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Definition

A strategic game (normal form game) consists of

- A finite set N of players
- A non-empty strategy set A_i for each player $i \in N$
- A payoff function $u_i: A_1 \times A_2 \times \cdots \times A_N \to R$ for $i \in N$

$$G=\{N, \{A_i\}_{i=1}^{N}, \{u_i\}_{i=1}^{N}\}$$

- An outcome $a^* = (a_1^*, a_2^*, \ldots, a_N^*)$ is a **Nash equilibrium (NE)** if for each players i

$$u_i(a_i^*, a_{-i}^*) \geq u_i(a_i, a_{-i}^*) \text{ for all } a_i \in A_i.$$

How to Find Nash Equilibria

- One way of finding Nash equilibrium for continuous strategies A_i:

(1) Find the best response correspondence for each player

Best response correspondence

$$B_i(a_{-i}) = \{a_i \in \text{argmax}_{a_i \in A_i} u_i(a_i, a_{-i})\}$$

(2) Find all Nash Equilibria $(a_1^*, a_2^*, \dots, a_N^*)$ such that

$$a_i^* \in B_i(a_{-i}^*) \text{ for each player}$$

Example

- Find all Nash equilibria

P2

		h	i	j	k	l	m
P1	a	7, 5	8, 6	2, 2	2, 3	6, 9	6, 5
	b	6, 5	9, 6	5, 8	6, 7	8, 8	7, 4
	c	9, 7	1, 1	7, 9	3, 2	9, 6	9, 2
	d	2, 14	10, 12	6, 5	6, 3	7, 2	9, 12
	e	8, 6	5, 9	3, 9	7, 5	13, 15	8, 9

Cournot Competition(古偌竞争, 1838)

- Two firms compete by choosing how much to produce

$$G = \{\{1,2\},\{q_1,q_2\},\{u_1,u_2\}\}$$

  - Price

$$p(q_1+q_2) = \max(0, a - b(q_1+q_2))$$

  - Costs $(i = 1,2)$

$$c_i(q_i) = cq_i$$

  - Payoffs $(i = 1,2)$

$$u_i(q_1,q_2) = (\max\big(0, a - b(q_1+q_2)\big) - c)q_i$$

  - Condition $a > \mathrm{b}, c > 0, q_1 \geq 0, q_2 \geq 0$

Cournot: Best Response Correspondence

Best Response Correspondence is given by

$$B_i(q_{-i}) = \max(0, (a - c - bq_{-i})/2b)$$

Proof. We will prove for i=1 (similarly for i=2)

If $q_2 \geq (a-c)/b$, then $u_1(q_1, q_2) \leq 0$ for any $q_1 > 0$. $q_1 = 0$.

If $q_2 < (a-c)/b$, then

$$u_i(q_1, q_2) = (a - c - b(q_1 + q_2))q_i$$

$$\frac{\partial u_1(q_1, q_2)}{\partial q_1} = a - c - bq_2 - 2bq_1 = 0$$

$$q_1 = (a - c - bq_2)/2b$$

Cournot: Nash Equilibrium

The Nash equilibria is give by

$$\left\{\left(\frac{a-c}{3b},\frac{a-c}{3b}\right)\right\}$$

Proof. Assume that (q_1^*, q_2^*) is a Nash equilibrium.

1) Prove $q_1^* > 0$ and $q_2^* > 0$ by contradiction

2) (q_1^*, q_2^*) is such that $q_1^* > 0, q_2^* > 0$

$$q_1^* = B_1(q_2^*) = (a - c - bq_2^*)/2b$$

$$q_2^* = B_2(q_1^*) = (a - c - bq_1^*)/2b$$

Mixed Strategies

Strategy game

$$G = \{N, \{A_1, A_2, \dots, A_N\}, \{u_1, u_2, \dots, u_N\}\}$$

Pure strategy: each strategy in A_i

Mixed strategy: a probability over the set A_i of strategies

Pure strategy can be viewed as a special mixed strategy

Nash Theorem Every finite strategic game has a mixed strategy Nash equilibrium

How to calculate Mixed Nash Equilibria

Theorem If a mixed strategy is a best response, then each of the pure strategies (positive prob.) involved in the mixed strategy must be a best response. Particularly, each must yield the same expected payoff

		Player 2	
		L, π_2	$R, 1-\pi_2$
Player 1	U, π_1	1 , 2	0 , 4
	$D, 1-\pi_1$	0 , 5	3 , 2

Dominant Strategies and Nash Equilibrium

A pure strategy a_i **strictly dominates** a_i' if

$$u_i(a_i, a_{-i}) > u_i(a_i', a_{-i}) \text{ for all } a_{-i} \in A_{-i}$$

Theorem A **strictly** dominated strategy is never used with positive probability in a mixed strategy Nash equilibrium

How to find NE:

Step 1: eliminate all strictly dominated strategies

Step 2: Find all Nash Equilibria

Two-Player zero-sum game

Maxmin (最大化最小原则)

Maxmin (最小化最大原则)

		Player 2 L	R
Player 1	U	3, -3	-1, 1
	D	-2, 2	1, -1

The Minmax Theorem For two-player zero-sum finite game $G = \{\{1,2\}, \{A_1, A_2\}, u\}$, we have

$$\max_{p\in\Delta_1}\min_{q\in\Delta_2} pMq^\top = \min_{q\in\Delta_2}\max_{p\in\Delta_1} pMq^\top .$$

How to solve Nash Equilibrium? Player 1:

$$\max \ \ v$$
$$3x_1 - 2x_2 \geq v,$$
$$-x_1 + x_2 \geq v$$
$$x_1 + x_2 = 1, x_1 > 0, x_2 > 0$$

Stackelberg game (主 从 博 弈)

		Player 2	
		Left	Right
Player 1	Up	1, 1	3, 0
	Down	0, 0	2, 1

Player 1: "leader" **Player 2 "follower"**

1) Leader selects a (possibly mixed) strategy x_1
2) Follower learns about x_1, selects the best response x_2
3) The payoff matrix is of size $d_1 \times d_2$

Content

Bayesian Games

A Bayesian game consists of

- A set of players N
- A set of strategies A_i for each player i
- A set of types Θ_i for each player i
  - Type set Θ_i includes all private information for player i
  - The types on payoff are adequate (Payoff types)
- Probability distribution $p = p(\theta_1, \ldots, \theta_N)$ on $\times_{i=1..n} \Theta_i$
- A payoff function $u_i : \times_{i=1..N} A_i \times \times_{i=1..n} \Theta_i \rightarrow R$

$$u_i(a_1, \ldots, a_N, \theta_1, \ldots, \theta_N) \text{ for } a_i \in A_i \text{ and } \theta_i \in \Theta_i$$

$$\boldsymbol{G = \{N, \{A_i\}, \{\Theta_i\}, \{u_i\}, p\}}$$

Bayesian Games (cont.)

Definition The outcome $(a_1, a_2, \dots, a_N)$ is a **Bayesian Nash Equilibrium** if for each type θ_i, we have

$$U_i(a_i(\theta_i), a_{-i}) \geq U_i(a'_i(\theta_i), a_{-i}) \text{ for all } a'_i(\theta_i) \in A_i$$

How to find Bayesian Nash Equilibrium

1) Find the best response function for each player and type
2) Find Bayesian NE by $a_i(\theta_i) \in B_i(a_{-i}, \theta_i)$

Bank Runs

- Two players
- Strategies $A_1 = A_2 = \{W, N\}$
- Types $\Theta_1 = \{1\}; \Theta_2 = \{\mathrm{G}, \mathrm{B}\}$
- A probability distribution $p_1(\theta_2 = \mathrm{G}) = p$
- Payoffs

Player 2 (G, p)

Player 1	W	N
W	50 50	100 0
N	0 100	150 150

Player 2 (B, $1-p$)

Player 1	W	N
W	50 50	100 0
N	0 100	0 0

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Extensive Game

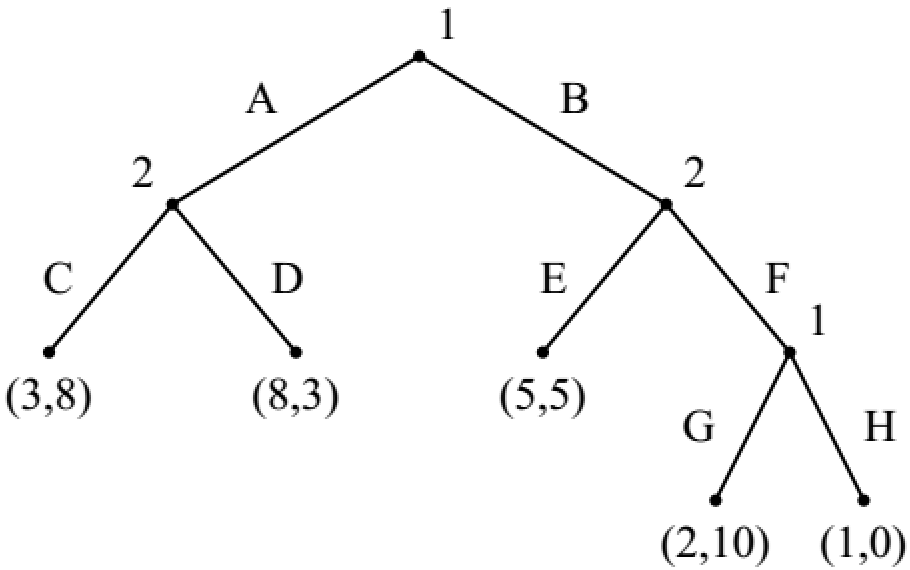
An extensive game with perfect information is defined by

- Players N is the set of N players
- Histories H is a set of sequence (finite or infinite)
- Player function
  - P assigns to each non-terminal history a player of N
  - $P(h)$ denotes the player who takes action after the history h
- Payoff function $u_i: Z \to R$

$$\boldsymbol{G} = \left\{\boldsymbol{N}, \boldsymbol{H}, \boldsymbol{P}, \{\boldsymbol{u_i}\}\right\}$$

Induced Strategic Game and NE

Every extensive game can be converted to a strategy game



	CE	CF	DE	DF
AG	$3,8$	$3,8$	$8,3$	$8,3$
AH	$3,8$	$3,8$	$8,3$	$8,3$
BG	$5,5$	$2,10$	$5,5$	$2,10$
BH	$5,5$	$1,0$	$5,5$	$1,0$

Subgame

Definition A subgame is a set of nodes, strategies and payoffs, following from a single node to the end of game.

Definition An outcome is $a = (a_1^*, a_2^*, \ldots, a_N^*)$ is a subgame perfect (子博弈完美) if it is Nash Equilibrium in every subgame

- Subgame perfect is a Nash Equilibrium
- This definition rules out “non-credible threat”

Theorem Every extensive game with perfect information has a subgame perfect

Back Induction (后向归纳)

How to find subgame perfect Equilibria (SPE)

Back induction is the process of "pruning the game tree" described as follows:

- Step 1: start at each of the final subgame in the game, and solve for the player's equilibrium. Remove that subgame and replace it with payoff of the player's choice
- Step 2: Repeat step 1 until we arrive at the first node in the extensive game

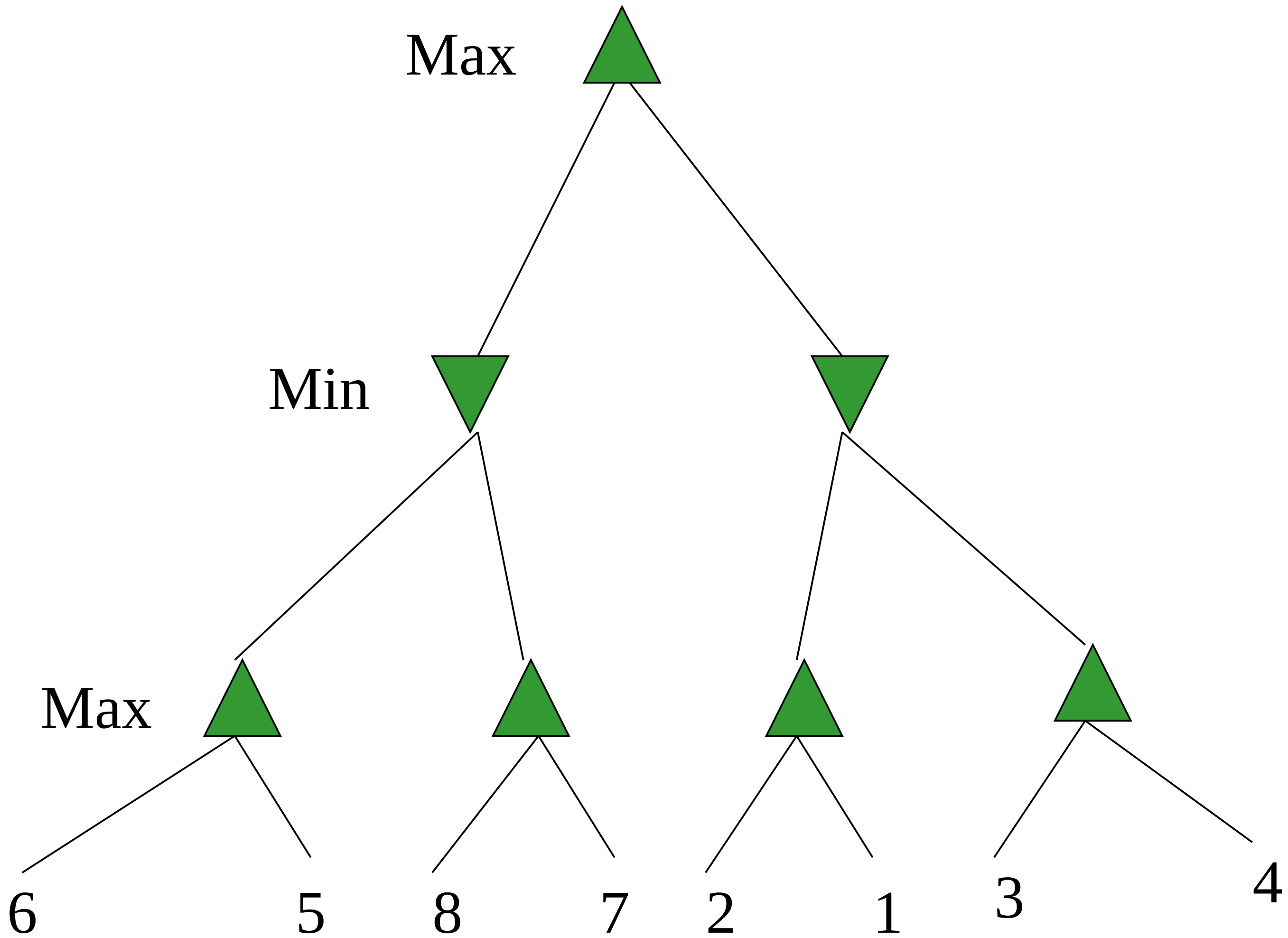
Theorem The set of strategy game constructed by backwards induction is equivalent to the set of SPE

Game Tree Search

- Two players made adversary actions alternately
- Game problem -> search problem
- Minimax algorithm
- Evaluation function
- Alpha-Beta Pruning

Example

What nodes can be pruned?



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Definition of Extensive Game with Imperfect Information

An extensive game with imperfect information is defined by $G = \{N, H, P, I, \{u_i\}\}$

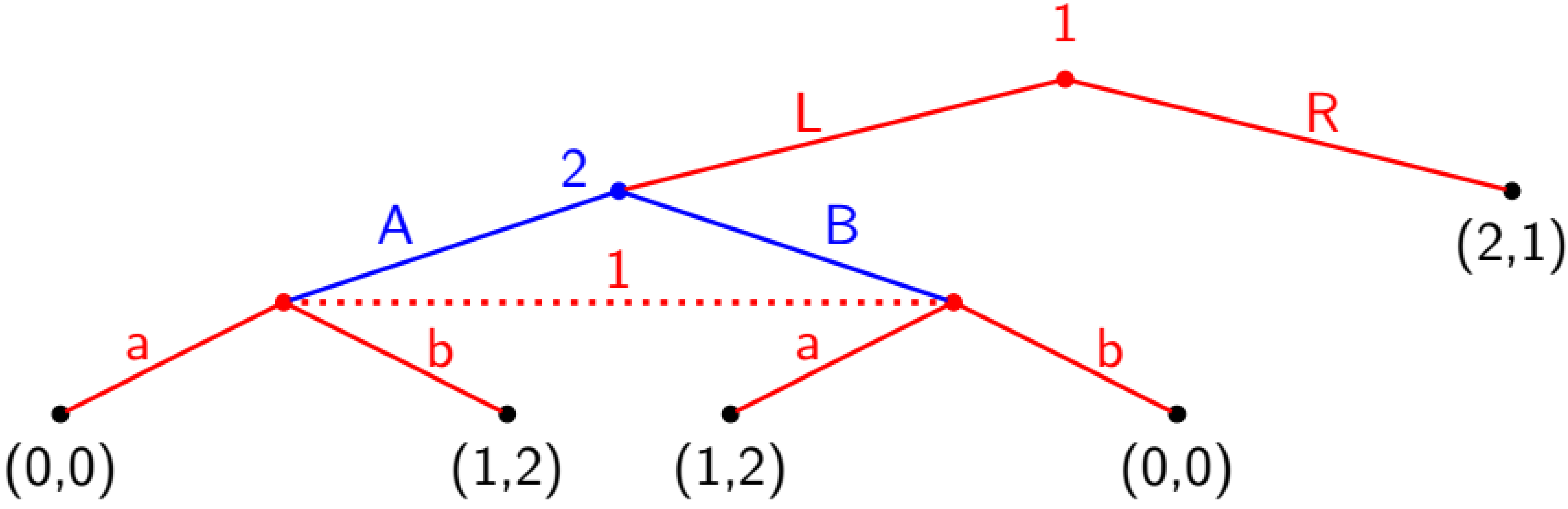
- Information set $I = \{I_1, I_2, \ldots I_N\}$ is the set of information partition of all players' strategy nodes, where the nodes in an information set are indistinguishable to player
  - $I_i = \{I_{i1}, \ldots, I_{ik_i}\}$ is the information partition of player i
  - $I_{i1} \cup \cdots \cup I_{ik_i}$ ={all nodes of player i}
  - $I_{ij} \cap I_{ik} = \emptyset$ for all $j \neq$ k
  - Action set $A(h) = A(h')$ for $h, h' \in I_{ij}$, denote by $A(I_{ij})$
  - $P(I_{ij})$ be the player who plays at information set I_{ij}
- An extensive game with perfect information is a special case where each I_{ij} contains only one node

Pure Strategies

- A pure strategy for player i selects an available action at each of i's information sets $I_{i1}, \ldots, I_{im}$
- All pure strategies for player i is

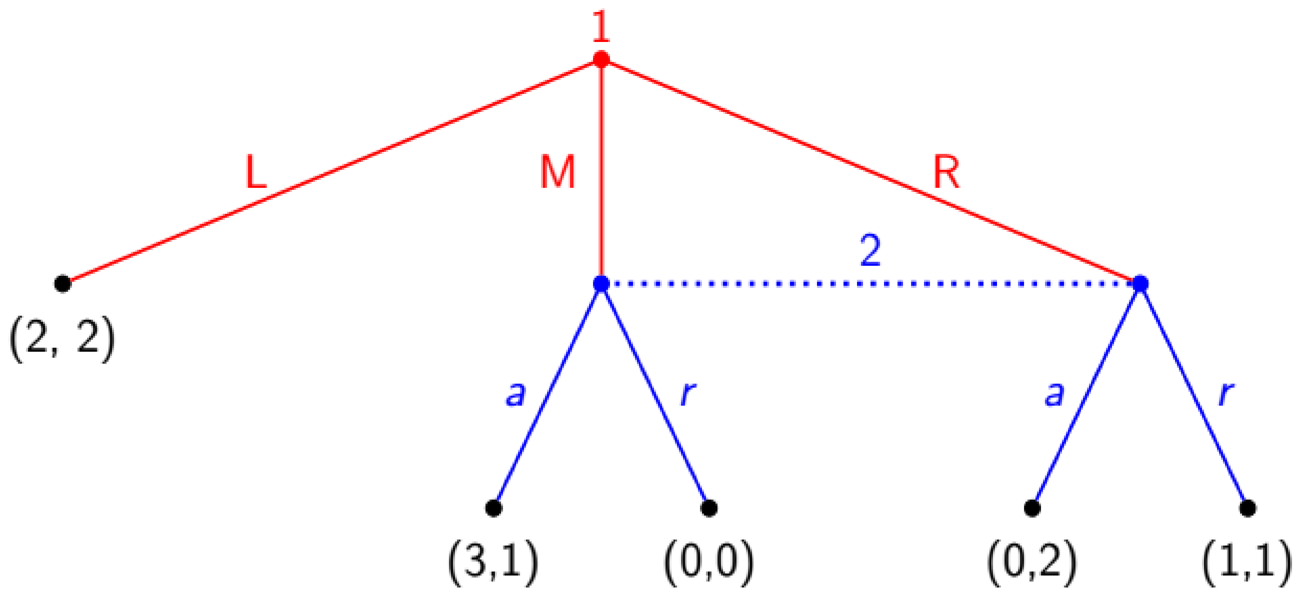
$$A(I_{i1}) \times A(I_{i2}) \times A(I_{im})$$

where $A(I_{ij})$ denotes the strategies available in I_{ij}



What's the pure strategies for players 1 and 2?

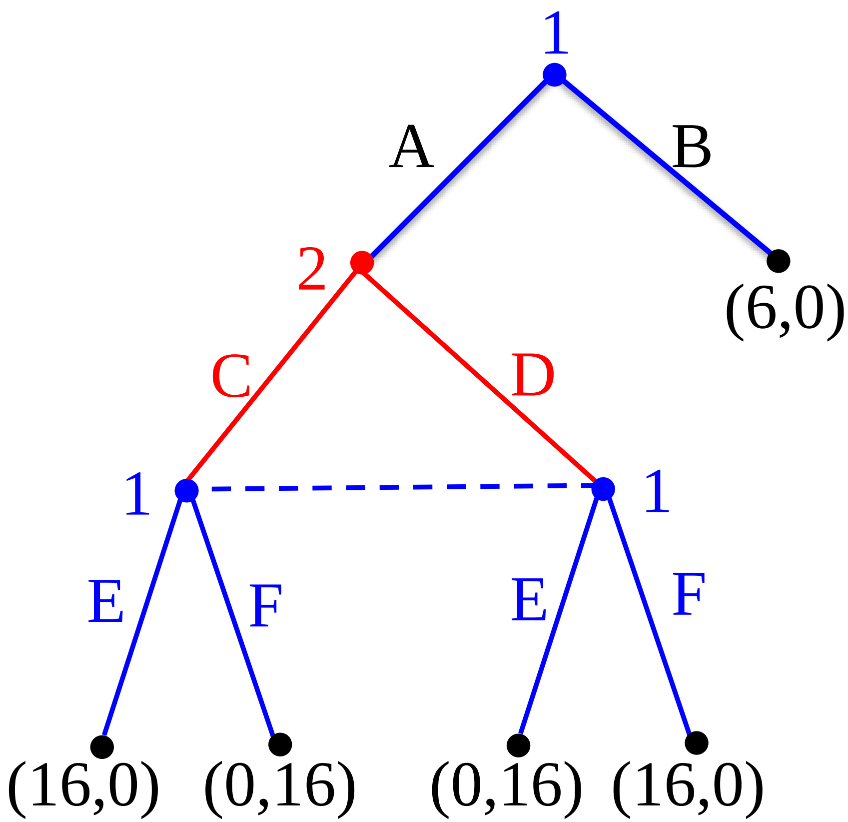
Normal-Form Representation of Extensive Imperf. Game



	a	r
L	2,2	2,2
M	3,1	0,0
R	0,2	1,1

- The pure and mixed strategy Nash Equilibrium remains?
- What's the difference from the extensive game with perfect information game?

Example



How to calculate the sequential equilibrium?

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Definition and Folk Theorem

- A repeated game $G^{\mathrm{T}}(\delta)$ consists of stage game G, terminal date T and discount factor δ

- Folk Theorem
  - An infinitely repeated game with a stage game equilibrium $a^* = (a_1^*, a_2^*, \dots, a_N^*)$ with payoffs $u^* = (u_1^*, u_2^*, \dots, u_N^*)$.
  - Suppose there is another $\hat{a} = (\hat{a}_1, \hat{a}_2, \dots, \hat{a}_N)$ with payoffs $\hat{u} = (\hat{u}_1, \hat{u}_2, \dots, \hat{u}_N)$, where, $\hat{u}_i \geq u_i^*$ for every player i
  - There is a Subgame Perfect Nash Equilibrium for some discount factor δ

Solving for Equilibria in Repeated Games

1. Solve all equilibria of the stage game (Competition)
2. Find an outcome (equilibrium/not) where all the players do at least as well as in the stage game (Cooperation)
3. Design trigger strategies that support cooperation and punish with competition
4. Compute the maximum discount factor so that cooperation is an equilibrium
5. The trigger strategies are an SPEN of the infinitely repeated game for some larger discount factor