1}, 0) - F_{k-1}\hat{x}_{k-1}^+\right] + L_{k-1}w_{k-1} \\
&= F_{k-1}x_{k-1} + \tilde{u}_{k-1} + \tilde{w}_{k-1}
\end{aligned} \tag{13.39}
$$

F_{k-1} and L_{k-1} are defined by the above equation. The known signal $\tilde{u}_k$ and the noise signal $\tilde{w}_k$ are defined as follows:

$$
\begin{aligned}
\tilde{u}_k &= f_k(\hat{x}_k^+, u_k, 0) - F_k\hat{x}_k^+ \\
\tilde{w}_k &\sim (0, L_kQ_kL_k^T)
\end{aligned} \tag{13.40}
$$

We linearize the measurement equation around $x_k = \hat{x}_k^-$ and $v_k = 0$ to obtain

$$
\begin{aligned}
y_k &= h_k(\hat{x}_k^-, 0) + \left.\frac{\partial h_k}{\partial x}\right|_{\hat{x}_k^-} (x_k - \hat{x}_k^-) + \left.\frac{\partial h_k}{\partial v}\right|_{\hat{x}_k^-} v_k \\
&= h_k(\hat{x}_k^-, 0) + H_k(x_k - \hat{x}_k^-) + M_kv_k \\
&= H_kx_k + \left[h_k(\hat{x}_k^-, 0) - H_k\hat{x}_k^-\right] + M_kv_k \\
&= H_kx_k + z_k + \tilde{v}_k
\end{aligned} \tag{13.41}
$$

H_k and M_k are defined by the above equation. The known signal z_k and the noise signal $\tilde{v}_k$ are defined as

$$
\begin{aligned}
z_k &= h_k(\hat{x}_k^-, 0) - H_k\hat{x}_k^- \\
\tilde{v}_k &\sim (0, M_kR_kM_k^T)
\end{aligned} \tag{13.42}
$$

We have a linear state-space system in Equation (13.39) and a linear measurement in Equation (13.41). That means we can use the standard Kalman filter equations to estimate the state. This results in the following equations for the discrete-time extended Kalman filter.

$$
\begin{aligned}
P_k^- &= F_{k-1}P_{k-1}^+F_{k-1}^T + L_{k-1}Q_{k-1}L_{k-1}^T \\
K_k &= P_k^-H_k^T(H_kP_k^-H_k^T + M_kR_kM_k^T)^{-1} \\
\hat{x}_k^- &= f_{k-1}(\hat{x}_{k-1}^+, u_{k-1}, 0) \\
z_k &= h_k(\hat{x}_k^-, 0) - H_k\hat{x}_k^- \\
\hat{x}_k^+ &= \hat{x}_k^- + K_k(y_k - H_k\hat{x}_k^- - z_k) \\
&= \hat{x}_k^- + K_k[y_k - h_k(\hat{x}_k^-, 0)] \\
P_k^+ &= (I - K_kH_k)P_k^-
\end{aligned} \tag{13.43}
$$

The discrete-time EKF can be summarized as follows.

The discrete-time extended Kalman filter

1. The system and measurement equations are given as follows:

$$\begin{aligned} x_k &= f_{k-1}(x_{k-1}, u_{k-1}, w_{k-1}) \\ y_k &= h_k(x_k, v_k) \\ w_k &\sim (0, Q_k) \\ v_k &\sim (0, R_k) \end{aligned} \tag{13.44}$$

2. Initialize the filter as follows:

$$\begin{aligned} \hat{x}_0^+ &= E(x_0) \\ P_0^+ &= E[(x_0 - \hat{x}_0^+)(x_0 - \hat{x}_0^+)^T] \end{aligned} \tag{13.45}$$

3. For $k = 1, 2, \cdots$, perform the following.

   (a) Compute the following partial derivative matrices:

$$\begin{aligned} F_{k-1} &= \left.\frac{\partial f_{k-1}}{\partial x}\right|_{\hat{x}_{k-1}^+} \\ L_{k-1} &= \left.\frac{\partial f_{k-1}}{\partial w}\right|_{\hat{x}_{k-1}^+} \end{aligned} \tag{13.46}$$

   (b) Perform the time update of the state estimate and estimation-error covariance as follows:

$$\begin{aligned} P_k^- &= F_{k-1} P_{k-1}^+ F_{k-1}^T + L_{k-1} Q_{k-1} L_{k-1}^T \\ \hat{x}_k^- &= f_{k-1}(\hat{x}_{k-1}^+, u_{k-1}, 0) \end{aligned} \tag{13.47}$$

   (c) Compute the following partial derivative matrices:

$$\begin{aligned} H_k &= \left.\frac{\partial h_k}{\partial x}\right|_{\hat{x}_k^-} \\ M_k &= \left.\frac{\partial h_k}{\partial v}\right|_{\hat{x}_k^-} \end{aligned} \tag{13.48}$$

   (d) Perform the measurement update of the state estimate and estimation-error covariance as follows:

$$\begin{aligned} K_k &= P_k^- H_k^T (H_k P_k^- H_k^T + M_k R_k M_k^T)^{-1} \\ \hat{x}_k^+ &= \hat{x}_k^- + K_k[y_k - h_k(\hat{x}_k^-, 0)] \\ P_k^+ &= (I - K_k H_k) P_k^- \end{aligned} \tag{13.49}$$

   Note that other equivalent expressions can be used for K_k and P_k^+, as is apparent from Equation (5.19).

13.3 HIGHER-ORDER APPROACHES

More refined linearization techniques can be used to reduce the linearization error in the EKF for highly nonlinear systems. In this section, we will derive and illustrate two such approaches: the iterated EKF, and the second-order EKF. We will also briefly discuss other approaches, including Gaussian sum filters and grid filters.

13.3.1 The iterated extended Kalman filter

In this section, we will discuss the iterated EKF. We will confine our discussion here to discrete-time filtering, although the concepts can easily be extended to continuous or hybrid filters.

When we derived the discrete-time EKF in Section 13.2.3, we approximated $h(x_k, v_k)$ by expanding it in a Taylor series around $\hat{x}_k^-$, as shown in Equation (13.41):

$$\begin{aligned} h(x_k, v_k) &= h(\hat{x}_k^-, 0) + \left.\frac{\partial h}{\partial x}\right|_{\hat{x}_k^-} (x_k - \hat{x}_k^-) + \left.\frac{\partial h}{\partial v}\right|_{\hat{x}_k^-} v_k \\ &= h(\hat{x}_k^-, 0) + H_k(x_k - \hat{x}_k^-) + M_k v_k \end{aligned} \tag{13.50}$$

Based on this linearization, we then wrote the measurement-update equations as shown in Equation (13.43):

$$\begin{aligned} K_k &= P_k^- H_k^T (H_k P_k^- H_k^T + M_k R_k M_k^T)^{-1} \\ P_k^+ &= (I - K_k H_k) P_k^- \\ \hat{x}_k^+ &= \hat{x}_k^- + K_k [y_k - h_k(\hat{x}_k^-, 0)] \end{aligned} \tag{13.51}$$

The reason that we expanded $h(x_k)$ around $\hat{x}_k^-$ was because that was our best estimate of x_k before the measurement at time k is taken into account. But after we implement the discrete EKF equations to obtain the *a posteriori* estimate $\hat{x}_k^+$, we have a better estimate of x_k. So we can reduce the linearization error by reformulating the Taylor series expansion of $h(x_k)$ around our new estimate. If we then use that new Taylor series expansion of $h(x_k)$ and recalculate the measurement-update equations, we should get a better a posteriori estimate of $\hat{x}_k^+$. But then we can repeat the previous step; since we have an even better estimate of x_k, we can again reformulate the expansion of $h(x_k)$ around this even better estimate to get an even *better* estimate. This process can be repeated as many times as desired, although for most problems the majority of the possible improvement is obtained by only relinearizing one time.

We use the notation $\hat{x}_{k,i}^+$ to refer to the *a posteriori* estimate of x_k after i relinearizations have been performed. So $\hat{x}_{k,0}$ is the *a posteriori* estimate that results from the application of the standard EKF. Likewise, we use $P_{k,i}^+$ to refer to the approximate estimation-error covariance of $\hat{x}_{k,i}^+$, $K_{k,i}$ to refer to the Kalman gain that is used during the ith relinearization step, and $H_{k,i}$ to refer to the partial derivative matrix evaluated at the $x_k = \hat{x}_{k,i}^+$.

With this notation, we can describe an algorithm for the iterated EKF as follows. First, at each time step k we initialize the iterated EKF estimate to the standard EKF estimate:

$$\begin{aligned} \hat{x}_{k,0}^+ &= \hat{x}_k^+ \\ P_{k,0}^+ &= P_k^+ \end{aligned} \tag{13.52}$$

Second, for $i = 0, 1, \cdots, N$, evaluate the following equations:

$$\begin{aligned} H_{k,i} &= \left.\frac{\partial h}{\partial x}\right|_{\hat{x}_{k,i}^+} \\ K_{k,i} &= P_k^- H_{k,i}^T (H_{k,i} P_k^- H_{k,i}^T + M_k R_k M_k^T)^{-1} \\ P_{k,i+1}^+ &= (I - K_{k,i} H_{k,i}) P_k^- \\ \hat{x}_{k,i+1}^+ &= \hat{x}_k^- + K_{k,i}[y_k - h_k(\hat{x}_k^-)] \end{aligned} \tag{13.53}$$

This is done for as many steps as desired to improve the linearization. If $N = 0$ then the iterated EKF reduces to the standard EKF.

We still have to make one more modification to the above equations to obtain the iterated Kalman filter. Recall that in the derivation of the EKF, the $\hat{x}$ measurement update equation was originally derived from the following first-order Taylor series expansion of the measurement equation:

$$\begin{aligned} y_k &= h(x_k) \\ &\approx h(\hat{x}_k^-) + H|_{\hat{x}_k^-} (x_k - \hat{x}_k^-) \end{aligned} \tag{13.54}$$

To derive the measurement-update equation for $\hat{x}$ we evaluated the right side at the *a priori* estimate $\hat{x}_k^-$ and subtracted from y_k to get our correction term (the residual):

$$\begin{aligned} r_k &= y_k - h(\hat{x}_k^-) - H|_{\hat{x}_k^-} (\hat{x}_k^- - \hat{x}_k^-) \\ &= y_k - h(\hat{x}_k^-) \end{aligned} \tag{13.55}$$

With the iterated EKF we instead want to expand the measurement equation around $\hat{x}_{k,i}^+$ as follows:

$$y_k \approx h(\hat{x}_{k,i}^+) + H|_{\hat{x}_{k,i}^+} (x_k - \hat{x}_{k,i}^+) \tag{13.56}$$

To derive the iterated EKF measurement-update equation for $\hat{x}$, we evaluate the right side of the above equation at the *a priori* estimate $\hat{x}_k^-$ and subtract from y_k to get our correction term:

$$r_k = y_k - h(\hat{x}_{k,i}^+) - H_{k,i}(\hat{x}_k^- - \hat{x}_{k,i}^+) \tag{13.57}$$

This gives the iterated EKF update equation for $\hat{x}$ as

$$\hat{x}_{k,i+1}^+ = \hat{x}_k^- + K_{k,i}[y_k - h(\hat{x}_{k,i}^+) - H_{k,i}(\hat{x}_k^- - \hat{x}_{k,i}^+)] \tag{13.58}$$

The iterated EKF can then be summarized as follows.

The iterated extended Kalman filter

1. The nonlinear system and measurement equations are given as follows:

$$\begin{aligned} x_k &= f_{k-1}(x_{k-1}, u_{k-1}, w_{k-1}) \\ y_k &= h_k(x_k, v_k) \\ w_k &\sim (0, Q_k) \\ v_k &\sim (0, R_k) \end{aligned} \tag{13.59}$$

2. Initialize the filter as follows.

$$
\begin{aligned}
\hat{x}_0^+ &= E(x_0) \\
P_0^+ &= E\left[(x_0 - \hat{x}_0)(x_0 - \hat{x}_0)^T\right]
\end{aligned} \tag{13.60}
$$

3. For $k = 1, 2, \cdots$, do the following.

   (a) Perform the following time-update equations:

$$
\begin{aligned}
P_k^- &= F_{k-1} P_{k-1}^+ F_{k-1}^T + L_{k-1} Q_{k-1} L_{k-1}^T \\
\hat{x}_k^- &= f_{k-1}(\hat{x}_{k-1}^+, u_{k-1}, 0)
\end{aligned} \tag{13.61}
$$

   where the partial derivative matrices F_{k-1} and L_{k-1} are defined as follows:

$$
\begin{aligned}
F_{k-1} &= \left.\frac{\partial f_{k-1}}{\partial x}\right|_{\hat{x}_{k-1}^+} \\
L_{k-1} &= \left.\frac{\partial f_{k-1}}{\partial w}\right|_{\hat{x}_{k-1}^+}
\end{aligned} \tag{13.62}
$$

   Up to this point the iterated EKF is the same as the standard discrete-time EKF.

   (b) Perform the measurement update by initializing the iterated EKF estimate to the standard EKF estimate:

$$
\begin{aligned}
\hat{x}_{k,0}^+ &= \hat{x}_k^- \\
P_{k,0}^+ &= P_k^-
\end{aligned} \tag{13.63}
$$

   For $i = 0, 1, \cdots, N$, evaluate the following equations (where N is the desired number of measurement-update iterations):

$$
\begin{aligned}
H_{k,i} &= \left.\frac{\partial h}{\partial x}\right|_{\hat{x}_{k,i}^+} \\
M_{k,i} &= \left.\frac{\partial h}{\partial v}\right|_{\hat{x}_{k,i}^+} \\
K_{k,i} &= P_k^- H_{k,i}^T \left(H_{k,i} P_k^- H_{k,i}^T + M_{k,i} R_k M_{k,i}^T\right)^{-1} \\
P_{k,i+1}^+ &= (I - K_{k,i} H_{k,i}) P_k^- \\
\hat{x}_{k,i+1}^+ &= \hat{x}_k^- + K_{k,i}\left[y_k - h(\hat{x}_{k,i}^+) - H_{k,i}(\hat{x}_k^- - \hat{x}_{k,i}^+)\right]
\end{aligned} \tag{13.64}
$$

   (c) The final *a posteriori* state estimate and estimation-error covariance are given as follows:

$$
\begin{aligned}
\hat{x}_k^+ &= \hat{x}_{k,N+1}^+ \\
P_k^+ &= P_{k,N+1}^+
\end{aligned} \tag{13.65}
$$

An illustration of the iterated EKF will be presented in Example 13.3.

13.3.2 The second-order extended Kalman filter

The second-order EKF is similar to the iterated EKF in that it attempts to reduce the linearization error of the EKF. In the iterated EKF of the previous section, we refined the point at which we performed a first-order Taylor series expansion of the measurement equation $h(\cdot)$. In the second-order EKF we instead perform a *second-order* Taylor series expansion of $f(\cdot)$ and $h(\cdot)$. The second-order EKF presented in this section is based on [Ath68, Gel74].

In this section, we will consider the hybrid system with continuous-time system dynamics and discrete-time measurements:

$$\begin{aligned} \dot{x} &= f(x,u,w,t) \\ y_k &= h(x_k,t_k)+v_k \\ w(t) &\sim (0,Q) \\ v_k &\sim (0,R_k) \end{aligned} \tag{13.66}$$

In the standard EKF, we expanded $f(x,u,w,t)$ using a first-order Taylor series. In this section, we will consider only the expansion around a nominal x, ignoring the expansion around nominal u and w values. This is done so that we can present the main ideas of the second-order EKF without getting too bogged down in notation. The development in this section can be easily extended to second-order expansions around u and w once the main idea is understood.

The first-order expansion of $f(x,u,w,t)$ around $x=\hat{x}$ is given as

$$f(x,u,w,t) = f(\hat{x},u_0,w_0,t) + \left.\frac{\partial f}{\partial x}\right|_{\hat{x}} (x-\hat{x}) \tag{13.67}$$

In the standard EKF, we evaluated this expression at $x=\hat{x}$ to obtain our time-update equation for $\hat{x}$ as

$$\dot{\hat{x}} = f(\hat{x},u_0,w_0,t) \tag{13.68}$$

In the second-order EKF we expand $f(x,u,w,t)$ with an additional term in the Taylor series:

$$f(x,u,w,t) = f(\hat{x},u_0,w_0,t) + \left.\frac{\partial f}{\partial x}\right|_{\hat{x}} (x-\hat{x}) + \frac{1}{2}\sum_{i=1}^{n}\phi_i (x-\hat{x})^T \left.\frac{\partial^2 f_i}{\partial x^2}\right|_{\hat{x}} (x-\hat{x}) \tag{13.69}$$

where n is the dimension of the state vector, f_i is the ith element of $f(x,u,w,t)$, and the ϕ_i vector is defined as an $n\times 1$ vector with all zeros except for a one in the ith element. The quadratic term in the summation can be written as

$$(x-\hat{x})^T \left.\frac{\partial^2 f_i}{\partial x^2}\right|_{\hat{x}} (x-\hat{x}) = \mathrm{Tr}\left[\left.\frac{\partial^2 f_i}{\partial x^2}\right|_{\hat{x}} (x-\hat{x})(x-\hat{x})^T\right] \tag{13.70}$$

Since we do not know the value of $(x-\hat{x})(x-\hat{x})^T$ in the above equation, we replace it with its expected value, which is the covariance of the Kalman filter, to obtain

$$(x-\hat{x})^T \left.\frac{\partial^2 f_i}{\partial x^2}\right|_{\hat{x}} (x-\hat{x}) \approx \mathrm{Tr}\left[\left.\frac{\partial^2 f_i}{\partial x^2}\right|_{\hat{x}} P\right] \tag{13.71}$$

We then evaluate Equation (13.69) at $x = \hat{x}$ and substitute the above expression in the summation to obtain the time-update equation for $\hat{x}$ as

$$\dot{\hat{x}} = f(\hat{x}, u_0, w_0, t) + \frac{1}{2}\sum_{i=1}^{n} \phi_i \mathrm{Tr}\left[\left.\frac{\partial^2 f_i}{\partial x^2}\right|_{\hat{x}} P\right] \tag{13.72}$$

The time-update equation for P remains the same as in the standard hybrid EKF as shown in Equation (13.28):

$$\dot{P} = FP + PF^T + LQL^T \tag{13.73}$$

Now we will derive the measurement-update equations. Suppose that the measurement-update equation for the state estimate is given as

$$\hat{x}_k^+ = \hat{x}_k^- + K_k\left[y_k - h(\hat{x}_k^-, t_k)\right] - \pi_k \tag{13.74}$$

where K_k is the Kalman gain to be determined, and π_k is a correction term to be determined. We will choose π_k so that the estimate $\hat{x}_k^+$ is unbiased, and we will then choose K_k to minimize the trace of the covariance of the estimate.

If we define the estimation errors as

$$\begin{aligned} e_k^- &= x_k - \hat{x}_k^- \\ e_k^+ &= x_k - \hat{x}_k^+ \end{aligned} \tag{13.75}$$

we can see from Equations (13.66) and (13.74) that

$$e_k^+ = e_k^- - K_k\left[h(x_k, t_k) - h(\hat{x}_k^-, t_k)\right] - K_k v_k + \pi_k \tag{13.76}$$

Now we perform a second-order Taylor series expansion of $h(x_k, t_k)$ around the nominal point $\hat{x}_k^-$ to obtain

$$\begin{aligned} h(x_k, t_k) &= h(\hat{x}_k^-, t_k) + \left.\frac{\partial h}{\partial x}\right|_{\hat{x}_k^-}(x_k - \hat{x}_k^-) + \\ &\quad \frac{1}{2}\sum_{i=1}^{m}\phi_i(x_k - \hat{x}_k^-)^T \left.\frac{\partial^2 h(i)}{\partial x^2}\right|_{\hat{x}_k^-}(x_k - \hat{x}_k^-) \\ &= h(\hat{x}_k^-, t_k) + H_k(x_k - \hat{x}_k^-) + \frac{1}{2}\sum_{i=1}^{m}\phi_i(x_k - \hat{x}_k^-)^T \left.\frac{\partial^2 h_i}{\partial x^2}\right|_{\hat{x}_k^-}(x_k - \hat{x}_k^-) \end{aligned} \tag{13.77}$$

where H_k is defined by the above equation, m is the dimension of the measurement vector, and h_i is the ith element of $h(x_k, t_k)$. This gives the *a posteriori* estimation error as

$$e_k^+ = e_k^- - K_k H_k e_k^- - \frac{1}{2}K_k\sum_{i=1}^{m}\phi_i(e_k^-)^T D_{k,i} e_k^- - K_k v_k + \pi_k \tag{13.78}$$

where $D_{k,i}$ is defined as

$$D_{k,i} = \left.\frac{\partial^2 h_i}{\partial x^2}\right|_{\hat{x}_k^-} \tag{13.79}$$

Taking the expected value of both sides of Equation (13.78), assuming that $E(e_k^-) = 0$, and making the same approximation as in Equation (13.71), we can see that in order to have $E(e_k^+) = 0$ we must set

$$\pi_k = \frac{1}{2} K_k \sum_{i=1}^{m} \phi_i \mathrm{Tr}\left[D_{k,i} P_k^-\right] \tag{13.80}$$

Defining P_k^+ as

$$P_k^+ = E\left[e_k^+ (e_k^+)^T\right] \tag{13.81}$$

and using the above equations, it can be shown after some involved algebraic calculations [Ath68] that

$$P_k^+ = (I - K_k H_k) P_k^- (I - K_k H_k)^T + K_k (R_k + \Lambda_k) K_k^T \tag{13.82}$$

where the matrix Λ_k is defined as

$$\Lambda_k = \frac{1}{4} E\left\{ \left[\sum_{i=1}^{m} \phi_i \mathrm{Tr}[D_{k,i}(e_k^-(e_k^-)^T - P_k^-)] \right] \left[\cdots \right]^T \right\} \tag{13.83}$$

Now we define a cost function J_k that we want to minimize as a weighted sum of estimation errors:

$$\begin{aligned} J_k &= E[(e_k^+)^T S_k e_k^+] \\ &= \mathrm{Tr}[S_k P_k^+] \end{aligned} \tag{13.84}$$

where S_k is any positive definition weighting matrix. The K_k that minimizes this cost function can be found as

$$K_k = P_k^- H_k^T \left(H_k P_k^- H_k^T + R_k + \Lambda_k\right)^{-1} \tag{13.85}$$

This gives the P_k^+ matrix from Equation (13.82) as

$$P_k^+ = P_k^- - P_k^- H_k^T \left(H_k P_k^- H_k^T + R_k + \Lambda_k\right)^{-1} H_k P_k^- \tag{13.86}$$

Now we need to figure out how to evaluate the Λ_k matrix in Equation (13.83). Note that Λ_k can be written as the double summation

$$\Lambda_k = \frac{1}{4} E\left\{ \sum_{i,j=1}^{m} \phi_i \phi_j^T \mathrm{Tr}\left[D_{k,i}(e_k^-(e_k^-)^T - P_k^-)\right] \mathrm{Tr}\left[D_{k,j}(e_k^-(e_k^-)^T - P_k^-)\right] \right\} \tag{13.87}$$

The product $\phi_i \phi_j^T$ is an $m \times m$ matrix whose elements are all zero except for the element in the ith row and jth column. Therefore, the element in the ith row and jth column of Λ_k can be written as

$$\Lambda_k(i,j) = \frac{1}{4} E\left\{ \mathrm{Tr}\left[D_{k,i}(e_k^-(e_k^-)^T - P_k^-)\right] \mathrm{Tr}\left[D_{k,j}(e_k^-(e_k^-)^T - P_k^-)\right] \right\} \tag{13.88}$$

This expression can be evaluated with the following lemma [Ath68].

Lemma 6 *Suppose we have the n-element random vector $x \sim N(0, P)$. Then*

$$\begin{aligned} E\left[x\,Tr(Axx^T)\right] &= 0 \\ E\left[Tr(Axx^TBxx^T)\right] &= E\left[Tr(Axx^T)\,Tr(Bxx^T)\right] \\ &= 2\,Tr(APBP) + Tr(AP)\,Tr(BP) \end{aligned} \tag{13.89}$$

where A and B are arbitrary $n \times n$ matrices.

Using this lemma with Equation (13.88) we can see that

$$\Lambda_k(i,j) = \frac{1}{2}\mathrm{Tr}(D_{k,i}P_k^- D_{k,j}P_k^-) \tag{13.90}$$

This equation, along with Equations (13.74), (13.80), (13.82), and (13.85), specify the measurement-update equations for the second-order EKF. The second-order EKF can be summarized as follows.

The second-order hybrid extended Kalman filter

1. The system equations are given as follows:

$$\begin{aligned} \dot{x} &= f(x,u,w,t) \\ y_k &= h(x_k,t_k) + v_k \\ w(t) &\sim (0,Q) \\ v_k &\sim (0,R_k) \end{aligned} \tag{13.91}$$

2. The estimator is initialized as follows:

$$\begin{aligned} \hat{x}_0^+ &= E(x_0) \\ P_0^+ &= E\left[(x_0-\hat{x}_0^+)(x_0-\hat{x}_0^+)^T\right] \end{aligned} \tag{13.92}$$

3. The time-update equations are given as

$$\begin{aligned} \dot{\hat{x}} &= f(\hat{x},u,0,t) + \frac{1}{2}\sum_{i=1}^{n}\phi_i \mathrm{Tr}\left[\left.\frac{\partial^2 f_i}{\partial x^2}\right|_{\hat{x}} P\right] \\ \dot{P} &= FP + PF^T + LQL^T \\ \phi_i &= \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{array}{l} \\ \\ \\ \leftarrow i\text{th element} \\ \\ \\ \\ \end{array} \\ F &= \left.\frac{\partial f}{\partial x}\right|_{\hat{x}} \\ L &= \left.\frac{\partial f}{\partial w}\right|_{\hat{x}} \end{aligned} \tag{13.93}$$

4. The measurement update equations are given as

$$\begin{aligned}
\hat{x}_k^+ &= \hat{x}_k^- + K_k \left[y_k - h(\hat{x}_k^-) \right] - \pi_k \\
\pi_k &= \frac{1}{2} K_k \sum_{i=1}^{m} \phi_i \mathrm{Tr} \left[D_{k,i} P_k^- \right] \\
D_{k,i} &= \left. \frac{\partial^2 h_i(x_k, t_k)}{\partial x^2} \right|_{\hat{x}_k^-} \\
K_k &= P_k^- H_k^T \left(H_k P_k^- H_k^T + R_k + \Lambda_k \right)^{-1} \\
H_k &= \left. \frac{\partial h(x_k, t_k)}{\partial x} \right|_{\hat{x}_k^-} \\
\Lambda_k(i,j) &= \frac{1}{2} \mathrm{Tr}(D_{k,i} P_k^- D_{k,j} P_k^-) \\
P_k^+ &= P_k^- - P_k^- H_k^T \left(H_k P_k^- H_k^T + R_k + \Lambda_k \right)^{-1} H_k P_k^- \qquad (13.94)
\end{aligned}$$

Note that setting the second partial derivative matrices in this algorithm to zero matrices results in the standard hybrid EKF.

■ EXAMPLE 13.3

In this example, we compare the performance of the EKF, the second-order EKF, and the iterated EKF for the falling body problem described in Example 13.2. A similar comparison was shown in [Wis69], where it was concluded that the iterated EKF had better RMS error performance, but the second-order filter had smaller bias. The system equations are the same as those shown in Example 13.2:

$$\begin{aligned}
\dot{x}_1 &= x_2 + w_1 \\
\dot{x}_2 &= \rho_0 \exp(-x_1/k) x_2^2 x_3/2 - g + w_2 \\
\dot{x}_3 &= w_3 \qquad (13.95)
\end{aligned}$$

In this example, we change the measurement system so that it does not measure the altitude of the falling body, but instead measures the range to the measuring device. The measuring device is located at an altitude a and at a horizontal distance M from the body's vertical line of fall. The measurement equation is therefore given by

$$\begin{aligned}
y_k &= \sqrt{M^2 + (x_1(t_k) - a)^2} + v_k \\
&= h(x_k) + v_k \qquad (13.96)
\end{aligned}$$

This makes the problem more nonlinear and hence more difficult to estimate (i.e., in Example 13.2 we had a nonlinear system but a linear measurement, whereas in this example we have nonlinearities in both the system and the measurement equations). The partial derivative F matrix for the EKFs are given in Example 13.2. The other partial derivative matrices used in the second-order EKF are given as follows:

$$
\begin{aligned}
H &= \frac{\partial h}{\partial x} \\
&= \begin{bmatrix} (x_1 - a)(M^2 + (x_1 - a)^2)^{-1/2} & 0 & 0 \end{bmatrix} \\
L &= \frac{\partial f}{\partial w} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
D_1 &= \frac{\partial^2 h_1}{\partial x^2} \\
&= \begin{bmatrix} h^{-1}(1 - (x_1 - a)^2 h^{-2}) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
\frac{\partial^2 f_1}{\partial x^2} = \frac{\partial^2 f_3}{\partial x^2} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
\frac{\partial^2 f_2}{\partial x^2} &= \rho_0 \exp(-x_1/k) \begin{bmatrix} x_2^2 x_3/2k^2 & -x_2 x_3/k & -x_2^2/2k \\ -x_2 x_3/k & x_3 & x_2 \\ -x_2^2/2k & x_2 & 0 \end{bmatrix}
\end{aligned}
\tag{13.97}
$$

Table 13.2 shows the performances of the EKFs (averaged over 20 simulation runs). It is seen that second-order EKF provides significant improvement over the first-order EKF for altitude and velocity estimation, but for some reason it actually provides worse performance for ballistic coefficient estimation. Also note that the iterated EKF provides only slight improvement over the first-order EKF, and (as expected) the iterated EKF performs better when more iterations are executed for the linearization refinement.

Table 13.2 Example 13.3 results. A comparison of the estimation errors of different EKF approaches for tracking a falling body.

Filter	Altitude	Velocity	Ballistic Coefficient
First-order EKF	758 feet	518 feet/sec	0.091 feet^3/lb/sec^2
Second-order EKF	356	483	0.129
Iterated EKF ($N = 2$)	755	517	0.091
Iterated EKF ($N = 3$)	745	516	0.091
Iterated EKF ($N = 4$)	738	509	0.091
Iterated EKF ($N = 5$)	733	506	0.091
Iterated EKF ($N = 6$)	723	506	0.091

We conclude from this that the second-order filter has better estimation performance. However, the implementation is much more difficult and requires the computation of second-order derivatives. In this example, the second-order derivatives could be taken analytically because we have explicit analytical system and measurement equations. In many applications second-order derivatives will not be available analytically, and approximations will inevitably be subject to error.

These results are different than reported in [Wis69], where it was shown that the iterated EKF performed better than the second-order EKF. The different conclusions between this book and [Wis69] show that comparisons between different algorithms are often subjective. Perhaps the discrepancies are due to differences in implementations of the filtering algorithms, differences in implementations of the system dynamics or random noise generation, differences in the way that the estimation errors were measured, or even differences in the computing platforms that were used.

$\nabla\nabla\nabla$

The second-order filter was initially developed by Bass [Bas66] and Jazwinski [Jaz66]. A Gaussian second-order filter was developed by Athans [Ath68] and Jazwinski [Jaz70], in which fourth-order terms in Taylor series approximations are retained and approximated by assuming that the underlying probabilities are Gaussian. A small correction in the original derivations of the second-order EKF was reported by Rolf Henriksen [Hen82]. Although the second-order filter often provides improved performance over the extended Kalman filter, nothing definitive can be said about its performance, as evidenced by an example of an unstable second-order filter reported in [Kus67]. Additional comparison and analysis of some nonlinear Kalman filters can be found in [Sch68, Wis69, Wis70, Net78]. A simplified version of Henriksen's discrete-time second-order filter can be summarized as follows.

The second-order discrete-time extended Kalman filter

1. The system equations are given as follows:

$$\begin{aligned} x_{k+1} &= f(x_k, u_k, k) + w_k \\ y_k &= h(x_k, k) + v_k \\ w_k &\sim (0, Q_k) \\ v_k &\sim (0, R_k) \end{aligned} \tag{13.98}$$

2. The estimator is initialized as follows:

$$\begin{aligned} \hat{x}_0^+ &= E(x_0) \\ P_0^+ &= E\left[(x_0 - \hat{x}_0^+)(x_0 - \hat{x}_0^+)^T\right] \end{aligned} \tag{13.99}$$

3. The time update equations are given as follows:

$$\begin{aligned} \hat{x}_{k+1}^- &= f(\hat{x}_k^+, u_k, k) + \frac{1}{2}\sum_{i=1}^{n} \phi_i \mathrm{Tr}\left[\left.\frac{\partial^2 f_i}{\partial x}\right|_{\hat{x}_k^+} P_k^+ \right] \\ P_{k+1}^- &= F P_k^+ F^T + Q_k \end{aligned}$$

$$
\begin{aligned}
\phi_i &= \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \leftarrow i\text{th element} \\ \\ \\ \\ \end{matrix} \\
F &= \left. \frac{\partial f}{\partial x} \right|_{\hat{x}_k^+}
\end{aligned} \tag{13.100}
$$

4. The measurement update equations are given as follows:

$$
\begin{aligned}
\hat{x}_k^+ &= \hat{x}_k^- + K_k[y_k - h(\hat{x}_k^-, k)] - \pi_k \\
\pi_k &= \frac{1}{2} K_k \sum_{i=1}^{m} \phi_i \mathrm{Tr}\left[D_{k,i} P_k^-\right] \\
D_{k,i} &= \left. \frac{\partial^2 h_i(x_k, k)}{\partial x^2} \right|_{\hat{x}_k^-} \\
K_k &= P_k^- H_k^T \left(H_k P_k^- H_k^T + R_k\right)^{-1} \\
H_k &= \left. \frac{\partial h(x_k, k)}{\partial x} \right|_{\hat{x}_k^-} \\
P_k^+ &= (I - K_k H_k) P_k^-
\end{aligned} \tag{13.101}
$$

A more general version of the above algorithm can be found in [Hen82]. Similar to the hybrid second-order EKF presented earlier in this section, we note that setting the second-order partial derivative matrices in this algorithm to zero matrices results in the standard discrete-time EKF.

13.3.3 Other approaches

We have considered a couple of higher-order approaches to reducing the linearization error of the EKF. We looked at the iterated EKF and the second-order EKF, but other approaches are also available. For example, Gaussian sum filters are based on the idea that a non-Gaussian pdf can be approximated by a sum of Gaussian pdfs. This is similar to the idea that any curve can be approximated by a piecewise constant function. Since the true pdf of the process noise and measurement noise can be approximated by a sum of M Gaussian pdfs, we can run M Kalman filters in parallel on M Gaussian filtering problems, each of them optimal filters, and then combine them to obtain an approximately optimal estimate. The number of filters M is a trade-off between approximation accuracy (and hence optimality) and computational effort. This idea was first mentioned in [Aok65] and was explored in [Cam68, Sor71b, Als74, Kit89]. The Gaussian sum filter algorithm presented in [Als72] can be summarized as follows.

The Gaussian sum filter

1. The discrete-time n-state system and measurement equations are given as follows:

$$\begin{aligned} x_k &= f_{k-1}(x_{k-1}, u_{k-1}, w_{k-1}) \\ y_k &= h_k(x_k, v_k) \\ w_k &\sim (0, Q_k) \\ v_k &\sim (0, R_k) \end{aligned} \tag{13.102}$$

2. Initialize the filter by approximating the pdf of the initial state as follows:

$$\text{pdf}(\hat{x}_0^+) = \sum_{i=1}^{M} a_{0i} N(\hat{x}_{0i}^+, P_{0i}^+) \tag{13.103}$$

The a_{0i} coefficients (which are positive and add up to 1), the $\hat{x}_{0i}^+$ means, and the P_{0i}^+ covariances, are chosen by the user to provide a good approximation to the pdf of the initial state.

3. For $k = 1, 2, \cdots$, do the following.

(a) The *a priori* state estimate is obtained by first executing the following time-update equations for $i = 1, \cdots, M$:

$$\begin{aligned} \hat{x}_{ki}^- &= f_{k-1}(\hat{x}_{k-1,i}^+, u_{k-1}, 0) \\ F_{k-1,i} &= \left.\frac{\partial f_{k-1}}{\partial x_{k-1}}\right|_{\hat{x}_{k-1,i}^+} \\ P_{ki}^- &= F_{k-1,i} P_{k-1,i}^+ F_{k-1,i}^T + Q_{k-1} \\ a_{ki} &= a_{k-1,i} \end{aligned} \tag{13.104}$$

The pdf of the *a priori* state estimate is obtained by the following sum:

$$\text{pdf}(\hat{x}_k^-) = \sum_{i=1}^{M} a_{ki} N(\hat{x}_{ki}^-, P_{ki}^-) \tag{13.105}$$

(b) The *a posteriori* state estimate is obtained by first executing the following measurement update equations for $i = 1, \cdots, M$:

$$\begin{aligned} H_{ki} &= \left.\frac{\partial h_k}{\partial x_k}\right|_{\hat{x}_{ki}^-} \\ K_{ki} &= P_{ki}^- H_{ki}^T (H_{ki} P_{ki}^- H_{ki}^T + R_k)^{-1} \\ P_{ki}^+ &= P_{ki}^- - K_{ki} H_{ki} P_{ki}^- \\ \hat{x}_{ki}^+ &= \hat{x}_{ki}^- + K_{ki} \left[y_k - h_k(\hat{x}_{ki}^-, 0) \right] \end{aligned} \tag{13.106}$$

The weighting coefficients a_{ki} for the individual estimates are obtained as follows:

$$
\begin{aligned}
r_{ki} &= y_k - h_k(\hat{x}_{ki}^-, 0) \\
S_{ki} &= H_{ki} P_{ki}^- H_{ki}^T + R_k \\
\beta_{ki} &= \frac{\exp\left[-r_{ki}^T S_{ki}^{-1} r_{ki}/2\right]}{(2\pi)^{n/2} |S_{ki}|^{1/2}} \\
a_{ki} &= \frac{a_{k-1,i}\beta_{ki}}{\sum_{j=1}^{M} a_{k-1,j}\beta_{kj}}
\end{aligned} \tag{13.107}
$$

Note that the weighting coefficient a_{ki} is computed by using the measurement y_k to obtain the relative confidence β_{ki} of the estimate $\hat{x}_{ki}^-$. The pdf of the *a posteriori* state estimate is obtained by the following sum:

$$
\text{pdf}(\hat{x}_k^+) = \sum_{i=1}^{M} a_{ki} N(\hat{x}_{ki}^+, P_{ki}^+) \tag{13.108}
$$

This approach can also be extended to smoothing [Kit94]. Similar approaches can be taken to expand the pdf using non-Gaussian functions [Aok67, Sor68, Sri70, deF71, Hec71, Hec73, Mcr75, Wil81, Kit87, Kra88]. A related filter has been derived for the case where either the process noise or the measurement noise is strictly Gaussian, but the other noise is Gaussian with heavy tails [Mas75, Tsa83]. This is motivated by the observation that many instances of noise in nature have pdfs that are approximately Gaussian but with heavier tails [Mas77].

Another approach to nonlinear filtering is called grid-based filtering. In grid-based filtering, the value of the pdf of the state is approximated, stored, propagated, and updated at discrete points in state space [Buc69, Buc71]; [Spa88, Chapter 6]. This is similar to particle filtering (discussed in Chapter 15), except in particle filtering we choose the particles to be distributed in state space according to the pdf of the state. Grid-based filtering does not distribute the particles in this way, and hence has computational requirements that increase exponentially with the dimension of the state. Grid-based filtering is even more computationally expensive than particle filtering, and this has limited its application. Furthermore, particle filtering is a type of "intelligent" grid-based filtering. This seems to portend very little further work in grid-based filtering.

Richard Bucy suggested yet another approach to nonlinear filtering [Buc65]. Instead of linearizing the system dynamics, compute the theoretically optimal nonlinear filter, and then linearize the nonlinear filter. However, the theoretically optimal nonlinear filter is very difficult to compute except in special cases.

13.4 PARAMETER ESTIMATION

State estimation theory can be used to not only estimate the states of a system, but also to estimate the unknown parameters of a system. This may have first been suggested in [Kop63]. Suppose that we have a discrete-time system model, but the system matrices depend in a nonlinear way on an unknown parameter vector p:

$$
\begin{aligned}
x_{k+1} &= F_k(p)x_k + G_k(p)u_k + L_k(p)w_k \\
y_k &= H_k x_k + v_k \\
p &= \text{unknown parameter vector}
\end{aligned} \tag{13.109}
$$

In this model, we are assuming that the measurement is independent of p, but this is only for notational convenience. The discussion here can easily be extended to include a dependence of y_k on p. Assume that p is a constant parameter vector. We do not really care about estimating the state, but we are interested in estimating p. This is the case, for example, in the aircraft engine health estimation problem [Kob03, Sim05a]. In those papers it was assumed that we want to estimate aircraft engine health (for the purpose of maintenance scheduling), but we do not really care about estimating the states of the engine.

In order to estimate the parameter p, we first augment the state with the parameter to obtain an augmented state vector x':

$$
x'_k = \begin{bmatrix} x_k \\ p_k \end{bmatrix} \tag{13.110}
$$

If p_k is constant then we model $p_{k+1} = p_k + w_{pk}$, where w_{pk} is a small artificial noise term that allows the Kalman filter to change its estimate of p_k. Our augmented system model can be written as

$$
\begin{aligned}
x'_{k+1} &= \begin{bmatrix} F_k(p_k)x_k + G_k(p_k)u_k + L_k(p_k)w_k \\ p_k + w_{pk} \end{bmatrix} \\
&= f(x'_k, u_k, w_k, w_{pk}) \\
y_k &= \begin{bmatrix} H_k & 0 \end{bmatrix} \begin{bmatrix} x_k \\ p_k \end{bmatrix} + v_k
\end{aligned} \tag{13.111}
$$

Note that $f(x'_k, u_k, w_k, w_{pk})$ is a nonlinear function of the augmented state x'_k. We can therefore use an extended Kalman filter (or any other nonlinear filter) to estimate x'_k.

■ EXAMPLE 13.4

This example is taken from [Ste94]. Suppose we have a second-order system governed by the following equations:

$$
\ddot{x}_1 + 2\zeta\omega_n\dot{x}_1 + \omega_n^2 x_1 = \omega_n^2 w \tag{13.112}
$$

where ω_n is the natural frequency of the system, ζ is the damping ratio, and the input w is zero-mean noise. A state-space model for this system can be written as

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\omega_n^2 x_1 - 2\zeta\omega_n x_2 + \omega_n^2 w \\
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix} w
\end{aligned} \tag{13.113}
$$

Suppose that $-2\zeta\omega_n$ is known, but ζ and ω_n are unknown. We want to estimate $-\omega_n^2$. Suppose that both x_1 and x_2 are available for measurement. We define the known parameter as b; that is, $b = -2\zeta\omega_n$. We define a new state element equal to the parameter that we want to estimate. That is, $x_3 = -\omega_n^2$. We then form an augmented system model as follows:

$$\begin{aligned} \dot{x}' &= \begin{bmatrix} x_2 \\ x_3x_1 + bx_2 - x_3w \\ w_p \end{bmatrix} \\ &= f(x', w') \\ w' &= \begin{bmatrix} w \\ w_p \end{bmatrix} \\ y &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x' + v \end{aligned} \tag{13.114}$$

where w_p is an artificial noise term that we add to the system that allows the Kalman filter to modify its estimate of x_3. We can use an extended Kalman filter to estimate the augmented state. First we need to find the partial derivative matrices:

$$\begin{aligned} F &= \left.\frac{\partial f}{\partial x'}\right|_{\hat{x}', w_0'} \\ &= \left.\begin{bmatrix} 0 & 1 & 0 \\ x_3 & b & x_1 - w \\ 0 & 0 & 0 \end{bmatrix}\right|_{\hat{x}', w_0'} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ \hat{x}_3 & b & \hat{x}_1 \\ 0 & 0 & 0 \end{bmatrix} \\ L &= \left.\frac{\partial f}{\partial w'}\right|_{\hat{x}', w_0'} \\ &= \begin{bmatrix} 0 & 0 \\ -\hat{x}_3 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned} \tag{13.115}$$

The continuous-time extended Kalman filter can be written as

$$\begin{aligned} \dot{\hat{x}}' &= f(\hat{x}', 0) + K(y - H\hat{x}') \\ K &= PH^TR^{-1} \\ \dot{P} &= FP + PF^T + LQL^T - PH^TR^{-1}HP \end{aligned} \tag{13.116}$$

Figure 13.4 illustrates the results of a typical simulation of the extended Kalman filter that is used to estimate $-\omega_n^2$ for this system. The true system parameters are $\omega_n = 2$ and $\zeta = 0.1$, so $-\omega_n^2 = -4$. Suppose that we begin by estimating $-\omega_n^2$ as -8 with an initial estimation variance of 20. Figure 13.4 shows that the error in our estimate of $-\omega_n^2$ gradually decreases toward zero, and the estimation variance gradually decreases. We set the variance of the artificial noise w_p equal to 0.1 in this example. This allows the Kalman filter

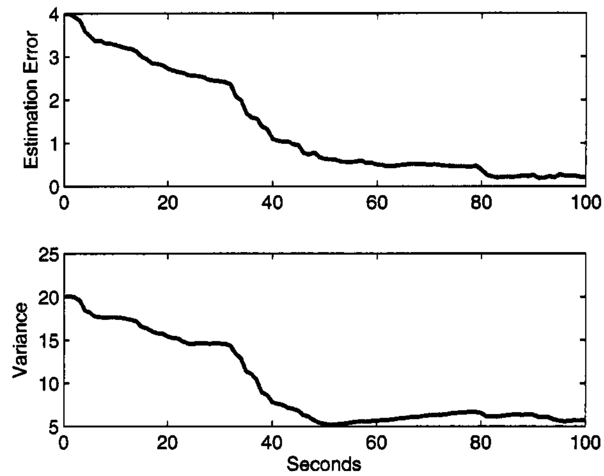

Figure 13.4 Example 13.4 results. Typical parameter estimation performance and parameter uncertainty for an extended Kalman filter estimating $-\omega_n^2$ for a second-order system. The estimation error of the unknown parameter and its variance gradually decrease toward zero.

to more readily adjust its estimate of $-\omega_n^2$, but also may prevent the filter from converging to the true value (see Problem 13.23).

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13.5 SUMMARY

Optimal state estimators can be derived for general classes of nonlinear systems as shown in [Kus67], but the filters are generally infinite dimensional, which makes them impractical for implementation. Finite-dimensional, optimal, nonlinear state estimators can be derived for more restricted classes of nonlinear systems [Liu80], but the restriction on the classes of applicable systems are significant enough to prevent wide applicability. Because of these factors, nonlinear Kalman filtering is the most widespread approach to state estimation for nonlinear systems.

It is interesting to note that the first applications of Kalman filtering were on nonlinear orbit-estimation problems [Bat62]. Some early investigations in nonlinear Kalman filtering can be found in [Cox64, Fri66]. Whereas stability and convergence results are readily available for the linear Kalman filter, such results are much more difficult to obtain for nonlinear Kalman filtering. Some convergence results for nonlinear Kalman filtering are found in [Urs80]. If the nonlinearities have known bounds then the Riccati equation can be modified in a simple way to guarantee stability for the continuous-time EKF [Rei98]. Conditions needed to guarantee the boundedness of the discrete-time EKF error covariance can be related to the observability of the underlying nonlinear system [Dez92, Son95].

PROBLEMS

Written exercises

13.1 Consider the scalar system

$$\begin{aligned} \dot{x} &= -x + w \\ y &= x + v \end{aligned}$$

The process noise has a mean value of 2, and the measurement noise has a mean value of 3. Redefine the noise quantities and the state to obtain an equivalent system of the form

$$\begin{aligned} \dot{x}' &= Ax' + Bu + w' \\ y &= Cx' + v' \end{aligned}$$

so that the new noise quantities w' and v' both have mean values of 0.

13.2 Consider the scalar system

$$\dot{x} = -x + u + w$$

w is zero-mean process noise with a variance of Q. The control has a mean value of u_0, an uncertainty of 2 (one standard deviation), and is uncorrelated with w. Rewrite the system equations to obtain an equivalent system with a normalized control that is perfectly known. What is the variance of the new process noise term in the transformed system equation?

13.3 Suppose that x is a constant scalar, and $y_k = \sqrt{x}(1 + v_k)$ are noisy measurements, where $v_k \sim N(0, R)$.

a) An intuitive way to estimate x is to set $\hat{x}_k = y_k^2$. Compute the mean and variance of the estimation error for this estimate. Your answer should be a function of x and R. Hint: recall that $E(v_k^3) = 0$ and $E(v_k^4) = 3R^2$.

b) Perhaps a better estimate for x_k could be obtained by averaging all previous values of y_k^2. That is,

$$\hat{x}_k = \frac{1}{k} \sum_{i=1}^{k} y_i^2$$

Compute the mean and variance of the estimation error for this estimate. Your answer should be a function of k, x, and R. Note that if you substitute $k = 1$ into your solution, you should get the same answer as part (a). What is the variance as $k \to \infty$?

c) Write the extended Kalman filter equations to estimate x. What is the theoretical mean and variance of the EKF estimate as $k \to \infty$?

13.4 Consider the system

$$\begin{aligned} x_{k+1} &= x_k + w_k \\ y_k &= x_k + v_k^2 \end{aligned}$$

where w_k and v_k are uniformly distributed, uncorrelated, zero-mean white noise processes with variances Q and R, respectively.

a) What is the mean of the *a posteriori* estimation error for the discrete EKF?

b) Modify the measurement equation by subtracting the known bias of the measurement noise so that the modified measurement noise is zero-mean. What is the variance of the modified measurement noise?

13.5 Consider the nonlinear system

$$\begin{aligned} x_{k+1} &= -x_k^2 + u_k + w_k \\ y_k &= 4x_k^2 + v_k \end{aligned}$$

Find the nominal values for x_k and y_k when $x_0 = 0$ and $u_k = 1$.

13.6 Consider the system $x_{k+1} = x_k^2 + w_k$, where w_k is zero-mean. The initial state x_0 is uniformly distributed between 0 and 1. An EKF is initialized with $\hat{x}_0^+ = E(x_0)$. What is $E(x_1)$? What is $\hat{x}_1^-$? This problem illustrates the fact that the state estimate of an EKF is not always equal to the expected value of the state.

13.7 Find the terminal velocity of the falling body of Example 13.2 if the terminal velocity occurs at an altitude of 1 mile.

13.8 Consider the hybrid scalar system

$$\begin{aligned} \dot{x} &= f(x) + w, \qquad w \sim N(0, Q) \\ y_k &= h(x_k) + v_k, \qquad v_k \sim N(0, R) \end{aligned}$$

The estimator that is used for the system is

$$\hat{x}_k = a + by_k + cy_k^2$$

Suppose that the state $x(t)$ is normally distributed with a mean of zero and a variance of P_x.

a) Find an equation relating a, b, and c that must be satisfied in order for $\hat{x}_k$ to be an unbiased estimate of $x(t_k)$ [Gel74].

b) Find values of a, b, and c so that $\hat{x}_k$ is the minimum-variance estimate. Assume that $h(x)$ is an odd function of x.

13.9 Suppose for a scalar system that $P_k^- = 1$, $R = 1$, and $H = 3$. What is the value of P_k^+ as given by Equation (5.19)? What will be the computed value of P_k^+ if $H = 2$ is used instead? What will be the computed value of P_k^+ if $H = 1$ is used instead? This illustrates how the iterated Kalman filter gets a more accurate estimate of P_k^+ by using a more accurate value for H_k.

13.10 Consider a system with the measurement equation $y_k = x_k^2 + v_k$. At time k the *a priori* state estimate is $\hat{x}_k^- = 1$, the true state is $x_k = 5$, and the measurement is $y_k = 25$. The *a priori* estimation-error variance is $P_k^- = 1$, and the measurement noise variance is $R_k = 4$. Use the iterated EKF algorithm to find $\hat{x}_{k,1}^+$ and $\hat{x}_{k,2}^+$. Although the iterated EKF does not always improve the *a posteriori* state estimate, this problem illustrates how it usually does.

13.11 Prove Lemma 6 for scalar random variables x.

13.12 Suppose you have the process equation $\dot{x} = x^2 + w$ and the state estimate $\hat{x}_k^+ = 0$. What is the differential equation for propagating $\hat{x}$ to the next measurement time using the first-order EKF? What is the differential equation using the second-order EKF?

13.13 Consider the measurement equation $y_k = x_k^2 + v_k$, where $v_k \sim (0, R)$. Suppose that $P_k^- = 1$, and $\hat{x}_k^- = 1$ is unbiased.

a) What is the expected value of $\hat{x}_k^+$ if the first-order EKF is used for the measurement update? Based on your expression for $E(\hat{x}_k^+)$, how does the bias of the state estimate change with R? Does this make intuitive sense?

b) What is the expected value of $\hat{x}_k^+$ if the second-order EKF is used for the measurement update?

13.14 Consider the system

$$\begin{aligned} z_{k+1} &= a z_k + w_k, \quad w_k \sim (0, Q) \\ y_k &= z_k + v_k, \quad v_k \sim (0, R) \end{aligned}$$

with unknown parameter a. Suppose that an EKF is used to estimate the state z_k and the parameter a. Further suppose that the artificial noise term used in the estimation of a is zero, and the EKF converges to the correct value of a with zero variance. Show that the EKF in this situation is equivalent to the standard Kalman filter for the scalar system when a is known.

Computer exercises

13.15 Write a program that implements the moving average filter and the extended Kalman filter for the system described in Problem 13.3. Use $R = 1$, $x = 1$, $P_0^+ = 1$, and $\hat{x}_0 = 2$. Which filter appears to perform better?

13.16 A planar model for a satellite orbiting around the earth can be modeled as

$$\begin{aligned} \ddot{r} &= r\dot{\theta}^2 - \frac{GM}{r^2} + w \\ \ddot{\theta} &= \frac{-2\dot{\theta}\dot{r}}{r} \end{aligned}$$

where r is the distance of the satellite from the center of the earth, θ is the angular position of the satellite in its orbit, $G = 6.6742 \times 10^{-11} \text{m}^3/\text{kg/s}^2$ is the universal gravitational constant, $M = 5.98 \times 10^{24}$ kg is the mass of the earth, and $w \sim (0, 10^{-6})$ is random noise due to space debris, atmospheric drag, outgassing, and so on.

a) Write a state-space model for this system with $x_1 = r$, $x_2 = \dot{r}$, $x_3 = \theta$, and $x_4 = \dot{\theta}$.

b) What must $\dot{\theta}$ be equal to in order for the orbit to have a constant radius when $w = 0$?

c) Linearize the model around the point $r = r_0$, $\dot{r} = 0$, $\theta = \omega_0 T$, $\dot{\theta} = \omega_0$. What are the eigenvalues of the system matrix for the linearized system when $r_0 = 6.57 \times 10^6$ m? What would you estimate to be the largest integration step size that could be used to simulate the system? (Hint: recall that for a second-order transfer function with imaginary poles $\pm ja$, the time constant is equal to $1/a$.)

d) Suppose that measurements of the satellite radius and angular position are obtained every minute, with error standard deviations of 100 meters and 0.1 radians, respectively. Simulate the linearized Kalman filter for three hours. Initialize the system with $x(0) = \begin{bmatrix} r_0 & 0 & 0 & 1.1\omega_0 \end{bmatrix}$, $\hat{x}(0) = x(0)$, and $P(0) = \text{diag}(0,0,0,0)$. Plot the radius estimation error as a function of time. Why is the performance so poor? How could you modify the linearized Kalman filter to get better performance?

e) Implement an extended Kalman filter and plot the radius estimation error as a function of time. How does the performance compare with the linearized Kalman filter?

13.17 Implement the hybrid EKF with a measurement period of $0.1s$ for the system described in Example 13.1. Assume that the winding current measurement noises have a standard deviation of 0.1 amps. Create a table showing the experimental standard deviation of the motor velocity estimation error as a function of the standard deviation of the control input uncertainties q_1 and q_2. Use control input standard deviations from 0 to 0.1 volts in steps of 0.01 (i.e., $\sigma_q = 0$, $\sigma_q = 0.01$, $\cdots$, $\sigma_q = 0.1$). In order to make a fair comparison, you should either run several simulations for each value of σ_q and average the results, or else initialize the random seed in your software so that each simulation runs with the same random noise history.

13.18 Derive the first-order EKF, second-order EKF, and iterated EKF (with one iteration) for the scalar system

$$\begin{aligned} x_{k+1} &= x_k^2 + w_k \\ y_k &= x_k^2 + v_k \end{aligned}$$

where w_k and v_k are independent zero-mean white noise terms with variances 0.1 and 1, respectively. Simulate the first-order, second-order, and iterated extended Kalman filters for five time steps. Set the initial state to 1, the initial estimation-error variance to 1, and the initial state estimate to 2. Compute the RMS error of the filter estimates. How does the performance of the filters compare? (Note that you need more than one simulation, in general, to obtain a fair comparison of filter performance.)

13.19 Use the following procedure [Sor71b] to approximate a uniform pdf that is defined on ± 1 with M Gaussian pdfs; that is, $U(-1,1) \approx \sum_{i=1}^{M} a_i N(\mu_i, \sigma_i^2)$.

- Select the weighting coefficients so that $a_i = 1/M$ for all i.
- Select the means of the Gaussian pdfs to be equally spaced on the range $[-1,1]$ with $\mu_{i+1} - \mu_i = 2/(M+1)$.
- Select the variances σ_i of the Gaussian pdfs to all be the same and to minimize the RMS difference between $U(-1,1)$ and $\sum_{i=1}^{M} a_i N(\mu_i, \sigma_i^2)$ over the range $[-1,1]$.

The above approach reduces the approximation problem to a one-dimensional optimization problem, which can be solved in a number of different ways (for example, using the golden search method [Pre92]). Plot the true pdf and the approximate pdf for $M = 3$, 5, and 10, and compare the RMS errors.

13.20 Suppose you have a scalar system given as

$$\begin{aligned} x_{k+1} &= x_k \\ y_k &= x_k^2 + v_k \end{aligned}$$

where v_k is white Gaussian noise with a variance of 0.01. The pdf of the initial state x_0 is uniform between -1 and $+1$. Note from the measurement equation that there is no way to distinguish between a positive state and a negative state.

a) What will the extended Kalman filter estimate of the system be equal to?

b) The pdf of x_0 can be approximated with two Gaussian pdfs, each with a variance of 0.43, and with respective means of $-1/3$ and $+1/3$. Suppose that $x_0 = -1/2$. Plot the true state and the individual state estimates of a two-term Gaussian sum filter for 20 time steps. Plot the Gaussian pdfs at the final time for each estimate of the two-term Gaussian sum filter.

13.21 Consider the problem of tracking a moving vehicle in two dimensions (north is one dimension and east is the other dimension). The vehicle's acceleration in the north and east directions consists of independent white noise. Two tracking stations, located at north–east coordinates (N_1, E_1) and (N_2, E_2), respectively, measure the range to the vehicle. The system model can therefore be written as

$$\begin{aligned} \begin{bmatrix} n_{k+1} \\ e_{k+1} \\ \dot{n}_{k+1} \\ \dot{e}_{k+1} \end{bmatrix} &= \begin{bmatrix} 1 & 0 & T & 0 \\ 0 & 1 & 0 & T \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_k \\ e_k \\ \dot{n}_k \\ \dot{e}_k \end{bmatrix} + w_k \\ y_k &= \begin{bmatrix} \sqrt{(n_k - N_1)^2 + (e_k - E_1)^2} \\ \sqrt{(n_k - N_2)^2 + (e_k - E_2)^2} \end{bmatrix} + v_k \end{aligned}$$

where n_k and e_k are the vehicle's north and east coordinates at time step k, T is the time step of the system, w_k is the zero-mean process noise, and v_k is the zero-mean measurement noise. Suppose that the time step $T = 0.1s$, the process noise covariance $Q = \text{diag}(0, 0, 4, 4)$, and the measurement noise covariance $R = \text{diag}(1, 1)$. The tracking stations are located at $(N_1, E_1) = (20, 0)$, and $(N_2, E_2) = (0, 20)$. The initial state of the vehicle $x_0 = \begin{bmatrix} 0 & 0 & 50 & 50 \end{bmatrix}^T$ and is perfectly known. Design an extended Kalman filter to estimate the state of the vehicle. Run the simulation for 60 s. Plot the estimation error for the four states. What is the experimental standard deviation of the estimation error for each of the four states? Based on the steady-state covariance matrix of the filter, what is the theoretical standard deviation of the estimation error for each of the four states?

13.22 Consider the system

$$\begin{aligned} x_{k+1} &= \phi x_k + w_k \\ y_k &= x_k \end{aligned}$$

where $w_k \sim (0, 1)$, and $\phi = 0.9$ is an unknown constant. Design an extended Kalman filter to estimate ϕ. Simulate the filter for 100 time steps with $x_0 = 1$,

$P_0 = I$, $\hat{x}_0 = 0$, and $\hat{\phi}_0 = 0$. Hand in your source code and a plot showing $\hat{\phi}$ as a function of time.

13.23 Simulate Example 13.4 with artificial parameter noise variance values $\sigma_p^2 = 0$, 1, and 100. How does a change in the artificial parameter noise variance affect the filter's estimate of $-\omega_n^2$?