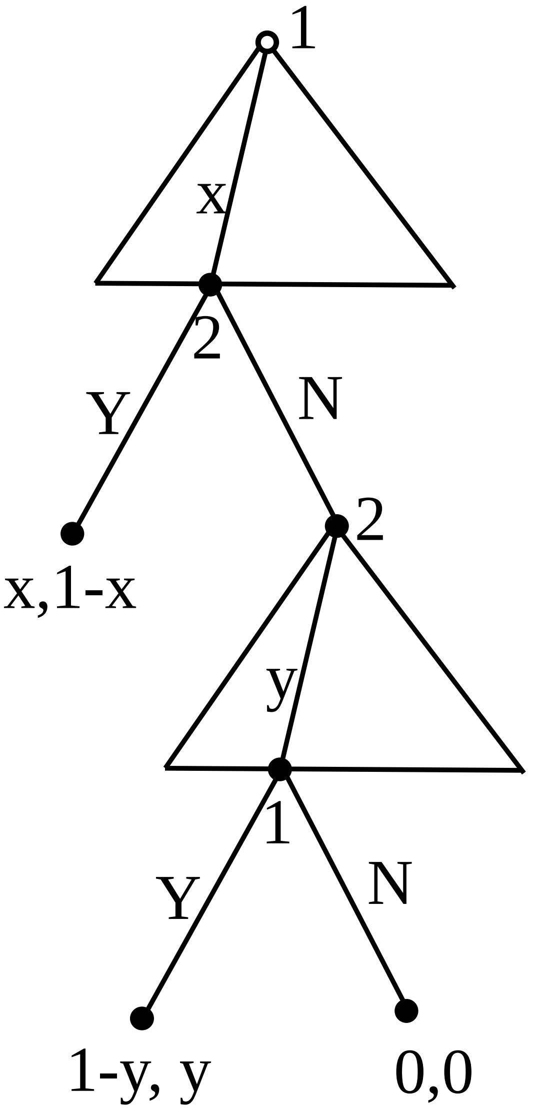
Chapter 8: Stackelberg Game

- Applications to Security

Recap on previous chapter

- The existence and computation of SPE
- Back induction
- One deviation principle
- Formal definition of sub-game
- Bargaining game



Supplements for Nash Equilibrium

Payoffs Matrix

Player 2

Player 1

	X	Y
U	6 6	0 -2
D	7 1	0 -2

$$L = \begin{bmatrix} 6 & 0 \\ 7 & 0 \end{bmatrix}$$

$$R = \begin{bmatrix} 6 & -2 \\ 1 & -2 \end{bmatrix}$$

Player 1: $P_1 = (p_{11}, p_{12})$

Player 2: $P_2 = (p_{21}, p_{22})$

Payoff for Player 1: $U_1 = P_1 L P_2^{\top}$

Payoff for Player 2: $U_2 = P_1 R P_2^{\top}$

Nash Equilibrium

Two players with payoff matrices L and R of $m \times n$

$$P_1 = (p_{11}, p_{12}, \dots, p_{1m})$$
$$P_2 = (p_{21}, p_{22}, \dots, p_{2n})$$

(P_1, P_2) is a NE if and only if

$$P_1 L P_2^\top \geq P_1' L P_2^\top \text{ for all mixture } P_1'$$
$$P_1 R P_2^\top \geq P_1^\top R [P_2']^\top \text{ for all mixture } P_2'$$

(P_1, P_2) is a NE if and only if

$$P_1 L P_2^\top \geq e_i L P_2^\top \text{ for every } i \in [m] \qquad e_i = (0, \dots, 0,1,0, \dots, 0)$$
$$P_1 R P_2^\top \geq P_1 R e_i^\top \text{ for every } i \in [n] \qquad e_i = (0, \dots, 0,1,0, \dots, 0)$$

Strategy game

		Player 2	
		Left	Right
Player 1	Up	1 1	3 0
	Down	0 0	2 1

$$M = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$$

$$N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Player 1's mixed strategy: $\boldsymbol{x} = (x_1, 1 - x_1)$

Player 2's mixed strategy: $\boldsymbol{y} = (y_1, 1 - y_1)$

Player 1's payoff: $\boldsymbol{x} M \boldsymbol{y}^\top$

Player 2's payoff: $\boldsymbol{x} N \boldsymbol{y}^\top$

Stackelberg game (主 从 博 弈)

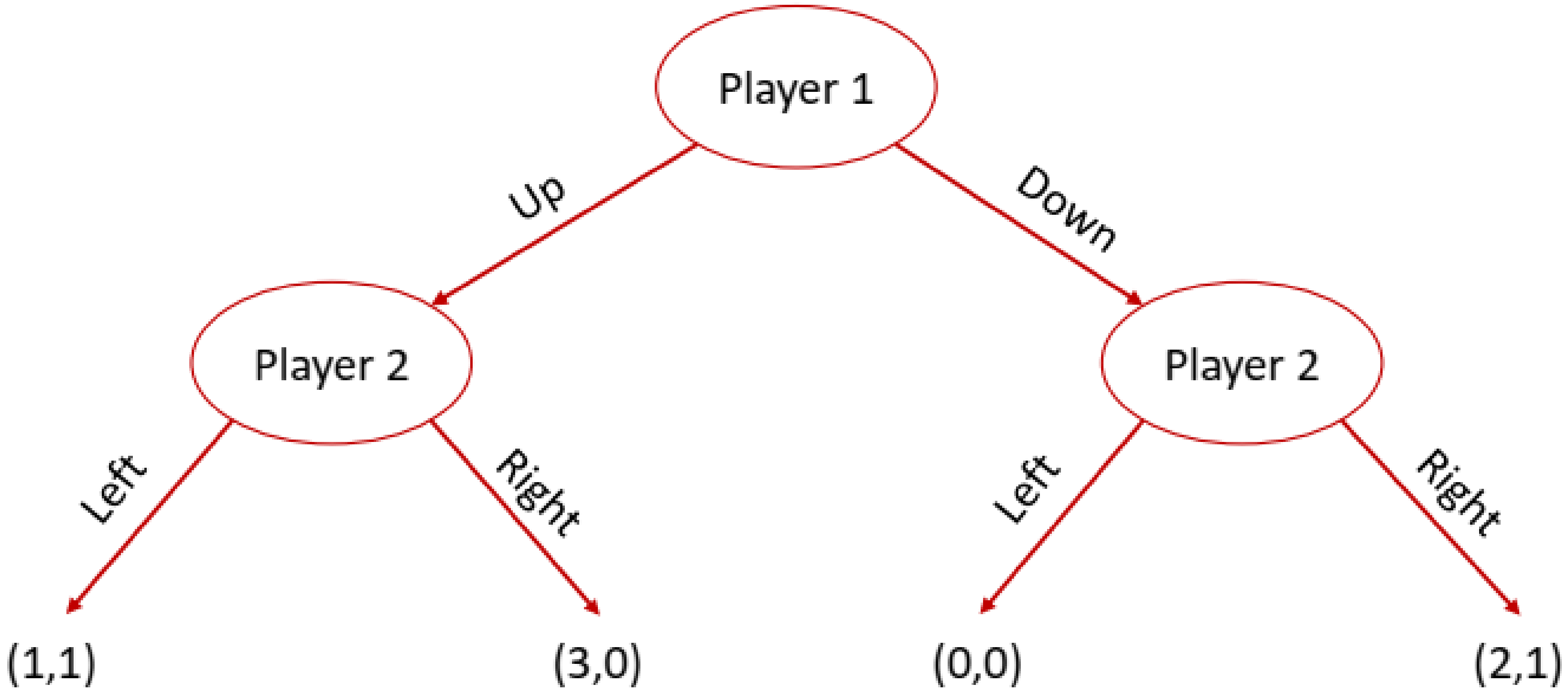
		Player 2	
		Left	Right
Player 1	Up	1 1	3 0
	Down	0 0	2 1

Player 1: "leader" **Player 2 "follower"**

1) Leader selects a (possibly mixed) strategy x_1
2) Follower learns about x_1, selects the best response x_2

How to present the game?

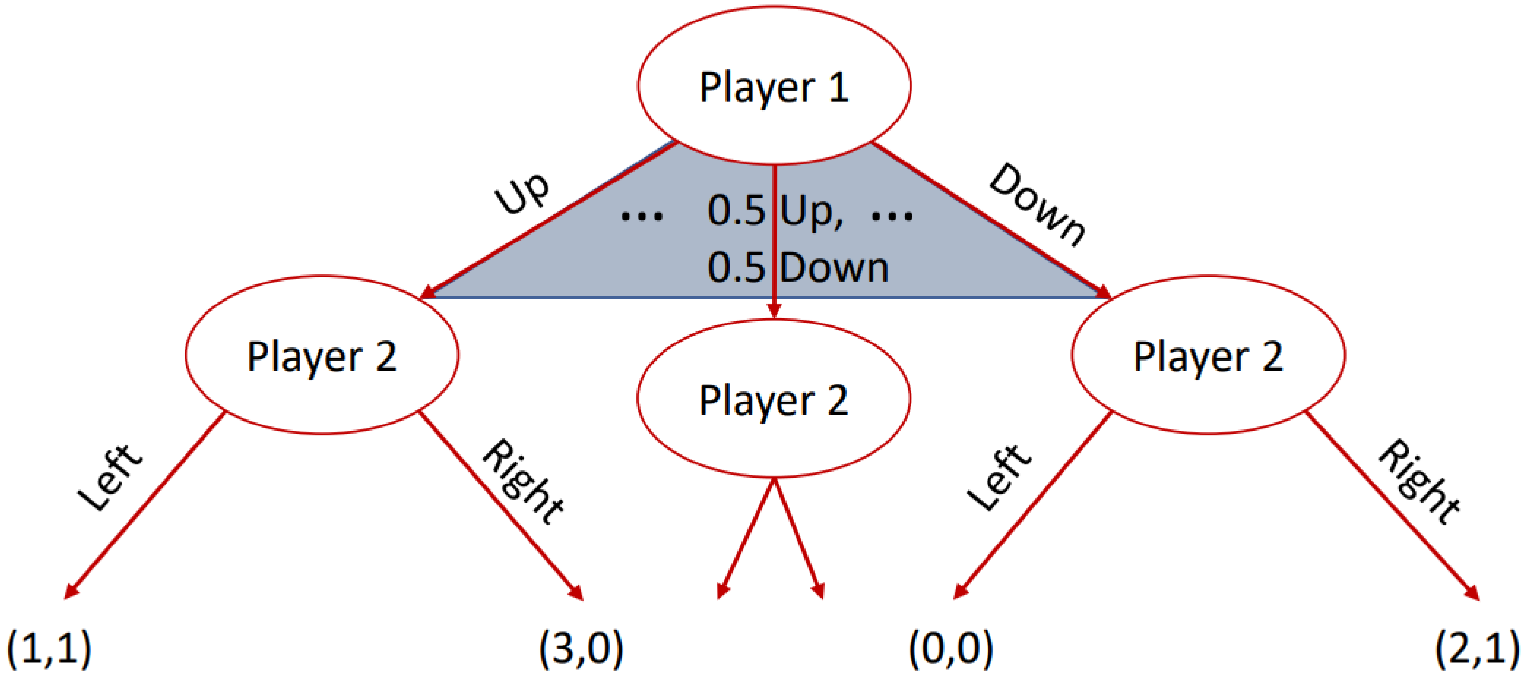
Extensive game



How to present the game?

Mixed strategies are hard to visually represent

- Continuous spectrum of possible actions



Stackelberg game (主从博弈)

		Player 2	
		Left	Right
Player 1	Up	1 1	3 0
	Down	0 0	2 1

What are the Nash equilibrium of this game?

You are player 1, What is your reward in Nash equilibrium?

Stackelberg game (主从博弈)

		Player 2	
		Left	Right
Player 1	Up	1, 1	3, 0
	Down	0, 0	2, 1

As Player 1, you want to take a pure strategy.

What's your strategy?

What would your reward be now?

Stackelberg game (主 从 博 弈)

		Player 2			
		Left		Right	
Player 1	Up	1	1	3	0
	Down	0	0	2	1

Player 1 takes mixed strategies, the advantage is more:

- If Player 1 plays Up and Down with probabilities 0.49 and 0.51, respectively
- Player 2 is still better off playing Right than Left, in expectation
- The Reward of Player 1 >2

Stackelberg vs strategy game

- Can the leader lose in Stackelberg equilibrium compared to a Nash equilibrium?
  - In Stackelberg, the leader must take action in advance, while in Nash, he can change his strategy at any point.

- The answer: No
  - The optimal reward for the leader in the Stackelberg game is always greater than or equal to his maximum reward under any Nash equilibrium.
  - If $(\boldsymbol{x}, \boldsymbol{y})$ is a NE, then Player 1 can always select $\boldsymbol{x}$, ensure that Player 2 will play $\boldsymbol{y}$ and achieve the reward in NE
  - Player 1 may be able to select a better strategy than $\boldsymbol{x}$

Stackelberg vs Nash

- It is important to note that:
  - the leader can take mixed strategies
  - the follower knows (and trusts) the leader's strategies
  - the leader knows the follower's reward structure

- Applications to security

Defender vs Attacher

Stackelberg in zero-sum game

- Recall the minimax theorem

$$\max_{\boldsymbol{x}} \min_{\boldsymbol{y}} \boldsymbol{x} M \boldsymbol{y}^\top = \min_{\boldsymbol{y}} \max_{\boldsymbol{x}} \boldsymbol{x} M \boldsymbol{y}^\top$$

Player 1 goes first $\rightarrow$ player 1 selects the maxmin strategy

Player 2 goes first $\rightarrow$ player 2 selects the minmax strategy

The minimax theorem: make no difference

Strategy game and stackelberg game in zero-sum are essentially identical.

Stackelberg for general 2-persons game

2-persons game: the payoff matrices are M and N for player 1 and 2, respectively.

$$\max_{\boldsymbol{x}} \boldsymbol{x} M \boldsymbol{y}_*^\top$$

subject to

$$\boldsymbol{y}_* = \arg\max_{\boldsymbol{y}} \boldsymbol{x} N \boldsymbol{y}^\top$$

How to compute?

Example

		Player 2: Left	Player 2: Right
Player 1	Up	1, 1	3, 0
Player 1	Down	0, 0	2, 1

Mixed strategy $\boldsymbol{x} = (p, 1 - p)$ over 'Up' and 'Down'.
Consider two cases:

- Player 2 selects 'Left'
- Player 2 selects 'Right'

Example

		Player 2	
		Left	Right
Player 1	Up	1 1	3 0
	Down	0 0	2 1

Player 2 selects ‘Left’

$$\max_{p\in[0,1]} 1p + 0(1-p)$$

s.t.

$$p \cdot 1 + 0 \cdot (1-p) \geq 0 \cdot p + 1 \times (1-p)$$

Example

		Player 2	
		Left	Right
Player 1	Up	1, 1	3, 0
	Down	0, 0	2, 1

Player 2 selects 'Left'

$$\max_{p \in [0,1]} p$$

s.t.

$$p \geq 1 - p$$

Solution:

p=1

the payoff: 1

Example

		Player 2	
		Left	Right
Player 1	Up	1 1	3 0
	Down	0 0	2 1

Player 2 selects 'Right'

$$\max_{p\in[0,1]} 3p + 2(1-p)$$

s.t.

$$p \cdot 1 + 0 \cdot (1-p) \leq 0 \cdot p + 1 \times (1-p)$$

Example

		Player 2	
		Left	Right
Player 1	Up	1 1	3 0
	Down	0 0	2 1

Player 2 selects 'Right'

$$\max_{p\in[0,1]} p + 2$$

s.t.

$$p \leq 1 - p$$

Solution:

p=1/2

the payoff: 2.5

Stackelberg via linear programs

- **High-level idea**
  - Fixed an strategy $a_2^* \in A_2$ for Player 2
  - Write a linear program with mixed strategy $\boldsymbol{x}$ of Player 1
  - Maximizes the payoff of player 1 w.r.t. mixed strategy $\boldsymbol{x}$, and player 2 selects the best response a_2^*
  - Solve this linear program
- **Solve linear programs for every $\boldsymbol{a_2^* \in A_2}$**

Stackelberg via linear programs (formally)

A, B for player 1 (Leader) and player 2 (Follower)

$|A| = m, |B| = n$

$\boldsymbol{x}$ is a mixed strategy over A

$\boldsymbol{x}(a_i)$ denotes the probability of Leader selecting $a_i \in A$

$u_1(a, b)$ and $u_2(a, b)$: payoff for Leader and Follower

Stackelberg via linear programs (formally)

One LP for fixed $\boldsymbol{b^*} \in \boldsymbol{A_2}$

Fixed strategy $b^* \in A_2$ for player 2 (Follower)

$$\mathrm{U}_1(b^*) = \max \sum_{a \in A} \boldsymbol{x}(a) u_1(a, b^*)$$

subject to

- $\forall b \in B, \quad \sum_{a \in A} \boldsymbol{x}(a) u_2(a, b^*) \geq \sum_{a \in A} \boldsymbol{x}(a) u_2(a, b)$
- $\sum_{a \in A} \boldsymbol{x}(a) = 1, \boldsymbol{x}(a) \geq 0$

Taking the maximum over $b \in A_2$

$$\max_b \mathrm{U}_1(b)$$

Computing Stackelberg

Theorem [Conitzer and Sandholm 2006]: In 2-player normal form games, an optimal Stackelberg strategy can be found in polynomial time

Theorem: In (>)3-player normal form games, an optimal Stackelberg strategy is an NP-Hard problem

Cournot Competition (Strategy game)

Two firms compete by choosing how much to produce

$$G = \{\{1, 2\}, \{q_1, q_2\}, \{u_1, u_2\}\}$$

- Price $p(q_1 + q_2) = \max(0, a - b(q_1 + q_2))$
- Costs $c_i(q_i) = cq_i$
- Payoffs $u_i(q_1, q_2) = (\max\left(0, a - b(q_1 + q_2)\right) - c)q_i$
- Condition $a > \mathrm{b}, c > 0, q_1 \geq 0, q_2 \geq 0$

The Nash equilibria is give by $\left\{\left(\frac{a-c}{3b}, \frac{a-c}{3b}\right)\right\}$

Stackelberg Competition (主从博弈)

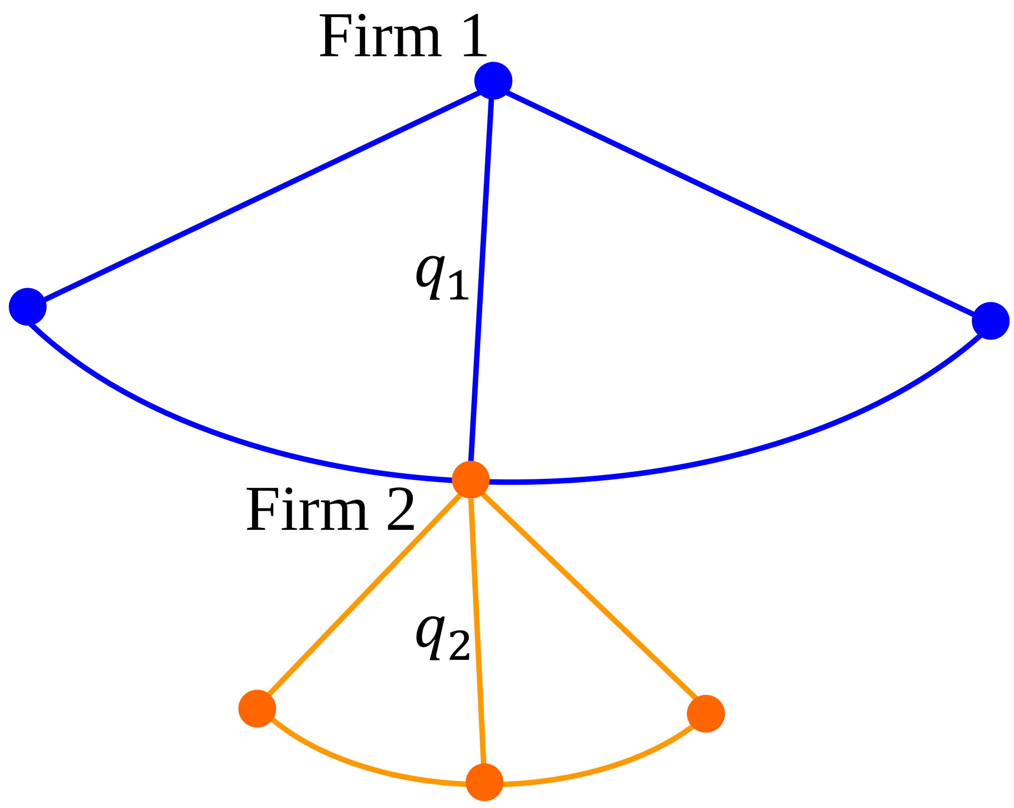
Two firms compete by choosing how much to produce

$$G = \{\{1,2\},\{q_1,q_2\},\{u_1,u_2\}\}$$

- Price $p(q_1 + q_2) = \max(0, a - b(q_1 + q_2))$
- Costs $c_i(q_i) = cq_i$
- Payoffs $u_i(q_1, q_2) = (\max\big(0, a - b(q_1 + q_2)\big) - c)q_i$
- Condition $a > \mathrm{b}, c > 0, q_1 \geq 0, q_2 \geq 0$

Difference: player 1 choose q_1 first, then player 2 choose q_2 after observe q_1

Extensive game



There are infinite sub-game

Stackelberg Competition (Continued)

- This is an extensive game, and we look for SPE
- Back Induction - Not a finite game but with finite length
- Look at a subgame by player 1 with q_1. Then, player 2's maximization problem is to

$$\max_{q_2 \geq 0} \quad u_2(q_1, q_2) = (a - b(q_1 + q_2) - c)q_2$$

- This gives the best response for player 2

$$q_2 = (a - c - bq_1)/2b$$

No difference

Stackelberg Competition (Continued)

The difference: player 1 will choose q_1 after the recognition of player 2's best response.

Player 1 is the leader; player 2 is the follower

The problem of player 1 is

$$\max_{q_1 \geq 0} \quad u_1(q_1, q_2) = (a - b(q_1 + q_2) - c)q_1$$

$$\text{subject to} \quad q_2 = (a - c - bq_1)/2b$$

This implies that

$$\max_{q_1 \geq 0} \quad (a - b(q_1 + (a - c - bq_1)/2b) - c)q_1$$

Stackelberg Competition (Continued)

We get the best response for player 1

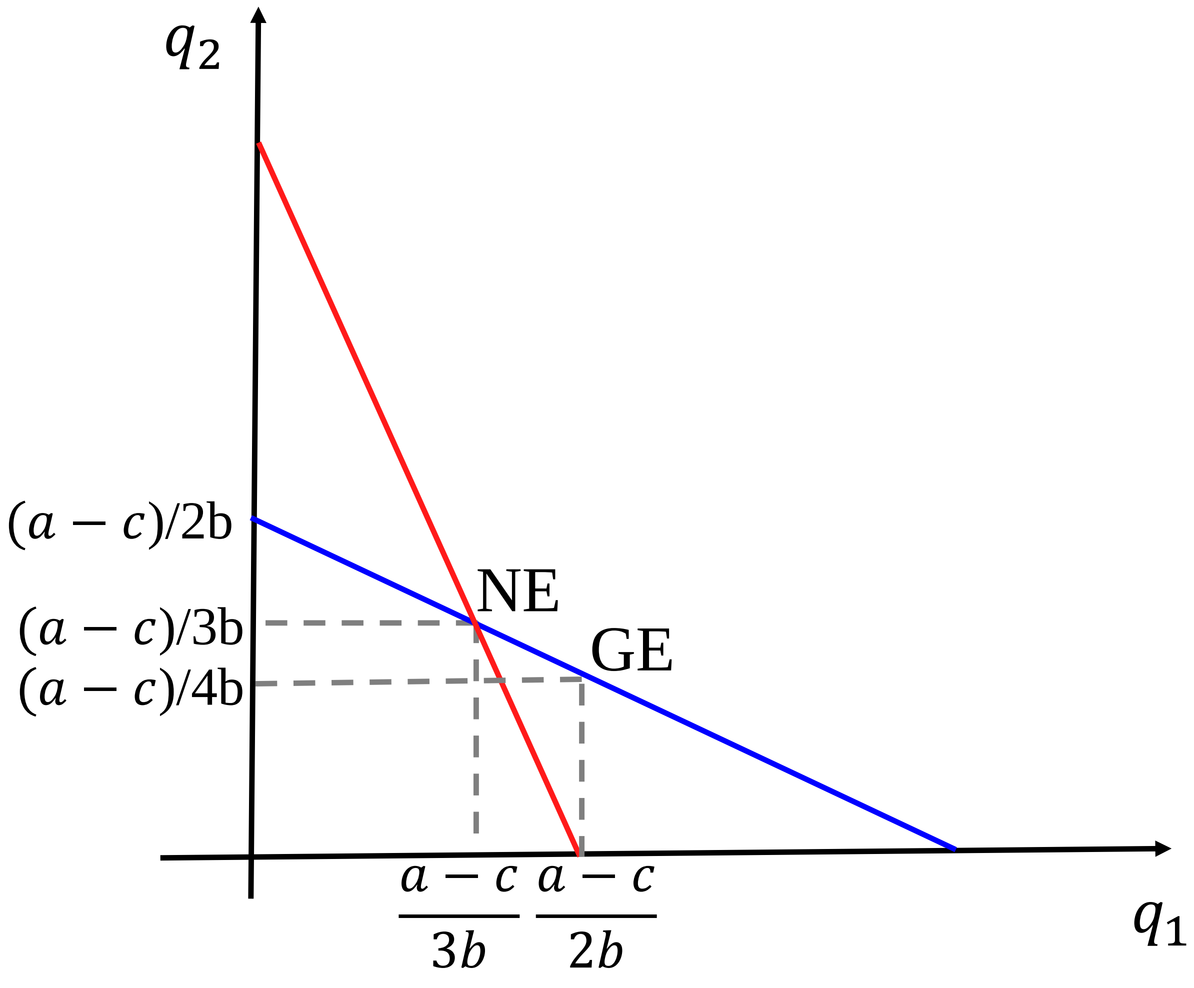
$$q_1 = (a - c)/2b$$

This gives the best response for player 2

$$q_2 = (a - c)/4b$$

SPE: The player 1 has advantages

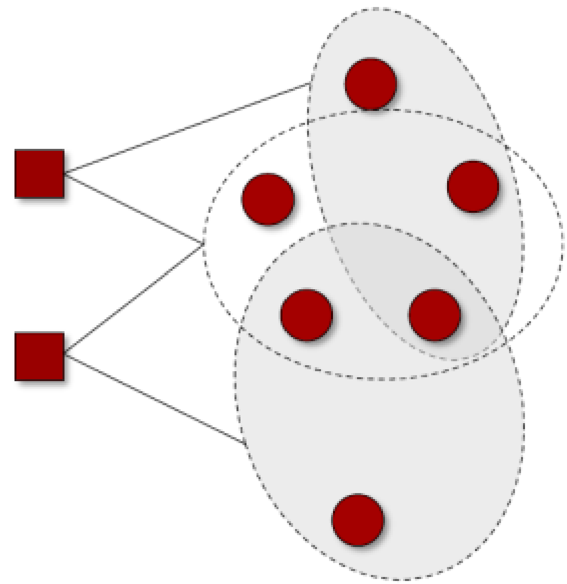
Stackelberg Competition (Continued)



Real-world Applications

Security Game

- Defender (leader) has k identical patrol units
- Defender wants to defend a set of n targets T
- In a pure strategy, each patrol unit can protect a subset of targets $S \subseteq T$ from a given collection $\mathcal{S}$

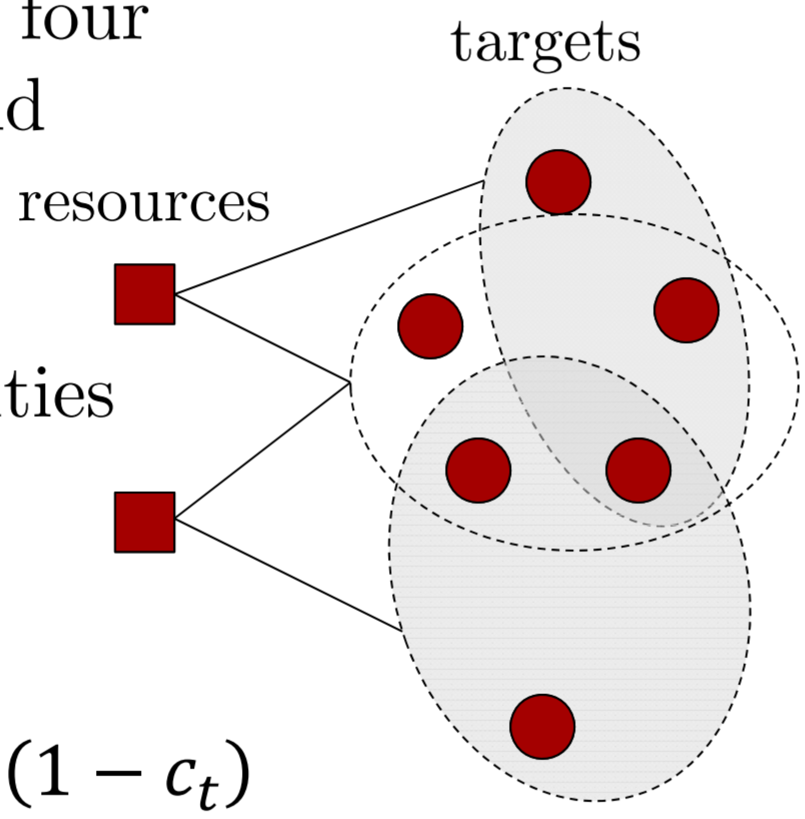


- A target is covered if it is protected by at least one patrol unit
- Attacker wants to select a target to attack

Security Game

Security Game: Defender selects a mixed strategy; attacker follows by choosing a target to attack.

- For each target t, there are four numbers: $u_d^+(t) \geq u_d^-(t)$, and
  $$u_a^+(t) \leq u_a^-(t)$$
- Let $\boldsymbol{c} = (c_1, \dots, c_n)$ be the vector of coverage probabilities
- The utilities to the defender/attacker under $\mathbf{c}$ if target t is attacked are
  $$u_d(t, \boldsymbol{c}) = u_d^+(t) \cdot c_t + u_d^-(t)(1 - c_t)$$
  $$u_a(t, \boldsymbol{c}) = u_a^+(t) \cdot c_t + u_a^-(t)(1 - c_t)$$



Security Game

- Consider the case of $\Sigma = T$, i.e., resources are assigned to individual targets, i.e., schedules have size 1
- Nevertheless, number of leader strategies is exponential
- Theorem [Korzhyk et al. 2010]: Optimal leader strategy can be computed in poly time

A compact LP

- LP formulation similar to previous one
- Advantage: logarithmic in #leader strategies
- Problem: do probabilities correspond to strategy?

$$
\begin{aligned}
\textbf{max}\quad & u_d(t^*, c) \\
\textbf{s.t.}\quad & \forall \omega \in \Omega, \forall t \in A(\omega), 0 \le c_{\omega,t} \le 1 \\
& \forall t \in T, c_t = \sum_{\omega \in \Omega : t \in A(\omega)} c_{\omega,t} \le 1 \\
& \forall \omega \in \Omega, \sum_{t \in A(\omega)} c_{\omega,t} \le 1 \\
& \forall t \in T, u_a(t, \boldsymbol{c}) \le u_a(t^*, \boldsymbol{c})
\end{aligned}
$$

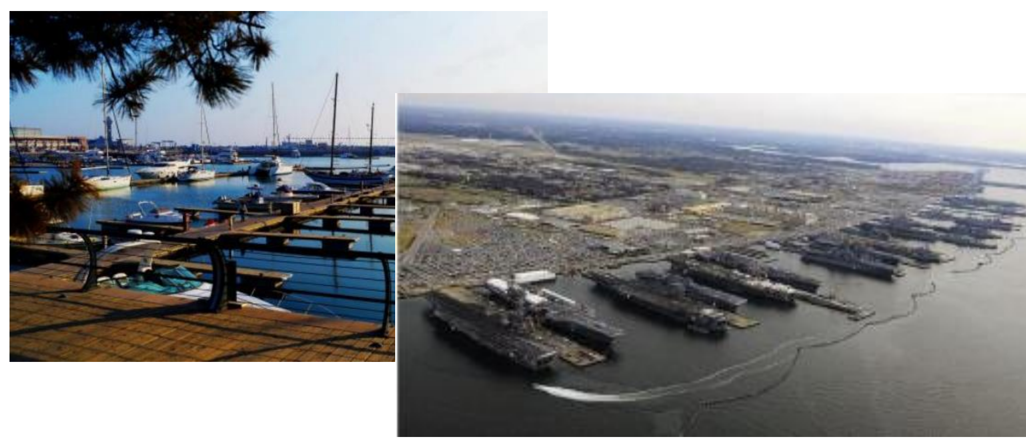
Example

Train

Airport

School

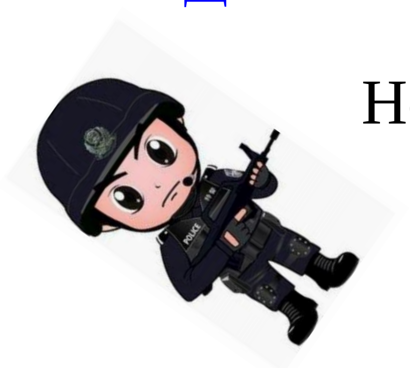
Port



Homework

Terrorist

Police		School	Hospital
	School	5, -2	-1, 1
	Hospital	-5, 5	2, -1



Solve the optimal Stackelberg strategy

Homework

Follower

Leader

	L	M	R
U	1, 1	3, 0	2, 2
M	0, 0	2, 1	1, 1
D	3, 4	2, 3	3, 0

Present a solution for the optimal Stackelberg strategy