作业

Homework1

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A Game Theory Homework Assignment

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题目一

Let a_n be a sequence of positive real number. Denote by $S_n = \sum_{i=1}^{n} a_i$. If $S_{n+1} \geq 2S_n$, then there exists a constant $c > 0$, such that $a_n \geq 2^n c$ for every positive n.

解答： 因为 $S_{n+1} \geq 2S_n$，所以 $a_{n+1} \geq S_n$；
令 $c = a_1/4$, 则 $a_1 = 4c \geq 2c$
$a_2 \geq S_1 = a_1 = 2^2 c$;
$a_3 \geq S_2 = a_1 + a_2 \geq (2^2 + 2^2)c = 2^3 c$;
$a_4 \geq S_3 = a_1 + a_2 + a_3 \geq (2^2 + 2^2 + 2^3)c = 2^4 c$;
$\cdots$
$a_n \geq S_n = a_1 + a_2 + \cdots + a_{n-1} \geq (2^2 + 2^2 \cdots + 2^{n-1})c = 2^n c$。

题目二

Suppose that $(1, 1, -1)$ is an eigenvector of matrix

$$\begin{bmatrix} 2 & -1 & b \\ 5 & a & 3 \\ -1 & 2 & -1 \end{bmatrix}$$

Solve a, b, and the corresponding eigenvalue.

解答： 令 $A = \begin{bmatrix} 2 & -1 & b \\ 5 & a & 3 \\ -1 & 2 & -1 \end{bmatrix}$, $x = (1, 1, -1)^T$, 则 $Ax = \lambda x$;

化简得，$\begin{bmatrix} 2 & -1 & b \\ 5 & a & 3 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$，$a = -4$，$b = 3$;

$A - \lambda I = \begin{bmatrix} 2-\lambda & -1 & 3 \\ 5 & -4-\lambda & 3 \\ -1 & 2 & -1-\lambda \end{bmatrix}$，得到特征多项式为 $-\lambda^3 - 3\lambda^2 + 4\lambda + 12 = 0$
即 $(2-\lambda)(2+\lambda)(\lambda+3) = 0$
即 $a = -4$，$b = 3$，特征向量 x 对应的特征值为-2，全部特征值为 2，-2，-3。

题目三

For $\epsilon \in [0, 1]$, prove that

$$\frac{1}{2}(1 + \sqrt{1+4\epsilon^2})e^{1-\sqrt{1+4\epsilon^2}} \leq e^{-(\epsilon^2 - \epsilon^3)/2}$$

解答：
因为

$$(1-\epsilon)(1 + \sqrt{1+4\epsilon^2}) \leq 1 + \sqrt{1+4\epsilon^2} \leq 1 + \sqrt{5} \leq 4$$

所以

$$\frac{4}{1+\sqrt{1+4\epsilon^2}} \geq 1-\epsilon$$

所以

$$\frac{-4\epsilon^2}{1+\sqrt{1+4\epsilon^2}} \leq -(\epsilon^2-\epsilon^3)$$

又因为

$$-4\epsilon^2 = (1+\sqrt{1+4\epsilon^2})(1-\sqrt{1+4\epsilon^2})$$

所以

$$1-\sqrt{1+4\epsilon^2} \leq -(\epsilon^2-\epsilon^3)$$

所以

$$\frac{1}{2}(1-\sqrt{1+4\epsilon^2}) \leq -(\epsilon^2-\epsilon^3)/2$$

所以

$$\frac{1}{2}(1+\sqrt{1+4\epsilon^2}) \leq -(\epsilon^2-\epsilon^3)/2+\sqrt{1+4\epsilon^2}$$

又因为

$$x \leq e^{x-1}$$

所以

$$\frac{1}{2}(1+\sqrt{1+4\epsilon^2}) \leq e^{-(\epsilon^2-\epsilon^3)/2-1+\sqrt{1+4\epsilon^2}}$$

所以

$$\frac{1}{2}(1+\sqrt{1+4\epsilon^2})e^{1-\sqrt{1+4\epsilon^2}} \leq e^{-(\epsilon^2-\epsilon^3)/2}$$

题目 n-Cournot Competition

n firms compete by choosing how much to produce

$$G=\{\{1,\ldots,n\},\{q_1,\ldots,q_n\},\{u_1,\ldots,u_n\}\}$$

- Price

$$p\left(q_1+\cdots+q_n\right)=a-b\left(q_1+\cdots+q_n\right)$$

- Costs $(i=1,\ldots,n)$

$$c_i\left(q_i\right)=cq_i$$

- Payoffs $(i=1,\ldots,n)$

$$u_i\left(q_1,\ldots,q_n\right)=\left(a-b\left(q_1+\cdots+q_n\right)-c\right)q_i$$

- Condition $a>c>0, b>0, q_i \geq 0$

解答： **竞争关系**：

最佳响应为

$$B_i(q_{-i}) = \max(0, \frac{a-c-b\sum_{k=1,k\neq i}^{n} q_k}{2b})$$

证明：对于第 i 个厂商，

$$\max_{q_i\geq 0} u_i\left(q_1,\cdots,q_n\right) = \max_{q_i\geq 0}\left(a-b\left(q_1+\cdots+q_n\right)-c\right)q_i$$
$$\frac{\partial u_i\left(q_1,\cdots,q_n\right)}{\partial q_i} = a-c-b\sum_{k=1,k\neq i}^{n} q_k - 2bq_i = 0$$
$$q_i = \frac{\left(a-c-b\sum_{k=1,k\neq i}^{n} q_k\right)}{2b}$$

纳什均衡点为

$$\{(\frac{a-c}{(n+1)b},\cdots,\frac{a-c}{(n+1)b})\}$$

证明：假设 $(q_1^\star,\cdots,q_n^\star)$ 是纳什均衡点，那么利用反证法易知 $q_i^\star > 0$；$(q_1^\star,\cdots,q_n^\star)$ 满足如下关系：

$$q_1^\star = \frac{\left(a-c-b\sum_{k=1,k\neq 1}^{n} q_k^\star\right)}{2b}$$
$$q_i^\star = \frac{\left(a-c-b\sum_{k=1,k\neq i}^{n} q_k^\star\right)}{2b}$$
$$q_n^\star = \frac{\left(a-c-b\sum_{k=1,k\neq n}^{n} q_k^\star\right)}{2b}$$

联立即可解得纳什均衡点为 $(\frac{a-c}{(n+1)b},\cdots,\frac{a-c}{(n+1)b})$，此时收益为 $(\frac{(a-c)^2}{(n+1)^2b},\cdots,\frac{(a-c)^2}{(n+1)^2b})$。

合作关系：

$$\sum_{i=1}^{n} u_i(q_1,\cdots,q_n) = \left(a-b\left(q_1+\cdots+q_n\right)-c\right)\left(q_1+\cdots+q_n\right)$$

$$\sum_{i=1}^{n} u_i(q_1,\cdots,q_n) = \left(a-b\left(x\right)-c\right)x, \quad x=\left(q_1+\cdots+q_n\right)$$

$$x = \frac{a-c}{2b}$$

即纳什均衡点为 $(\frac{a-c}{2nb},\cdots,\frac{a-c}{2nb})$，此时收益为 $(\frac{(a-c)^2}{4nb},\cdots,\frac{(a-c)^2}{4nb})$。