Game Theory and Applications (博弈论及其应用)

Chapter 6： Strategic Games with Incomplete Information

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Recap on Previous Chapter

- two-player zero-sum game $u_1(a_1, a_2) + u_2(a_1, a_2) = 0$
- Player 1 method: $\max\limits_{\boldsymbol{a_1} \in \boldsymbol{A_1}} \min\limits_{\boldsymbol{a_2} \in \boldsymbol{A_2}} \boldsymbol{u}(\boldsymbol{a_1}, \boldsymbol{a_2})$
- Player 2 method: $\min\limits_{\boldsymbol{a_2} \in \boldsymbol{A_2}} \max\limits_{\boldsymbol{a_1} \in \boldsymbol{A_1}} \boldsymbol{u}(\boldsymbol{a_1}, \boldsymbol{a_2})$
- The strategy outcome (a_1^*, a_2^*) is a Nash Equilibrium if and only if $\max\limits_{a_1 \in A_1} \min\limits_{a_2 \in A_2} u(a_1, a_2) = \min\limits_{a_2 \in A_2} \max\limits_{a_1 \in A_1} u(a_1, a_2)$

The Minmax Theorem For two-player zero-sum finite game $G = \{\{1,2\}, \{A_1, A_2\}, u\}$, we have

$$\max_{p \in \Delta_1} \min_{q \in \Delta_2} pMq^\top = \min_{q \in \Delta_2} \max_{p \in \Delta_1} pMq^\top .$$

Recap on Previous Chapter

Theorem The optimization problem of $\max_{p\in\Delta_1}\min_{q\in\Delta_2} pMq^\top$ is equivalent to

$$\max v$$

s.t.

$$pM \geq v\mathbf{1}$$
$$p = (p_1, \dots, p_m) \in \Delta_1$$
$$\mathbf{1} = (1, \dots, 1)^\top$$

Linear programming: can be solved in polynomial time

Strategy Game with Complete Information

Previous strategy game with complete information

- Who are playing
- All players are rational
- What strategies are possible for each player
- What are the payoffs
- All players know the complete information
- All players knows that all players know complete information

  …

Games with Incomplete Information

- In many games, some players are not sure the payoff
- Players may have incomplete inform. about components
- Bidder does not know value of other bidders in auction
- Some players may have private information
- Despite of different types of incomplete information, we consider incomplete payoffs

Incomplete information motivates additional interaction

Battle of Sexes with Incomplete Information

Complete information:

- both boy and girl like to meet

		Girl: Ballet	Girl: Football
Boy	Ballet	1, 2	0, 0
Boy	Football	0, 0	2, 1

There are 2 pure strategy NE and 1 mixed strategy NE

Both boy and girl know each other in a short time.

The boy is not sure if the girl wishes to meet or not

Incomplete information

Battle of Sexes with Incomplete Information (cont.)

The girl has two types: like or dislike

The boy assumes that girl like with probability p

Girl (like, p)

Boy \ Girl	F	B
F	2, 1	0, 0
B	0, 0	1, 2

Girl (dislike $1-p$)

Boy \ Girl	F	B
F	2, 0	0, 2
B	0, 1	1, 0

The girl knows the complete information

The boy does not know

What strategies in this game? how to reason?

Battle of Sexes with Incomplete Information (cont.)

Girl (like, p)

Boy	F		B	
F	2	1	0	0
B	0	0	1	2

Girl (dislike $1-p$)

Boy	F		B	
F	2	0	0	2
B	0	1	1	0

If the boy selects B, then the girl selects (B,F). {B, (B,F)}

$$U_1\big(B,(B,F)\big) = p + (1-p) * 0 = p$$

$$U_1\big(F,(B,F)\big) = 0 + (1-p) * 2 = 2(1-p)$$

if $p \geq 2/3$, {B, (B,F)} is a NE;

if $p < 2/3$, {B, (B,F)} is not a NE;

Battle of Sexes with Incomplete Information (cont.)

Girl (like, p)

Boy \ Girl	F	B
F	2, 1	0, 0
B	0, 0	1, 2

Girl (dislike $1-p$)

Boy \ Girl	F	B
F	2, 0	0, 2
B	0, 1	1, 0

If the boy selects F, then the girl selects (F,B). {F, (F,B)}

$$U_1\big(F,(F,B)\big) = 2p + (1-p) * 0 = 2p$$

$$U_1\big(B,(F,B)\big) = 0 + (1-p) = 1-p$$

if $p \geq 1/3$, {F, (F,B)} is a NE;

if $p < 1/3$, {F, (F,B)} is not a NE;

Battle of Sexes with Incomplete Information (cont.)

Girl (like, p)

Boy	F	B
F	2, 1	0, 0
B	0, 0	1, 2

Girl (dislike $1-p$)

Boy	F	B
F	2, 0	0, 2
B	0, 1	1, 0

If the boy selects B, then the girl selects (B,F). {B, (B,F)}

$$U_1\big(B,(B,F)\big) = p + (1-p) * 0 = p$$

$$U_1\big(F,(B,F)\big) = 0 + (1-p) * 2 = 2(1-p)$$

if $p \geq 2/3$, {B, (B,F)} is a NE;

if $p < 2/3$, {B, (B,F)} is not a NE;

Battle of Sexes with Incomplete Information (cont.)

Girl (like, p)

Boy	F	B
F	2, 1	0, 0
B	0, 0	1, 2

Girl (dislike $1-p$)

Boy	F	B
F	2, 0	0, 2
B	0, 1	1, 0

In a summary:

if $p \geq 1/3$, {F, (F,B)} is a NE;

if $p \geq 2/3$, {F, (F,B)} and {B, (B,F)} are NEs;

Bayesian Games

A Bayesian game consists of

- A set of players N
- A set of strategies A_i for each player i
- A set of types Θ_i for each player i
  - Type set Θ_i includes all private information for player i
  - The types on payoff are adequate (Payoff types)
- Probability distribution $p = p(\theta_1, \ldots, \theta_N)$ on $\times_{i=1..n} \Theta_i$

Bayesian Games (cont.)

- For player i, a pure strategy is a map $a_i : \Theta_i \rightarrow A_i$, which prescribes an strategy for each type

$$a_i = \left(a_i(\theta_i^1), a_i(\theta_i^2), \dots, a_i(\theta_i^{n_i})\right)$$

- A payoff function $u_i : \times_{i=1..N} A_i \times \times_{i=1..n} \Theta_i \rightarrow R$

$$u_i(a_1, \dots, a_N, \theta_1, \dots, \theta_N) \text{ for } a_i \in A_i \text{ and } \theta_i \in \Theta_i$$

$$\boldsymbol{G = \{N, \{A_i\}, \{\Theta_i\}, \{u_i\}, p\}}$$

Battle of Sexes with Incomplete Information

Girl (like, p) — Boy (like q)

	F	B
F	2 1	0 0
B	0 0	1 2

Girl (dislike $1-p$) — Boy (like q)

	F	B
F	2 0	0 2
B	0 1	1 0

Girl (like, p) — Boy (dislike 1-q)

	F	B
F	0 1	2 0
B	1 0	0 2

Girl (dislike $1-p$) — Boy (dislike 1-q)

	F	B
F	0 0	2 2
B	1 1	0 0

Bayesian Games (cont.)

- The set of types Θ_i for each player i
  - Player i does not know the selection of Θ_j
  - All types are drawn from the prior dis. $p(\theta_1, \dots, \theta_N)$
  - $p(\theta_1, \dots, \theta_N) = p(\theta_1)p(\theta_2) \dots p(\theta_N)$(independent types)
- Given $p(\theta_1, \dots, \theta_N)$, we have, by Bayes rule,

$$p(\theta_{-i}|\theta_i) = p(\theta_i, \theta_{-i})/p(\theta_i)$$

where $\theta_{-i} = (\theta_1, \dots, \theta_{i-1}, \theta_{i+1} \dots, \theta_N)$

Such game is called Bayes Game

Outcome and Payoff Functions

A pure strategy for player i

$$\left(a_i(\theta_i^1), a_i(\theta_i^2), \ldots, a_i(\theta_i^{n_i})\right)$$

An outcome of Bayes game is given by

$$\left(\left(a_1(\theta_1^1), \ldots, a_1(\theta_1^{n_1})\right), \ldots, \left(a_N(\theta_N^1), \ldots, a_N(\theta_N^{n_N})\right)\right)$$

Given a_{-i}, the expected payoff of player i and type θ_i is

$$U_i(a_i(\theta_i), a_{-i}) = \sum_{\theta_{-i}} p(\theta_{-i}|\theta_i) u_i(a_{-i}(\theta_{-i}), a_i, \theta_{-i}, \theta_i)$$

Bayesian Nash Equilibrium

Definition The outcome $(a_1, a_2, \dots, a_N)$ is a **Bayesian Nash Equilibrium** if for each type θ_i, we have

$$U_i(a_i(\theta_i), a_{-i}) \geq U_i(a'_i(\theta_i), a_{-i}) \text{ for all } a'_i(\theta_i) \in A_i$$

Given a_{-i} and type θ_i, the **best response for player *i*** is

$$B_i(a_{-i}, \theta_i) = \{a_i(\theta_i): U_i(a_i(\theta_i), a_{-i}) \geq U_i(a'_i(\theta_i), a_{-i}) \text{ for all } a'_i(\theta_i)\}$$

Theorem The outcome $(a_1, a_2, \dots, a_N)$ is a Bayesian NE if and only if for every player i and each type θ_i, we have

$$a_i(\theta_i) \in B_i(a_{-i}, \theta_i)$$

How to find Bayesian Nash Equilibrium

How to find Bayesian Nash Equilibrium

1) Find the best response function for each player and type

2) Find Bayesian NE by $a_i(\theta_i) \in B_i(a_{-i}, \theta_i)$

Bank Runs (银行挤脱)

- Both players 1 and 2 have a deposit of $100 in bank
- If the bank manager is good, each player get $150; if the manager is bad, then they lose all money
- Players can withdraw money but bank has only $100
  - If only one player withdraws, he get $100 and the other gets 0
  - If two players both withdraw, each get $50
- Player 1 believes a good manager with probability p
- Player 2 knows whether the manager is good or bad
- Two players simultaneously make strategy: withdraw/not

Bank Runs (cont.)

- Two players
- Strategies $A_1 = A_2 = \{W, N\}$
- Types $\Theta_1 = \{1\}; \Theta_2 = \{\mathrm{G}, \mathrm{B}\}$
- A probability distribution $p_1(\theta_2 = \mathrm{G}) = p$
- Payoffs

Player 2 (G, p)

Player 1	W	N
W	50 50	100 0
N	0 100	150 150

Player 2 (B, $1 - p$)

Player 1	W	N
W	50 50	100 0
N	0 100	0 0

Bank Runs (cont.)

Player 2 (G, p)

Player 1	W	N
W	50 50	100 0
N	0 100	150 150

Player 2 (B, $1-p$)

Player 1	W	N
W	50 50	100 0
N	0 100	0 0

Player 1: W or N

Player 2: W(G) N(G) W(B) N(B)

Bayesian Nash Equilibrium of Bank Runns

Player 2 (G, p)

		W	N
Player 1	W	50, 50	100, 0
	N	0, 100	150, 150

Player 2 (B $1-p$)

		W	N
Player 1	W	50, 50	100, 0
	N	0, 100	0, 0

If Player 1 selects W, then

$$B_2(\mathrm{W},\mathrm{G}) = \{\mathrm{W}\},\ B_2(\mathrm{W},\mathrm{B}) = \{\mathrm{W}\}$$

Outcome $(\mathrm{W},(\mathrm{W}(\mathrm{G}),\mathrm{W}(\mathrm{B})))$: best strategy for Player 2

Is W a best response to W(G),W(B)?

$$U_1(W,(W(G),W(B))) = 50p + 50(1-p) = 50$$

$$U_1(N,(W(G),W(B))) = 0p + 0(1-p) = 0$$

(W,(W(G), W(B))) is a Bayesian NE

Bayesian Nash Equilibrium of Bank Runs

Player 2 (G, p)

Player 1	W	N
W	50 50	100 0
N	0 100	150 150

Player 2 (B $1-p$)

Player 1	W	N
W	50 50	100 0
N	0 100	0 0

If Player 1 selects N, then

$$B_2(\mathrm{N},\mathrm{G}) = \{\mathrm{N}\},\ B_2(\mathrm{N},\mathrm{B}) = \{\mathrm{W}\}$$

Outcome (N,(N(G),W(B))): Player 2 makes best strategy

Is N a best response to (W(G),W(B))?

$$U_1\big(N, N(G), W(B)\big) = 150p + 0(1-p) = 150p$$

$$U_1\big(W, N(G), W(B)\big) = 100p + 50(1-p) = 50(1+p)$$

If $150p > 50 + 50p$, then (N,(N(G),W(B))) is a BNE

Cournot Duopoly with Incomplete Information

Two firms $N = \{1,2\}$

Firm 1 has a cost c_H;

Firm 2 has two costs c_L and c_H

Firm 1 believes that Firm 2 selects c_H with probability p

Firm 1’s strategy $\{q_1 : q_1 \geq 0\}$

Firm 2’s strategy $\{q_{2,L}, q_{2,H} : q_{2,L} \geq 0$ and $q_{2,H} \geq 0\}$

Price: $a - q_1 - q_{2,L}$ or $a - q_1 - q_{2,H}$

Cournot Duopoly with Incomplete Information

For player 1, the expected payoff function is

$$U_1(q_1, q_{2,L}, q_{2,H}, c_L, c_H) = pq_1(a - q_1 - q_{2,H}) +$$
$$+(1-p)q_1(a - q_1 - q_{2,L}) - c_H q_1$$

For player 2, the expected payoff function of type c_H is

$$U_2(q_1, q_{2,H}, c_H) = (a - q_1 - q_{2,H})q_{2,H} - c_H q_{2,H}$$

For player 2, the expected payoff function of type c_L is

$$U_2(q_1, q_{2,L}, c_L) = (a - q_1 - q_{2,L})q_{2,L} - c_L q_{2,L}$$

Best Response for Player 1

For player 1, the expected payoff function is

$$U_1\left(q_1, q_{2,L}, q_{2,H}, c_L, c_H\right) = pq_1\left(a - q_1 - q_{2,H}\right) +$$

$$+(1-p)q_1\left(a - q_1 - q_{2,L}\right) - c_H q_1$$

Maximizing $U_1\left(q_1, q_{2,L}, q_{2,H}, c_L, c_H\right)$ gives

$$B_1\left(q_{2,L}, q_{2,H}\right) = \left\{\frac{a - pq_{2,H} - (1-p)q_{2,L} - c_H}{2}\right\}$$

Best Response for Player 2

For player 2, the expected payoff function of type c_H is

$$U_2(q_1, q_{2,H}, c_H) = (a - q_1 - q_{2,H})q_{2,H} - c_H q_{2,H}$$

Maximizing $U_2(q_1, q_{2,H}, c_H)$ gives

$$B_2(q_1, c_H) = \left\{\frac{a - q_1 - c_H}{2}\right\}$$

Similarly, we have

$$B_2(q_1, c_L) = \left\{\frac{a - q_1 - c_L}{2}\right\}$$

Bayesian Nash Equilibrium

$$B_1(q_{2,L}, q_{2,H}) = \left\{\frac{a - pq_{2,H} - (1-p)q_{2,L} - c_H}{2}\right\}$$

$$B_2(q_1, c_H) = \left\{\frac{a - q_1 - c_H}{2}\right\}$$

$$B_2(q_1, c_L) = \left\{\frac{a - q_1 - c_L}{2}\right\}$$

We solve the Bayesian Nash Equilibrium by

$$q_1 \in B_1(q_{2,L}, q_{2,H})$$

$$q_{2,H} \in B_2(q_1, c_H)$$

$$q_{2,L} \in B_2(q_1, c_L)$$

Bayesian Nash Equilibrium

The Bayesian Nash Equilibrium is $(q_1, (q_{2,L}, q_{2,H}))$

$$q_1 = \frac{a - c_H - (1-p)(c_H - c_L)}{3}$$

$$q_{2,\mathrm{H}} = \frac{a}{3} - \frac{c_H + c_L}{6} - p\frac{c_H - c_L}{6}$$

$$q_{2,L} = \frac{a}{3} - \frac{c_L}{3} - p\frac{c_H - c_L}{6}$$

Discussions

- Incomplete information affects the outputs of players
- $q_{2,L} > q_{2,H}$ implies player 2 produce more for lower price

Discussions on Bayesian Nash Equilibrium

- If player 1 knows that player 2 selects c_H $(p = 1)$ then

$$q_1 = q_{2,H} = \frac{a - c_H}{3}$$

- If player 1 does not know the choices of player 2

$$q_1 = \frac{a - c_H - (1 - p)(c_H - c_L)}{3}$$

Player 1 produces less with the incomplete information

Discussions on Bayesian Nash Equilibrium

- If player 1 knows that player 2 selects c_L ($p = 0$) then

$$q_1 = q_{2,L} = \frac{a - c_L + c_H - c_L}{3}$$

- If player 1 does not know the choices of player 2

$$q_1 = \frac{a - c_L + c_H - c_L - (3 - p)(c_H - c_L)}{3}$$

Player 1 produces less with the incomplete information

Discussions on Bayesian Nash Equilibrium

- If player 1 knows that player 2 selects c_H $(p = 1)$ then

$$q_{2,H} = \frac{a - c_H}{3};$$

otherwise,

$$q_{2,H} = \frac{a - c_H}{3} + (1 - p)\frac{c_H - c_L}{6}$$

Player 2 benefits from the incomplete information

The firm will benefit by keeping cost secrets

First Price Auction with Incomplete information

$N = \{1,2\}$: players bid a building

v_i: the true value for player i

v_i: a uniform distribution over [0,1]

$b_i \geq 0$: the bid price for player i

$$b_i = b_i(v_i) = av_i \quad (a > 0)$$

The payoff functions for player i

$$u_i(\mathrm{b_i}, \mathrm{b_j}) = \begin{cases} v_i - b_i & \text{if } \ b_i > b_j \\ v_i/2 - b_i & \text{if } \ b_i = b_j \\ 0 & \text{otherwise} \end{cases}$$

Bayesian NE of First Price Auction

For player i, the expected payoff function is

$$U_i\left(\mathrm{b_i}, \mathrm{b_j}(v_j), v_i\right) = (v_i - b_i)\Pr[b_i > \mathrm{b_j}(v_j)]$$
$$+(v_i/2 - b_i)\Pr[b_i = \mathrm{b_j}(v_j)]$$

$$\Pr\left[b_i > \mathrm{b_j}\left(v_j\right)\right] = \Pr\left[b_i > av_j\right] = \Pr\left[b_i/a > v_j\right]$$

v_j is a uniform distribution over [0,1]

$$U_i\left(\mathrm{b_i}, \mathrm{b_j}, v_i\right) = (v_i - b_i)b_i/a$$

Maximizing $U_i\left(\mathrm{b_i}, \mathrm{b_j}, v_i\right)$ with respect to $\mathrm{b_i}$ gives

$$\mathrm{b_i}(v_i) = \frac{v_i}{2}$$

Homework: First Price Auction with Incomplete information

$N = \{1, \ldots, \mathrm{N}\}$: players bid a building

v_i: the true value for player i

v_i: a uniform distribution over [0,1]

$b_i \geq 0$: belief that the bid price for player i

$$b_i = b_i(v_i) = av_i \quad (a > 0)$$

The payoff functions for player i

$$u_i(\mathrm{b}_1, \ldots \mathrm{b}_\mathrm{N}) = \begin{cases} v_i - b_i & \text{if } \; b_i > \max_{j \neq i} b_j \\ v_i/2 - b_i & \text{if } \; b_i = \max_{j \neq i} b_j \\ 0 & \text{otherwise} \end{cases}$$

Conclusions

- Strategy game with incomplete information
- Bayes game $G = \{N, \{A_i\}, \{\Theta_i\}, \{u_i\}, p\}$
- Bayes Nash Equilibrium
- How to find Bayes Nash equilibrium

Bayesian NE of First Price Auction (N Players)

For player i, the expected payoff function is

$$U_i\left(\mathrm{b_i}, \mathrm{b_j}(v_j), v_i\right) = (v_i - b_i)\Pr[b_i > \max_{j \neq i} \mathrm{b_j}(v_j)]$$

$$U_i\left(\mathrm{b_i}, \mathrm{b_j}, v_i\right) = (v_i - b_i)(b_i/a)^{\mathrm{N}-1}$$

Maximizing $U_i\left(\mathrm{b_i}, \mathrm{b_j}, v_i\right)$ with respect to $\mathrm{b_i}$ gives

$$\mathrm{b_i}(v_i) = \frac{\mathrm{N} - 1}{N} v_i$$