Game Theory and Applications (博弈论及其应用)

Chapter 6: Strategic Games with Incomplete Information

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Recap on Previous Chapter

- two-player zero-sum game $u_1(a_1, a_2) + u_2(a_1, a_2) = 0$
- Player 1 method: $\max_{a_1 \in A_1} \min_{a_2 \in A_2} u(a_1, a_2)$
- Player 2 method: $\min_{a_2 \in A_2} \max_{a_1 \in A_1} u(a_1, a_2)$
- The strategy outcome (a_1^*, a_2^*) is a Nash Equilibrium if and only if $\max_{a_1 \in A_1} \min_{a_2 \in A_2} u(a_1, a_2) = \min_{a_2 \in A_2} \max_{a_1 \in A_1} u(a_1, a_2)$

The Minmax Theorem For two-player zero-sum finite game $G = \{\{1,2\}, \{A_1,A_2\}, u\}$, we have $\max_{p \in \Delta_1} \min_{q \in \Delta_2} pMq^{\mathsf{T}} = \min_{q \in \Delta_2} \max_{p \in \Delta_1} pMq^{\mathsf{T}}.$

Theorem The optimization problem of $\max_{p \in \Delta_1} \min_{q \in \Delta_2} pMq^{\top}$ is equivalent to

max v

s.t.

$$pM \ge v\mathbf{1}$$

$$p = (p_1, ..., p_m) \in \Delta_1$$

$$\mathbf{1} = (1, ..., 1)^{\mathsf{T}}$$

Linear programming: can be solved in polynomial time

Strategy Game with Complete Information

Previous strategy game with complete information

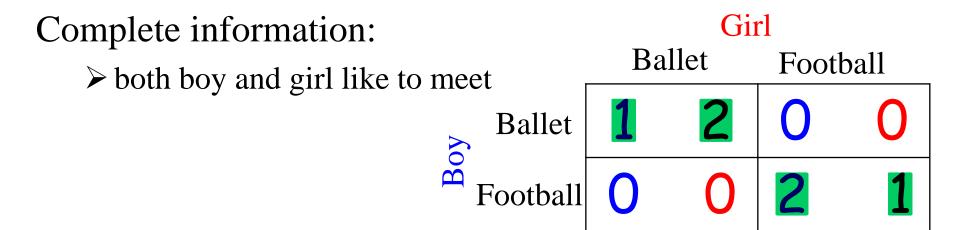
- Who are playing
- All players are rational
- What strategies are possible for each player
- What are the payoffs
- All players know the complete information
- All players knows that all players know complete information

. . .

Games with Incomplete Information

- In many games, some players are not sure the payoff
- Players may have incomplete inform. about components
- Bidder does not know value of other bidders in auction
- Some players may have private information
- Despite of different types of incomplete information, we consider incomplete payoffs

Incomplete information motivates additional interaction



There are 2 pure strategy NE and 1 mixed strategy NE

Both boy and girl know each other in a short time.

The boy is not sure if the girl wishes to meet or not

Incomplete information

The girl has two types: like or dislike

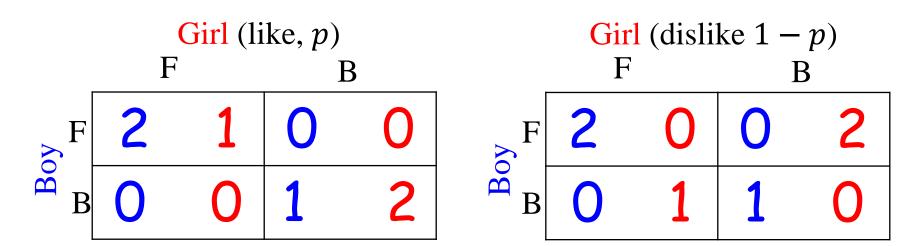
The boy assumes that girl like with probability p

	Girl (like, p)					Gir	p)		
	F	7	В			F		B	
F	2	1	0	0	F	2	0	0	2
В	0	0	1	2	В	0	1	1	0

The girl knows the complete information

The boy does not know

What strategies in this game? how to reason?



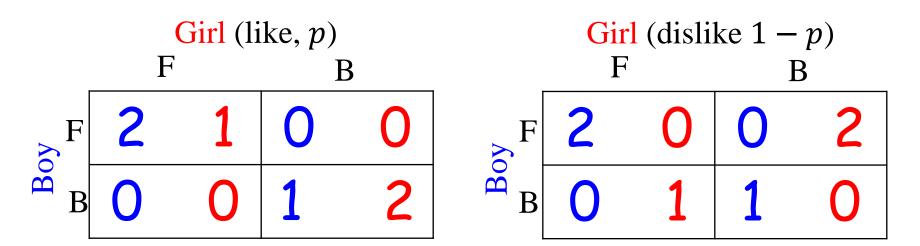
If the boy selects B, then the girl selects (B,F). {B, (B,F)}

$$U_1(B, (B, F)) = p + (1 - p) * 0 = p$$

 $U_1(F, (B, F)) = 0 + (1 - p) * 2 = 2(1 - p)$

if $p \ge 2/3$, {B, (B,F)} is a NE;

if p < 2/3, {B, (B,F)} is not a NE;



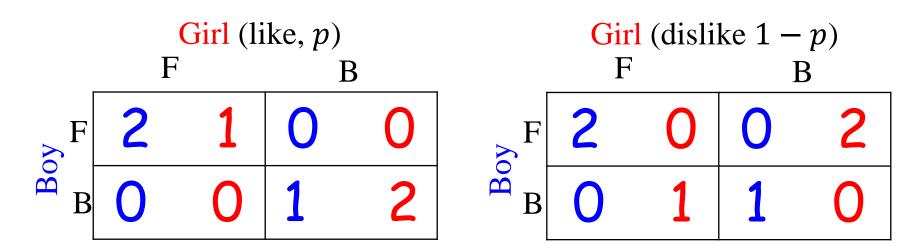
If the boy selects F, then the girl selects (F,B). {F, (F,B)}

$$U_1(F,(F,B)) = 2p + (1-p) * 0 = 2p$$

$$U_1(B,(F,B)) = 0 + (1-p) = 1-p$$

if $p \ge 1/3$, {F, (F,B)} is a NE;

if p < 1/3, {F, (F,B)} is not a NE;



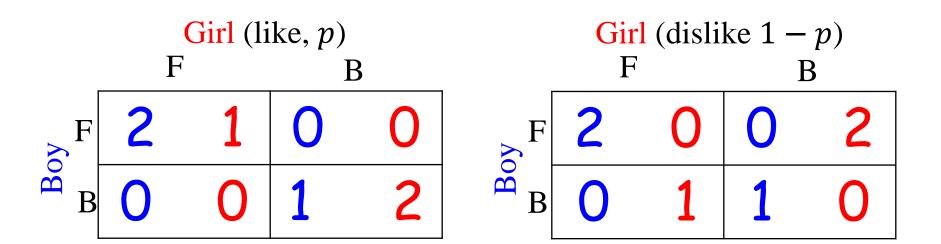
If the boy selects B, then the girl selects (B,F). {B, (B,F)}

$$U_1(B,(B,F)) = p + (1-p) * 0 = p$$

 $U_1(F,(B,F)) = 0 + (1-p) * 2 = 2(1-p)$

if $p \ge 2/3$, {B, (B,F)} is a NE;

if p < 2/3, {B, (B,F)} is not a NE;



In a summary:

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if p \ge 1/3, {F, (F,B)} is a NE;
if p \ge 2/3, {F, (F,B)} and {B, (B,F)} are NEs;
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Bayesian Games

A Bayesian game consists of

- A set of players N
- A set of strategies A_i for each player i
- A set of types Θ_i for each player i
 - Type set Θ_i includes all private information for player i
 - The types on payoff are adequate (Payoff types)
- Probability distribution $p = p(\theta_1, ..., \theta_N)$ on $\times_{i=1...n} \Theta_i$

Bayesian Games (cont.)

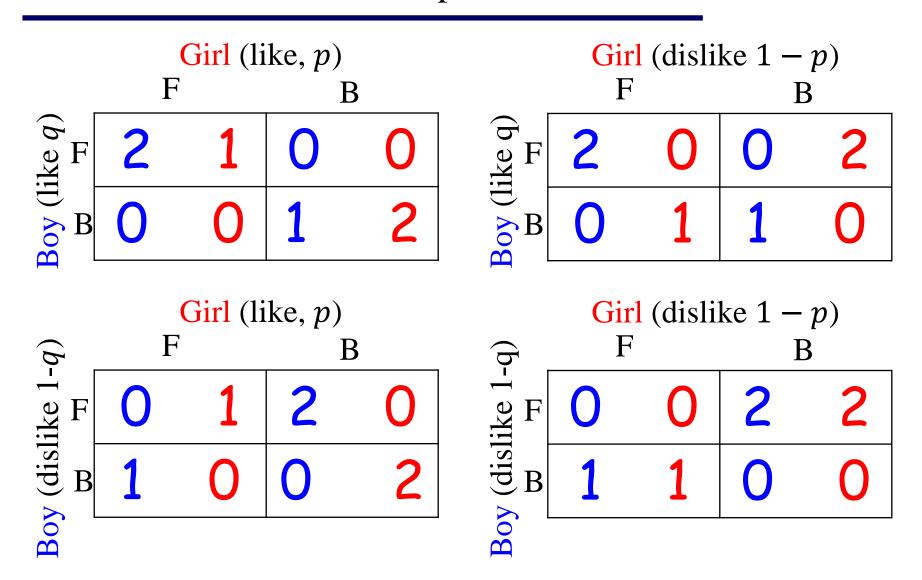
• For player i, a pure strategy is a map $a_i: \Theta_i \to A_i$, which prescribes an strategy for each type

$$a_i = \left(a_i(\theta_i^1), a_i(\theta_i^2), \dots, a_i(\theta_i^{n_i})\right)$$

• A payoff function $u_i:\times_{i=1..N} A_i \times \times_{i=1..n} \Theta_i \to R$

$$u_i(a_1, ..., a_N, \theta_1, ..., \theta_N)$$
 for $a_i \in A_i$ and $\theta_i \in \Theta_i$

$$G = \{N, \{A_i\}, \{\Theta_i\}, \{u_i\}, p\}$$



Bayesian Games (cont.)

- The set of types Θ_i for each player i
 - Player i does not know the selection of Θ_j
 - All types are drawn from the prior dis. $p(\theta_1, ..., \theta_N)$
 - $p(\theta_1, ..., \theta_N) = p(\theta_1)p(\theta_2) ... p(\theta_N)$ (independent types)
- Given $p(\theta_1, ..., \theta_N)$, we have, by Bayes rule,

$$p(\theta_{-i}|\theta_i) = p(\theta_i, \theta_{-i})/p(\theta_i)$$

where
$$\theta_{-i} = (\theta_1, \dots, \theta_{i-1}, \theta_{i+1} \dots, \theta_N)$$

Such game is called Bayes Game

Outcome and Payoff Functions

A pure strategy for player i

$$\left(a_i(\theta_i^1), a_i(\theta_i^2), \dots, a_i(\theta_i^{n_i})\right)$$

An outcome of Bayes game is given by

$$\left(\left(a_1(\theta_1^1), \dots, a_1(\theta_1^{n_1})\right), \dots, \left(a_N(\theta_N^1), \dots, a_N(\theta_N^{n_N})\right)\right)$$

Given a_{-i} , the expected payoff of player i and type θ_i is

$$U_i(a_i(\theta_i), a_{-i}) = \sum_{\theta_{-i}} p(\theta_{-i}|\theta_i) u_i(a_{-i}(\theta_{-i}), a_i, \theta_{-i}, \theta_i)$$

Definition The outcome $(a_1, a_2, ..., a_N)$ is a **Bayesian** Nash Equilibrium if for each type θ_i , we have

$$U_i(a_i(\theta_i), a_{-i}) \ge U_i(a'_i(\theta_i), a_{-i})$$
 for all $a'_i(\theta_i) \in A_i$

Given a_{-i} and type θ_i , the best response for player *i* is

$$B_i(a_{-i}, \theta_i) = \{a_i(\theta_i) : U_i(a_i(\theta_i), a_{-i}) \}$$

$$\geq U_i(a'_i(\theta_i), a_{-i}) \text{ for all } a'_i(\theta_i) \}$$

Theorem The outcome $(a_1, a_2, ..., a_N)$ is a Bayesian NE if and only if for every player i and each type θ_i , we have

$$a_i(\theta_i) \in B_i(a_{-i}, \theta_i)$$

How to find Bayesian Nash Equilibrium

How to find Bayesian Nash Equilibrium

- 1) Find the best response function for each player and type
- 2) Find Bayesian NE by $a_i(\theta_i) \in B_i(a_{-i}, \theta_i)$

Bank Runs (银行挤脱)

- Both players 1 and 2 have a deposit of \$100 in bank
- If the bank manager is good, each player get \$150; if the manager is bad, then they lose all money
- Players can withdraw money but bank has only \$100
 - If only one player withdraws, he get \$100 and the other gets 0
 - If two players both withdraw, each get \$50
- Player 1 believes a good manager with probability p
- Player 2 knows whether the manager is good or bad
- Two players simultaneously make strategy: withdraw/not

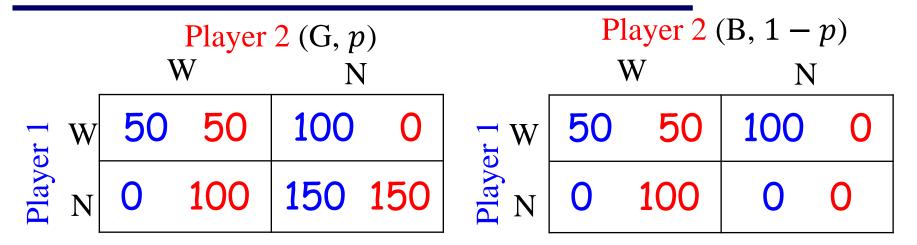
Bank Runs (cont.)

- Two players
- Strategies $A_1 = A_2 = \{W, N\}$
- Types $\Theta_1 = \{1\}; \Theta_2 = \{G, B\}$
- A probability distribution $p_1(\theta_2 = G) = p$
- Payoffs

	Player 2 (G, p)							
	1	V	N					
T W	50	50	100	0				
Playe Z	0	100	150	150				

	Player 2 (B, $1-p$)									
	7	W	N							
W W	50	50	100	0						
Player N	0	100	0	0						

Bank Runs (cont.)



Player 1: W or N

Player 2: W(G) N(G) W(B) N(B)

Bayesian Nash Equilibrium of Bank Runs

	Player 2 (G, p)						Player 2 (B $1 - p$)			
		W		N			W		N	
r 1	W	50	50	100	0	W W	50	50	100	0
Playe	N	0	100	150	150		0	100	0	0

If Player 1 selects W, then

$$B_2(W,G) = \{W\}, B_2(W,B) = \{W\}$$

Outcome (W, (W(G), W(B))): best strategy for Player 2

Is W a best response to W(G), W(B)?

$$U_1(W, (W(G), W(B))) = 50p + 50(1 - p) = 50$$

 $U_1(N, (W(G), W(B))) = 0p + 0(1 - p) = 0$

(W,(W(G), W(B))) is a Bayesian NE

Bayesian Nash Equilibrium of Bank Runs

	Player 2 (G, p)						Player 2 (B $1 - p$)			
		W		N			W		N	
er 1	W	50	50	100	0	T W	50	50	100	0
Playe	N	0	100	150	150	Playe Z	0	100	0	0

If Player 1 selects N, then

$$B_2(N,G) = \{N\}, B_2(N,B) = \{W\}$$

Outcome (N,(N(G),W(B))): Player 2 makes best strategy

Is N a best response to (W(G), W(B))?

$$U_1(N, N(G), W(B)) = 150p + 0(1-p) = 150p$$

 $U_1(W, N(G), W(B)) = 100p + 50(1-p) = 50(1+p)$

If 150p > 50 + 50p, then (N,(N(G),W(B))) is a BNE

Cournot Duopoly with Incomplete Information

Two firms $N = \{1,2\}$

Firm 1 has a cost c_H ;

Firm 2 has two costs c_L and c_H

Firm 1 believes that Firm 2 selects c_H with probability p

Firm 1's strategy $\{q_1: q_1 \ge 0\}$

Firm 2's strategy $\{q_{2,L}, q_{2,H}: q_{2,L} \ge 0 \text{ and } q_{2,H} \ge 0\}$

Price: $a - q_1 - q_{2,L}$ or $a - q_1 - q_{2,H}$

Cournot Duopoly with Incomplete Information

For player 1, the expected payoff function is

$$U_1(q_1, q_{2,L}, q_{2,H}, c_L, c_H) = pq_1(a - q_1 - q_{2,H}) +$$

$$+ (1 - p)q_1(a - q_1 - q_{2,L}) - c_Hq_1$$

For player 2, the expected payoff function of type c_H is

$$U_2(q_1, q_{2,H}, c_H) = (a - q_1 - q_{2,H})q_{2,H} - c_H q_{2,H}$$

For player 2, the expected payoff function of type c_L is

$$U_2(q_1, q_{2,L}, c_L) = (a - q_1 - q_{2,L})q_{2,L} - c_L q_{2,L}$$

For player 1, the expected payoff function is

$$U_1(q_1, q_{2,L}, q_{2,H}, c_L, c_H) = pq_1(a - q_1 - q_{2,H}) +$$

$$+ (1 - p)q_1(a - q_1 - q_{2,L}) - c_H q_1$$

Maximizing $U_1(q_1, q_{2,L}, q_{2,H}, c_L, c_H)$ gives

$$B_1(q_{2,L},q_{2,H}) = \left\{ \frac{a - pq_{2,H} - (1-p)q_{2,L} - c_H}{2} \right\}$$

For player 2, the expected payoff function of type c_H is

$$U_2(q_1, q_{2,H}, c_H) = (a - q_1 - q_{2,H})q_{2,H} - c_H q_{2,H}$$

Maximizing $U_2(q_1, q_{2,H}, c_H)$ gives

$$B_2(q_1, c_H) = \left\{ \frac{a - q_1 - c_H}{2} \right\}$$

Similarly, we have

$$B_2(q_1, c_L) = \left\{ \frac{a - q_1 - c_L}{2} \right\}$$

Bayesian Nash Equilibrium

$$B_{1}(q_{2,L}, q_{2,H}) = \left\{ \frac{a - pq_{2,H} - (1 - p)q_{2,L} - c_{H}}{2} \right\}$$

$$B_{2}(q_{1}, c_{H}) = \left\{ \frac{a - q_{1} - c_{H}}{2} \right\}$$

$$B_{2}(q_{1}, c_{L}) = \left\{ \frac{a - q_{1} - c_{L}}{2} \right\}$$

We solve the Bayesian Nash Equilibrium by

$$q_{1} \in B_{1}(q_{2,L}, q_{2,H})$$

$$q_{2,H} \in B_{2}(q_{1}, c_{H})$$

$$q_{2,L} \in B_{2}(q_{1}, c_{L})$$

Bayesian Nash Equilibrium

The Bayesian Nash Equilibrium is $(q_1, (q_{2,L}, q_{2,H}))$

$$q_{1} = \frac{a - c_{H} - (1 - p)(c_{H} - c_{L})}{3}$$

$$q_{2,H} = \frac{a}{3} - \frac{c_{H} + c_{L}}{6} - p \frac{c_{H} - c_{L}}{6}$$

$$q_{2,L} = \frac{a}{3} - \frac{c_{L}}{3} - p \frac{c_{H} - c_{L}}{6}$$

Discussions

- ➤ Incomplete information affects the outputs of players
- $> q_{2,L} > q_{2,H}$ implies player 2 produce more for lower price

Discussions on Bayesian Nash Equilibrium

- If player 1 knows that player 2 selects c_H (p=1) then $q_1 = q_{2,H} = \frac{a c_H}{3}$
- If player 1 does not know the choices of player 2

$$q_1 = \frac{a - c_H - (1 - p)(c_H - c_L)}{3}$$

Player 1 produces less with the incomplete information

Discussions on Bayesian Nash Equilibrium

- If player 1 knows that player 2 selects c_L (p=0) then $q_1 = q_{2,L} = \frac{a c_L + c_H c_L}{3}$
- If player 1 does not know the choices of player 2 $q_1 = \frac{a c_L + c_H c_L (3 p)(c_H c_L)}{3}$

Player 1 produces less with the incomplete information

Discussions on Bayesian Nash Equilibrium

• If player 1 knows that player 2 selects c_H (p=1) then

$$q_{2,H}=\frac{a-c_H}{3};$$

otherwise,

$$q_{2,H} = \frac{a - c_H}{3} + (1 - p) \frac{c_H - c_L}{6}$$

Player 2 benefits from the incomplete information

The firm will benefit by keeping cost secrets

First Price Auction with Incomplete information

 $N = \{1,2\}$: players bid a building

 v_i : the true value for player i

 v_i : a uniform distribution over [0,1]

 $b_i \ge 0$: the bid price for player i

$$b_i = b_i(v_i) = av_i \quad (a > 0)$$

The payoff functions for player i

$$u_i(\mathbf{b_i}, \mathbf{b_j}) = \begin{cases} v_i - b_i & \text{if } b_i > b_j \\ v_i/2 - b_i & \text{if } b_i = b_j \\ 0 & \text{otherwise} \end{cases}$$

Bayesian NE of First Price Auction

For player *i*, the expected payoff function is

$$U_i(b_i, b_j(v_j), v_i) = (v_i - b_i) \Pr[b_i > b_j(v_j)]$$

$$+ (v_i/2 - b_i) \Pr[b_i = b_j(v_j)]$$

$$\Pr[b_i > b_i(v_j)] = \Pr[b_i > av_j] = \Pr[b_i/a > v_j]$$

 v_i is a uniform distribution over [0,1]

$$U_i(b_i, b_j, v_i) = (v_i - b_i)b_i/a$$

Maximizing $U_i(b_i, b_j, v_i)$ with respect to b_i gives

$$\mathbf{b_i}(v_i) = \frac{v_i}{2}$$

Homework: First Price Auction with Incomplete information

 $N = \{1, ..., N\}$: players bid a building

 v_i : the true value for player i

 v_i : a uniform distribution over [0,1]

 $b_i \ge 0$: belief that the bid price for player i

$$b_i = b_i(v_i) = av_i \quad (a > 0)$$

The payoff functions for player *i*

$$u_i(\mathbf{b}_1, \dots \mathbf{b}_N) = \begin{cases} v_i - b_i & \text{if } b_i > \max_{j \neq i} b_j \\ v_i/2 - b_i & \text{if } b_i = \max_{j \neq i} b_j \\ 0 & \text{otherwise} \end{cases}$$

Conclusions

- Strategy game with incomplete information
- Bayes game $G = \{N, \{A_i\}, \{\Theta_i\}, \{u_i\}, p\}$
- Bayes Nash Equilibrium
- How to find Bayes Nash equilibrium

Bayesian NE of First Price Auction (N Players)

For player i, the expected payoff function is

$$U_{i}(b_{i}, b_{j}(v_{j}), v_{i}) = (v_{i} - b_{i}) \Pr[b_{i} > \max_{j \neq i} b_{j}(v_{j})]$$
$$U_{i}(b_{i}, b_{j}, v_{i}) = (v_{i} - b_{i})(b_{i}/a)^{N-1}$$

Maximizing $U_i(b_i, b_j, v_i)$ with respect to b_i gives

$$\mathbf{b_i}(v_i) = \frac{\mathbf{N} - 1}{N} v_i$$