

Game Theory and Applications (博弈论及其应用)

Chapter 2: Mixed Strategy Game and Nash Equilibrium

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Recap

- **Strategy game**

Each player selects his strategy **simultaneously** with **full information**, without knowledge of strategy choices by the other players (**non-cooperate**)

- **Formulation**

$$G = \{N, \{A_i\}_{i=1}^N, \{u_i\}_{i=1}^N\}$$

$$G = \{N, \{A_i\}_{i=1}^N, \{\succeq_i\}_{i=1}^N\}$$

Recap

- **Nash equilibrium**

An outcome $a^* = (a_1^*, a_2^*, \dots, a_N^*)$ is a **Nash equilibrium (NE)** if for each players i

$$u_i(a_i^*, a_{-i}^*) \geq u_i(a_i, a_{-i}^*) \text{ for all } a_i \in A_i.$$

- **How to find Nash equilibria - Payoff matrix**

		Prisoner 2			
		Confess (c)		Don't confess(d)	
Prisoner 1	Confess(c)	-6	-6	0	-12
	Don't confess(d)	-12	0	-1	-1

Recap

- **How to find Nash equilibria** - Payoff function

(1) Find the best response correspondence for each player

Best response correspondence

$$B_i(a_{-i}) = \{a_i \in \operatorname{argmax}_{a_i \in A_i} u_i(a_i, a_{-i})\}$$

(2) Find all Nash Equilibria $(a_1^*, a_2^*, \dots, a_N^*)$ such that

$$a_i^* \in B_i(a_{-i}^*) \text{ for each player}$$

An Example

- $G = \left\{ \{1,2\}, \left\{ \{U, L\}, \{L, R\} \right\}, \{u_1, u_2\} \right\}$

		Player 2	
		L	R
Player 1	U	1 2	0 4
	D	0 5	3 2

Mixed Strategy

- Player 1 is mixing over the pure strategies U and D
- Player 2 is mixing over the pure strategies L and R

		Player 2	
		L, π_2	$R, 1 - \pi_2$
Player 1	U, π_1	1 2	0 4
	$D, 1 - \pi_1$	0 5	3 2

- Mixed strategy keeps the guess of player's strategies, keep unpredictable on pure strategies
- Pure strategy can be viewed as a special mixed strategy

Pure and Mixed Strategies

Strategy game

$$G = \{N, \{A_1, A_2, \dots, A_N\}, \{u_1, u_2, \dots, u_N\}\}$$

Pure strategy: strategies set of player i

$$A_i = \{a_{i1}, a_{i2}, \dots, a_{in_i}\}$$

Mixed strategy: a probability over the set A_i of strategies

$$A_i = \{a_{i1}, a_{i2}, \dots, a_{in_i}\}$$

$$p_i = (p_{i1}, p_{i2}, \dots, p_{in_i})$$

Denote by $\Delta(A_i)$ the set of all prob. distributions over A_i

$$\Delta(A_i) = \left\{ p_i = \{p_{i1}, p_{i2}, \dots, p_{in_i}\}, p_{ij} \geq 0, \sum_j p_{ij} = 1 \right\}$$

Pure and Mixed Strategies

Strategy game

$$G = \{N, \{A_1, A_2, \dots, A_N\}, \{u_1, u_2, \dots, u_N\}\}$$

Pure strategy: $a_i \in A_i = \{a_{i1}, a_{i2}, \dots, a_{in_i}\}$

Mixed strategy: $p_i = (p_{i1}, p_{i2}, \dots, p_{in_i}) \in \Delta(A_i)$

An **mixed outcome** $p = (p_1, p_2, \dots, p_N)$ $p_i \in \Delta(A_i)$

p_1, p_2, \dots, p_N are independent

For any p_i , we define

$$p_{-i} = (p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_N)$$

$$p = (p_i, p_{-i})$$

Pure and Mixed Strategies (cont.)

	Player i	Outcome	Players	Outcome
Pure strategy	$a_i \in A_i$	$a = (a_1, a_2 \dots a_N)$	a_{-i}	$a = (a_i, a_{-i})$
Mixed strategy	$p_i \in \Delta(A_i)$	$p = (p_1, p_2 \dots p_N)$	p_{-i}	$p = (p_i, p_{-i})$

Pure strategy game

$$G = \{N, \{A_1, A_2, \dots, A_N\}, \{u_1, u_2, \dots, u_N\}\}$$

Mixed strategy game

$$G = \{N, \{\Delta(A_1), \Delta(A_2), \dots, \Delta(A_N)\}, \{?, ?, \dots, ?\}\}$$

The Expected Payoff

Given $G = \{N, \{A_i\}, \{u_i\}\}$ and mixed $p = (p_1, p_2, \dots, p_N)$, the expected payoff of player i is given by

$$\begin{aligned} U_i(p) &= \sum_{a \in A} p(a) u_i(a) \\ &= \sum_{a=(a_1, \dots, a_N) \in A} p_1(a_1) \times \dots \times p_N(a_N) u_i(a) \end{aligned}$$

Pure strategy game

$$G = \{N, \{A_1, A_2, \dots, A_N\}, \{u_1, u_2, \dots, u_N\}\}$$

Mixed strategy game

$$G = \{N, \{\Delta(A_1), \Delta(A_2), \dots, \Delta(A_N)\}, \{U_1, U_2, \dots, U_N\}\}$$

Example

		Player 2	
		L, π_2	$R, 1 - \pi_2$
Player 1	U, π_1	1 2	0 4
	$D, 1 - \pi_1$	0 5	3 2

$$p = (p_1, p_2) = ((0.4, 0.6), (0.5, 0.5))$$

$$\begin{aligned} U_1(p) = & p_1(U)p_2(L)u_1(U, L) + p_1(U)p_2(R)u_1(U, R) \\ & + p_1(D)p_2(L)u_1(D, L) + p_1(D)p_2(R)u_1(D, R) = 1.1 \end{aligned}$$

$$U_2(p) = \dots = 3.3$$

Continuous Expected Payoff Function

Lemma $U_i(p)$ is a **continuous** function for each variable.

proof. $U_i(p) = \sum_{a=(a_1, \dots, a_N) \in A} p_1(a_1) \times \dots \times p_N(a_N) u_i(a)$

Given $p = (p_1, p_2, \dots, p_N)$, write

$$U_i(p_{-i}, a_i) = \sum_{a_{-i} \in A_{-i}} p_{-i}(a_{-i}) u_i(a_i, a_{-i})$$

Then

$$U_i(p) = \sum_{a_i \in A_i} p_i(a_i) U_i(p_{-i}, a_i)$$

Multi-linear Payoff Function

Lemma: The expected payoff function U_i is **multi-linear**

For mixed outcome $p = (p_1, p_2, \dots, p_N)$ and p'_i , we have

$$U_i(\lambda p_i + (1 - \lambda)p'_i, p_{-i}) = \lambda U_i(p_i, p_{-i}) + (1 - \lambda)U_i(p'_i, p_{-i})$$

$$\lambda \in [0,1].$$

Proof. See board from the definition

$$U_i(p) = \sum_{a_i \in A_i} p_i(a_i) U_i(p_{-i}, a_i).$$

Mixed Strategy Nash Equilibrium

An mixed strategy outcome $p = (p_1, p_2, \dots, p_N)$ is a **Nash equilibrium (NE)** if for each i , we have

$$U_i(p_i, p_{-i}) \geq U_i(p'_i, p_{-i}) \text{ for } p'_i \in \Delta(A_i)$$

Given $G = \{N, \{\Delta(A_i)\}, \{U_i\}\}$ and $p = (p_1 \dots p_N)$, the **best response correspondence** of player i is given by

$$B_i(p_{-i}) = \{p_i : U(p_i, p_{-i}) \geq U(p'_i, p_{-i}) \text{ for all } p'_i \in \Delta(A_i)\}$$

Theorem A mixed outcome $p = (p_1 \dots p_N)$ is a NE if and only if $p_i \in B_i(p_{-i})$

Nash Theorem

Theorem Every finite strategic game has a mixed strategy Nash equilibrium

Here finite strategic game means

- **finite players**
- each player has **finite pure strategies**

Why is this important

- Difficult to understand properties (NE) without existence
- Find the NE if we know the existence of NE

An Property of MNE

Theorem If a mixed strategy is a best response, then each of the pure strategies (positive prob.) involved in the mixed strategy must be a best response. Particularly, each must yield the same expected payoff.

If a mixed strategy p_i is a best response to the strategies of the others p_{-i} , then each pure strategy a_i s.t. $p_i(a_i) > 0$ is itself a best response to p_{-i} .

Particularly, all $U_i(a_i, p_{-i})$ must be equal

An Property of MNE (cont.)

Theorem $G = \{N, \{A_i\}, \{u_i\}\}$, $p = (p_1, p_2, \dots, p_N)$ is a mixed Nash equilibrium if and only if every pure strategy of player i with positive probability is a best response to p_{-i}

$p = (p_1, p_2, \dots, p_N)$ is an MNE



$p_i(a) \geq 0$, a is a best response to p_{-i}

Proof. See board by contradiction and from

$$U_i(p) = \sum_{a_i \in A_i} p_i(a_i) U_i(p_{-i}, a_i).$$

Mixed Strategy Nash Equilibrium: Example 1

Nash Equilibrium		Player 2	
		L, π_2	$R, 1 - \pi_2$
Player 1	U, π_1	1 2	0 4
	$D, 1 - \pi_1$	0 5	3 2

Fixed Player 1, the expectation payoff of Player 2 on L

$$2\pi_1 + 5(1 - \pi_1)$$

the expectation payoff of Player 2 on R

$$4\pi_1 + 2(1 - \pi_1)$$

Nash Equilibrium implies

$$2\pi_1 + 5(1 - \pi_1) = 4\pi_1 + 2(1 - \pi_1) \rightarrow \pi_1 = 3/5$$

Mixed Strategy Nash Equilibrium: Example 1

Nash Equilibrium		Player 2	
		L, π_2	$R, 1 - \pi_2$
Player 1	U, π_1	1 2	0 4
	$D, 1 - \pi_1$	0 5	3 2

Fixed Player 2, the expectation payoff of Player 1 on U

$$\pi_2$$

the expectation payoff of Player 1 on D

$$3(1 - \pi_2)$$

Nash Equilibrium implies

$$\pi_2 = 3(1 - \pi_2) \rightarrow \pi_2 = 3/4$$

Mixed Strategy Nash Equilibrium: Example 1

		Player 2	
		$L, 3/4$	$R, 1/4$
Player 1	$U, 3/5$	1 2	0 4
	$D, 2/5$	0 5	3 2

Nash Equilibrium:

Player 1 selects the mixed strategy $p_1 = (3/5, 2/5)$ over $\{U, D\}$

Player 2 selects the mixed strategy $p_2 = (3/4, 1/4)$ over $\{L, R\}$

The expected payoff of Player 1 on mixed strategy $p = (p_1, p_2)$

$$3/5 * 3/4 * 1 + 2/5 * 1/4 * 3 = 3/4$$

The expected payoff of Player 2 on mixed strategy $p = (p_1, p_2)$

$$3/5 * 3/4 * 2 + 3/5 * 1/4 * 4 + 2/5 * 3/4 * 5 + 2/5 * 1/4 * 2 = 16/5$$

Prisoners' Dilemma: Mixed Strategy NE

		Prisoner 2	
		Confess(c)	Don't confess(d)
Prisoner 1	Confess(c)	-6 -6	0 -12
	Don't confess(d)	-12 0	-1 -1

PNE and MNE ?

Exercise: Primitive Hunting

Find all Nash Equilibria (pure and mix NE)

		Hunter 2	
		Rabbit (r)	Deer (d)
Hunter 1	Rabbit (r)	3 3	3 0
	Deer (d)	0 3	9 9

Mixed Strategy Nash Equilibrium: Example 2

Rock-Paper-Scissors

		Player 2					
		Rock		Paper		Scissors	
Player 1	Rock	0	0	-1	1	1	-1
	Paper	1	-1	0	0	-1	1
	Scissors	-1	1	1	-1	0	0

Nash Equilibrium (Proof on board):

$$p = (p_1, p_2) = \left(\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right), \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) \right)$$

General Method for MNE

General Method for 2-player games with payoff matrix

		Player 2	
Player 1	0 0	...	1 -1

	-1 1	...	0 0

Step 1: Conjecture some rows and columns (positive prob.)

Step 2: Calculate the mixed strategy

Step 3: Check Nash Equilibria

The running time is exponential with # of strategies

How Many Nash Equilibria?

- A game with a finite number of players, each with a finite number of pure strategies, has at least one Nash equilibrium.
- So if the game has no pure strategy Nash equilibrium, then it must have at least one mixed strategy Nash equilibrium.
- In the worst case, the running time for find MNE is exponential in the # of strategies

Nash Theorem

Theorem Every finite strategic game has a mixed strategy Nash equilibrium

Theorem A mixed outcome p is a NE iff $p \in B(p)$

$$B(p) = (B_1(p_{-1}), B_2(p_{-2}), \dots, B_N(p_{-N}))$$

$$B(p): \Delta(A_1) \times \dots \times \Delta(A_N) \rightarrow \Delta(A_1) \times \dots \times \Delta(A_N)$$

Proof Sketch

Kakutani Fixed Point Theorem Let $f: A \rightarrow A$ be a correspondence with $f(x) \subset A$ for $x \in A$. If

- 1) A is compact, convex and non-empty (finite space);
- 2) $f(x)$ is non-empty for all $x \in A$;
- 3) $f(x)$ is a convex set;
- 4) $f(x)$ has a closed graph: if $\{x_n, y_n\} \rightarrow \{x, y\}$ and $y_n \in f(x_n)$ then $y \in f(x)$,

then, there is a $x \in A$ such that $x \in f(x)$

Proof Sketch

$$B(p) = (B_1(p_{-1}), B_2(p_{-2}), \dots, B_N(p_{-N}))$$

$$B(p): \Delta(A_1) \times \dots \times \Delta(A_N) \rightarrow \Delta(A_1) \times \dots \times \Delta(A_N)$$

- 1) $\Delta(A_1) \times \dots \times \Delta(A_N)$ is compact, convex and non-empty;
- 2) $B(p)$ is non-empty for all p ;
- 3) $B(p)$ is a convex set;
- 4) $B(p)$ has a closed graph: if $\{p^n, \hat{p}^n\} \rightarrow \{p, \hat{p}\}$ and $\hat{p}^n \in B(p^n)$ then $\hat{p} \in B(p)$,

then, there is a $p \in \Delta(A_1) \times \dots \times \Delta(A_N)$ s.t. $p \in B(p)$

Proof of condition 1

$\Delta(A_1) \times \cdots \times \Delta(A_N)$ is compact, convex and non-empty

Pf. It suffices to prove $\Delta(A_i)$ is compact, convex and non-empty. Let $n = |A_i|$. Then

$$\Delta(A_i) = \{(x_1, \dots, x_n) : x_i \in [0,1], \sum x_i = 1\}$$

is a simplex of dimension $n - 1$.

Proof of condition 2

$B(p) = \{(p'_1, p'_2, \dots, p'_N) : p'_i \in B_i(p_{-i})\}$ is non-empty

Pf. It suffices to prove $B_i(p_{-i})$ is non-empty.

$$B_i(p_{-i}) = \operatorname{argmax}_{p'_i \in \Delta(A_i)} U_i(p'_i, p_{-i})$$

Let $f(x) = U_i(x, p_{-i}) = \sum_k x_k U_i(p_{-i}, a_k)$ for $x \in \Delta(A_i)$.

$f(x)$ is continuous and $\Delta(A_i)$ is a nonempty compact set. By [Weierstrass Theorem](#), $f(x)$ has maximum in $\Delta(A_i)$.

$$B_i(p_{-i}) = \operatorname{argmin}_{x \in \Delta(A_i)} f(x) \text{ is not-empty}$$

Proof of condition 3

$B(p) = \{(p'_1, p'_2, \dots, p'_N) : p'_i \in B_i(p_{-i})\}$ is a convex set

Pf. It suffices to prove $B_i(p_{-i})$ is convex. For any $\lambda \in [0,1]$, if $p'_i, p''_i \in B_i(p_{-i})$ then we need to prove

$$\lambda p'_i + (1 - \lambda)p''_i \in B_i(p_{-i}).$$

From $p_i, p'_i \in B_i(p_{-i})$, we have

$$U_i(p_i, p_{-i}) \geq U_i(p_i^*, p_{-i}) \text{ for } p_i^* \in \Delta(A_i)$$

$$U_i(p'_i, p_{-i}) \geq U_i(p_i^*, p_{-i}) \text{ for } p_i^* \in \Delta(A_i)$$

$$U_i(\lambda p_i + (1 - \lambda)p'_i, p_{-i}) \geq U_i(p_i^*, p_{-i}) \text{ for } p_i^* \in \Delta(A_i)$$

Proof of condition 4

$B(p) = \{(p'_1, p'_2, \dots, p'_N) : p'_i \in B_i(p_{-i})\}$ has a closed graph

Pf. Assume $(p^n, \hat{p}^n) \rightarrow (p, \hat{p})$, $\hat{p}^n \in B(p^n)$ but $\hat{p} \notin B(p)$.
There exists $\hat{p}_i \notin B_i(p_{-i})$, i.e., there exist \bar{p}_i and $\epsilon > 0$ s.t.

$$U_i(\bar{p}_i, p_{-i}) \geq U_i(\hat{p}_i, p_{-i}) + 3\epsilon$$

For continuous U_i , $p_{-i}^n \rightarrow p_{-i}$ and $(\hat{p}_i^n, p_{-i}^n) \rightarrow (\hat{p}_i, p_{-i})$

$$U_i(\bar{p}_i, p_{-i}^n) > U_i(\bar{p}_i, p_{-i}) - \epsilon$$

$$U_i(\hat{p}_i, p_{-i}) > U_i(\hat{p}_i^n, p_{-i}^n) - \epsilon$$

We have $U_i(\bar{p}_i, p_{-i}^n) > U_i(\hat{p}_i^n, p_{-i}^n) + \epsilon$. Thus $\hat{p}_i^n \notin B_i(p_{-i}^n)$

Summary on Mixed Strategy

- Mixed strategy, mixed strategy game
- Nash Theorem
- How to find mixed strategy Nash Theorem

Homework

$$G = \{\{1,2\}, \{\{a_1, a_2\}, \{b_1, b_2\}\}, \{u_1, u_2\}\}$$

		Player 2	
		b_1, π_2	$b_2, 1 - \pi_2$
Player 1	a_1, π_1	a c	e g
	$a_2, 1 - \pi_1$	b d	f h

PNE and MNE ?