Testing for long and semi-long alternatives using R/S type test.

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Abstract

Persistent dependence arises naturally in different fields of study like hydrology and finance, and several tools for detecting potential long memory properties have been developed in the literature. Recent work introduces the notion of semi-long memory, meaning that the process exhibits a long-range dependence up to a certain point and begins to decay exponentially fast.

We contribute to the literature by verifying the behavior of tests based on the classical R/S KPSS and V/S statistics under specific conditions for memory and tempering parameters that lead to short-range, semi-long, or long-range dependence. We first reproduce previous results found in the literature, focusing on convergence in distribution and compute the empirical size and power of the test when appropriate. Our second and third contributions aim to state some conjectures about possible extensions of the test in the case of semi-long dependence and testing over the value of the tempering parameter not covered in the literature.

We show that, with appropriate scaling, these statistics have some potential uses for testing hypotheses involving the tempering parameter, but some work is still needed to improve our conjectures.

Keywords: Long memory, Semi-Long memory, ARTFIMA.

1 Introduction

Long memory or long-range dependence (LRD) arises in several spatial or time series applications where the dependence decays slowly compared to the exponential decay observed in short-range dependence (SRD). This phenomenon appears in several disciplines like econometric analysis of asset return in finance, traffic modeling, and hydrology.

Recent work in time series analysis has introduced and focused on semi-long dependence (SLRD)(Giraitis, Kokoszka, and Leipus 2000; Giraitis, Kokoszka, Leipus, and Teyssiere 2003; De Brabanter and Sabzikar 2021). The main difference between SLRD and LRD can summarize by the rate of dependence. Under SLRD, the process exhibits long memory up to a certain point, and afterward, it presents exponential decay.

As noted by De Brabanter and Sabzikar 2021, in the case of ARTFIMA processes, this type of dependence is mainly captured on the fractional parameter d and tempering parameter λ where the tempering parameter attenuates the long memory properties of the process.

Previous work has focused on developing tests to assess whether a process exhibits LRD or SRD by conducting a hypothesis test over d (see Lo 1991; Giraitis, Kokoszka, Leipus, and Teyssière 2003; Kwiatkowski et al. 1992) and asses its empirical size and power under different alternatives (Giraitis, Kokoszka, Leipus, and Teyssière 2003), however less attention has been paid to semi-long tests, meaning focusing the hypothesis on λ.

We first contribute to this literature by conducting an empirical size and power assessment of the SRD versus LRD scenarios under contiguous alternative as in Giraitis, Kokoszka, Leipus, and Teyssiere 2003. Next, we propose testing alternatives for LRD vs. SLRD, and SRD vs. LRD when the parameter is present.

We organize our work as follows, Section 2 introduces the ARTFIMA process and fractional tempered innovations (TFI) we have used as our data generating process and how the parameters determine whether we are dealing with SRD,SLRD or LRD. Section 3 presents the test statistic used in the literature, while 4 establishes the hypothesis of interest and the distribution of the statistic under them. Then we introduce the process for obtaining critical values in 5. Lastly, we present the results of our simulation in 6 and our conclusions in 7.

Our results show that with appropriate scaling, these statistics have potential uses for testing hypotheses related to the tempering parameter but still some work is needed to improve our conjectures.

2 ARTFIMA and fractional tempered innovations

sec:ARTFIMA

A discrete Stochastic process $\{X_t\}_{t\in\mathbb{Z}}$ follows an auto regressive tempered fractional integrated moving average time series, denoted by ARTFIMA (p, d, λ, q), if after applying the tempered fractional difference operator it follows the ARMA(p, q) model, meaning

$$Y_t = \Delta^{d,\lambda} X_t = \left(I - e^{-\lambda} B\right)^d X_t$$
$$Y_t - \sum_{j=1}^{p} \phi_j Y_{t-j} = Z_t + \sum_{i=1}^{q} \theta_i Z_{t-i}$$

In this case $\{Z_t\}_{t\in\mathbb{Z}}$ denotes i.i.d. WN, meaning $\mathbb{E}\left[Z_t\right] = 0$ and $\mathbb{E}\left[Z_t^2\right] = \sigma^2$.

We will focus on a special case, the ARTFIMA $(0, d, \lambda, 0)$ or TFI(d, λ).

$$\left(I - e^{-\lambda} B\right)^d X_t = Z_t \tag{2.1}$$
$$Z_t \sim WN(0, \sigma^2) \tag{2.2}$$

Or alternative can be written as:

eq:TFI

$$X_t = \left(I - e^{-\lambda}B\right)^{-d} Z_t \tag{2.3}$$

$$Z_t \sim WN(0, \sigma^2) \tag{2.4}$$

The tempering parameter plays an important role in defining the length of the memory. When $\lambda = 0$, the process becomes the standard ARFIMA(0,d,0) where larger values of $d > 0$ are associated with stronger correlations. In the ARTFIMA case, affect the rate at which the autocovariance decays, for small values of $\lambda > 0$ the covariance function exhibits long memory up to a certain point and then it began to decay exponentially fast. This case is referred as semi-long memory Sabzikar, McLeod, and Meerschaert 2019.

To illustrate the role that each parameter plays, we present the auto covariance (Figure 1) and auto correlation function (Figure 2). For $d > 0$ it is easy to appreciate the dependence and it increases for larger values of d. When d is kept constant, increasing the value of the tempering parameter λ results in a shorter dependence.

As in Sabzikar and De Brabanter 2020, we are interested in tempering parameters that are sample size dependent and their limiting behavior, that is,

$$N \times \lambda \to \lambda_* \in [0, \infty] \quad \text{as} \quad N \to \infty \tag{2.5}$$

And following their work we distinguishes three cases for λ_*

- If $\lambda_* = \infty$ the process is strongly tempered
- If $0 < \lambda_* < \infty$ the process is moderately tempered
- If $\lambda_* = 0$ the process is weakly tempered

We focus on the case where is a function of N, in particular $\lambda = cN^{-\delta}$, with $c > 0$.Hence:

$$N\lambda = cN^{1-\delta}$$

And under fix $d > 0$, we can state our three conditions in terms of δ and what this entails for the dependence of the process.

$$cN^{1-\delta} \to \lambda_* \in [0, \infty] asN \to \infty \tag{2.6}$$

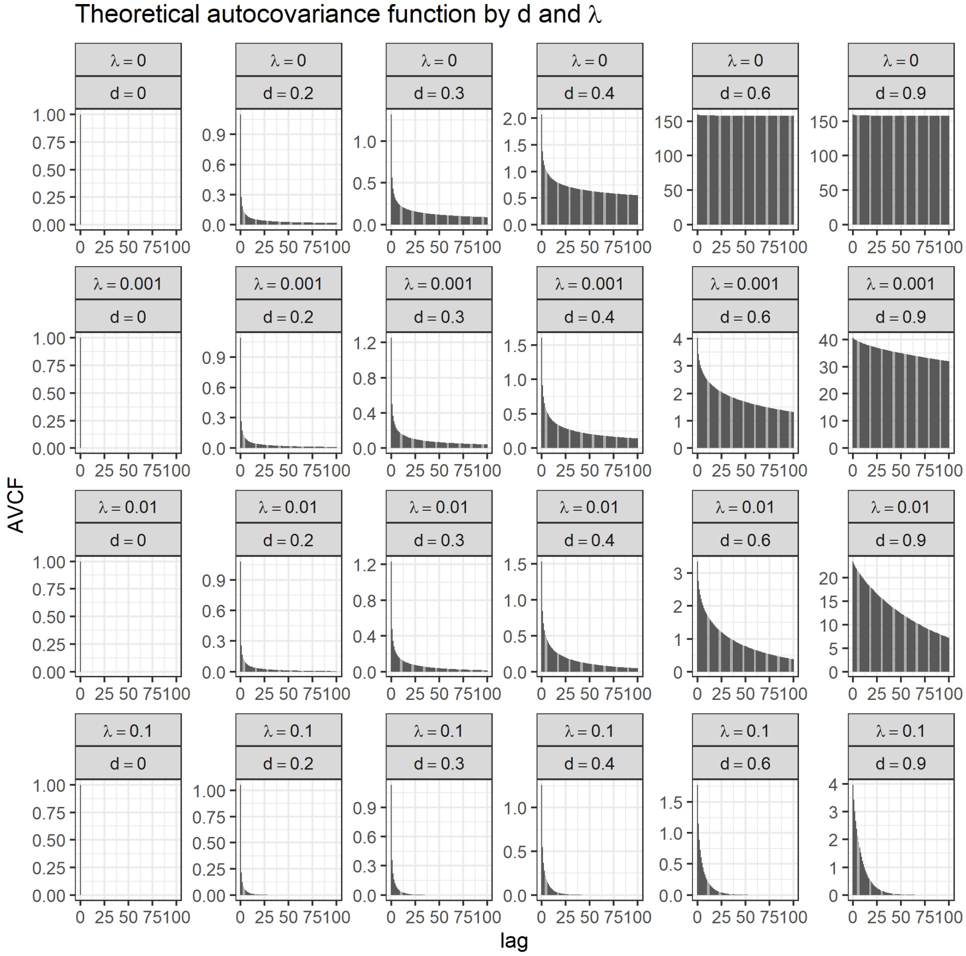
- If $\delta < 1$, $\lambda_* = \infty$, hence it is long range dependent (SRD).
- If $\delta = 1$, $\lambda_* \in (0, \infty)$, hence it is semi-long-range dependent (SLRD).
- If $\delta > 1$, $\lambda_* = 0$, hence it is long range dependent (LRD).

3 Test Statistics

secc:Test

The following sections present an introduction to the statistics used in our work. The interested reader can refer to a more extended review presented in chapter 8 of Giraitis, Koul, and Surgailis 2012

Figure 1: Auto covariance function for TFI fig:ACVF_ARTF



Note: y axis vary by plots.

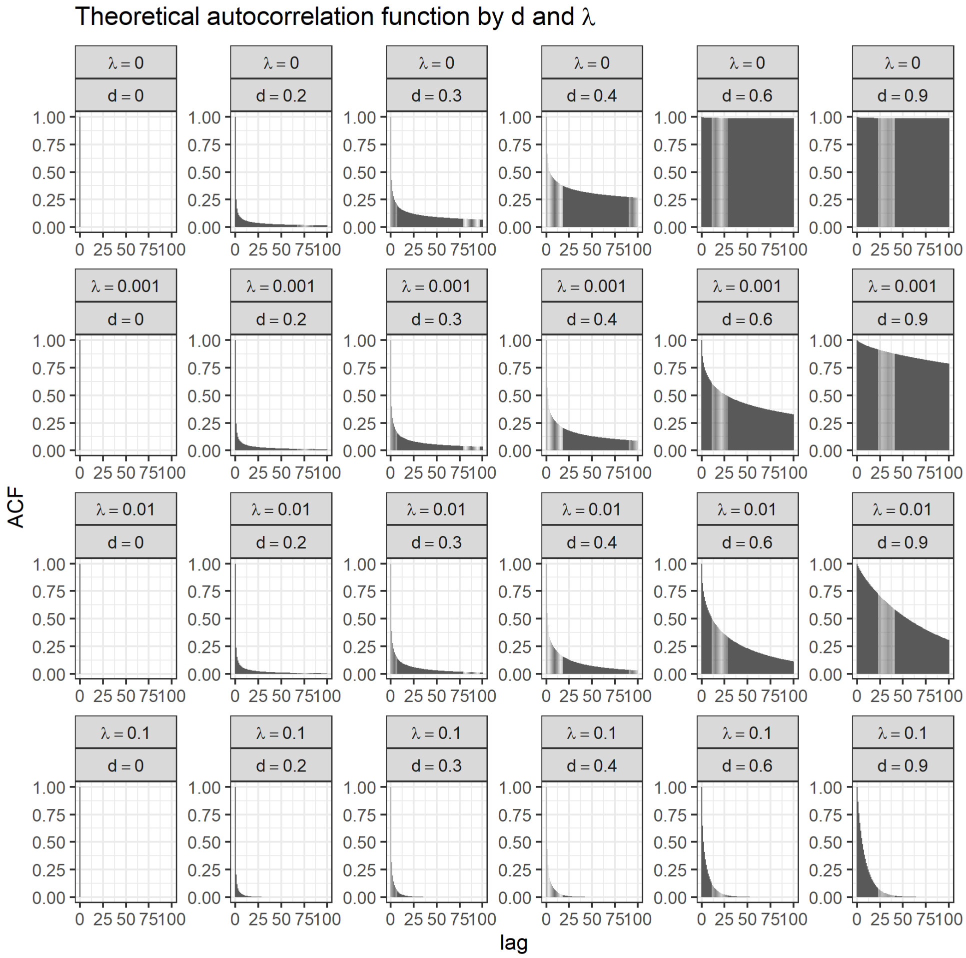
Let $X_1, \ldots, X_n$ be a sample of the stochastic process defined by 2.3. We will consider the partial sum of the centered process,

$$S_k = \sum_{j=1}^{k} X_j - \bar{X}_N \tag{3.1}$$

This partial sums play a key role in the different version of the re scaled statistic defined next.

The first version of the static for detecting long-range dependence was proposed by Hurst 1951, and is referenced in the literature as the re-scaled range statistic (RS). We denote this version as RS_N, and it can be computed by:

Figure 2: Auto correlation function for TFI fig:ACVF_ARTF



Note: y axis vary by plots.

$$RS_N = \frac{1}{\hat{S}_N}\left[\max_{1kN}\sum_{j=1}^{k}\left(X_j - \bar{X}_N\right) - \min_{1kN}\sum_{j=1}^{k}\left(X_j - \bar{X}_N\right)\right] \tag{3.2}$$

$$= \frac{1}{\hat{S}_N}\left[\max_{1\leq k\leq N}\sum_{j=1}^{k} S_k - \min_{1\leq k\leq N}\sum_{j=1}^{k} S_k\right] \tag{3.3}$$

With $\hat{S}_N$ being the square root of:

$$\hat{S}_N^2 = \frac{1}{N}\sum_{j=1}^{N}\left(X_j - \bar{X}_N\right)^2$$

As noted by Lo 1991, this $\hat{S}_N$ is the MLE for the standard deviation and this statistic is always non-negative. The author states that the original statistic is sensitive to short-range dependence and modifies the denominator introducing the estimator for the long-run variance. This statistic is often referred to in the literature as the modified RS statistic, we denote it by $RS_{N,q}$ and it can be computed as

$$RS_{N,q} = \frac{1}{\hat{S}^2_{N,q}} \left[\max_{1 \leq k \leq N} \sum_{j=1}^{k} S_k - \min_{1 \leq k \leq N} \sum_{j=1}^{k} S_k \right]$$

Where the denominator is the square root of an estimator for $\sigma^2 = \sum_j Cov(X_j, X_0)$ that can be estimated by

$$\hat{S}^2_{N,q} = \frac{1}{N} \sum_{j=1}^{N} \left(X_j - \bar{X}_N\right)^2 + 2 \sum_{j=1}^{q} \omega_j(q) \hat{\gamma}_j$$

where the weights $\omega_j(q)$ are the Bartlet weights given by

$$\omega_j(q) = 1 - \frac{j}{q+1},$$

and the estimation of $\hat{\gamma}_j$ is computed using the sample covariance meaning

$$\hat{\gamma}_j = \frac{1}{N} \sum_{i=1}^{N-j} \left(X_i - \bar{X}_N\right) \left(X_{i+j} - \bar{X}_N\right), \quad 0j < N$$

The second test static we consider for this project is the KPSS statistic. Developed by Kwiatkowski et al. 1992 as a test for trend stationary with unit root alternative, but as mentioned by Gromykov 2011 it has been used for other test as test statistic for short vs long memory and stationary vs long memory.

We denote this statistic as $KPSS_{N,q}$ and can be computed as

$$KPSS_{N,q} = \frac{1}{\hat{S}^2_{N,q} N^2} \sum_{k=1}^{N} \left(\sum_{j=1}^{k} \left(X_j - \bar{X}_N\right) \right)^2 = \frac{1}{\hat{s}^2_{N,q} N^2} \sum_{k=1}^{N} \left(S_k\right)^2$$

Note that the estimator of the variances remains the same as in the modified RS statistic and the range in the numerator has been replaced by the second moment of sums S_k.

The re esscaled variance statistic (VS) is modification over the KPSS introduce by Giraitis, Kokoszka, Leipus, and Teyssière 2003. The modification consist of centering the second moment to increase the sensitivity to shift in the variance in comparison to the KPSS statistic.

We denote the statistic by $VS_{N,q}$ and it can be computed as follows,

$$VS_{N,q} = \frac{1}{\hat{S}_{N,q}^2 N^2}\left[\sum_{k=1}^{N}\left(\sum_{j=1}^{k}\left(X_j - \bar{X}_N\right)\right)^2 - \frac{1}{N}\left(\sum_{k=1}^{N}\sum_{j=1}^{k}\left(X_j - \bar{X}_N\right)\right)^2\right] \tag{3.4}$$

$$= \frac{1}{\hat{S}_{N,q}^2 N^2}\left[\sum_{k=1}^{N}\left(S_k\right)^2 - \frac{1}{N}\left(\sum_{k=1}^{N} S_k\right)^2\right] \tag{3.5}$$

In their work Giraitis, Kokoszka, Leipus, and Teyssière 2003 run the simulations to test SRD vs LRD and observe that the V/S statistic is more powerful than the KPSS and RS alternatives.

4 Hypothesis and Distributions

Hypothesis

This section introduces the hypothesis of interest and the distribution of the statistic under the different hypotheses. Before doing so, we first establish some notation for the Brownian bridge (BB) and the fractional Brownian bridge (FBB). In the following section, we shall state that the distributions of our test statistics are functions of such distributions.

Let $W(t)$ denote a standard Brownian process and $W_{d+1/2}(t)$ the fractional Brownian process where $0 \le t \le 1$ and d represent the memory parameter. The tied Brownian bridge (BB) can be constructed $W^0(t) = W(t) - tW(1)$ with $0 \le t \le 1$. Similarly, we construct the fractional version of the Brownian bridge (FBB) by $W^0_{d+1/2}(t) = W_{d+1/2}(t) - tW_{d+1/2}(1)$ with $0 \le t \le 1$

4.1 SRD ($H_0^{(1)}$) VS LRD ($H_1^{(1)}$)

Traditionally testing involve the memory parameter has focus mainly on testing short vs long memory. In the case of the TFI with $\lambda = 0$, that could be expressed in terms of the model parameters.

- $H_0^{(1)}$ SRD : $d = 0$
- $H_a^{(1)}$ LRD: $d > 0$

On this project we first reproduce the results presented by Giraitis, Kokoszka, Leipus, and Teyssière 2003 that aimed at estimating the size and power for different test using fractional innovations as their main DGP. In a second paper Giraitis, Kokoszka, Leipus, and Teyssiere 2003, the authors replace the fractional difference operator $(I - B)^{-d}$ with what they call a mixed operator $(I - rB)^{-d}$ that matches the tempered fractional difference operator replacing r for $e^{-\lambda}$.We expand on their work by increasing the range of d values tested and also by extending the values of the tempering parameter to show how the test behaves.

Under SRD Giraitis, Kokoszka, Leipus, and Teyssière 2003 states the distribution of the test statistic, as functions of the Brownian bridges.

$$\frac{\mathrm{RS_{n,q}}}{n^{1/2}} \Rightarrow U_{RS} = \max_{0\le t\le 1} W^0(t) - \min_{0\le t\le 1} W^0(t) \tag{4.1}$$

$$\mathrm{KPSS_{n,q}} \Rightarrow U_{KPSS} = \int_0^1 (W^0(t))^2 dt \tag{4.2}$$

$$\mathrm{VS_{n,q}} \Rightarrow U_{VS} = \int_0^1 (W^0(t))^2 dt - \left(\int_0^1 (W^0(t)) \right)^2 \tag{4.3}$$

And for LRD,

$$\frac{\mathrm{RS_{n,q}}}{n^{1/2}} \Rightarrow \infty \tag{4.4}$$

$$\mathrm{KPSS_{n,q}} \Rightarrow \infty \tag{4.5}$$

$$\mathrm{VS_{n,q}} \Rightarrow \infty \tag{4.6}$$

The previous results show us that these three tests are right-sided tests. We would reject if we observe a value of the statistic that is larger than the a critical value associated with the Right tail of the null distribution.

4.2 LRD ($H_0^{(2)}$) VS SLRD ($H_1^{(2)}$)

Next we focus on the problem of testing the hypothesis of long range dependence vs semi long. In terms of our model parameter λ_*, these hypotheses can be expressed as

- $H_0^{(2)}$ LRD : $d > 0, \lambda_* = 0$
- $H_a^{(2)}$ SLRD: : $d > 0, 0 < \lambda_* < \infty$

Theoretical results and test statistic have been presented in the literature for decades [1] but testing have traditionally placed LRD as the alternative hypothesis so critical value for the test are scarce in the literature.

Under our second set of hypotheses, the null states that the data present long-range dependence and that a correctly scaled version of our test statistic will converge to functionals of the fractional Weiner process (Giraitis, Kokoszka, Leipus, and Teyssière 2003; Lo 1991). A summarize version of the proof is presented in Chapter 4 of Gromykov 2011.

Let $RS_{n,q}, KPSS_{n,q}, KPSS_{n,q}$ and $VS_{n,q}$ denote the test statistics indicated in Section 3, where n is the sample size and q the lag used in the estimation of the variance, d the memory parameter.

The distribution under the LRD hypothesis can be shown to be,

[1] Lo 1991 presents the results for the modified RS statistic and it limiting distribution under LRD

$$\left(\frac{q}{n}\right)^{d}\frac{\mathrm{RS_{n,q}}}{n^{1/2}} \Rightarrow FU_{RS} = \max_{0\le t\le 1} W^0_{d+1/2}(t) - \min_{0\le t\le 1} W^0_{d+1/2}(t) \tag{4.7}$$

$$\left(\frac{q}{n}\right)^{2d}\mathrm{KPSS_{n,q}} \Rightarrow FU_{KPSS} = \int_0^1 (W^0_{d+1/2}(t))^2 dt \tag{4.8}$$

$$\left(\frac{q}{n}\right)^{2d}\mathrm{VS_{n,q}} \Rightarrow FU_{VS} = \int_0^1 (W^0_{d+1/2}(t))^2 dt - \left(\int_0^1 (W^0_{d+1/2}(t))\right)^2 \tag{4.9}$$

The literature does not provide distribution under SLRD. Our first task is then to assess our conjecture that this statistic will converge to the limiting distribution under weak tempering and to determine whether the test would be able to differentiate between weakly and moderately tempered processes.

4.3 SRD ($H_0^{(3)}$) Vs LRD ($H_a^{(3)}$)

Our third hypothesis deals with testing SRD and LRD, but in this case we no longer assume that $\lambda = 0$, for a fixed d parameter. We differentiate between SRD and LRD based on the value of the limit behavior of λ_*. We can express this set up in terms of model parameters meaning,

- $H_0^{(3)}$ SRD : $d > 0, \lambda_* = \infty$
- $H_a^{(3)}$ LRD: : $d > 0, \lambda_* = 0$

Next, we present our conjectures, first by reviewing and adressing the difference with the statetements from $H^{(0)}$

4.3.1 SRD ($H_0^{(1)}$) VS SRD ($H_0^{(3)}$)

Recall that our first hypothesis also adressed short range depednence, however there is a difference in the definitios.

- Giraitis' definition:
  - Short-range dependence: $d = 0$,
- Our definition:
  - SRD(strong tempering process): $d > 0, \lambda_* = \infty$,

Both our hypotheses and Giraiti's consider the short-range dependence in the hypothesis. However, Giraitis, Kokoszka, Leipus, and Teyssière 2003 is testing the differences between SRD with $d = 0$ and LRD with $d > 0$. In our cases, d is fixed in $(0, 1/2)$. Therefore, we first need to verify the equivalent between strong tempering process with $d \in (0, 1/2), \lambda_* \to \infty$ and the short-range dependence process with $d = 0, \lambda = 0$.

Following the idea presented in Theorem 2 from De Brabanter and Sabzikar 2021, for a tempered linear process, the authors provide a convergence result for the local polynomial regression under a short-range-dependent error (strong tempering process) by introducing the parameter λ^d.

$$\frac{\lambda^d}{\sqrt{Nh}} \sum_{j=1}^{N} K(\frac{Nx-j}{Nh}) X_{d,\lambda}(j) \to N(0, \sigma^2 \int_{-1}^{1} K^2(u)du) \tag{4.10}$$

$$\frac{1}{\sqrt{h}} \sum_{j=1}^{N} K(\frac{Nx-j}{Nh}) \frac{\lambda^d}{\sqrt{N}} X_{d,\lambda}(j) \to N(0, \sigma^2 \int_{-1}^{1} K^2(u)du) \tag{4.11}$$

Where K is Epanechnikov kernel, h is the bandwidth choice and N is the sample size.

Our first conjecture is if, omitting the local polynomial regression, $\frac{\lambda^d}{\sqrt{N}} X_j$ also converges to the Brownian process, which means without long-range memory. The strong tempered process R/S will behave like the Giraitis SRD R/S.

Since for the short-range dependence process(strong tempering process) we already know

$$S_k = \sum_{j=1}^{k} X_j - \bar{X}_N, \quad \hat{S}_{N,q}^2 = \frac{1}{N} \sum_{j=1}^{N} \left(X_j - \bar{X}_N\right)^2 + 2 \sum_{j=1}^{q} \omega_j(q) \hat{\gamma}_j \tag{4.12}$$

$$RS_{N,q} = \frac{1}{\hat{S}_{N,q}} \left[\max_{1 \le k \le N} \sum_{j=1}^{k} S_k - \min_{1 \le k \le N} \sum_{j=1}^{k} S_k \right] \tag{4.13}$$

$$RS_{N,q} = \frac{\sqrt{N}}{\lambda^d \hat{S}_{N,q}} \left[\max_{1 \le k \le N} \frac{\lambda^d}{\sqrt{N}} \sum_{j=1}^{k} S_k - \min_{1 \le k \le N} \frac{\lambda^d}{\sqrt{N}} \sum_{j=1}^{k} S_k \right] \tag{4.14}$$

$$RS_{N,q} = \frac{\sqrt{N}}{\lambda^d \hat{S}_{N,q}} \left[\max_{1 \le k \le N} \frac{\lambda^d}{\sqrt{N}} \sum_{j=1}^{k} \left(X_j - \bar{X}_N\right) - \min_{1 \le k \le N} \frac{\lambda^d}{\sqrt{N}} \sum_{j=1}^{k} \left(X_j - \bar{X}_N\right) \right] \tag{4.15}$$

By taking the limit $q \to \infty$ and $N \to \infty$,

$$\omega_j(q) = 1 - \frac{j}{q+1} \to 1, \quad as \quad q \to \infty \tag{4.16}$$

$$\hat{S^2}_{N,q} \to \sum_{j=-\infty}^{\infty} \hat{\gamma}_j = 2\pi f(0) = (1 - e^{-\lambda})^{-2d}, \quad as \quad q \to \infty, N \to \infty \tag{4.17}$$

$$\frac{1}{\lambda^d \hat{S}_{N,q}} = \left[\frac{(1 - e^{-\lambda})}{\lambda}\right]^d \xrightarrow{p} c, \quad \text{Where } c \text{ is close to 1} \tag{4.18}$$

$$\frac{RS_{N,q}}{N^{1/2}} \to \left[\max_{1\le k\le N} \frac{\lambda^d}{\sqrt{N}} \sum_{j=1}^{k} S_k - \min_{1\le k\le N} \frac{\lambda^d}{\sqrt{N}} \sum_{j=1}^{k} S_k\right] \tag{4.19}$$

$$\frac{RS_{N,q}}{N^{1/2}} \to \left[\max_{1\le k\le N} \frac{\lambda^d}{\sqrt{N}} \sum_{j=1}^{k} \left(X_j - \bar{X}_N\right) - \min_{1\le k\le N} \frac{\lambda^d}{\sqrt{N}} \sum_{j=1}^{k} \left(X_j - \bar{X}_N\right)\right] \tag{4.20}$$

$$\frac{RS_{N,q}}{N^{1/2}} \to \left[\max_{1\le k\le N} \sum_{j=1}^{k} \left(\frac{\lambda^d}{\sqrt{N}} X_j - \frac{\lambda^d}{\sqrt{N}} \bar{X}_N\right) - \min_{1\le k\le N} \sum_{j=1}^{k} \left(\frac{\lambda^d}{\sqrt{N}} X_j - \frac{\lambda^d}{\sqrt{N}} \bar{X}_N\right)\right] \tag{4.21}$$

Therefore, the $\frac{\lambda^d}{\sqrt{N}}$ is implicit in the R/S formula. Based on the Giraitis, Koul, and Surgailis 2012 the SRD R/S will convergence to Brownian Bridge range distribution. If $\frac{\lambda^d}{\sqrt{N}} X_{d,\lambda}(j)$ also has no long memory property, it is rational to assume:

eq:4.11

$$\frac{\mathrm{RS_{N,q}}}{N^{1/2}} \to U_{RS} = \max_{0\le t\le 1} W^0(t) - \min_{0\le t\le 1} W^0(t), as \quad q \to \infty, N \to \infty \tag{4.22}$$

Similarly,

$$\mathrm{KPSS_{n,q}} \to U_{KPSS} = \int_0^1 (W^0(t))^2 dt, as \quad q \to \infty, N \to \infty \tag{4.23}$$

4.3.2 LRD ($H_a^{(0)}$) VS λ^d LRD ($H_a^{(3)}$)

In the case of long range dependence, we also need to adress the difference in definitions.

- Giraitis' definition:
  - Long-range dependence: $d > 0$.
- Our definition:
  - LRD (weak tempering process): $d > 0, \lambda_* = 0$.

For the long range dependence process (weakly tempered process) we already know,

$$\frac{\mathrm{RS^{(l)}_{n,q}}}{n^{1/2}} \Rightarrow \infty \tag{4.24}$$

$$\mathrm{KPSS^{(l)}_{n,q}} \Rightarrow \infty \tag{4.25}$$

Our second conjecture follows the idea presented in Theorem 2 of De Brabanter and Sabzikar 2021. We will omit the local polynomial regression and assess whether, under long-range dependence, the R/S, and KPSS statistic scaled by λ^d, the statistics distributions will converge to functions of Brownian bridges,

$$\lambda^d \times \frac{\mathrm{RS}^{(\mathrm{l})}_{\mathrm{n,q}}}{n^{1/2}} \Rightarrow U_{RS} = \max_{0\leq t\leq 1} W^0(t) - \min_{0\leq t\leq 1} W^0(t) \tag{4.26}$$

$$\lambda^d \times \mathrm{KPSS}^{(\mathrm{l})}_{\mathrm{n,q}} \Rightarrow U_{KPSS} = \int_0^1 (W^0(t))^2 dt \tag{4.27}$$

5 Critical values for functions of Brownian and Fractional Brownian Motions

c:Critical

In this section we present some results for our process of finding critical values for our hypothesis test. In our project, we found that other research papers may use different critical values for the same test, and we noted that the literature is more rich in terms of finding the distribution of functions of Brownian motions rather than fractional Brownian motions.

We first recall some results that allow us to find critical values for U_{RS}, U_{KPSS} and U_{VS} summarized in Giraitis, Kokoszka, Leipus, and Teyssière 2003 Giraitis, Koul, and Surgailis 2012 and Lo 1991.

Under the assumption of SRD the author cite the work of Kuiper that allows to find the theoretical distribution of the random variables U_{RS}, and its density is given by:

eq:URS

$$F_{U_{RS}}(x) = 1 + 2\sum_{k=1}^{\infty}\left(1 - 4k^2x^2\right)\mathrm{e}^{-2k^2x^2}. \tag{5.1}$$

The critical values can be calculated from this equation as in Lo 1991, and the authors also state the following results: $E_{U_{RS}}(x) = \sqrt{\pi/2}$ and $V_{U_{RS}}(x) = \pi(\pi-3)/6$.

Alternatively, critical values can be found by simulations using algorithm 2. We present our results in Table 1.

algo:URS

Algorithm 1 Pseudocode for Distribution of U_{RS}

Let M be the number of simulations and n the frequency of the Brownian bridge.
for i in 1:M **do**.
  $W^{(i)} \leftarrow$ Simulate tied down Brownian bridge with frequency n
  $U_{TS}^{(i)} \leftarrow \max(W^{(i)}) - \min(W^{(i)})$
end for
Compute ECDF of U_{TS}
Compute quantiles.

For U_{KPSS} Kwiatkowski et al. 1992 computes the critical value by simulating integral

Table 1: Critical values URS

	Simulated	Theo.aprox
1%	0.7367440	0.75503
5%	0.8440615	0.86131
10%	0.9078250	0.92746
90%	1.6060652	1.61696
95%	1.73272	1.74726
99%	2.01971	2.00092

ab:Crit_URS

values, but according to Giratis et al., those critical values are imprecise. The author follows Rosenblat and proposes using a series representation of the RV given by

$$U_{KPSS} = \frac{1}{\pi^2}\sum_j \frac{Y_j^2}{j^2},$$

where $Y_j \sim_{i.i.d.} N(0,1)$. Also they state that the expected value of the statistic and its variance are given by $E_{U_{KPSS}}(x) = 1/6$ and $V_{U_{KPSS}}(x) = 1/45$.

We can use this definition to simulate critical values, steps are described in Algorithm 2. Table 2 presents the results.

algo:URS

Algorithm 2 Pseudocode for the distribution of U_{KPSSS} using the series representation:

Let M be the number of simulations and N the number of Normal generated.
for i in 1:M **do**.
  for j in 1:N **do**.
    $y_j \leftarrow$ rnorm$(\mu = 1, \sigma = 1)$.
  end for
  $U_{KPSS}^{(i)} \leftarrow \frac{1}{\pi^2}\sum_{j=1}^{N}\frac{y_j^2}{j^2}$
end for
Compute ECDF of U_{KPSS}
Compute quantiles

Table 2: Critical Values U_{KPSS}

tab:Critical_

	Simulated series	Simulated integral
90%	0.34977	.
95%	0.46597	.
99%	0.73952	.

Lastly,the distribution function of the U_{VS} is presented in Giraitis, Kokoszka, Leipus, and Teyssière 2003 as,

$$F_{U_{VS}}(x) = 1 + 2\sum_{k=1}^{\infty}(-1)^k \mathrm{e}^{-2k^2\pi^2 x} \tag{5.2}$$

This function can be used to find critical values as before. Given a big enough K the contributions become smaller so an numeric approximation can be found and tabulated. We present the result in Table 3

Table 3: Critical Values U_{VS}

	Simulated	Theoretical aproximation
90%	.	0.15176
95%	.	0.18688
99%	.	0.26842

tab:UVS_Crit

Lastly, we present the critical values for FU_{VS} using the simulation approach presented for U_{VS} replacing the Brownian motion for the fractional version. Figure 3 presents the density and empirical CDF of the fractional Brownian bridge range. Critical values are presented in Table 4

Figure 3: Fractional Brownian Bridge range

fig:FBM_Stat

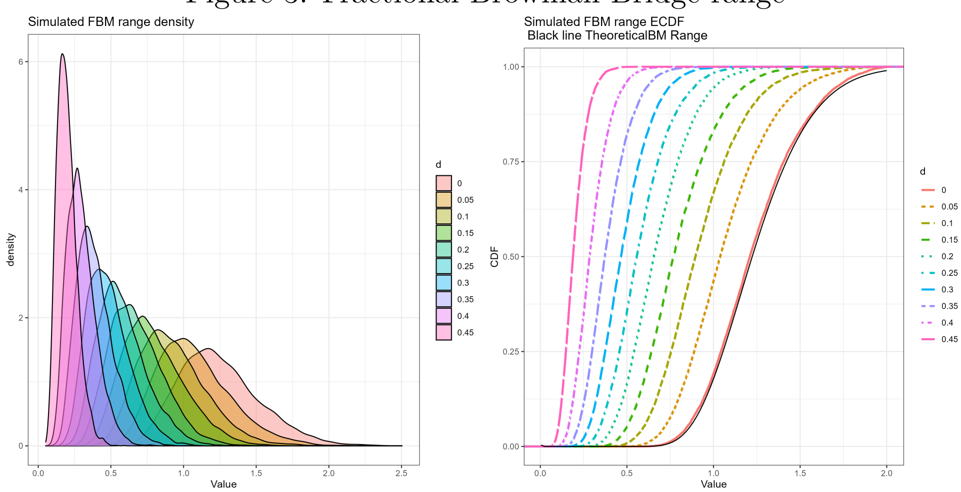


Table 4: Critical ValuesFU_{VS}

tab:Crit FURS

	d=0	d=0.05	d=0.1	d=0.15	d=0.2	d=0.25	d=0.3	d=0.35	d=0.4	d=0.45
1%	0.7384	0.6199	0.5077	0.4208	0.3476	0.2836	0.2268	0.1788	0.1304	0.0814
5%	0.8457	0.7153	0.5944	0.4983	0.4138	0.3428	0.2779	0.2165	0.1619	0.1042
10%	0.9142	0.7756	0.6501	0.5478	0.4585	0.3795	0.3114	0.2444	0.1816	0.1184
90%	1.6238	1.4103	1.2300	1.0864	0.9288	0.8094	0.6802	0.5630	0.4327	0.2880
95%	1.7513	1.5356	1.3346	1.1825	1.0126	0.8954	0.7518	0.6221	0.4841	0.3238
99%	1.9964	1.7754	1.5637	1.3707	1.1942	1.0447	0.8866	0.7419	0.5772	0.3947

6 Simulation results

secc:Simu

We organized our result sections into three sub-sections for each hypothesis tested. The fist subsection presents the results for the traditional set of hypothesis we denoted as $H^{(1)}$ and deals with testing d and we introduce tempering to examine the test behavior. The second section deals with the hypothesis set $H^{(2)}$ and the last section deals with $H^{(3)}$.

6.1 $H^{(1)}$:SRD vs LRD

This first section deals with testing the classical hypothesis of short range dependence vs long range dependence as in Giraitis, Kokoszka, Leipus, and Teyssière 2003.

To assess the empirical size and power of the test, we simulated data under different set-ups that could be associated with either the null or the alternative. To clarify, we present the pseudocode on Algorithm 3. Note that we added a tempering parameter in a manner similar to the r parameter in Giraitis, Kokoszka, Leipus, and Teyssiere 2003 to detect a change in size and power when the tempering parameter is involved. For our experiment we choose increasing tempering parameters meaning $\lambda = (0, 0.001, 0.01, 0.01)$, and in the context of the limiting behavior of λ_* we are still in the case of strong tempering.

Algorithm 3 Pseudocode for our simulation strategy

ulation:H1

Let $\boldsymbol{d}, \boldsymbol{\lambda}, \boldsymbol{N}$ be vector with simulation parameters:
  for each $d \in \boldsymbol{d}$ and $\lambda \in \boldsymbol{\lambda}$ **do**.
    for each each $n \in \boldsymbol{n}$ **do**
      for Repetition 1 to 10.0000 **do**
        Simulate an ARTFIMA$(0, d, \lambda, 0)$ of size n.
        for each each q in $\boldsymbol{q}$ **do**
          Compute: $RS_{N,q}, KPSS_{N,q}, VS_{N,q}$.
        end for
      end for
    end for

For each value of d, λ n a different sample is created and compared to the critical under SRD. Since in the literature there are different critical values presented we recap the critical values in table 5 to avoid confusions.

	RS	KPSS	VS
90%	1.60837	0.34977	0.15176
95%	1.73272	0.46597	0.18688

Table 5: Critical Values used in empirical power calculation

tab:Ctit_EMP

We first present the density of our simulated values in the Figures 4,5 and 6. Note that the axes are not fixed because we wanted to highlight changes in the distributions that are not easily noticed if we kept the plots on the same scale. To make plots comparable and have a meaningful interpretation, we added vertical lines representing the critical value that would lead to the rejection of the null. This representation allows for a visual interpretation of where the mass of the distribution lies.

We can begin by analyzing the first line of each plot, which is associated with traditional hypothesis testing. Under the scenario associated with the null hypothesis, meaning $d = 0$ we can see that the distribution is to the left of the critical values of the test. As we increase the values of d the values of the statistic shift to the right towards the the critical region. From these figures we could conjecture that that we will have more power in the case of larger d. Next, we examine the first column of the plots; increasing the value of will not affect the results. Finally, we assess the cases of $d > 0$ and > 0, fixing the value of d as we increase the tempering parameter, the density shifts to the acceptance region of the test.

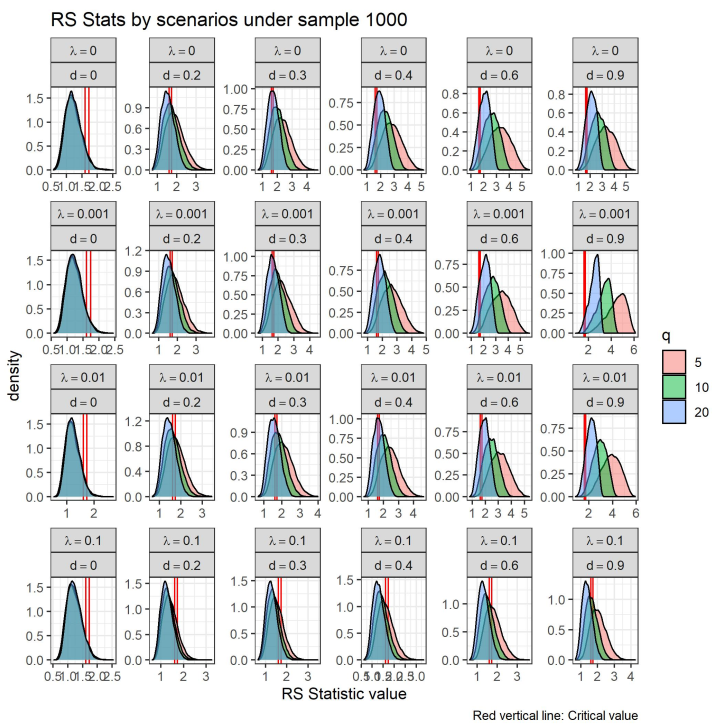


Figure 4: RS

fig:RS_D_h1

The following tables present our approximation for the empirical size and power of the test to quantify the intuition we gain from the previous plots. To compare these results more efficiently, we present a dot plot by sample size and $\alpha = .1$ (Figure 7) to complement the size and power tables. In summary, the rescaled RS-based test achieves a smaller size than the intended α, the closest result is achieved by the KPPS, followed by the VS static. As we have learned from the density plots, larger values of d are associated with larger power for the test, meaning when the DGP has LRD structure, we are more likely to reject the null of SRD. As we increase n from 500 to 1000 the power of the test increases

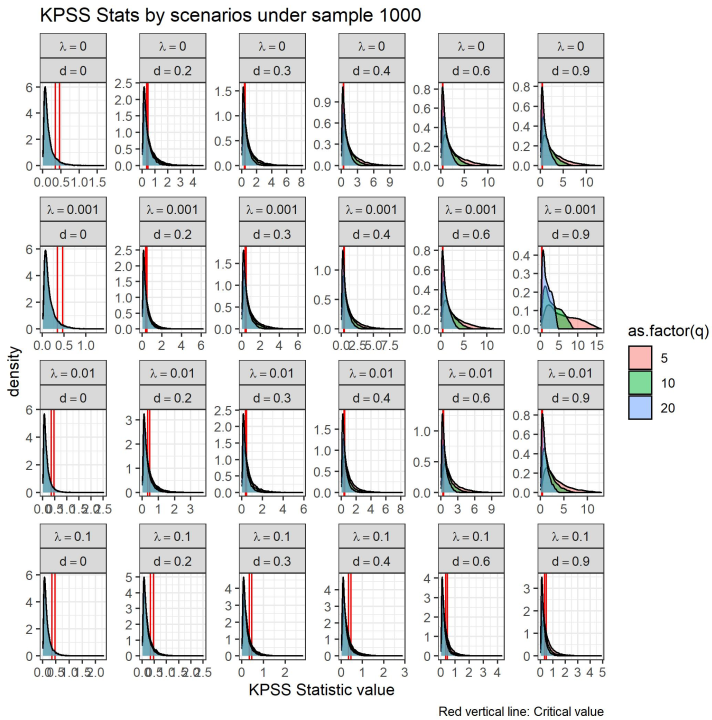


Figure 5: KPSS

fig:KPSS_D_h1

and the power decreases as we increase our q which is associated with the lags of the variance estimator.

We can conclude that the VS statistic is consistently more powerful than the remaining test and that the RS presents the lowest power. This result is consistent in our different scenarios.

Next, we explore the effect of increasing the tempering parameter. As we increase , the percentage of rejection drops, but we do not reach the level of H_0 if unless d is small enough.

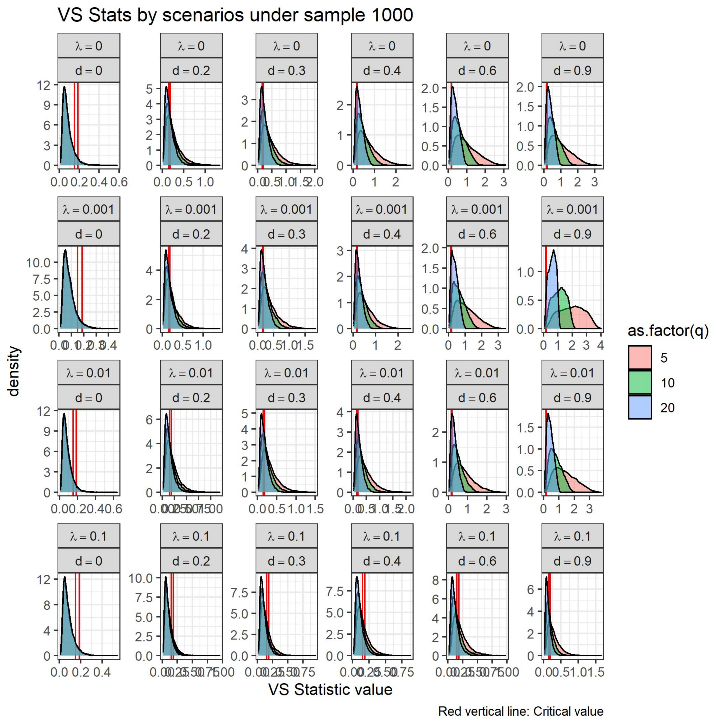


Figure 6: VS

fig:VS_D_h1

Table 6: Empirical size and power of the test as % for modified RS. d, λ denotes the parameters for $ARTFIMA(0, d, \lambda, 0)$, n the sample size and q the number of lags. Each cell is based on 10 000 replications, within d,λ and n the same sample is used for the different critical values.

			d=0		d=0.2		d=0.3		d=0.4		d=0.6		d=0.9	
Sample	λ	q	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%
500	0	5	7.40	3.24	53.85	41.87	75.07	66.08	88.48	82.56	94.68	91.17	94.96	91.43
		10	6.69	2.70	38.12	26.18	57.16	45.05	71.90	61.71	82.60	75.19	82.65	74.79
		20	5.89	1.62	21.27	10.63	33.07	19.88	44.16	29.38	56.16	41.03	55.19	40.27
	0.001	5	7.82	3.62	51.24	39.81	73.04	63.48	85.36	78.64	96.45	93.87	99.61	99.13
		10	7.23	2.93	36.37	24.72	54.83	42.19	67.53	56.14	84.98	77.79	95.94	93.29
		20	5.79	1.91	20.19	9.95	30.58	17.78	39.39	25.47	57.90	43.54	79.86	69.60
	0.01	5	6.91	3.20	50.38	38.33	70.66	60.07	84.42	77.13	96.38	93.61	99.53	98.87
		10	6.33	2.63	34.81	22.70	50.81	38.01	65.03	53.37	83.55	75.78	94.21	90.68
		20	5.32	1.69	18.38	8.75	26.73	15.24	35.79	21.31	53.48	38.40	72.85	59.87
	0.1	5	6.96	3.28	22.30	13.10	30.26	19.28	40.60	28.31	57.76	45.33	74.63	63.24
		10	6.30	2.55	13.29	6.48	16.51	8.76	21.15	12.21	30.00	18.38	40.47	27.41
		20	5.17	1.67	7.07	2.37	7.28	2.63	8.92	3.18	10.31	4.27	12.65	5.36
1000	0	5	8.51	4.00	71.82	61.70	90.81	85.19	97.36	95.20	99.41	98.70	99.30	98.67
		10	8.10	3.69	57.80	45.99	79.12	69.88	90.40	85.25	95.75	92.77	96.15	93.39
		20	7.59	3.21	40.43	28.25	59.49	47.58	74.41	64.40	83.57	75.62	84.09	76.95
	0.001	5	8.06	3.90	69.73	58.99	87.62	81.75	95.91	92.87	99.69	99.27	99.98	99.97
		10	7.69	3.64	55.08	43.40	75.08	65.35	86.63	80.00	97.16	94.51	99.75	99.42
		20	7.24	3.19	38.76	26.48	54.71	42.30	68.13	57.24	84.98	77.58	96.17	93.49
	0.01	5	8.57	3.80	62.69	51.05	83.73	75.23	94.19	89.78	99.38	98.57	99.95	99.91
		10	8.19	3.57	46.87	34.76	66.55	55.50	81.09	72.81	94.13	90.13	99.12	98.12
		20	7.21	3.08	29.97	18.39	44.24	31.32	57.92	45.37	75.83	65.74	90.23	84.03
	0.1	5	8.42	4.05	26.27	16.34	38.04	26.36	47.50	35.07	65.21	52.96	82.61	73.51
		10	8.01	3.49	16.97	9.69	22.16	13.59	26.92	17.09	37.44	25.71	52.06	39.37
		20	7.35	2.91	11.06	5.07	12.56	6.30	13.48	6.81	17.00	9.17	23.20	13.42

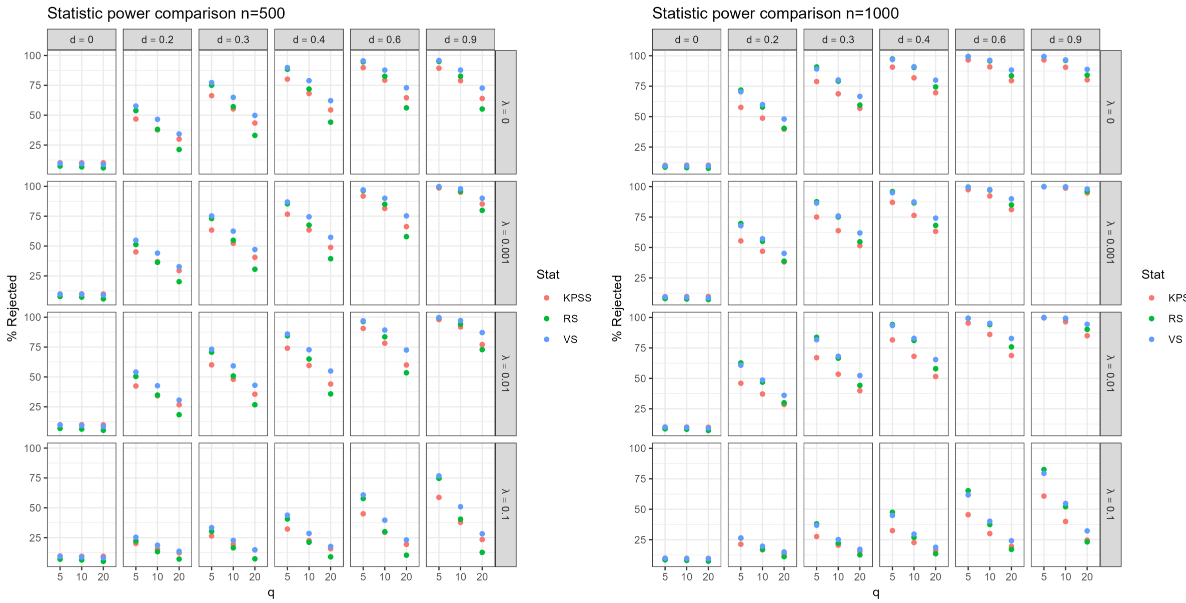
Table 7: Empirical size and power of the test as % for KPSS. d, λ denotes the parameters for $ARTFIMA(0, d, \lambda, 0)$, n the sample size and q the number of lags. Each cell is based on 10 000 replications, within d,λ and n the same sample is used for the different critical values.

			d=0		d=0.2		d=0.3		d=0.4		d=0.6		d=0.9	
Sample	λ	q	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%
500	0	5	10.15	4.87	46.83	34.10	66.34	55.14	80.19	70.74	89.81	82.67	89.29	82.34
		10	10.13	4.83	37.74	26.70	55.25	43.18	68.15	57.06	79.29	69.19	78.88	68.52
		20	10.13	4.50	29.95	19.70	43.43	31.59	54.38	41.65	64.61	52.37	63.98	51.27
	0.001	5	9.89	4.84	45.12	33.39	63.36	51.70	76.69	65.71	91.84	85.82	98.66	97.03
		10	9.94	4.77	36.91	25.73	52.30	40.15	63.46	51.13	81.49	72.18	95.04	90.29
		20	9.87	4.59	29.45	18.54	40.57	29.01	48.90	37.11	66.28	54.41	85.21	76.29
	0.01	5	10.01	4.81	42.39	30.38	60.08	47.64	74.03	62.25	90.61	83.33	97.99	95.75
		10	10.02	4.69	34.22	23.07	47.99	35.17	59.55	46.49	78.22	66.87	91.83	85.01
		20	9.91	4.38	26.70	16.50	35.52	24.27	44.07	31.38	59.99	46.64	77.13	64.67
	0.1	5	9.67	4.66	20.12	11.37	26.39	16.45	32.21	21.00	44.97	31.57	58.70	44.16
		10	9.43	4.54	15.60	8.10	19.42	10.95	22.51	13.04	29.49	18.37	37.84	25.50
		20	9.42	4.38	12.33	5.95	14.74	7.66	15.77	8.50	19.35	10.80	23.44	13.62
1000	0	5	9.96	4.77	57.61	45.59	78.82	69.08	90.65	84.21	96.51	93.29	96.56	93.22
		10	10.09	4.79	48.75	36.90	68.74	57.30	81.85	72.74	90.84	84.03	90.56	84.44
		20	10.07	4.77	39.72	28.44	56.81	44.10	69.62	58.81	79.53	70.00	80.24	70.94
	0.001	5	9.64	4.84	55.41	43.60	75.01	64.23	87.05	78.84	97.42	94.64	99.83	99.47
		10	9.68	4.68	46.92	34.85	63.80	51.65	76.38	66.20	92.32	86.28	98.75	97.42
		20	9.76	4.60	38.12	26.53	51.53	39.02	63.21	51.48	81.14	71.47	94.79	90.45
	0.01	5	9.99	4.98	45.99	33.60	66.97	53.67	81.57	71.18	95.41	90.39	99.62	98.52
		10	9.97	4.86	37.17	25.42	53.39	40.22	68.10	54.94	86.06	76.51	96.45	91.87
		20	9.85	4.68	28.54	18.01	39.82	27.58	51.51	38.01	68.73	55.86	85.02	75.40
	0.1	5	9.81	4.93	21.26	12.42	27.57	17.60	32.42	21.58	45.50	32.08	60.77	46.55
		10	9.66	4.92	16.70	9.47	20.48	12.04	22.66	14.01	30.01	19.14	39.96	27.18
		20	9.66	4.66	13.69	6.79	15.43	8.34	16.38	9.36	19.52	11.07	24.64	14.80

Table 8: Empirical size and power of the test as % for VS. d, λ denotes the parameters for $ARTFIMA(0, d, \lambda, 0)$, n the sample size, and q the number of lags. Each cell is based on 10,000 replications, within d,λ and n the same sample is used for the different critical values.

4-15			d=0		d=0.2		d=0.3		d=0.4		d=0.6		d=0.9	
Sample	λ	q	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%
500	0	5	9.60	4.48	57.68	45.29	77.28	67.91	89.85	83.55	95.51	91.96	95.79	92.12
		10	9.37	4.02	46.56	33.16	64.92	53.23	78.91	68.74	87.75	80.81	87.76	80.69
		20	8.69	3.25	34.29	21.61	49.80	36.36	62.16	49.60	72.94	61.88	72.71	60.96
	0.001	5	9.62	4.84	54.82	42.75	75.25	65.60	86.88	79.76	97.03	94.50	99.77	99.32
		10	9.58	4.36	44.09	31.80	62.48	50.21	74.49	63.98	90.04	83.65	97.92	95.87
		20	8.99	3.50	32.73	20.31	47.11	33.84	57.35	44.29	75.26	64.63	90.01	84.22
	0.01	5	9.75	4.45	54.10	41.22	73.18	62.73	85.94	78.34	96.94	94.21	99.71	99.21
		10	9.29	4.14	42.64	29.70	59.26	46.33	72.76	61.41	89.25	82.25	97.17	94.44
		20	8.52	3.45	30.59	18.70	43.02	29.08	54.95	40.31	72.51	60.56	87.13	79.30
	0.1	5	9.22	4.39	25.37	15.55	33.39	21.91	43.83	30.88	60.75	47.38	76.75	64.52
		10	8.76	4.04	18.70	10.18	22.68	12.98	28.53	17.27	39.64	26.46	50.89	37.08
		20	8.19	3.32	13.62	5.87	14.81	7.32	17.64	8.82	23.09	12.16	28.25	16.61
1000	0	5	9.57	4.55	70.43	59.67	89.14	82.92	96.90	93.90	99.17	98.16	99.27	98.27
		10	9.57	4.44	59.84	48.19	80.10	70.99	91.03	85.58	96.29	93.37	96.54	93.54
		20	9.50	4.28	48.02	34.75	66.61	54.50	79.95	70.75	88.25	81.20	88.81	82.05
	0.001	5	9.47	4.57	67.87	56.70	86.50	79.03	95.02	91.04	99.58	99.12	99.99	99.96
		10	9.15	4.37	57.17	45.41	75.88	66.23	87.43	80.27	97.55	94.84	99.79	99.44
		20	8.64	4.07	45.17	32.64	61.94	49.77	74.10	63.85	89.94	82.99	98.02	95.91
	0.01	5	10.03	4.74	60.76	48.35	81.74	71.74	93.36	87.81	99.20	98.04	99.97	99.90
		10	9.68	4.54	48.64	35.83	68.15	56.63	82.85	73.05	95.19	90.94	99.40	98.38
		20	9.16	3.97	36.05	23.42	52.28	38.75	65.41	52.92	82.76	73.38	94.42	89.29
	0.1	5	9.72	4.58	26.44	16.59	36.86	24.36	44.91	32.07	61.76	48.42	79.45	68.16
		10	9.40	4.49	19.64	11.37	25.08	14.93	29.57	18.50	40.16	27.35	54.77	41.14
		20	9.25	4.17	14.97	7.47	17.08	9.26	18.79	10.58	24.12	13.74	32.21	20.20

Figure 7: Statistical power and size comparison at $\alpha = 0.1$ fig:Dot plot



6.2 $H^{(2)}$:LRD vs SLRD

This sections deals with our second hypothesis test, which tries to differentiate between a long memory process and a semi long memory process. In this section, we select $d \in \{0, 1, 0.2, 0.3, 0.4\}$ and different values for that result in different λ_* as in De Brabanter and Sabzikar 2021.

- $LRD1$: $\lambda = N^{-1.1}$, which results in $N^{-0.1} \to \lambda_* = 0$, a weakly tempered LRD process.
- LRD_2: $\lambda = N^{-8}$ that also result in $\lambda_* = 0$ and LRD.
- $SLRD$ $\lambda = 2N^{-2}$ that results in $\lambda_* = 2$ a moderated tempered process that result in SLRD.

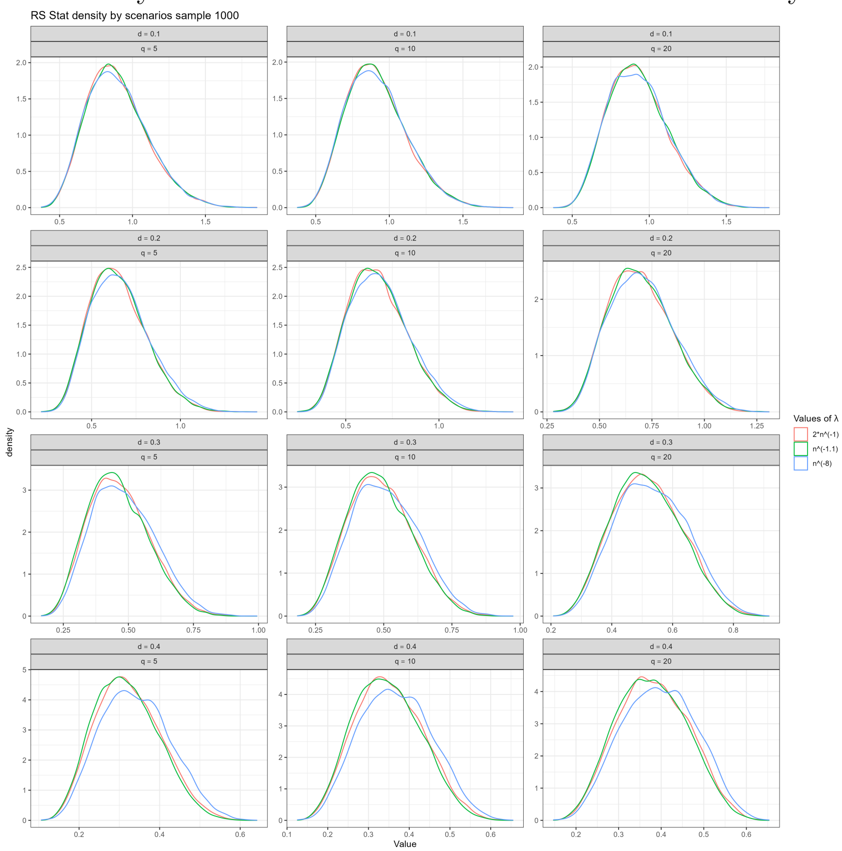
Although both the first and second cases result in LRD, they do have a different tempering rate; LRD2 results in a weaker tempering than LRD1. For this reason, we wanted to evaluate whether this has an impact on the distribution of the statistics.

As before, we first present the density distribution for each statistic (Figures 8 9 and 10)and empirical CDF (Figures 11,12 and 13) under the three values of chosen, where we faceted by scenarios resulting from a combination of values for d and q.

First, one can argue that under these scenarios there is little to no difference between the distribution for different values of q selected, which is a striking difference from our previous section. Since we have two values of λ associated with LRD and one with SLRD, we would expect the distribution of the statistic for the two LRD cases to closely match and be able to differentiate from the SLRD cases.

The figures show that for smaller values of the memory parameter d, there is little to no difference between the distributions associated with the three values of λ, however as we incrase d we observe a difference between LRD2 and SRD, but LRD remains close to SRD.

Figure 8: Density distribution for the test statistic RS faceted by scenarios fig:DensityH2



To further inspected this difference, we conducted Kolmogorov-Smirnov (KS) test over samples generated for each scenario and value of λ and present the resulting p-value in figure 14. Recall that the objective of the Kolmogorov-Smirnov test is to assess whether both samples come from the same unknown distribution.

The first row present the test comparing LRD_1 ($\lambda = N^{-1.1}$) vs LRD_2 ($\lambda = N^{-8}$), across all test statistic we would reject that data come from the same distribution for values of $d > .1$. In the case of LRD_1 ($\lambda = N^{-1.1}$) vs $SLRD$ ($\lambda = 2N^{-1}$) we reject for values larger than $d > .2$ in the case of RS ant VS statistic but that is not the case for KPSS which presented smaller p-values. Lastly, in the case of LRD_2 ($\lambda = N^{-8}$) versus $SLRD$ ($\lambda = 2N^{-1}$) it would seem that the distribution differs the most, and again the KPSS appears to have a more similar distribution.

Recall that we expected to see a difference between SLRD and both versions of LRD while finding little difference for samples generated by values of λ associated with LRD. Our results show that our test cannot discriminate correctly between LRD and SLRD

Figure 9: Density distribution for the test statistic KPSS faceted by scenarios fig:DensityH2

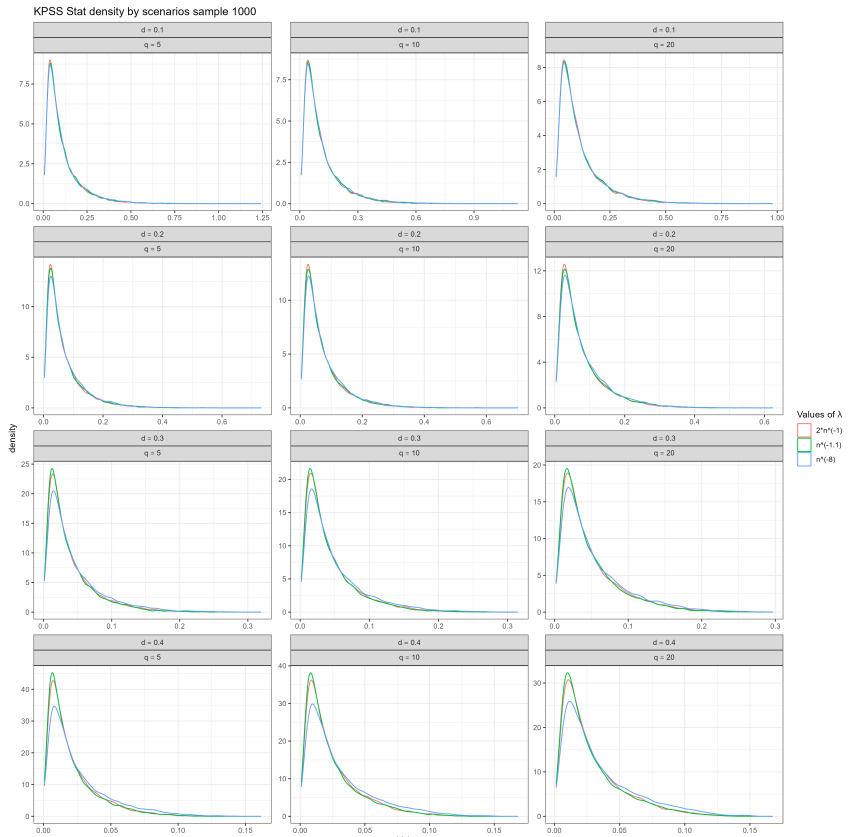


Figure 10: Density distribution for the test statistic VS faceted by scenarios

fig:DensityH2

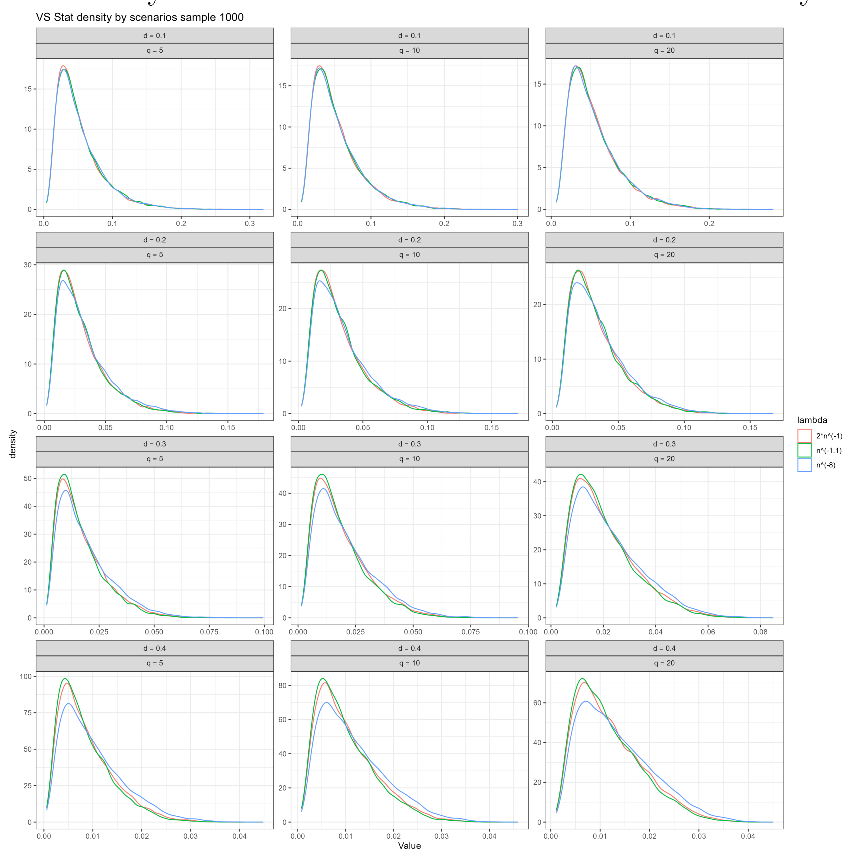
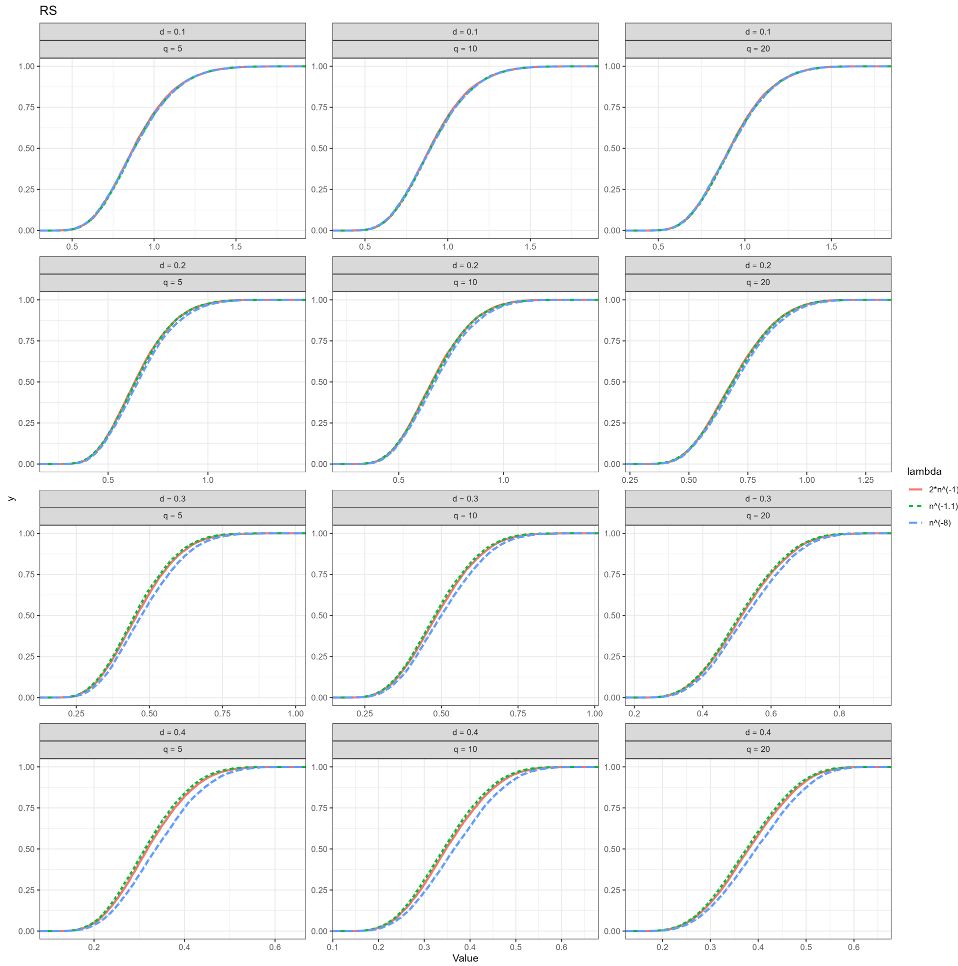


Figure 11: Empirical cumulative distribution for the test statistic RS faceted by scenarios fig:ECDFH2RS



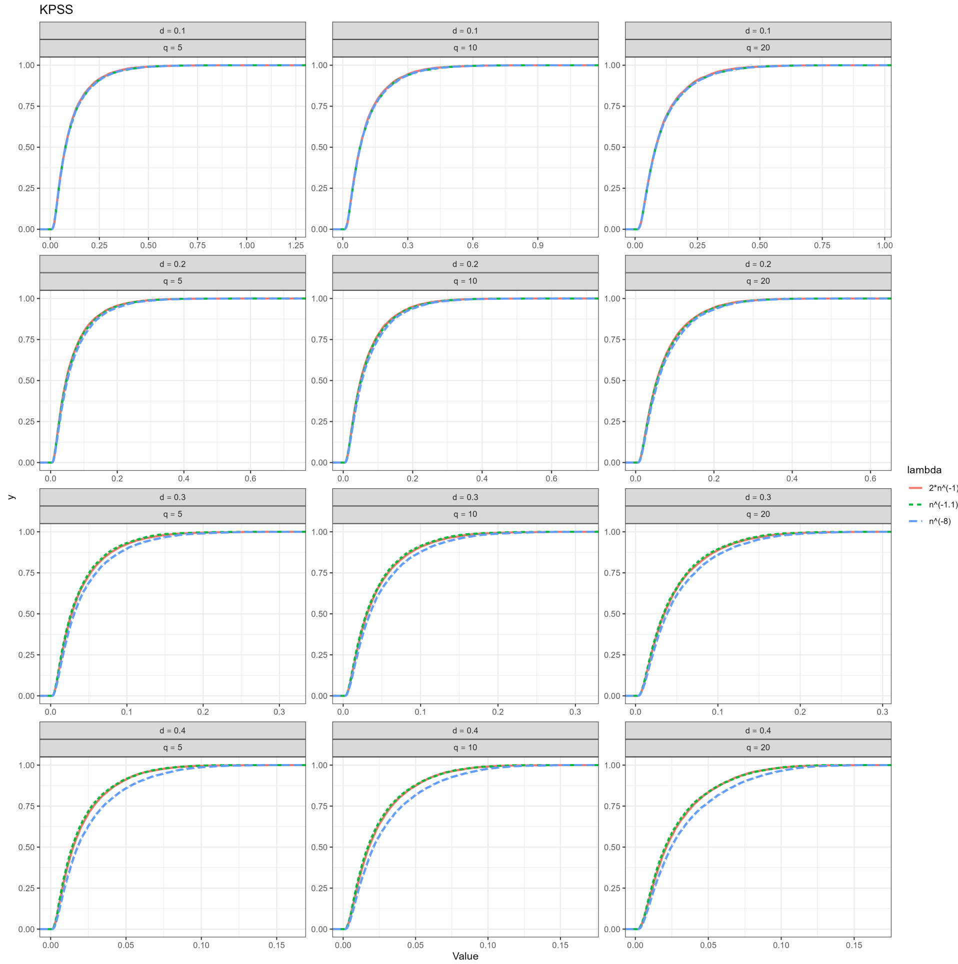
cases for our selected λ values, the value of λ plays a significant role and need to be chosen more carefully.

Lastly, we wanted to compare our simulation result with the expected theoretical limiting distribution. Since we do not have the tools to simulate all critical values for the test involving fractional Brownian motions, we next focus on the RS test with $q = 10$ for simplicity. Figures 15 present the density and overlay two vertical lines from the corresponding quantiles (5% and 95%) of the simulated limiting distribution, while table 9 presents the distribution of the mass of the simulated distribution compared to the quantiles found.

If our simulation matches the conjectures, we would expect to see a similar distribution of the mass of the statistic across the quantiles. Our result shows that for the case of $SLRD_2$ the mass of the distribution is shifted to the right, a fact that becomes more evident for larger values of d.

It is worth noting that the distribution does not exactlty match the simulated limiting distribution in all scenarios. To gain further insight into this problem, we plotted the

Figure 12: Empirical cumulative distribution for the test statistic KPSS faceted by scenarios fig:ECDFH2KPS



density and empirical CDF of our statistic for different values of and the simulated values of the fractional Brownian bridge range (Figure 16) . Our results show that there is a discrepancy, which is more notorious for larger values of the memory parameter.

In summary, our results show that our conjectures would not be able to differentiate between LRD and SLRD under the chosen values of λ. Furthermore, we could not verify the convergence to the correct expected limiting distribution, so we believe that this may not be appropriate for testing our hypothesis.

Figure 13: Empirical cumulative distribution for the test statistic VS faceted by scenarios fig:ECDFH2VS

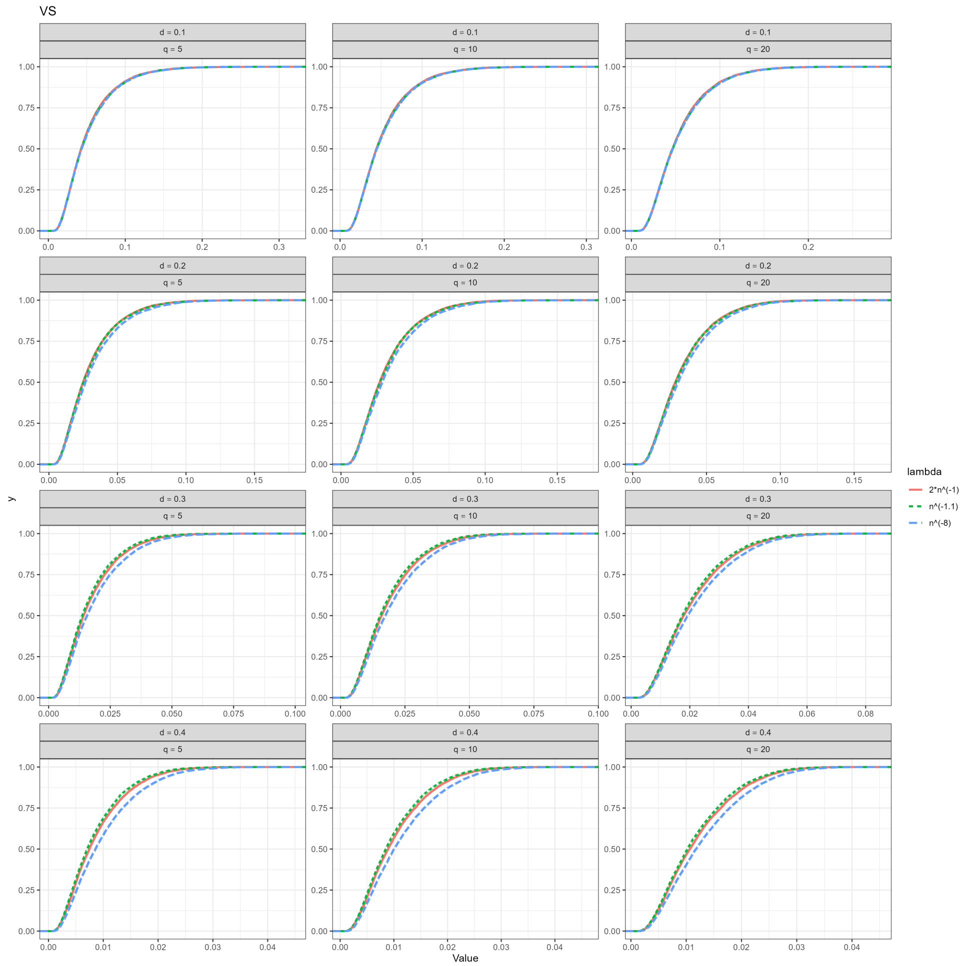


Table 9: Mass cover by Critical values of FMB bridge range tab:Crit_H2

d	λ	Left tail 5%	Center of distribution 90%	Right tail 5%
0.1	$2n^{-1}$	0.038	0.929	0.033
0.1	$n^{-1.1}$	0.037	0.930	0.033
0.1	n^{-8}	0.038	0.929	0.033
0.2	$2n^{-1}$	0.027	0.949	0.023
0.2	$n^{-1.1}$	0.030	0.946	0.024
0.2	n^{-8}	0.024	0.945	0.031
0.3	$2n^{-1}$	0.017	0.966	0.017
0.3	$n^{-1.1}$	0.017	0.969	0.014
0.3	n^{-8}	0.013	0.962	0.026
0.4	$2n^{-1}$	0.002	0.935	0.063
0.4	$n^{-1.1}$	0.001	0.946	0.052
0.4	n^{-8}	0.002	0.897	0.102

Figure 14: Kolmogorov-Smirnov test - Pvalues

fig:KSPval

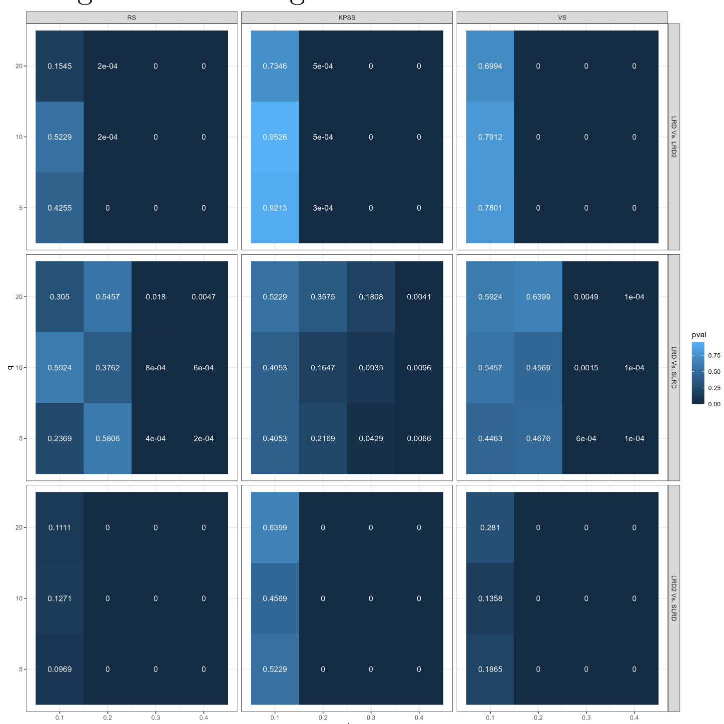


Figure 15: RS statistic distribution for $q = 10$ and FBB values

fig:RS_crit

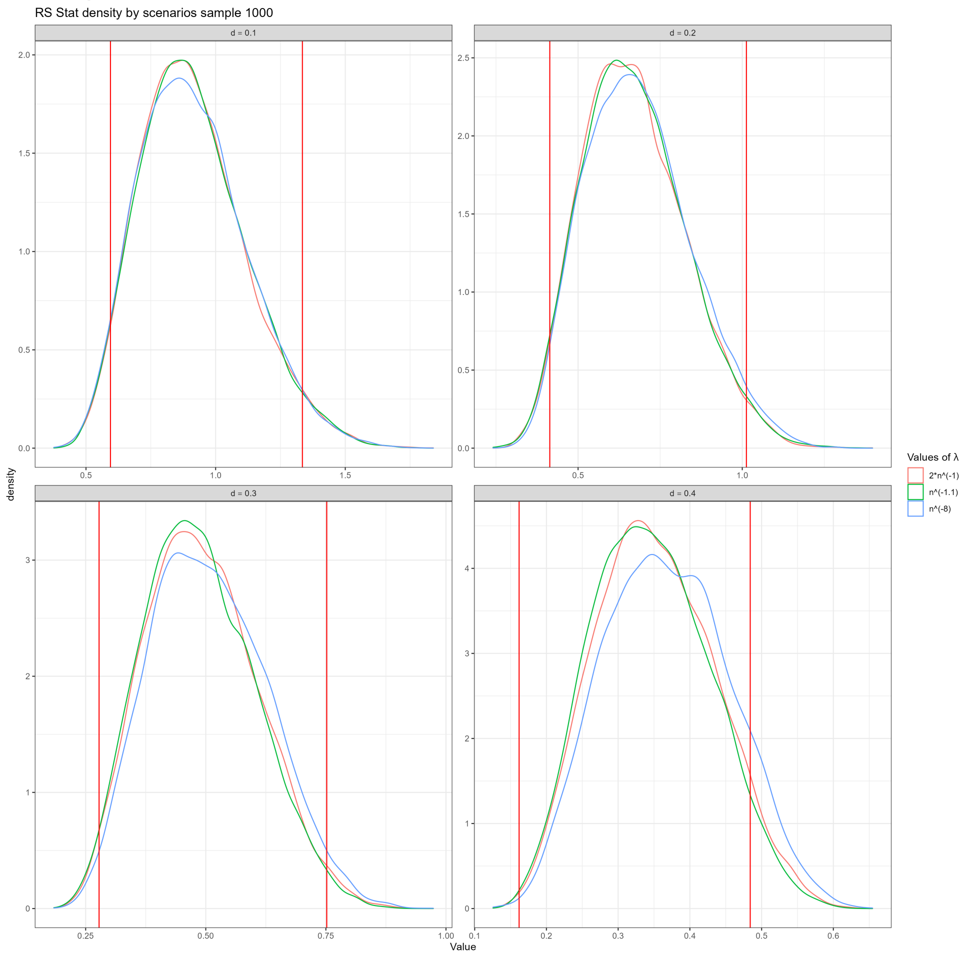
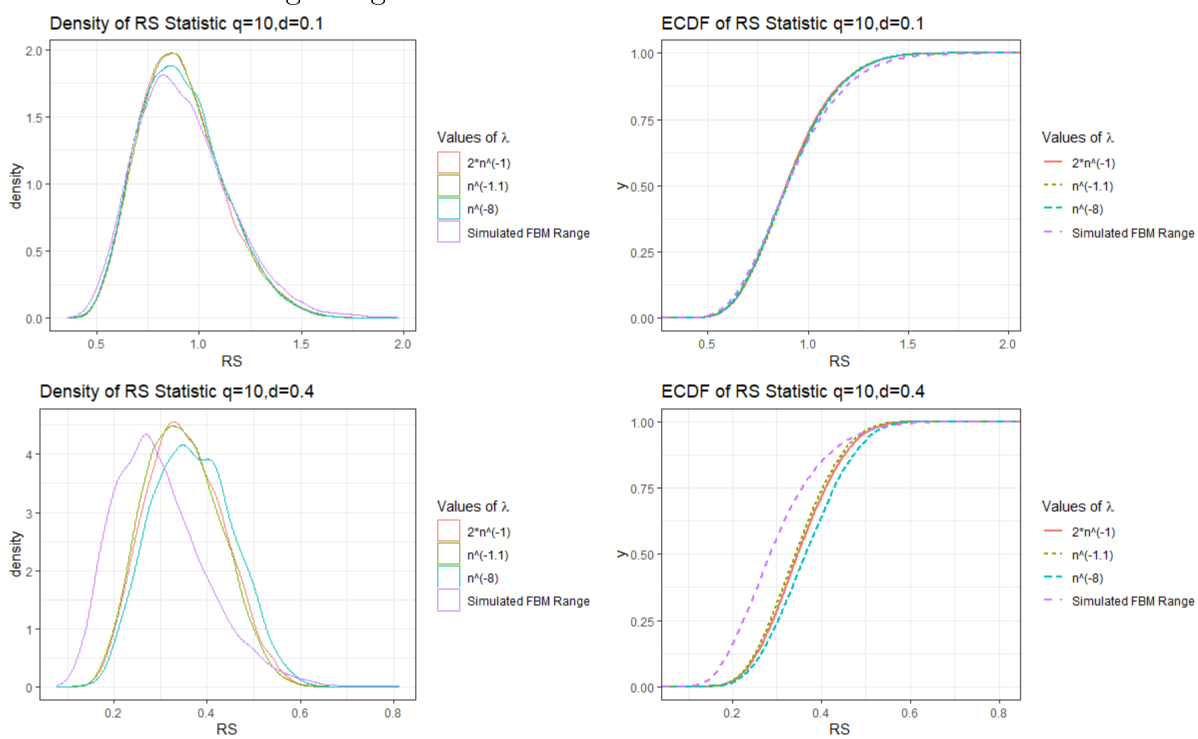


Figure 16: Density and empirical cumulative distribution for RS compared to the fractional Brownian bridge range

fig:ECDF_RS_T



6.3 $H^{(3)}$:SRD and LRD with λ parameter

Next, we evaluate our conjectures regarding testing for SRD and LRD when tempering is present. We divide our results into 6.3.1 where we examine the conjectures regarding short-range dependence and section 6.3.2 where long-range dependence is addressed. Lastly we present power calculation results in section 6.3.3

6.3.1 SRD with λ parameter

secc:h3SRD

To evaluate our conjecture, we will simulate short-range dependent using a similar setups as in $H^{(1)}$, with sample size $N = 1000$. Under this hypothesis, the fractional parameter d is fixed in $(0, 1/2)$, therefore we choose different values of $d \in \{0.2, 0.3, 0.4\}$, and choose a corresponding to strong tempering.

- $\lambda = 0.1$, which is strong tempering and result in $\lambda_* = 0.1N \to \infty$ for short range dependence as mentioned in equation 2.6.

As before, we will choose $q \in \{5, 10, 20\}$ for the lag parameters following Giraitis, Koul, and Surgailis 2012. We present an initial assessment of the convergence for our simulation results in figures and then provide a table about the overlap between the Monte Carlo simulations and the theoretical results.

For the RS statistic, Figure 17 presents the CDF of the simulated test statistic and the theoretical RS for comparison. Figure 18 presents the densities adding reference values from the theoretical distribution, and Table 10 presents the mass of the distribution that falls within these reference values.

Figure 17: $\frac{RS_{n,q}}{\sqrt{n}}$ CDF under SRD and theoretical Brownian Bridge range fig:H3A

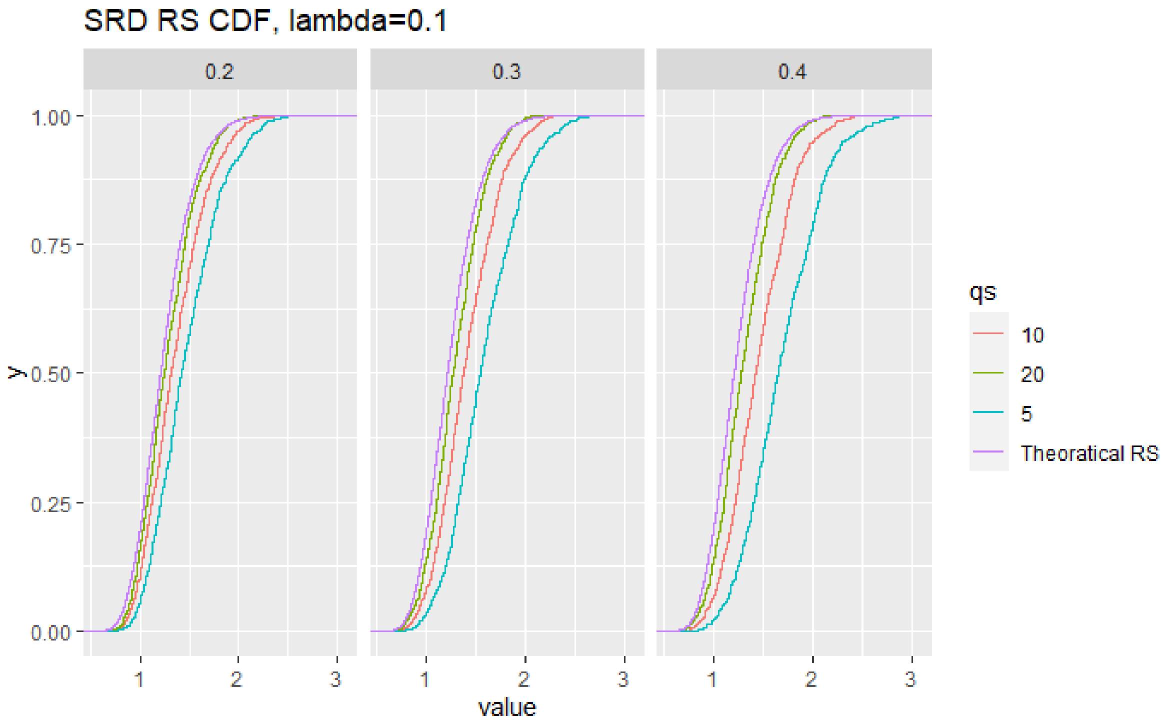
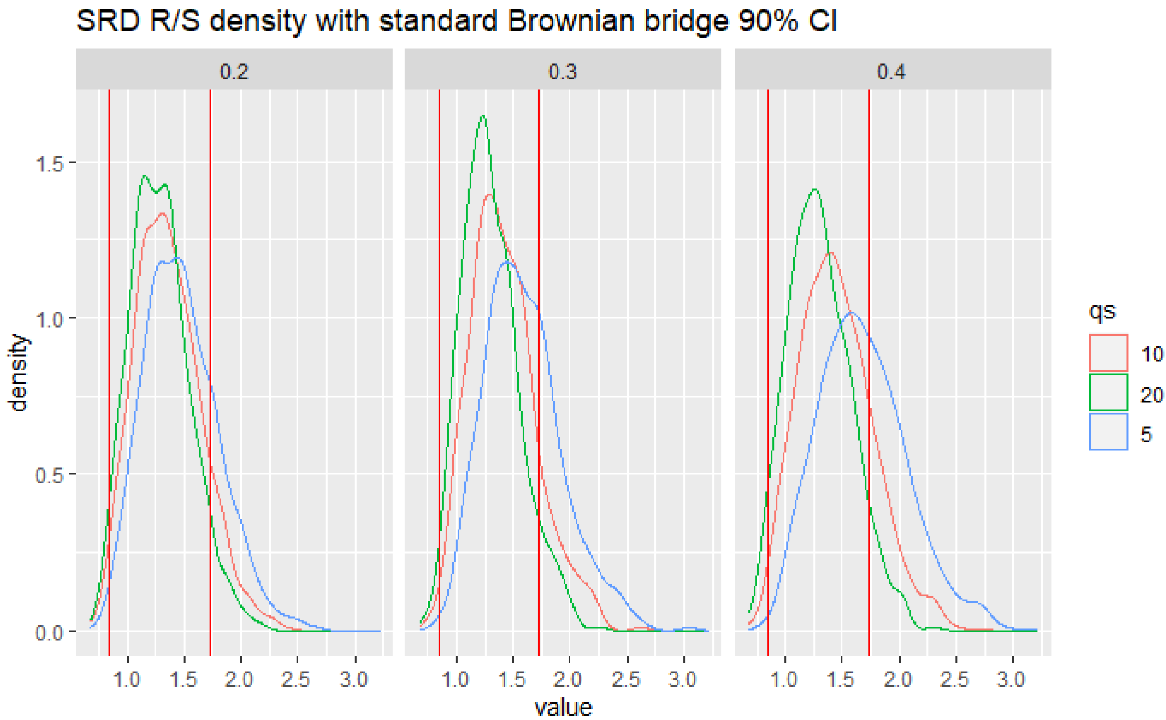


Figure 18: $\frac{RS_{n,q}}{\sqrt{n}}$ under SRD and theoretical reference values. fig:H3B



The red lines represent the 5% tail values of a Brownian Bridge range theoretical distribution.

Table 10: Mass cover by reference values of BB range with $\lambda = 0.1, N = 1000$

q	d=0.2	d=0.3	d=0.4
5	73.73	61.96	52.50
10	83.03	77.84	73.08
20	88.94	87.44	86.52

ambda_d_SRD

Figure 19: $KPSS$ CDF under SRD and theoretical results fig:H3A2

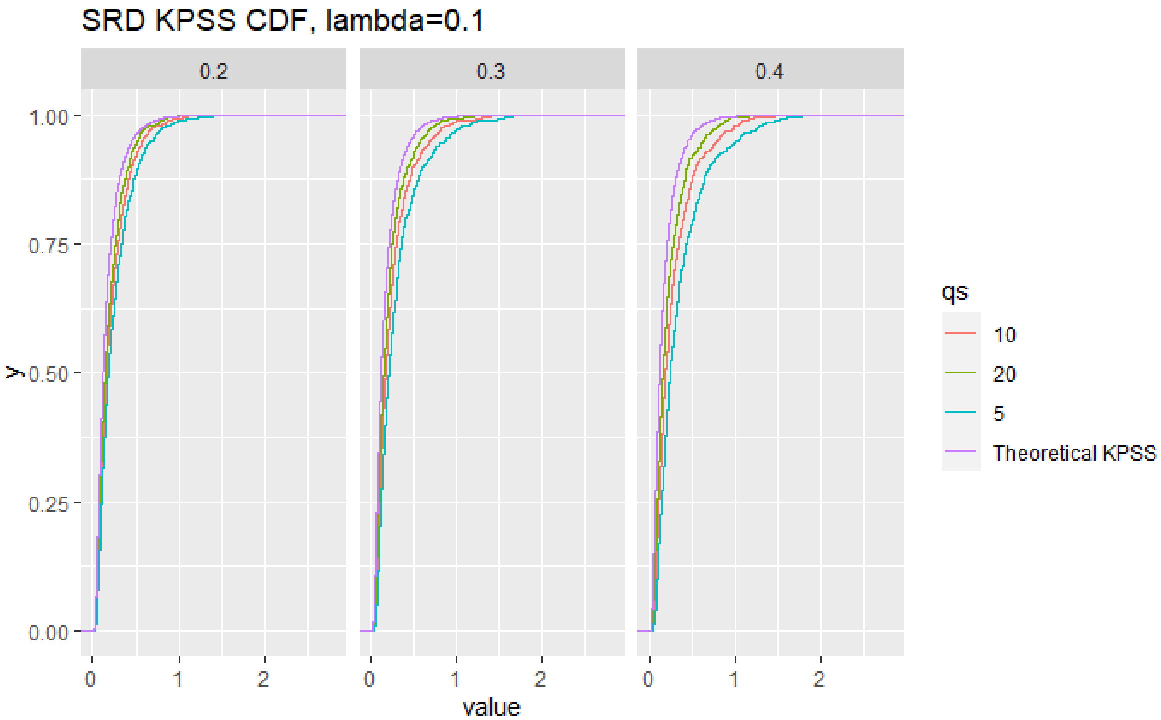
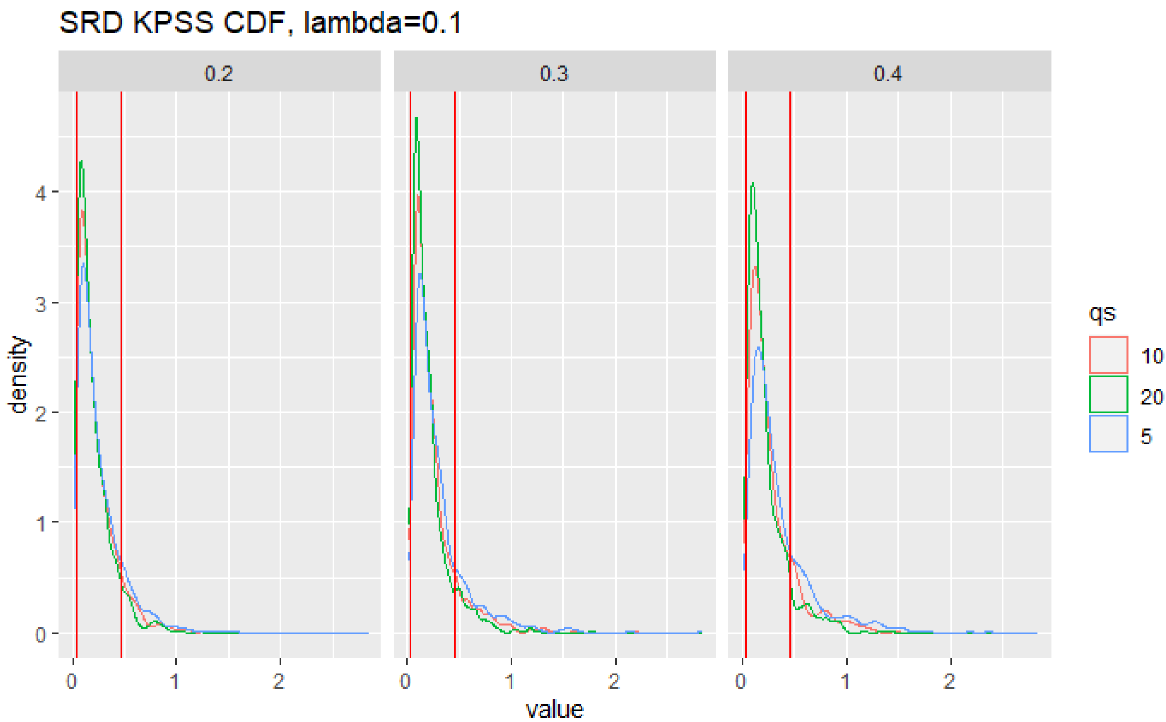


Figure 20: $KPSS$ under SRD and theoretical reference values. fig:H3B2



The red lines represent the 5% tail values of a theoretical KPSS distribution.

Table 11: Mass cover by reference values of theoretical KPSS with $\lambda = 0.1, N = 1000$

q	d=0.2	d=0.3	d=0.4
5	84.6	82.2	76.7
10	88.6	87.1	83.0
20	90.1	89.7	89.6

bda_d_SRD2

The results show that our simulated process for SRD ($\lambda = 0.1$) appears to converge to the expected limiting distribution. It is interesting to note that increasing q the simulations results are closer to the theoretical limiting distribution. This result matches with Giraitis, Kokoszka, Leipus, and Teyssière 2003's conclusions on the R/S type test.

We present the same analysis for the KPSS statistic. Empirical CDF and theoretical simulated values for KPSS in Figure 19 and densities with reference values in Figure 20. Lastly Table 11 presents the mass of the distribution that falls within those reference values.

As in the case of the RS statistic, our results show that the KPSS under SRD ($\lambda = 0.1$) appears to converge to the expected limiting distribution. The same conclusion regarding the values of q holds for KPSS.

secc:h3LRD

6.3.2 LRD with λ parameter

To evaluate our second conjecture regarding LRD, we will simulate long-range dependent data using a similar setups as before, but choose values of that would result in weak tempering and LRD.

- $\lambda = N^{-8}$, which is pretty weak tempering leading to $\lambda_* = N^{-7} \to 0$ for long range dependence.
- $\lambda = N^{-1.1}$, which is stronger than the previous one, but still weak tempering, resulting in $\lambda_* = N^{-0.1} \to 0$ and long-range dependence.
- $\lambda = N^{-1.0001}$, which is the strongest choice we made within weak tempering leading to $\lambda_* = N^{-0.0001} \to 0$ and long-range dependence, but note that it is very close to the definition of semi-long range dependence.

Now, we evaluate the LRD case under different values of selected, following the same strategy as in our previous section.

Under our weaker choice of tempering, $\lambda = N^{-8}$, we present the convergence result using empirical CDF plots for the RS and KPSS statistics in Figures 21 and 23, while Figures 22 and 24 present density estimation and theoretical percentiles. On the left, we present the results for the original test statistic, and on the right, we present the results of introducing ^{d} to the statistic.

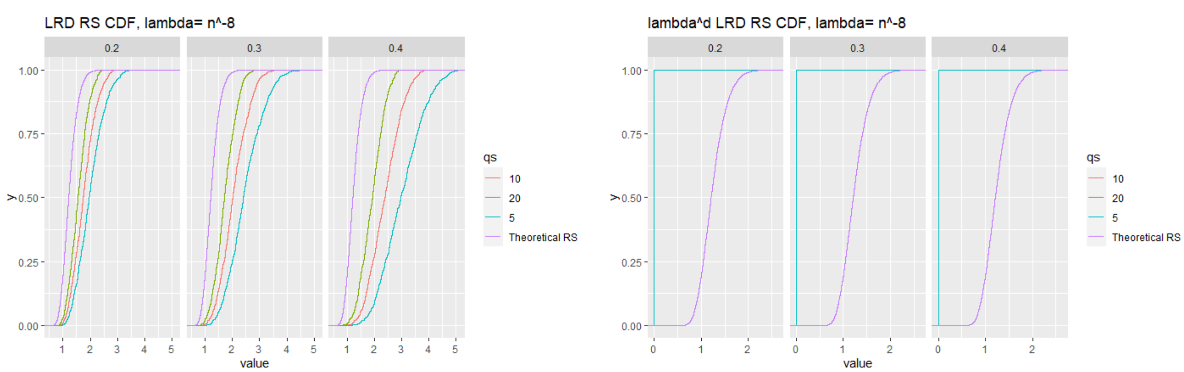


Figure 21: $R/S(q)/\sqrt{n}$ and $\lambda^d \times R/S(q)/\sqrt{n}$ CDF from simulations fig:LRD8_A

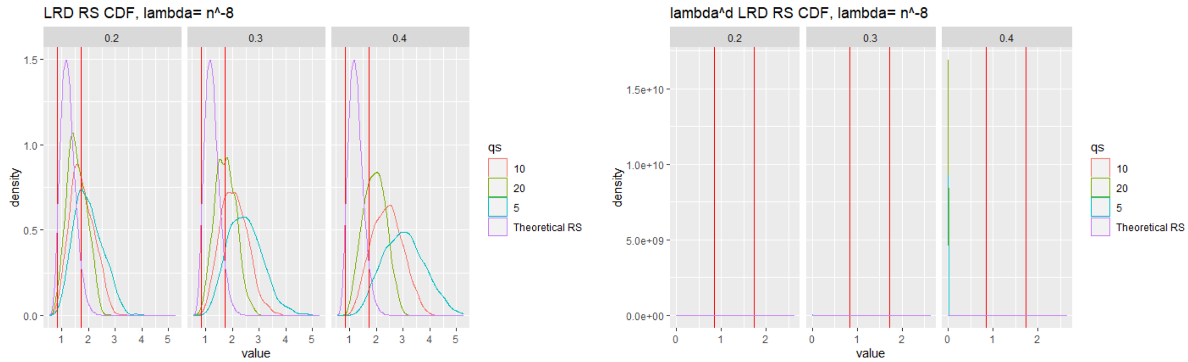


Figure 22: LRD RS density with Brownian Bridge 90% CI with $\lambda = N^{-8}$. The red lines represent the 90% CI for Theoretical results. fig:H3-CRS

Table 12: With $\lambda = N^{-8}, N = 1000$, the cumulative % for modified RS & λ^d RS for LRD will drop in the 90% CI of the Theoretical RS

bda_d_LRD1

q	d=0.2	d=0.3	d=0.4
5	35.5	11.6	2.7
10	50.3	27.8	12.4
20	66.7	49.3	32.7

q	d=0.2	d=0.3	d=0.4
5	0	0	0
10	0	0	0
20	0	0	0

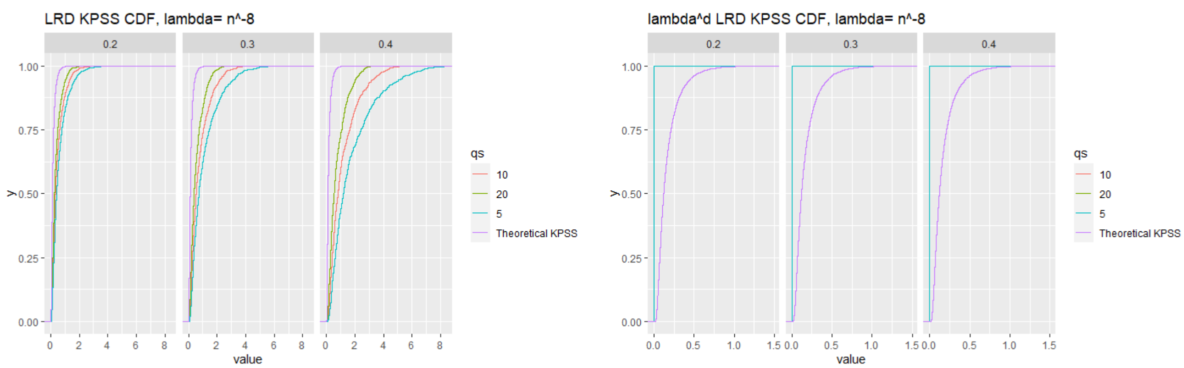


Figure 23: $KPSS$ and $\lambda^d \times KPSS$ CDF from simulations fig:LRD8_B

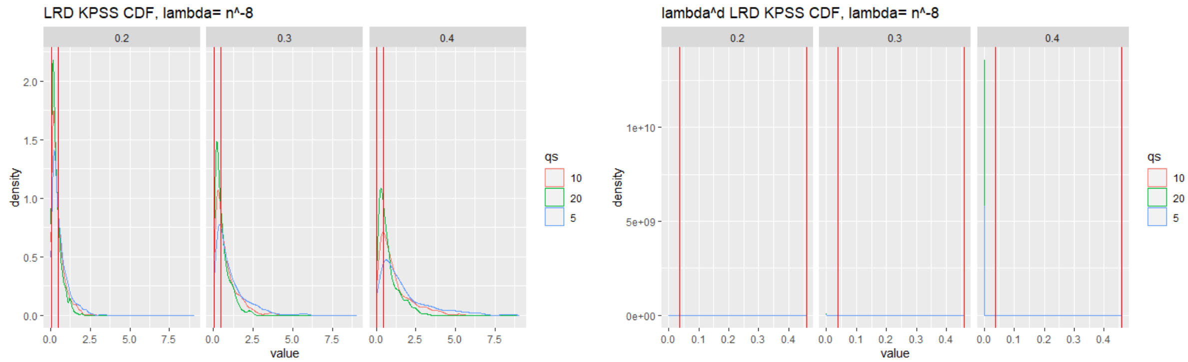


Figure 24: LRD KPSS density with Brownian Bridge 90% CI with $\lambda = N^{-8}$. The red lines represent the 90% CI for Theoretical results. fig:H3-CKPSS

Table 13: With $\lambda = N^{-8}, N = 1000$, the cumulative % for modified KPSS & λ^d KPSS for LRD will drop in the 90% CI of the Theoretical KPSS

bda_d_LRD2

q	d=0.2	d=0.3	d=0.4
5	53.5	32.1	16.5
10	61.6	43.8	28.4
20	70.1	55.0	43.4

q	d=0.2	d=0.3	d=0.4
5	0	0	0
10	0	0	0
20	0	0	0

Our results show that the difference between the simulations and the theoretical limiting distribution is significant in the case of the original statistic as expected. However, our results also show that the distribution obtained do not match with our expected theoretical results after introducing λ^d. The mass of the distribution is to the left of the expected outcome.We believe that $\lambda = N^{-8}$ may be too small, so next, we evaluate a stronger tempering setting $\lambda = N^{-1.1}$, which is still a weakly tempered process.

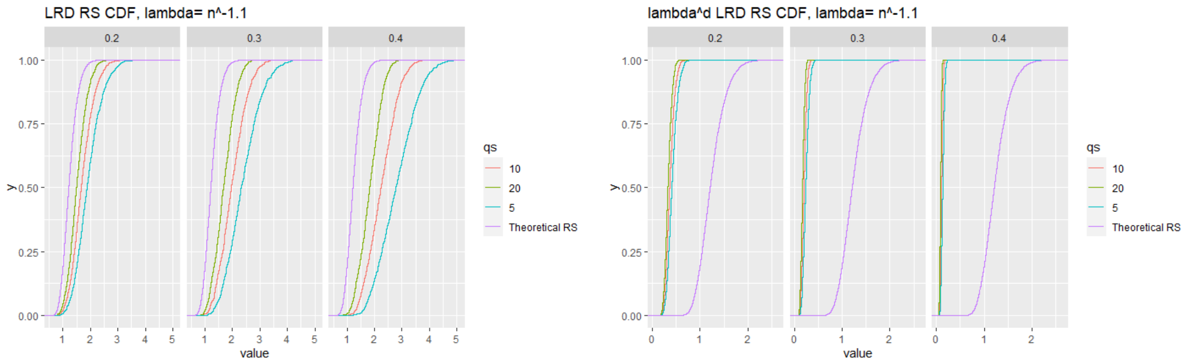


Figure 25: $R/S(q)/\sqrt{n}$ and $\lambda^d \times R/S(q)/\sqrt{n}$ CDF from simulations fig:LRD1.1_A

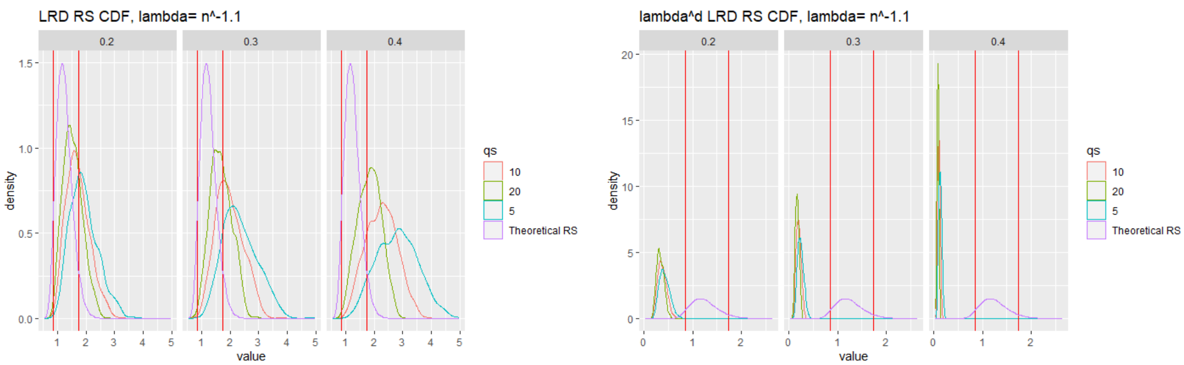


Figure 26: LRD RS density with Brownian Bridge 90% CI with $\lambda = N^{-1.1}$. The red lines represent the 90% CI for Theoretical results. fig:H3C

Table 14: With $\lambda = N^{-1.1}, N = 1000$, the cumulative % for modified RS & λ^d RS for LRD will drop in the 90% CI of the Theoretical RS

bda_d_LRD3

q	d=0.2	d=0.3	d=0.4
5	36.7	14.3	6.1
10	56.0	33.8	17.8
20	73.5	57.6	39.6

q	d=0.2	d=0.3	d=0.4
5	0	0	0
10	0	0	0
20	0	0	0

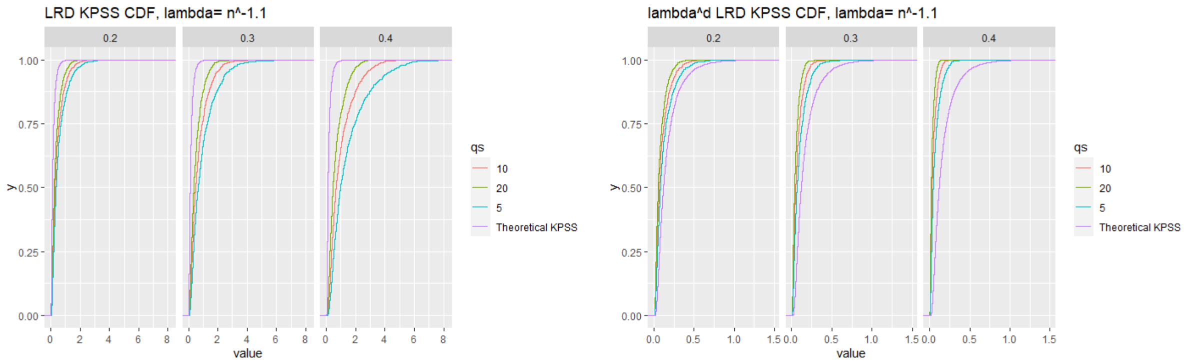


Figure 27: $KPSS$ and $\lambda^d \times KPSS$ CDF from simulations fig:LRD1.1_B

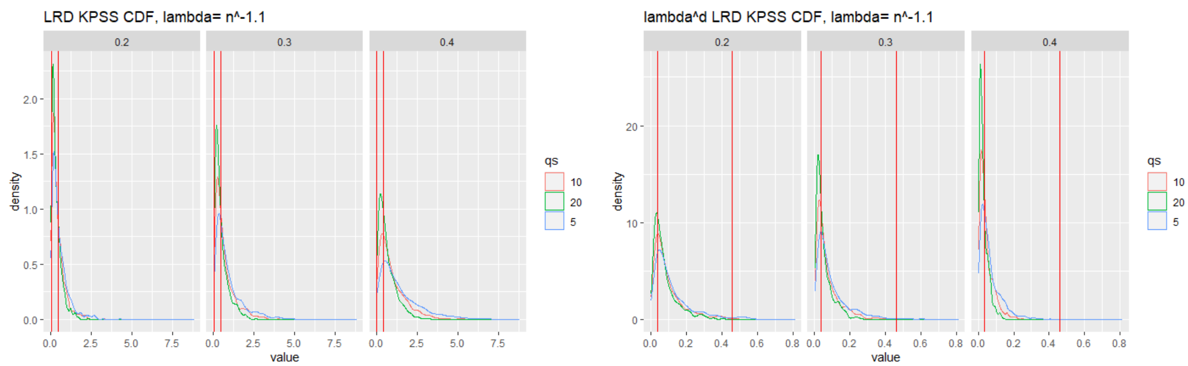


Figure 28: LRD KPSS density with Brownian Bridge 90% CI with $\lambda = N^{-1.1}$. The red lines represent the 90% CI for Theoretical results. fig:H3C

Table 15: With $\lambda = N^{-1.1}, N = 1000$, the cumulative % for modified KPSS & λ^d KPSS for LRD will drop in the 90% CI of the Theoretical KPSS

bda_d_LRD4

q	d=0.2	d=0.3	d=0.4
5	57.3	38.5	20.5
10	64.4	49.1	32.1
20	72.8	62.8	46.2

q	d=0.2	d=0.3	d=0.4
5	80.3	71.5	60.9
10	76.2	61.4	44.2
20	68.3	48.6	28.4

As before, we present our empirical CDF results in Figures 25 for RS and 27 for KPSS. Our results show that for this choice of λ our simulation results are closer to the theoretical expected values, and that the KPSS statistic is closer to its theoretical result than the R/S.

Since changing λ resulted in a closer distribution to the theoretical result, we want to see whether this improvement continues as $\lambda \to N^{-1}$. Therefore, we choose a new $\lambda = N^{-1.0001}$ and reproduce the R/S and KPSS CDF plots.

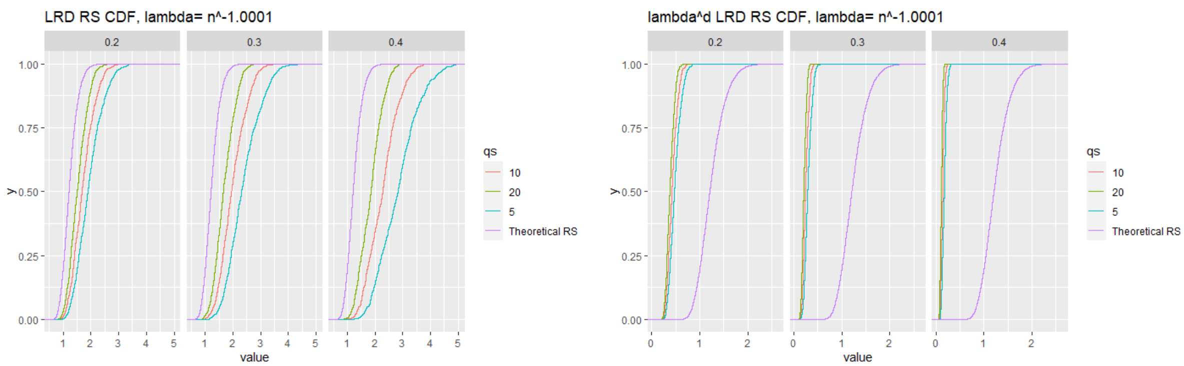


Figure 29: $R/S(q)/\sqrt{n}$ and $\lambda^d \times R/S(q)/\sqrt{n}$ CDF from simulations

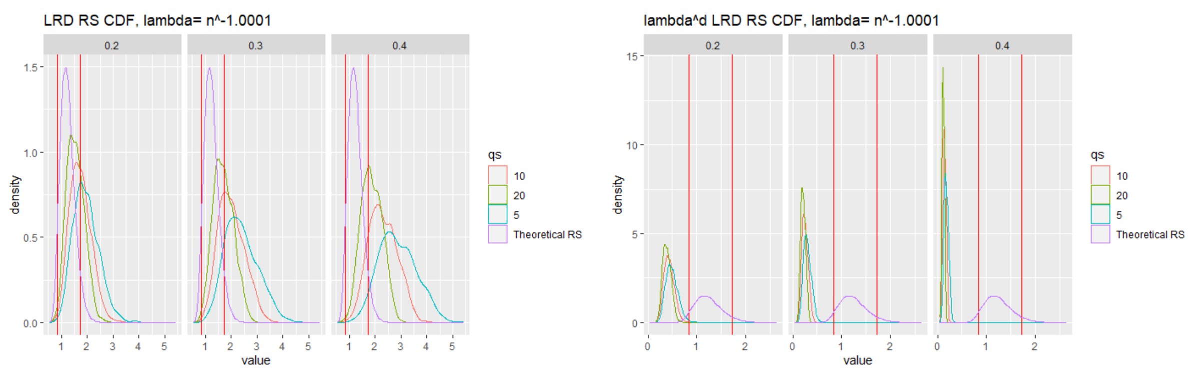


Figure 30: LRD RS density with Brownian Bridge 90% CI with $\lambda = N^{-1.0001}$. The red lines represent the 90% CI for Theoretical results. fig:H3D

Table 16: With $\lambda = N^{-1.0001}, N = 1000$, the cumulative % for modified RS & λ^d RS for LRD will drop in the 90% CI of the Theoretical RS

q	d=0.2	d=0.3	d=0.4
5	35.3	13.9	4.2
10	52.5	32.4	15.2
20	71.1	55.3	38.7

q	d=0.2	d=0.3	d=0.4
5	0.6	0	0
10	0	0	0
20	0	0	0

bda_d_LRD5

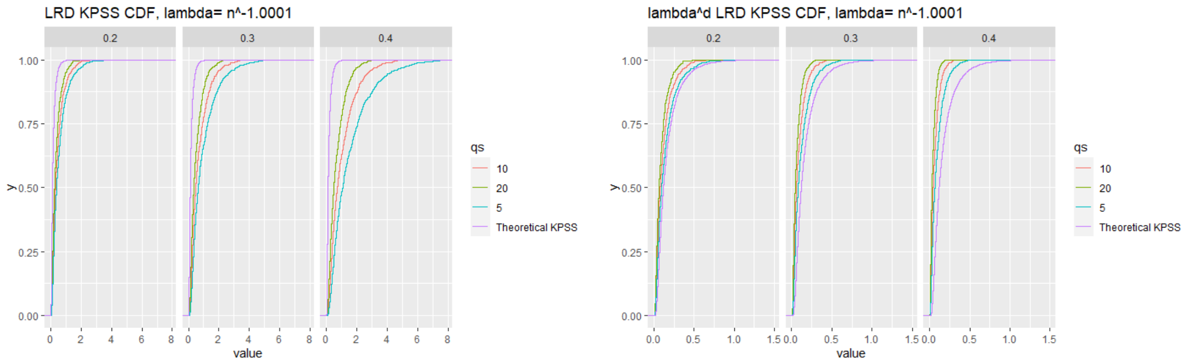


Figure 31: $KPSS$ and $\lambda^d \times KPSS$ CDF from simulations fig:LRD1.0001

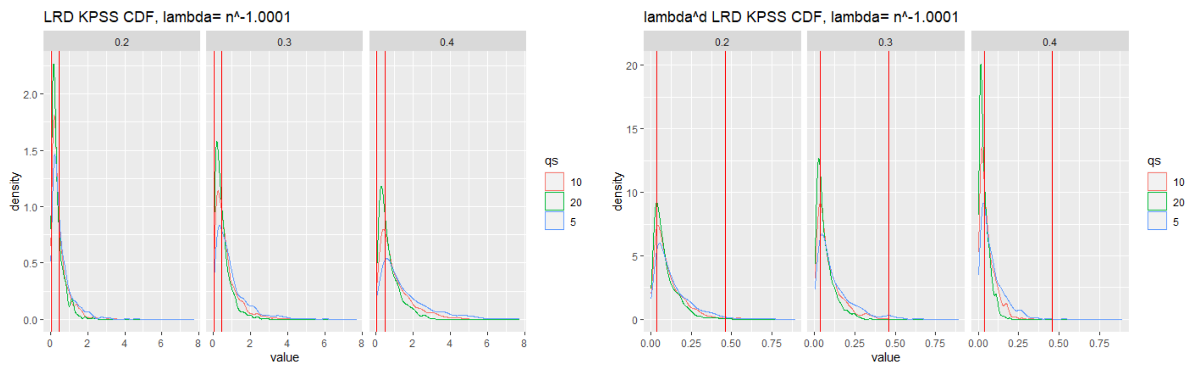


Figure 32: LRD KPSS density with Brownian Bridge 90% CI with $\lambda = N^{-1.0001}$. The red lines represent the 90% CI for Theoretical results. fig:H3D

Table 17: With $\lambda = N^{-1.0001}$, $N = 1000$, the cumulative % for modified KPSS & λ^d KPSS for LRD will drop in the 90% CI of the Theoretical KPSS

q	d=0.2	d=0.3	d=0.4
5	55.7	34.2	19.6
10	64.0	46.0	31.5
20	72.0	59.0	46.0

q	d=0.2	d=0.3	d=0.4
5	85.3	79.5	70.7
10	80.6	71.2	55.5
20	73.6	60.2	40.8

bda_d_LRD6

Improvements are difficult to observe from 27 and 31. However, based on the cumulative simulation results compared to the theoretical KPSS, 13, 15 and 17, it is rational to believe that LRD $\lambda^d KPSS$ will convergence to the theoretical KPSS with $\lambda_* \to c$. And for the q, different than the SRD, the smaller q, the more close to the theoretical KPSS. Therefor, we may think about the semi-long range dependence $\lambda_* = c$ cases with different q in future.

6.3.3 Empirical Power with λ parameter

ecc:h3Power

In section 6.3.1 we show the convergence in the case of SRD under strong tempering. Next we will use those test statistic to compute empirical size and power using the values of to check if they are suitable for testing procedures. Table 18 present the power calculation whille figure 33 summarize the results in a plot.

Our result show a similar pattern as in standar testing where the number of times we rejected the null decrese with q, however, as stated before we require to have a large enougth q to achieve the convergence in distribution. This can also be seen in the % of rejected cases under SRD, which is higher than expected for small values of q. As a trade off of increaseing q, we also note that would result in a reduce empirical power. Lastly we also note that contrary to our first $H^{(1)}$ result, the RS statistic appears to be more powerful.

Figure 33: Statistical power and size comparison at $\alpha = 0.1$

fig:PowerH3_d

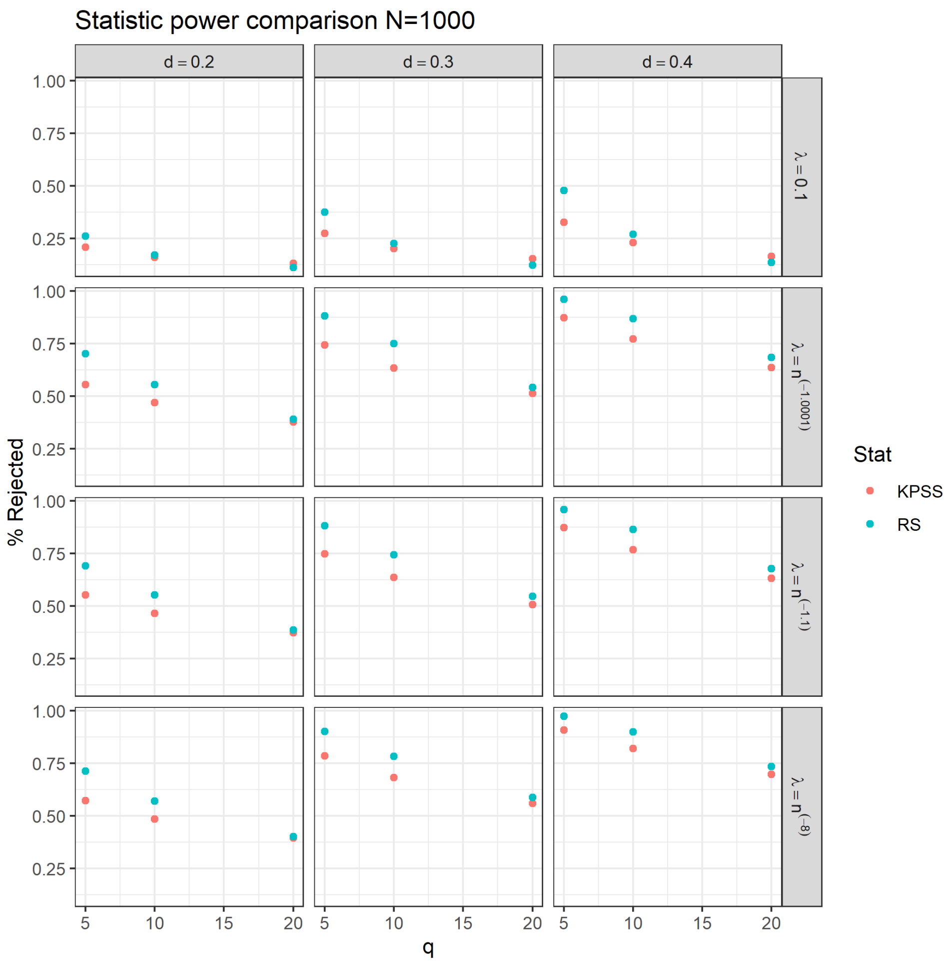


Table 18: Empirical size and power of the test as % for RS and KPSS under $N = 1000$. d and λ denotes the parameters for $ARTFIMA(0, d, \lambda, 0)$, and q the number of lags. Each cell is based on 10 000 replications, within d and λ the same sample is used for the different critical values.

tab:powerTabl

			RS						KPSS					
			$d = 0.2$		$d = 0.3$		$d = 0.4$		$d = 0.2$		$d = 0.3$		$d = 0.4$	
Type	λ	q	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%
SRD	0.1	5	16.57	26.15	26.02	37.69	35.45	47.99	12.3	20.89	17.4	27.48	21.84	32.79
		10	9.83	17.17	13.41	22.69	16.93	27.17	9.05	16.15	11.91	20.19	14.12	23.1
		20	5.23	11.19	6.08	12.33	6.72	13.66	6.82	13.17	8.09	15.34	8.75	16.48
LRD	$n^{(-1.0001)}$	5	59.58	70.21	81.95	88.09	92.98	95.96	42.91	55.5	63.43	74.28	79.47	87.19
		10	43.78	55.4	64.44	74.86	80.44	86.73	34.48	46.93	51.27	63.28	66.48	77.04
		20	27.17	38.97	41.58	54.02	57.57	68.41	25.86	37.67	38.39	51.16	51.25	63.42
	$n^{(-1.1)}$	5	58.55	69.06	81.49	88.16	92.59	95.75	43.05	55.21	63.84	74.81	78.99	87.32
		10	43.34	55.17	64.15	74.34	79.37	86.37	34.22	46.57	51.11	63.5	66.11	76.69
		20	26.32	38.64	41.8	54.54	56.33	67.78	25.87	37.35	38.44	50.71	50.45	63.13
	$n^{(-8)}$	5	61.08	71.39	84.8	90.17	95.06	97.4	45.3	57.27	68.55	78.54	84.21	90.87
		10	44.79	57.01	69.07	78.26	84.86	89.98	36.58	48.41	56.55	68.24	73.12	81.96
		20	28.7	40.15	46.72	58.81	63.63	73.48	28.03	39.5	43.67	56	57.65	69.75

7 Conclusion

Conclusions

This paper successfully reproduces previous work that dealt with testing for long-range dependence and proposes some conjectures to test hypotheses involving tempering parameters.

Reproducing the results is an essential first step since, during our work, we found some papers lacking in detail regarding how the process was simulated and different typos in their definition of the test statistics. Besides these shortcomings, we verify that simulations produced by `artsim` in the R package `artfima` behave as expected.

A lesser result was providing a critical value table, which we also found not being clearly established in other research papers. In particular, we deliver results for the fractional Brownian range. We expect these results will prove helpful in future work if conjecture regarding the distribution of the test statistic involves a fractional tempered process.

As in Giraitis, Kokoszka, Leipus, and Teyssiere 2003 we compute empirical power and size under for the SRD vs. LRD test, and by introducing positive values of λ, we assess what the author defines as contiguous scenarios. Under the presence of positive values of λ, the power decreases for all test statistics in accordance with the theory.

Next, we presented a conjecture for a long-range vs. semi long-range dependence test and assessed the convergence of the test statistics. Our simulation results show that for a particular set of λ, there is a discrepancy between the theoretical result and the distribution of our simulated statistics.

Lastly, we review the original test statistic but focus our hypothesis on values for λ_* given a positive memory parameter.

Under short range dependece, we first adrress our conjecture of implicit λ, the simulation result show that at least for the strong tempering process with d fixed in $(0, 1/2)$, the RS and KPSS statistic converge to the expected limiting distribution. Power calculation seems to agree with our initial assessment that larger values of q are required to obtain better results.

In the case of LRD, we found a discrepancy between the expected limiting distribution for the test stastic that incorporated λ^d .Our simulation results suggest that there is not enough evidence to support our conjecture for all values of λ. By modifying the values of λ we observed a trend in the case of the KPSS statistics. This is likely evidence for the different convergence conditions of the semi-long range dependence and long range dependence.

These results sugest that further reseacrh can be done to identify the relationship between λ and limiting distribution, especially for the case semi-long range dependence.

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