

Pricing Assets in Heterogeneous Agents Macro Economies

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December, 2023

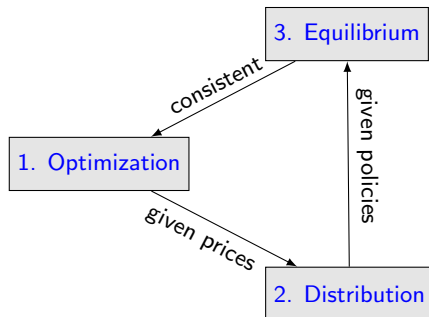
This Project

- ▶ We focus on **solving** macroeconomic models **not estimating** statistical models.
- ▶ We study continuous time DSGE models with the following features:
 1. A short-term risk free asset and a set of long-term risky assets,
 2. A large collection of price-taking agents who face idiosyncratic shocks/constraints
- ▶ We use deep learning to obtain a global solution.
- ▶ We are unaware of other global solution techniques for these models:
 1. *Traditional techniques*: are intractable unless the agent distribution can be approximated by low dimensional moments.
(e.g. [Krusell and Smith, 1998](#), [Fernández-Villaverde, Hurtado, and Nuno, 2023](#), [Kubler and Scheidegger, 2019](#))
 2. *Other deep learning techniques*: have struggled to solve models with wealth distribution, long-lived assets and complicated portfolio choice.
(e.g. [Duarte, 2018](#), [Gopalakrishna, 2021](#), [Sauzet, 2021](#), [Gu et al., 2023](#), [Fernández-Villaverde, Hurtado, and Nuno, 2023](#), [Huang, 2023](#), [Azinovic and Žemlička, 2023](#))

Three blocks

General equilibrium for this economy can be characterized by three blocks

1. **Optimization block:** High but finite dimensional PDE capturing agents optimization over consumption, portfolio holdings taking prices as given
2. **Distribution block:** High but finite dimensional law of motion of agent's wealth
3. **Equilibrium block:** Ensures that price processes are consistent with equilibrium



This is the first paper that can solve models with all three non-trivial blocks.

Roadmap

Illustrative model

Comparison to other models

Solution algorithm

Solutions to example models

Under the hood

General Equilibrium Endowment Asset Pricing Model

- ▶ Time is continuous with infinite horizon.
- ▶ Perishable consumption good created by a Lucas tree that produces output:

$$dy_t = \mu_y(y_t)dt + \sigma_y(y_t)dW_t$$

where W_t is a standard Brownian motion.

- ▶ Populated by price-taking households with flow utility $u(c_{i,t})$.
- ▶ Competitive markets for goods, risk-free bonds in net zero supply (with price r_t), and equity shares in Lucas tree (with price q_t). The price process follows

$$dq_t = \mu_t^q q_t dt + \sigma_t^q q_t dW_t$$

- ▶ **Financial frictions** restrict the agents' choice by $\Psi(a_{i,t}, b_{i,t}, \kappa_i) \geq 0$

- ▶ $a_{i,t}$ is agent wealth, $b_{i,t}$ is agent bond holdings.
- ▶ E.g. borrowing constraint is $a_{i,t} - \kappa_i \geq 0$.
- ▶ E.g. equity non-participation constraint is $a_{i,t} - b_{i,t} = 0$.
- ▶ Typically easier to model as penalty $\psi(a_{i,t}, b_{i,t}, \kappa_i) = |\Psi(a_{i,t}, b_{i,t}, \kappa_i)|^2$.

Optimization and Equilibrium

Given the belief about price processes \hat{r}, \hat{q} , agent i solves

$$\begin{aligned} \max_{c,b} \left\{ \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} (u(c_{i,t}) + \psi(a_{i,t}, b_{i,t}, \kappa_i)) dt \right] \right\} \\ \text{s.t.} \quad da_{i,t} = b_{i,t} \hat{r}_t - c_{i,t} + \left(\mu_t^q + \frac{y_t}{\hat{q}_t} \right) (a_{i,t} - b_{i,t}) + (a_{i,t} - b_{i,t}) \sigma_t^q dW_t \end{aligned} \quad (1)$$

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Equilibrium:

1. Individual i 's consumption decision c_t^i and bond holding b_t^i solves problem (1), given their believed price process (\hat{r}, \hat{q}) ;
2. Equilibrium prices (q_t, r_t) solves market clearing conditions: (i) goods market $\sum_i c_{i,t} = y$, (ii) stock market $\sum_i (a_{i,t} - b_{i,t}) = q_t$ and (iii) bond market $\sum_i b_{i,t} = 0$.
3. Agent beliefs about the price process are consistent with the optimal behaviour of other agents in the sense that $(\hat{r}, \hat{q}) = (r, q)$.

Recursive Characterization of Equilibrium

Individual state = a_i , Aggregate states = $(y, \{a_j\}_{j \neq i}) = (\cdot)$.

Given belief about evolution of other agents, $(\hat{\mu}_{a_j}(\cdot), \hat{\sigma}_{a_j}(\cdot))$, agent i solves:

$$\begin{aligned} \rho V_i(a_i, \cdot) = & \max_{c_i, b_i} \left\{ u(c_i) + \psi(a_i, s_i, \kappa_i) + \frac{\partial V_i}{\partial a_i} \mu_{a_i}(c_i, b_i, \cdot) + \frac{\partial V_i}{\partial y} \mu_y \right. \\ & + \frac{1}{2} \frac{\partial^2 V_i}{\partial a_i^2} \sigma_{a_i}^2(b_i, \cdot) + \frac{1}{2} \frac{\partial^2 V_i}{\partial y^2} \sigma_y^2 + \frac{\partial^2 V_i}{\partial a_i \partial y} \sigma_{a_i}(b_i, \cdot) \sigma_y + \sum_{j \neq i} \frac{\partial V_i}{\partial a_j} \hat{\mu}_{a_j}(\cdot) \\ & \left. + \frac{1}{2} \sum_{j \neq i, j' \neq i} \frac{\partial^2 V_i}{\partial a_j \partial a_{j'}} \hat{\sigma}_{a_j}(\cdot) \hat{\sigma}_{a_{j'}}(\cdot) + \sum_{j \neq i} \frac{\partial^2 V_i}{\partial a_j \partial y} \hat{\sigma}_{a_j}(\cdot) \sigma_y + \sum_{j \neq i} \frac{\partial^2 V_i}{\partial a_i \partial a_j} \sigma_{a_i}(b_i, \cdot) \hat{\sigma}_{a_j}(\cdot) \right\} \end{aligned}$$

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FOCs after defining $\xi_i := \frac{\partial V_i}{\partial a_i}$, and applying Ito's lemma to ξ_i are

$$\xi_i = u'(c_i) \quad \xi_i(r - r^q) = \sigma_{\xi_i} \sigma^q + (\partial \psi_i / \partial b_i)$$

In equilibrium, beliefs are consistent: $(\hat{\mu}_{a_j}(\cdot), \hat{\sigma}_{a_j}(\cdot)) = (\mu_{a_j}(\cdot), \sigma_{a_j}(\cdot))$

Imposing Equilibrium

We use aggregate state $(y, \{\eta_i\}_{1 \leq i \leq I})$, where $\eta_i := a_i/A$ is agent i 's wealth share.

Once equilibrium is imposed, we know ξ_i is a function of $(y, \{\eta_i\}_{1 \leq i \leq I})$

...so we can write μ_{ξ_i} and σ_{ξ_i} in terms of derivatives of $(y, \{\eta_i\}_{1 \leq i \leq I})$.

We group the resulting equilibrium equations into three blocks.

Block-1

1. Optimization

- ▶ Given prices $(r, r_q, q, \mu_q, \sigma_q)$, agent optimization implies:

Consumption: $\xi_i = u'(c_i)$

Euler equations:

$$\rho \xi_i = r \xi_i + \sum_j \frac{\partial \xi_i}{\partial \eta_j} \mu_{j,\eta} + \frac{\partial \xi_i}{\partial y} \mu_y + \frac{1}{2} \sum_{j,j'} \frac{\partial^2 \xi_i}{\partial \eta_j \partial \eta_{j'}} \sigma_{j,\eta} \sigma_{j',\eta} + \frac{1}{2} \frac{\partial^2 \xi_i}{\partial y^2} \sigma_y^2 + \sum_j \frac{\partial^2 \xi_i}{\partial \eta_j \partial y} \sigma_{j,\eta} \sigma_y + \frac{\partial \psi_i}{\partial a_i}, \quad \text{for all } i$$

Asset pricing: $\xi_i(r - r_q) = \left(\sum_j \frac{\partial \xi_i}{\partial \eta_j} \sigma_{j,\eta} + \frac{\partial \xi_i}{\partial y} \sigma_y \right) \sigma_q + \frac{\partial \psi_i}{\partial b_i}$

Block-2

2. Distribution

Given the prices $(r, r_q, q, \mu_q, \sigma_q)$ and (c, ξ, b) , the law of motion of wealth shares is given as

$$\begin{aligned}d\eta_{j,t} &= \mu_{j,\eta,t}dt + \sigma_{j,\eta,t}dW_t \\ \mu_{j,\eta,t} &= \frac{y_t}{q_t} + \frac{b_{j,t}}{a_{j,t}}(r_t - r_{q,t}) - \frac{(u')^{-1}(\xi_{j,t})}{\eta_{j,t}q_t} + \frac{b_{j,t}}{a_{j,t}}(\sigma_t^q)^2 \\ \sigma_{j,\eta,t} &= \left[\frac{1}{a_j} (a_{j,t} - b_{j,t}) \sigma_t^q - \sigma_t^q \right] = -\frac{b_{j,t}}{a_{j,t}} \sigma_t^q\end{aligned}$$

Block-3

3. Equilibrium

In equilibrium, the following market clearing conditions must hold:

$$\sum_i b_i = 0 \qquad \sum_i \eta_i = 1 \qquad \sum_i c_i = y$$

Clearing conditions pin down the prices but LOM of price, q , is implicit.

We impose consistency conditions on the LOM of the price q to close the model:

$$q\mu^q = \sum_j \frac{\partial q}{\partial \eta_j} \mu_{j,\eta} + \frac{\partial q}{\partial y} \mu_y + \frac{1}{2} \sum_{j,j'} \frac{\partial^2 q}{\partial \eta_j \partial \eta_{j'}} \sigma_{j,\eta} \sigma_{j',\eta} + \sum_j \frac{\partial^2 q}{\partial \eta_j \partial y} \sigma_{j,\eta} \sigma_y + \frac{1}{2} \frac{\partial^2 q}{\partial y^2} \sigma_y^2$$
$$q\sigma^q = \sum_j \frac{\partial q}{\partial \eta_j} \sigma_{j,\eta} + \frac{\partial q}{\partial y} \sigma_y$$

Roadmap

Illustrative model

Comparison to other models

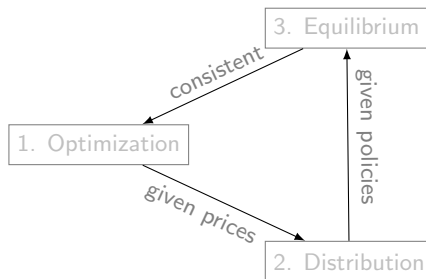
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Under the hood

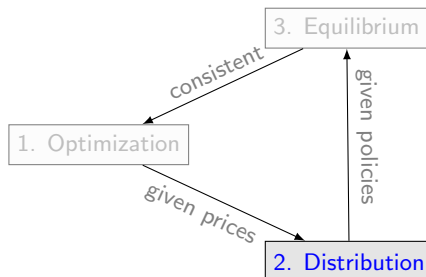
Comparison to other models

Models	Blocks			Method
	1	2	3	
Representative Agent (à la Lucas, 1978)	simple	NA	simple	Finite difference
Heterogeneous Agents (à la Krusell and Smith, 1998)	simple	✓	simple	Gu et al., 2023
Long-lived assets (à la Brunnermeier and Sannikov, 2014)	closed-form	low-dim	✓	Gopalakrishna, 2021
HA + Long-lived assets	✓	✓	✓	This paper



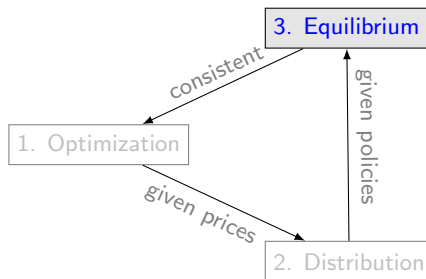
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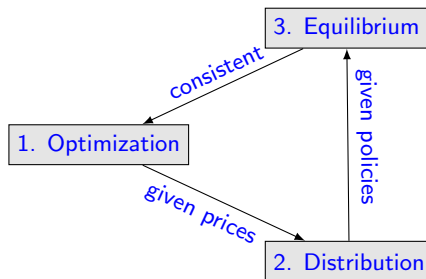
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Model summary

- Need to solve a system of PDEs + algebraic equations for (ξ_i, q) .

$$\begin{aligned} \text{Euler eqns.:} \quad 0 &= (r - \rho)\xi_i + \sum_j \frac{\partial \xi_i}{\partial \eta_j} \mu_{j,\eta} + \frac{\partial \xi_i}{\partial y} \mu_y + \frac{1}{2} \sum_{j,j'} \frac{\partial^2 \xi_i}{\partial \eta_j \partial \eta_{j'}} \sigma_{j,\eta} \sigma_{j',\eta} \\ &\quad + \frac{1}{2} \frac{\partial^2 \xi_i}{\partial y^2} \sigma_y^2 + \sum_j \frac{\partial^2 \xi_i}{\partial \eta_j \partial y} \sigma_{j,\eta} \sigma_y + \frac{\partial \psi_i}{\partial a_i} \end{aligned}$$

$$\text{Price:} \quad q = \frac{y}{\sum_i \omega_i \eta_i}$$

$$\begin{aligned} \text{Consistency-1: } q\mu^q &= \sum_j \frac{\partial q}{\partial \eta_j} \mu_{j,\eta} + \frac{\partial q}{\partial y} \mu_y + \frac{1}{2} \sum_{j,j'} \frac{\partial^2 q}{\partial \eta_j \partial \eta_{j'}} \sigma_{j,\eta} \sigma_{j',\eta} \\ &\quad + \frac{1}{2} \frac{\partial^2 q}{\partial y^2} \sigma_y^2 + \sum_j \frac{\partial^2 q}{\partial \eta_j \partial y} \sigma_{j,\eta} \sigma_y \end{aligned}$$

$$\text{Consistency-2: } q\sigma^q = \sum_j \frac{\partial q}{\partial \eta_j} \sigma_{j,\eta} + \frac{\partial q}{\partial y} \sigma_y$$

where $\omega_i := c_i/a_i$ is the consumption-wealth ratio of agent i .

Neural networks

We can parameterize functions (ξ_i, q) as neural network objects. Why?

- ▶ Automatic differentiation can be used for derivatives, and
- ▶ High-dimensional non-linear optimizers have been developed for neural networks

Let's start with the Euler equation of agent i and the unknown function ξ_i

$$\begin{aligned} 0 = & (r - \rho)\xi_i + \sum_j \frac{\partial \xi_i}{\partial \eta_j} \mu_{j,\eta} + \frac{\partial \xi_i}{\partial y} \mu_y + \frac{1}{2} \sum_{j,j'} \frac{\partial^2 \xi_i}{\partial \eta_j \partial \eta_{j'}} \sigma_{j,\eta} \sigma_{j',\eta} \\ & + \frac{1}{2} \frac{\partial^2 \xi_i}{\partial y^2} \sigma_y^2 + \frac{1}{2} \sum_j \frac{\partial^2 \xi_i}{\partial \eta_j \partial y} \sigma_{j,\eta} \sigma_y + \frac{\partial \psi_i}{\partial \mathbf{a}_i} \end{aligned}$$

- ▶ This is a high-dimensional PDE.
- ▶ In our model, the evolution of price function q also follows a PDE of the type above (the consistency conditions)

Neural networks

One way to solve this PDE is to parameterize the unknown function ξ_i as a neural network object $\hat{\xi}_i$ with parameters Θ_ξ .

The approximated function $\hat{\xi}_i$ takes as input a vector of training samples $\mathbf{X}^n = (y^n, \{\eta_i\}^n)_{n=1}^N$ drawn uniformly from the state space $\mathbf{X} := (y, \{\eta_i\}_{1 \leq i \leq I})$.

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Then the Euler equation residual (or PDE residual) can be computed as

$$\begin{aligned} f_i := & (r - \rho)\hat{\xi}_i + \sum_j \frac{\partial \hat{\xi}_i}{\partial \eta_j} \mu_{j,\eta} + \frac{\partial \hat{\xi}_i}{\partial y} \mu_y + \frac{1}{2} \sum_{j,j'} \frac{\partial^2 \hat{\xi}_i}{\partial \eta_j \partial \eta_{j'}} \sigma_{j,\eta} \sigma_{j',\eta} \\ & + \frac{1}{2} \frac{\partial^2 \hat{\xi}_i}{\partial y^2} \sigma_y^2 + \frac{1}{2} \sum_j \frac{\partial^2 \hat{\xi}_i}{\partial \eta_j \partial y} \sigma_{j,\eta} \sigma_y + \frac{\partial \psi_i}{\partial a_i} = 0 \end{aligned}$$

The Euler equation loss is given by $\frac{1}{N} \sum_{i=1}^N |f_i(\mathbf{X}^n)|$.

We can similarly compute the loss on consistency and goods market clearing conditions.

Parsimonious parameterization

Instead of parameterizing ξ_i and q , we parameterize the consumption-wealth ratio ω_i as $\hat{\omega}_i$ with Θ_ω and draw training sample \mathbf{X}^n

- ▶ Price q from the equilibrium condition $q = \frac{y}{\sum_i \hat{\omega}_i \eta_i}$
- ▶ Marginal value of wealth $\xi_i = u'(\hat{\omega}_i \eta_i q)$

We proceed to follow similar steps as before to compute Euler equation loss where the residual is computed as

$$\begin{aligned} f_i := & (r - \rho)\xi_i(\hat{\omega}_i) + \sum_j \frac{\partial \xi_i(\hat{\omega}_i)}{\partial \eta_j} \mu_{j,\eta} + \frac{\partial \xi_i(\hat{\omega}_i)}{\partial y} \mu_y + \frac{1}{2} \sum_{j,j'} \frac{\partial^2 \xi_i(\hat{\omega}_i)}{\partial \eta_j \partial \eta_{j'}} \sigma_{j,\eta} \sigma_{j',\eta} \quad (2) \\ & + \frac{1}{2} \frac{\partial^2 \xi_i(\hat{\omega}_i)}{\partial y^2} \sigma_y^2 + \sum_j \frac{\partial^2 \xi_i(\hat{\omega}_i)}{\partial \eta_j \partial y} \sigma_{j,\eta} \sigma_y + \frac{\partial \psi_i}{\partial \mathbf{a}_i} = 0 \end{aligned}$$

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$$+ \frac{1}{2} \frac{\partial^2 \xi_i(\hat{\omega}_i)}{\partial y^2} \sigma_y^2 + \sum_j \frac{\partial^2 \xi_i(\hat{\omega}_i)}{\partial \eta_j \partial y} \sigma_{j,\eta} \sigma_y + \frac{\partial \psi_i}{\partial \mathbf{a}_i} = 0$$

- ▶ The derivatives can still be easily computed using automatic differentiation, which leads to the same Euler equation loss as before.
- ▶ With parsimonious parameterization, we don't require goods market clearing condition loss

Algorithm

Algorithm Pseudo Code

- 1: Initialize neural network objects $\{\hat{\omega}_i\}_{1 \leq i \leq I}$ with parameters $\{\Theta_{\omega_i}\}_{1 \leq i \leq I}$.
 - 2: Initialize optimizer.
 - 3: **while** Loss > tolerance, at iteration t **do**
 - 4: Sample training points: $\mathbf{X}^n = (y^n, \{\eta_i\}^n)_{n=1}^N$.
 - 5: Compute $q^n = y^n / (\sum_i \hat{\omega}_i(\mathbf{X}^n) \eta_i^n)$, $c_i^n = \hat{\omega}_i(\mathbf{X}^n) \eta_i^n q^n$, and $\xi_i^n = u'(c_i^n)$ for each i
 - 6: Determine $\sigma_{\boldsymbol{\eta}}$, $r_q - r$ and then $\sigma^q, \mu^q, \mu_{\boldsymbol{\eta}}$.
 - 7: Construct loss as $\sum_i \frac{1}{N} \sum_n |f_i(\mathbf{X}^n)|$, where f_i is defined in (2).
 - 8: Update $\{\Theta_{\omega_i}\}$ using ADAM optimizer.
 - 9: **end while**
-

Approach Q & A

- ▶ **Q.** *What about imposing dimension reduction?*
 - ▶ Han, Yang, and E, 2021 and Kahou et al., 2021 suggest feeding the distribution through a preliminary neural network that reduces the dimension.
 - ▶ We have solved the model **without this approach** but agree this could be useful.
- ▶ **Q.** *Neural networks are slow to train so how do we calibrate?*
 - ▶ Could add parameters to neural network so we only have to train the model once.
 - ▶ Let ζ denote economic parameters needing external calibration to match moments.
 - ▶ Add ζ as inputs into the neural network:

$$V(\hat{X}_t, \zeta) \approx \hat{V}(\hat{X}_t, \zeta; \theta), \quad \hat{X}_t = \{x_t, z_t, \hat{g}_t\}$$

- ▶ Train the neural network using sampling from \hat{X} and ζ .
- ▶ Use the equilibrium \hat{V} to calculate moments for different parameters.

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Under the hood

Representative Agent Model (Lucas (1978))

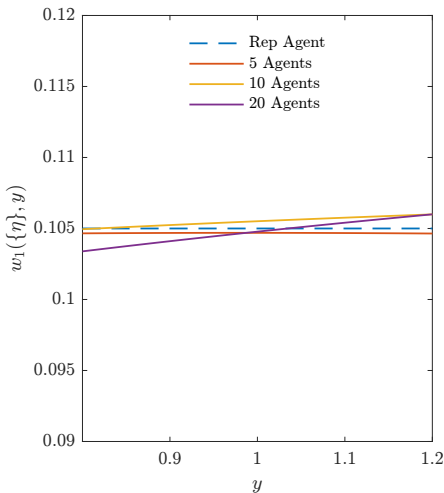
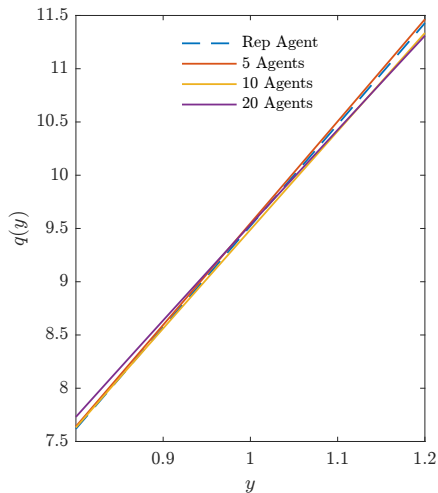
Suppose there are no financial frictions: $\psi(a_{i,t}, b_{i,t}, \kappa_i) = 0$.

In this case, households are identical so there is a “representative agent”.

The model has closed form solution:

$$q(y) = \frac{y}{\rho + (\gamma - 1)\mu - \frac{1}{2}\gamma(\gamma - 1)\sigma^2}$$
$$\omega(y) = \left[\rho + (\gamma - 1)\mu - \frac{1}{2}\gamma(\gamma - 1)\sigma^2 \right]$$

Model solution. MSE: $< 10^{-4}$



As-if Complete Market Model, $\gamma = 5$, $\mu = 2\%$, $\sigma = 5\%$, $\rho = 5\%$.

Limited Participation Model (Basak and Cuoco, 1998)

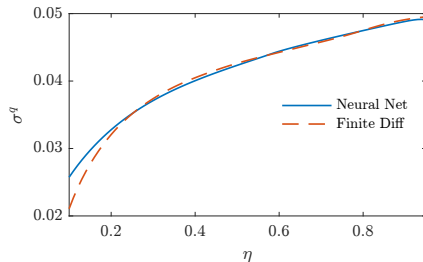
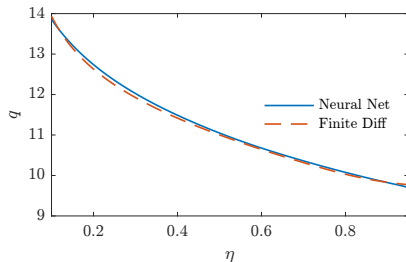
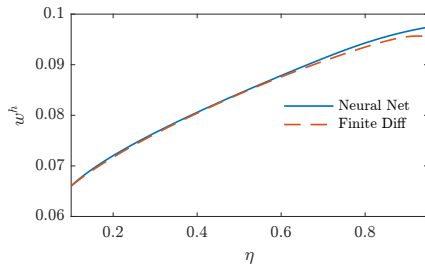
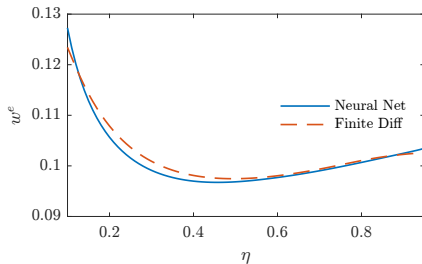
Two types of agents: experts (e) and households (h).

Expert sector can hold stocks and bonds.

Household sector can only hold bonds: $\Psi_h(a_{h,t}, b_{h,t}) = a_{h,t} - b_{h,t} = 0$.

State space is (y, η) , where η is expert's wealth share.

Model solution. L2 Loss: $< 10^{-5}$



2 Agents Limited Participation Model, $\gamma = 5, \rho_e = \rho_h = 5\%, \mu = 2\%, \sigma = 5\%$.

Production Economy With Idiosyncratic Risk

Production technology:

$$y_t = z_t K_t^\alpha L_t^{1-\alpha}$$

Capital accumulation (investment friction $\Phi(\iota) = \frac{1}{\phi} \log(\phi\iota + 1)$):

$$dK_t = (\Phi(\iota_t) - \delta) K_t dt$$

Dividend price ratio:

$$\frac{d_t}{q_t} = \frac{\alpha y_t - \iota_t K_t}{q_t K_t}$$

Labor endowment (Poisson switching):

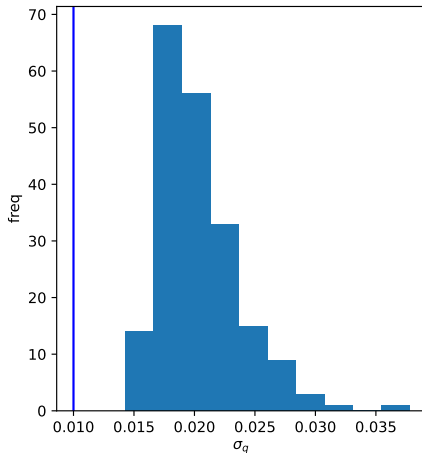
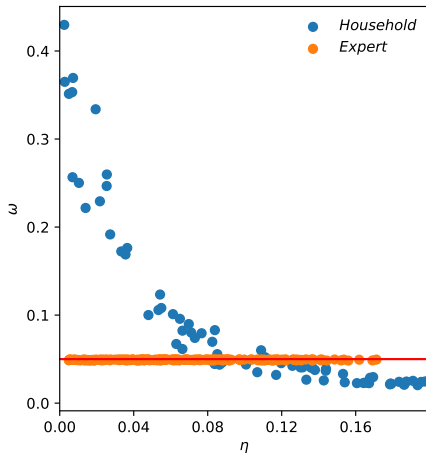
$$\ell_i \in \{\underline{\ell}, \bar{\ell}\}$$

Households wealth process:

$$da_{i,t} = [\hat{w}_t \ell_{i,t} + a_{i,t} \hat{r}_t - c_{i,t}] dt$$

Expert's wealth process: the same as before

Model solution. L2 loss: $< 10^{-4}$



11 Agents Limited Participation model with Production: $\gamma = 1, \sigma = 1\%$,
 $\alpha = 0.3, \phi = 10, \delta = 0.05, \lambda_1 = \lambda_2 = 0.4, \underline{\ell} = 0.3, \bar{\ell} = 1.7, \rho_e = \rho_h = 5\%$

Roadmap

Illustrative model

Comparison to other models

Solution algorithm

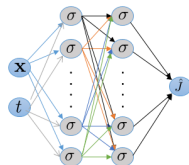
Solutions to example models

Under the hood

Automatic differentiation in practice

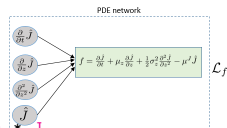
Approximating J using a neural network

```
def J(z,t):
    J = neural_net(tf.concat([z,t],1),weights,biases)
    return J
```



Constructing regularizer: 1D model

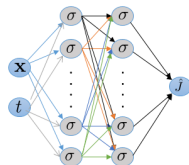
```
def f(z,t):
    J = J(z,t)
    J_t = tf.gradients(J,t)[0]
    J_z = tf.gradients(J,z)[0]
    J_zz = tf.gradients(J_z,z)[0]
    f = J_t + advection * J_z + diffusion * J_zz - linearTerm * J
    return f
```



Automatic differentiation in practice

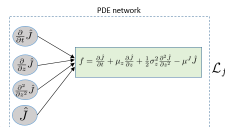
Approximating J using a neural network

```
def J(z,t):
    J = neural_net(tf.concat([z,t],1),weights,biases)
    return J
```

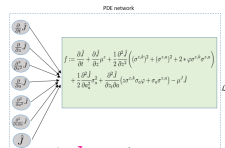


Constructing regularizer: 1D model

```
def f(z,t):
    J = J(z,t)
    J_t = tf.gradients(J,t)[0]
    J_z = tf.gradients(J,z)[0]
    J_zz = tf.gradients(J_z,z)[0]
    f = J_t + advection * J_z + diffusion * J_zz - linearTerm
    return f
```



```
def f(z,a,t):
    J = J(z,a,t)
    J_t = tf.gradients(J,t)[0]
    J_z = tf.gradients(J,z)[0]
    J_a = tf.gradients(J,a)[0]
    J_zz = tf.gradients(J_z,z)[0]
    J_aa = tf.gradients(J_a,a)[0]
    J_az = tf.gradients(J_a,z)[0]
    f = J_t + advection_z * J_z + advection_a * J_a + diffusion_z * J_zz +
        diffusion_a * J_aa + crossTerm * J_az - linearTerm * J
    return f
```



Conclusion

We have used deep learning to solve continuous time DSGE models with:






1. A short-term risk free asset and a set of long-term risky assets,
2. A large collection of price-taking agents who face idiosyncratic shocks/constraints

We believe this will allow the development of heterogeneous agent macro-finance.







Our current project uses this to solve a model of institutional asset pricing.

Thank You!

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Appendix: FOCs

FOCs are

$$\begin{aligned} [c_i] : \quad 0 &= u'(c_i) - \partial_a V_i(a_i) \\ [b_i] : \quad 0 &= -\frac{\partial V_i}{\partial a_i} (r(\cdot) - r_q(\cdot)) + \frac{\partial^2 V_i}{\partial a_i^2} (a_i - b_i)(\sigma^q)^2(\cdot) \\ &\quad + \frac{\partial^2 V_i}{\partial a_i \partial y} \sigma^q(\cdot) \sigma_y(\cdot) + \sum_{j \neq i} \frac{\partial^2 V_i}{\partial a_i \partial a_j} \sigma^q(\cdot) \hat{\sigma}_{a_j}(\cdot) + \frac{\partial \psi}{\partial b_i} \end{aligned} \quad (3)$$

► Back

Appendix: FOCs

$$\begin{aligned}[c_i] : \quad 0 &= u'(c_i) - \xi_i \\[b_i] : \quad 0 &= -\xi_i (r(\cdot) - r_q(\cdot)) + \frac{\partial \xi_i}{\partial a_i} (a_i - b_i) (\sigma^q)^2(\cdot) \\ &\quad + \frac{\partial^2 \xi_i}{\partial y^2} \sigma_y(\cdot)^2 + \frac{\partial \xi_i}{\partial y} \sigma^q(\cdot) \sigma_y(\cdot) + \sum_{j \neq i} \frac{\partial \xi_i}{\partial a_j} \sigma^q(\cdot) \hat{\sigma}_{a_j}(\cdot) + \frac{\partial \psi}{\partial b_i}\end{aligned} \tag{4}$$

Rearranging the asset pricing equation and using

$$\sigma_{\xi_i} = \frac{\partial \xi_i}{\partial a_i} \sigma_{a_i} + \frac{\partial \xi_i}{\partial y} \sigma_y + \sum_{j \neq i} \frac{\partial \xi_i}{\partial a_j} \sigma_{a_j}$$

we get

$$\xi_i (r - r_q) = \sigma_{\xi_i} \sigma^q + \frac{\partial \psi_i}{\partial b_i}$$

► Back

Appendix: Grid sampling vs Simulation method

- ▶ Our method draws training points from throughout the state space
- ▶ Sampling procedure is complementary to simulation based methods ([Azinovic et al \(2018\)](#), [Villaverde et al \(2020\)](#)), but also works for models with **rare events** and financial constraints that bind far away from the steady state

