Game Theory and Applications (博弈论及其应用)

# Chapter 10: Extensive Game with Imperfect Information

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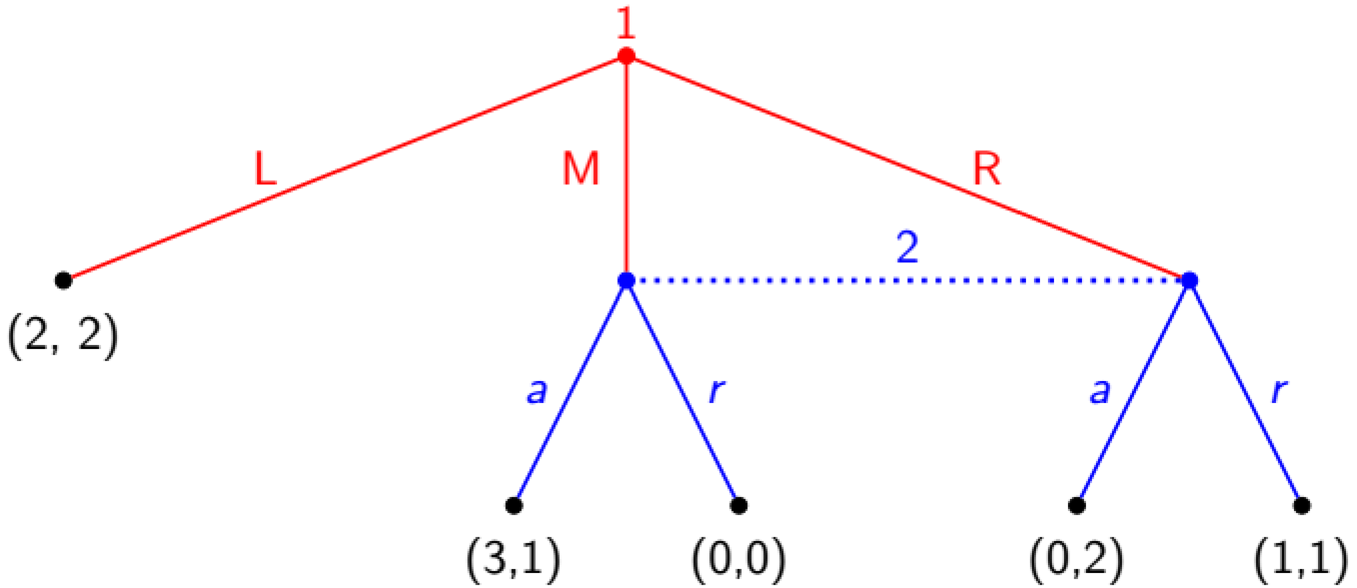
# Recap on Extensive Game

- The extensive game is an alternative representation that makes the temporal structure explicit
- Nash equilibrium
- Subgame perfect equilibrium (SPE): an outcome is SPE if it is Nash Equilibrium in every subgame
- How to find SPE – back induction and one deviation
- Two variants
  - Perfect information: game tree
  - **Imperfect information**

# Motivation

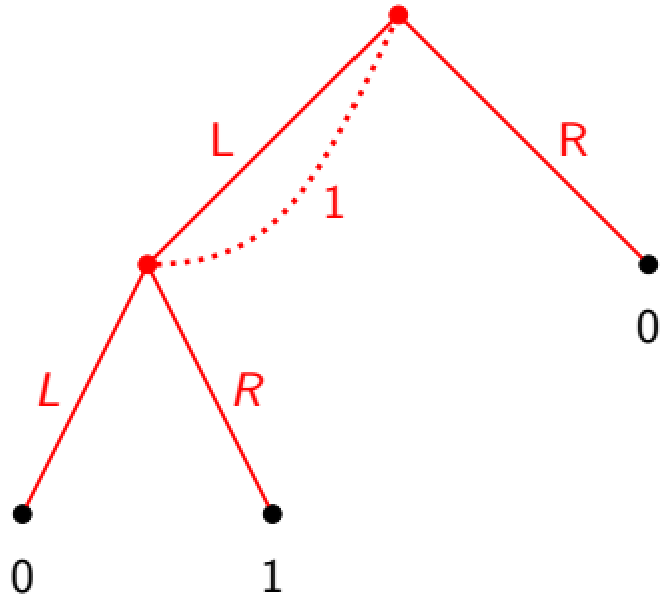
- Extensive game with perfect information
  - Know all previous strategies for all players
- Sometimes, players
  - Don't know all the strategies the other take or
  - Don't recall all their own past actions
- Extensive game captures some of this unknown
  - A later choice is made without knowledge of an earlier choice
- How to represent the case two players make choices at the same time, in mutual ignorance of each other

# Example




Player 2 does not know the choice of player 1 over M or R

# Example




Player 1 does not know if he has made a choice or not

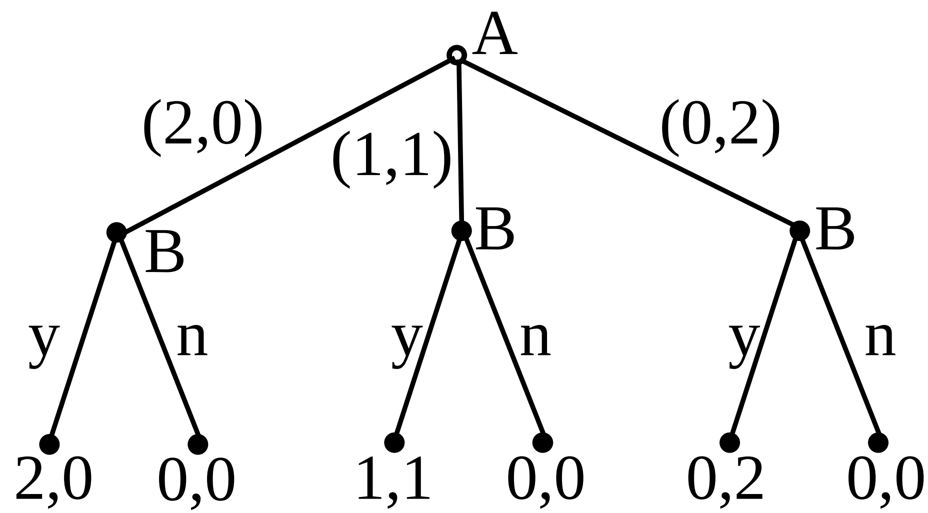
# Definition of extensive game with Perfect Information

An extensive game with perfect information is defined by
$G = \{N, H, P, \{u_i\}\}$

- Players $N$ is the set of $N$ players
- Histories $H$ is a set of sequence $a^1$ ... $a^k$, where each component $a^i$ is a strategy
- Player function $P(h): H \rightarrow N$ is the player who takes action after the history $h$
- Payoff function $u_i$
- Action set $A(h) = \{a: (h, a) \in H\}$ for non-terminal $h$

# Ultimatum Game

$G = \{N, H, P, \{u_i\}\}$




$N$={A,B}

$H$={∅,(2,0),(1,1),(0,2), ((2,0),y)}

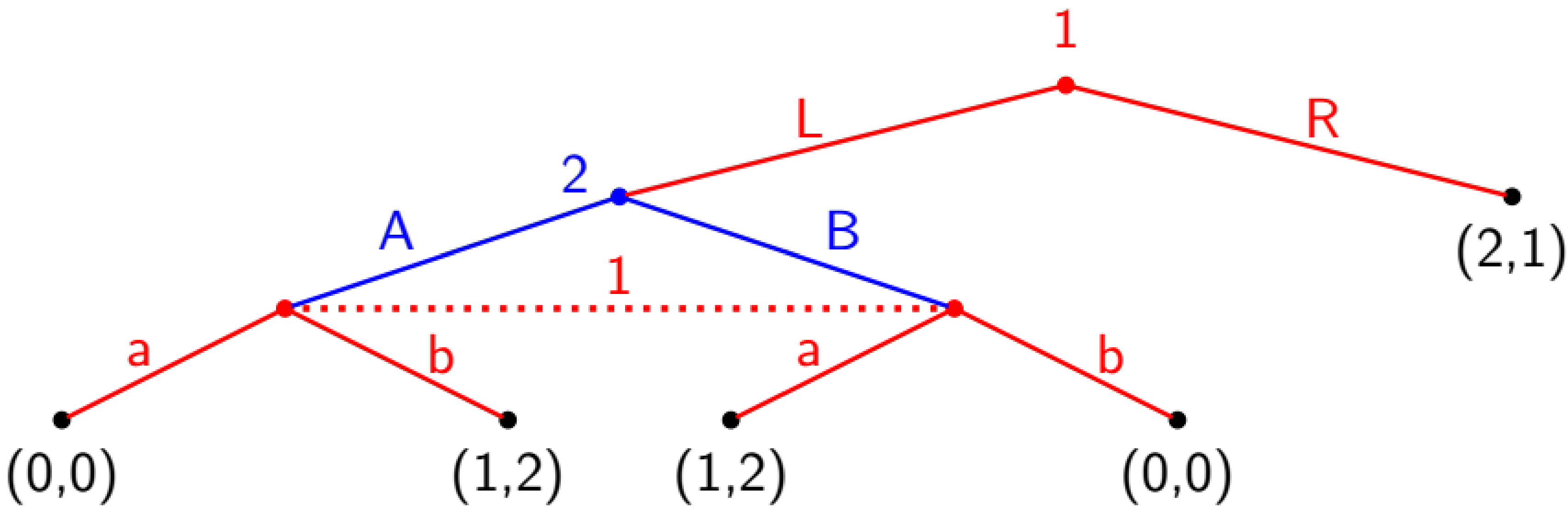
∪{((2,0),n), ((1,1),y), ((1,1),n)}

∪{((0,2),y), ((0,2),n)}

$P$: $P$(∅)=A; $P$((2,0))=B; $P$((1,1))=B; $P$((0,2))=B

$A$: $A$(∅)={(2,0),(0,2),(1,1)}; $A$((2,0))=$A$((0,2))=$A$((1,1))={y,n}

# Extensive Game with Imperfect Information

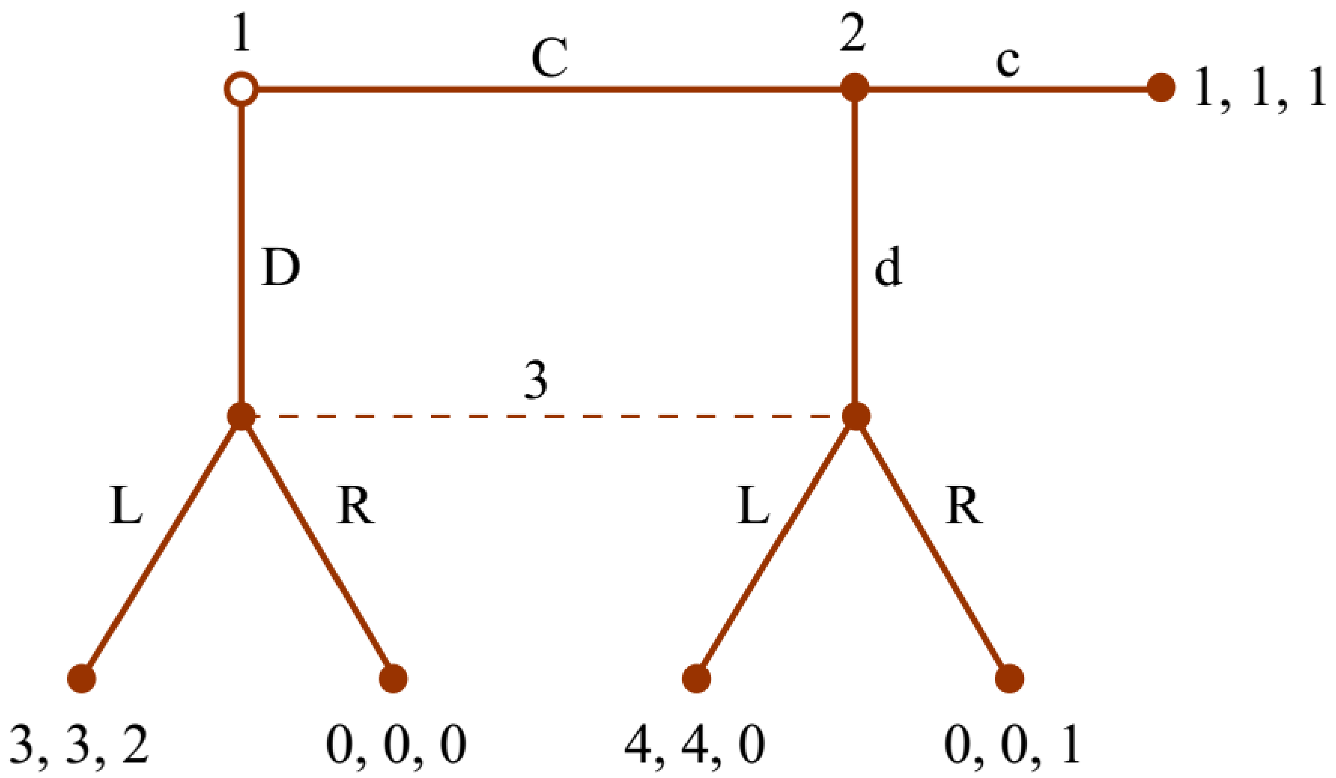



Player 1 does not know the choice of player 2 over LA or LB

Nonterminal histories: $\{\emptyset, \mathrm{L}, \mathrm{LA}, \mathrm{LB}\}$

- Player 1 has information set $I_1 = \{\emptyset, \{\mathrm{LA}, \mathrm{LB}\}\}$,
- Player 2 has information set $I_2 = \{\{\mathrm{L}\}\}$

# Example




- Player 1 has information set $I_{11} = \{\emptyset\}$
- Player 2 has information set $I_{21} = \{C\}$
- Player 3 has the information set $I_{31} = \{D, Cd\}$

# Definition of Extensive Game with Imperfect Information

An extensive game with imperfect information is defined by $G = \{N, H, P, I, \{u_i\}\}$

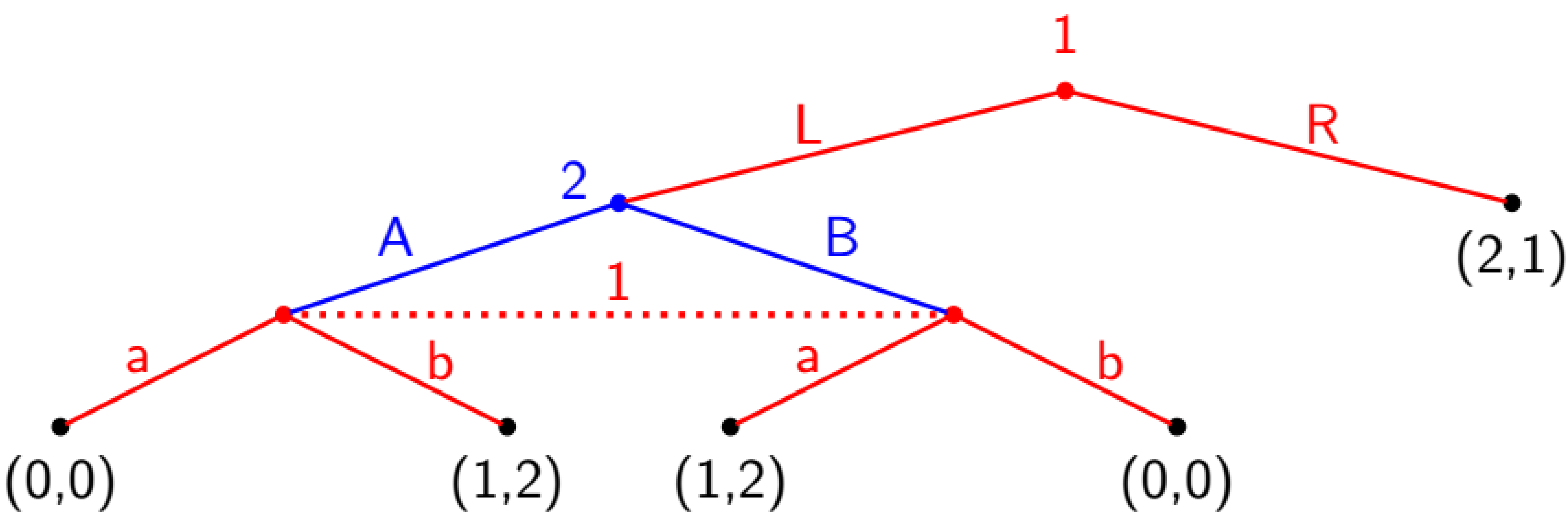
- Information set $I = \{I_1, I_2, \ldots I_N\}$ is the set of information partition of all players' strategy nodes, where the nodes in an information set are indistinguishable to player
  - $I_i = \{I_{i1}, \ldots, I_{ik_i}\}$ is the information partition of player $i$
  - $I_{i1} \cup \cdots \cup I_{ik_i}$ ={all nodes of player $i$}
  - $I_{ij} \cap I_{ik} = \emptyset$ for all $j \neq$ k
  - Action set $A(h) = A(h')$ for $h, h' \in I_{ij}$, denote by $A(I_{ij})$
  - $P(I_{ij})$ be the player who plays at information set $I_{ij}$
- An extensive game with perfect information is a special case where each $I_{ij}$ contains only one node

# Pure Strategies

- A pure strategy for player $i$ selects an available action at each of $i$'s information sets $I_{i1}, \dots, I_{im}$
- All pure strategies for player $i$ is

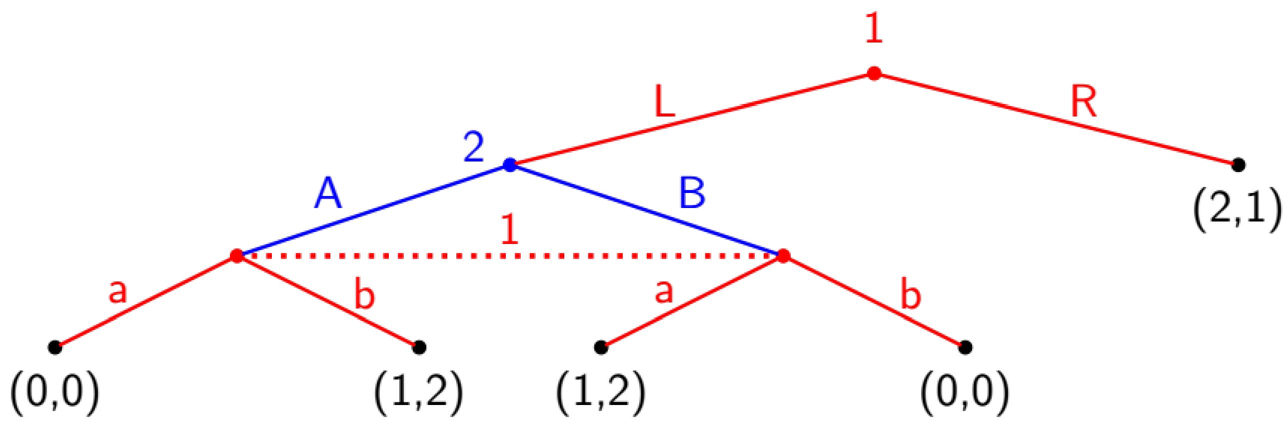
$$A(I_{i1}) \times A(I_{i2}) \times A(I_{im})$$

where $A(I_{ij})$ denotes the strategies available in $I_{ij}$




What's the pure strategies for players 1 and 2?

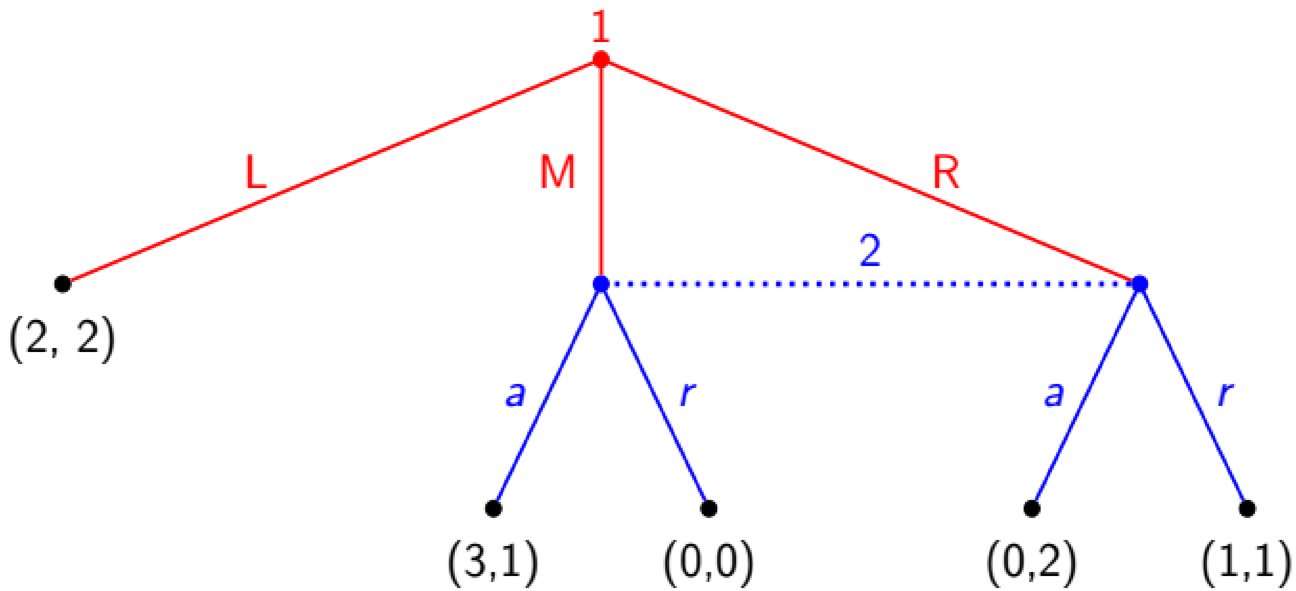
# Pure Strategies




What's the pure strategies for players 1 and 2?

- $I_1 = \{I_{11}, I_{12}\} = \{\emptyset, \{LA, LB\}\}$ $I_2 = \{I_{21}\} = \{\{L\}\}$
- $A(I_{21}) = \{A, B\}$
- The pure strategy for player 2: $A, B$
- $A(I_{11}) = \{L, R\}$, $A(I_{12}) = \{a, b\}$
- The pure strategy for player 1: $La, Lb, Ra, Rb$

# Normal-Form Representation of Extensive Imperf. Game




| | a | r |
|---|---|---|
| L | 2,2 | 2,2 |
| M | 3,1 | 0,0 |
| R | 0,2 | 1,1 |

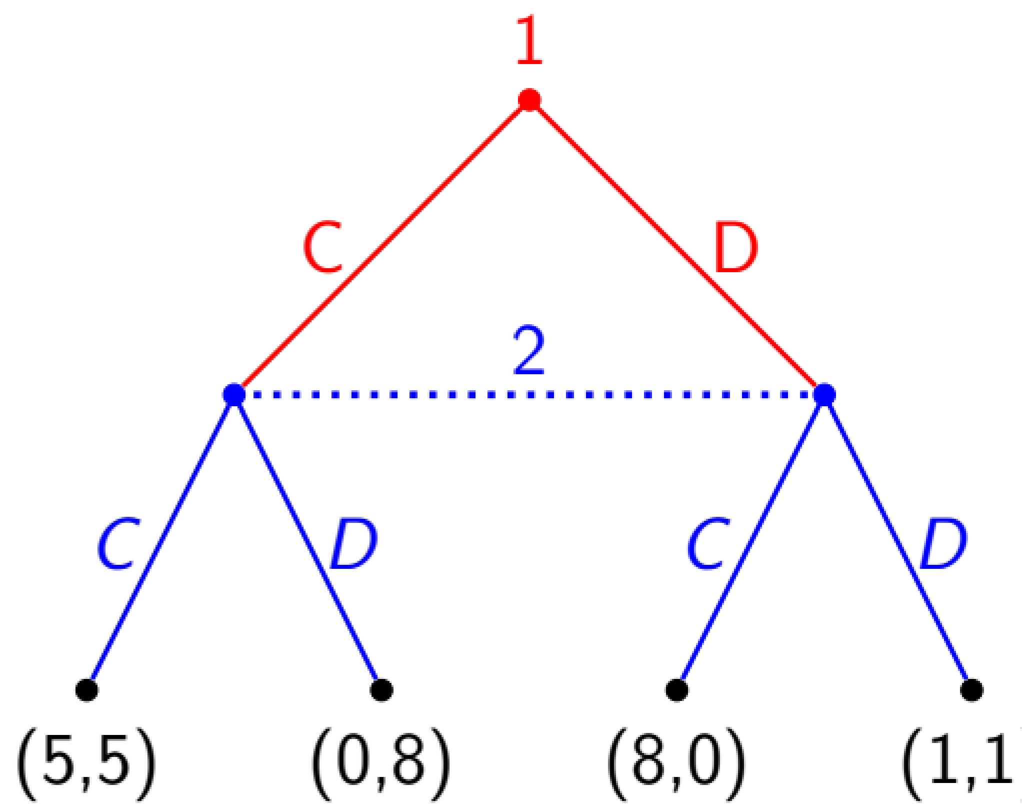
- The pure and mixed strategy Nash Equilibrium remains?
- What's the difference from the extensive game with perfect information game?

# Extensive Representation of Normal-Form Game

A strategy game $\Longrightarrow$ An extensive game with imperfect inf.

| | C | D |
|---|---|---|
| C | 5, 5 | 0, 8 |
| D | 8, 0 | 1, 1 |

Prisoner's Dilemma



# Exercise: 3-Players Game

$$G = \{\{1,2,3\}, \{\{a, b, c\}, \{x, y, z\}, \{L, R\}\}, \{u_i\}_{i=1}^{3}\}$$

**P3 chooses *L***

**P2**

| P1 | | $x$ | | | $y$ | | | $z$ | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | $a$ | 8 | 7 | 4 | 2 | 9 | 1 | 4 | 1 | 8 |
| | $b$ | 4 | 6 | 5 | 7 | 2 | 6 | 1 | 3 | 7 |
| | $c$ | 6 | 2 | 2 | 5 | 1 | 7 | 4 | 4 | 2 |

**P3 chooses *R***

**P2**

| P1 | | $x$ | | | $y$ | | | $z$ | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | $a$ | 5 | 3 | 2 | 6 | 5 | 4 | 1 | 2 | 4 |
| | $b$ | 8 | 6 | 2 | 2 | 8 | 10 | 5 | 2 | 6 |
| | $c$ | 6 | 9 | 4 | 1 | 1 | 3 | 9 | 7 | 8 |

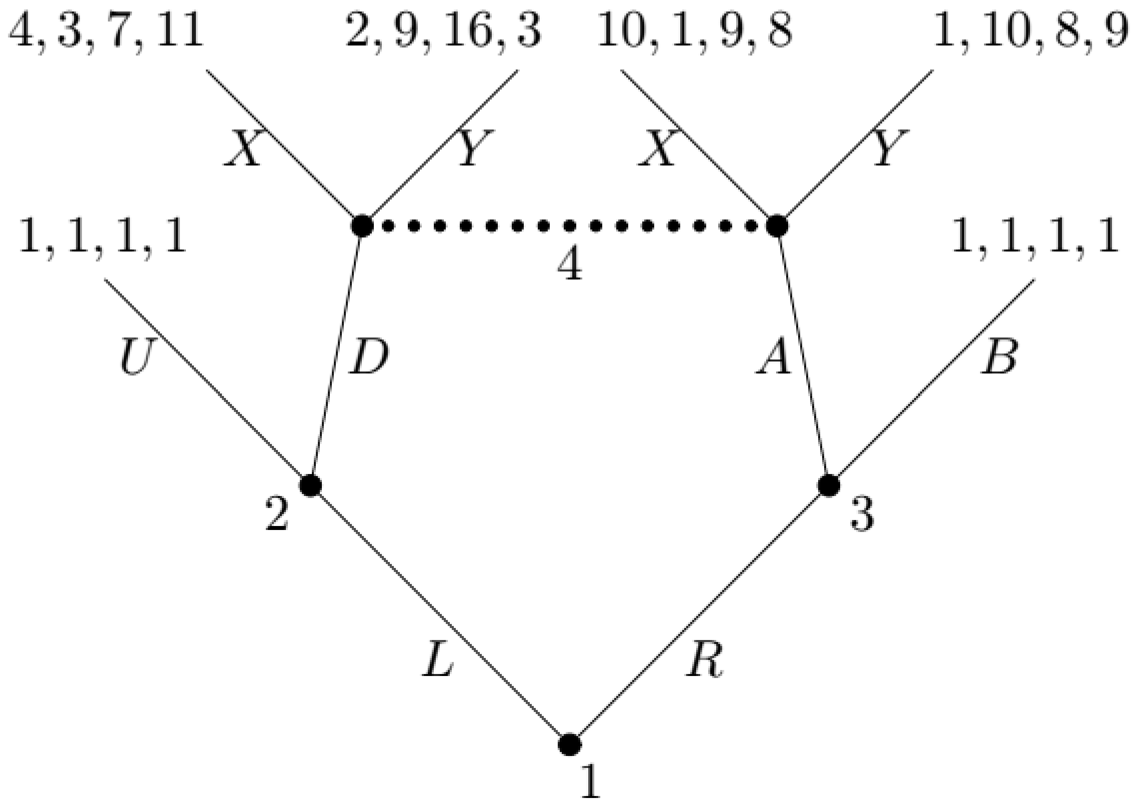
# Perfect Recall (完美回忆) and Imperfect Recall

- An extensive game has perfect information if each information set consist of only one nodes

- An extensive game has perfect recall if each player recalls exactly what he did in the past

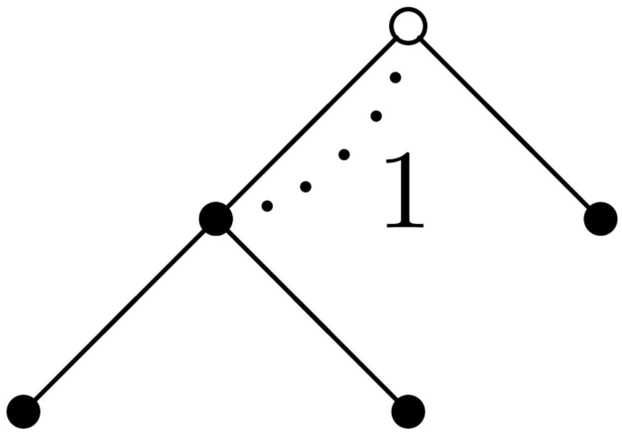
  – otherwise, this game has imperfect recall

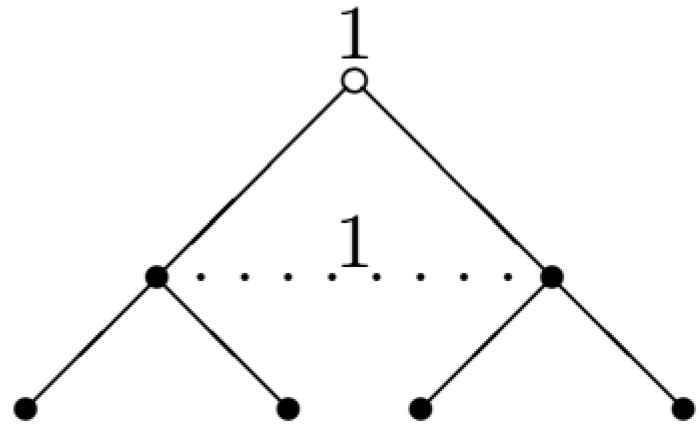
# Example

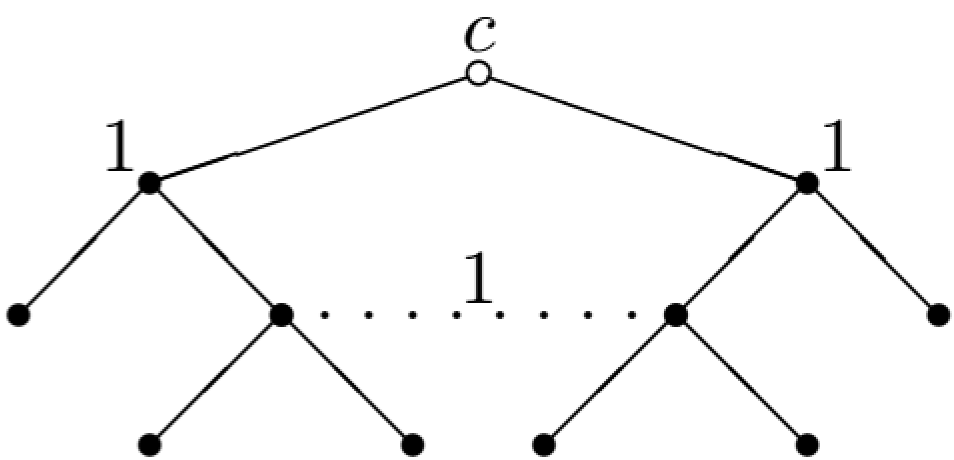



Perfect recall

# Example of Imperfect Recall



# Formal Definition of Perfect Recall

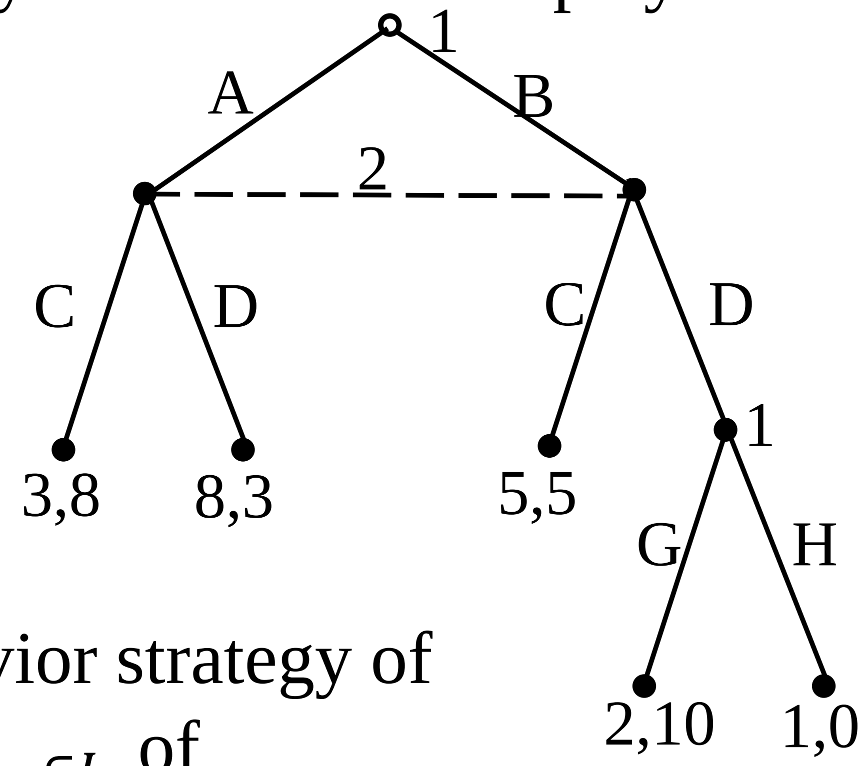
Player $i$ has **perfect recall** in game G if for any two history $h$ and $h$' that are in the same information set for player $i$, for any path $h_0, h_1, \dots h_n, h$ and $h'_0, h'_1, \dots h'_m, h'$ from the root to $h$ and $h'$with $P(h_k) = P(h'_k) = i$, we have

- $n = m$
- $h_i = h'_i$ for $1 \leq i \leq n$
- $a(h_i) = a(h'_i)$ where $a(h)$ denotes the action after $h$

G is **a game of perfect recall** if every player has perfect recall in it.

# Definition of Mixed and Behavioral Strategies

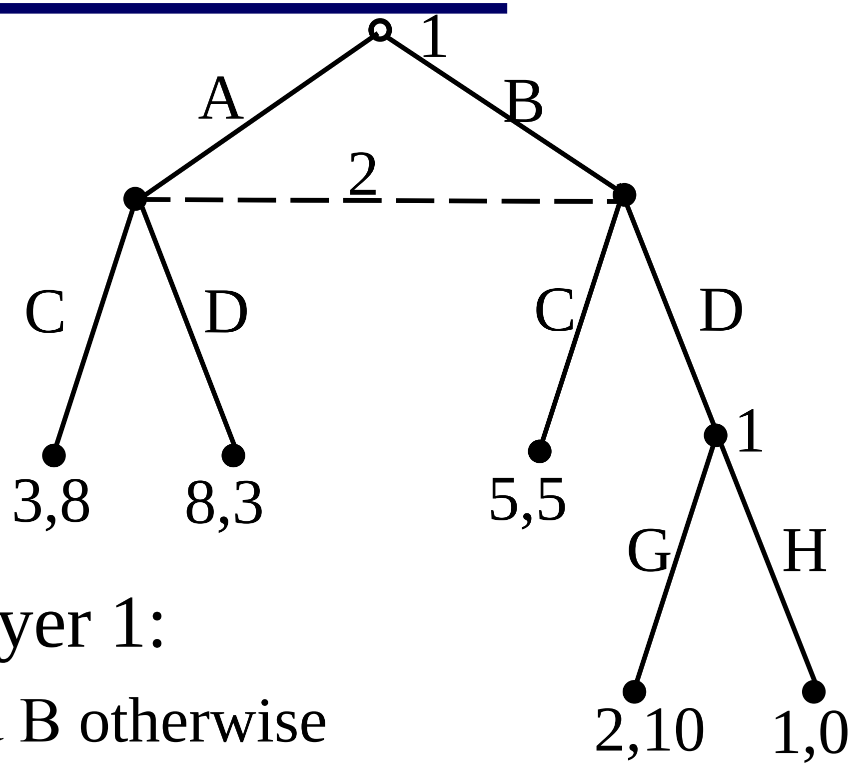
Mixed Strategies: A mixed strategy of player $i$ in an extensive game is a probability over the set of player $i$'s pure strategy




**Behavioral strategies:** A behavior strategy of player $i$ is a collection $\beta_{ik}(I_{ik})_{I_{ik}\in I_i}$ of independent probability measure, where $\beta_{ik}(I_{ik})$ is a probability measure over $\mathrm{A}(I_{ik})$

# Behavioral vs. Mixed Strategies

Behavioral strategies distinguish from mixed strategies




A behavioral strategy for player 1:

- Selects A with prob. 0.5, and B otherwise
- choose G with prob. 0.3, and H otherwise

Here's a mixed strategy that isn't a behavioral strategy

- Pure Strategy AG with probability 0.6, pure strategy BH 0.4
- The choices at the two nodes are not independent

# Behavioral vs. Mixed Strategies

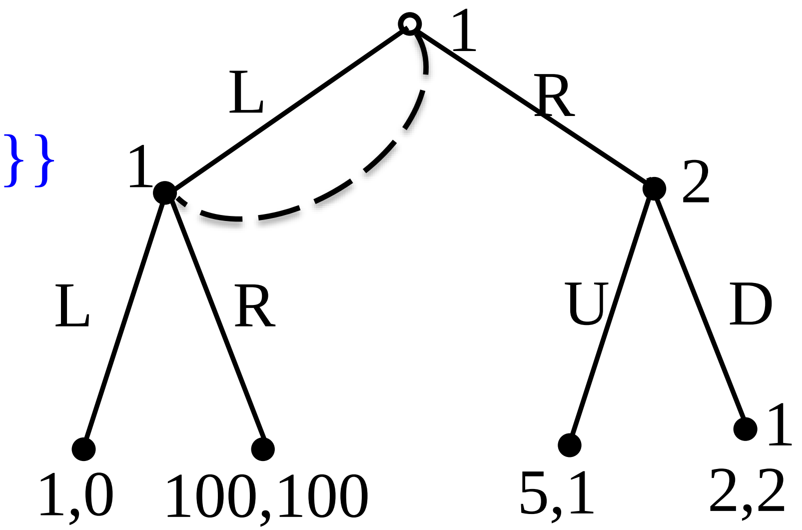
In imperfect-information games, mixed and behavioral strategies produce different sets of equilibria

- In some games, mixed strategies can achieve equilibria that aren’t achievable by any behavioral strategy
- In some games, behavioral strategies can achieve equilibria that aren’t achievable by any mixed strategy

# Behavioral vs. Mixed Strategies

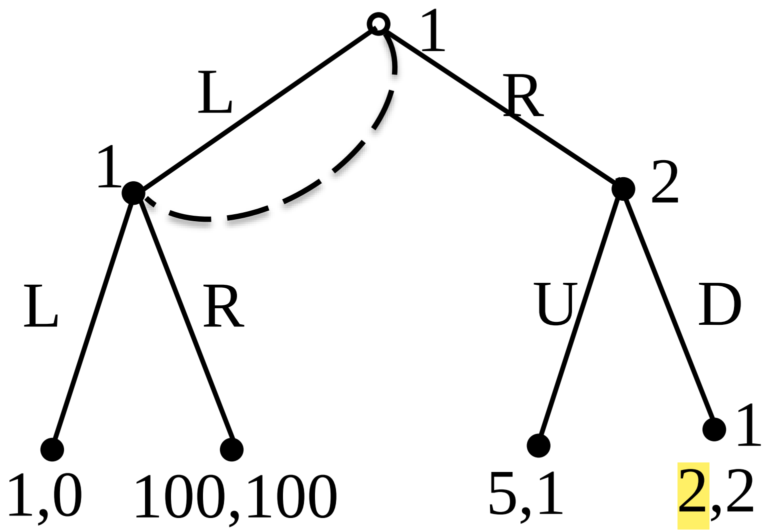
Consider game

Player 1 inform. set: {{Ø, L}}




- Player 1: R is a strictly dominant strategy
- Player 2: D is a strictly dominant strategy
  - (R, D) is the unique Nash equilibrium for mixed strategy

# Behavioral vs. Mixed Strategies




1: the information set is {(∅,L)}

2: D is a strictly dominant strategy

Player 2's best response to D:

- Player 1's the behavioral strategy $[L, p; R, 1-p]$ i.e., choose L with probability $p$
- The expected payoff of player 1 is
- $U_1 = p^2 + 100p(1-p) + 2(1-p) = -99p^2 + 98p + 2$
- To find the maximum, we have $p = 49/99$

(R,D) is not an equilibrium for behavioral strategy

# Kuhn Theorem (1953)

**Theorem** In an finite extensive game with perfect recall

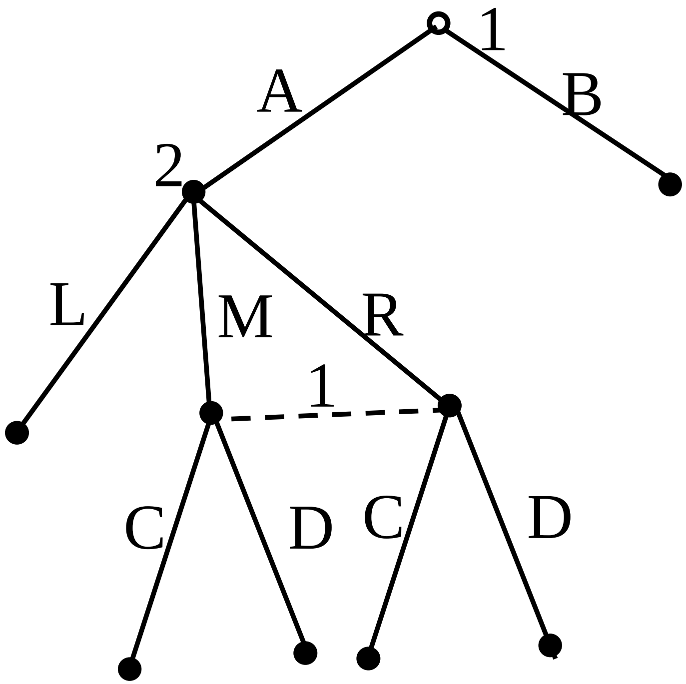
- any mixed strategy of a player can be replaced by an equivalent behavioral strategy
- any behavioral strategy can be replaced by an equivalent mixed strategy
- Two strategies are equivalent

**Corollary** In an finite extensive game with perfect recall, the set of Nash equilibrium does not change if we restrict ourselves to behavior strategies

# Example

What behavioral strategy is equivalent to mixed strategy $(p_{AC}, p_{AD}, p_{BC}, p_{BD})$

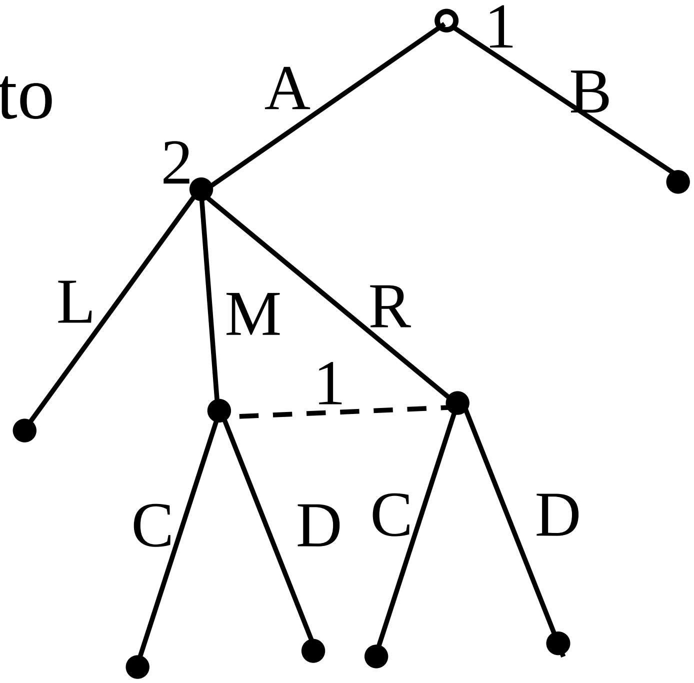
- $I_{11} = \{\emptyset\}$ $I_{12} = \{AM, AR\}$
- $A(I_{11}) = \{A, B\}$
- $A(I_{12}) = \{C, D\}$




- $\beta_{11}(I_{11})(A) = p_{AC} + p_{AD}$ $\beta_{11}(I_{11})(B) = p_{BC} + p_{BD}$
- $\beta_{12}(I_{12})(C) = \frac{p_{AC}}{p_{AC}+p_{AD}}$ $\beta_{12}(I_{12})(D) = \frac{p_{AD}}{p_{AC}+p_{AD}}$

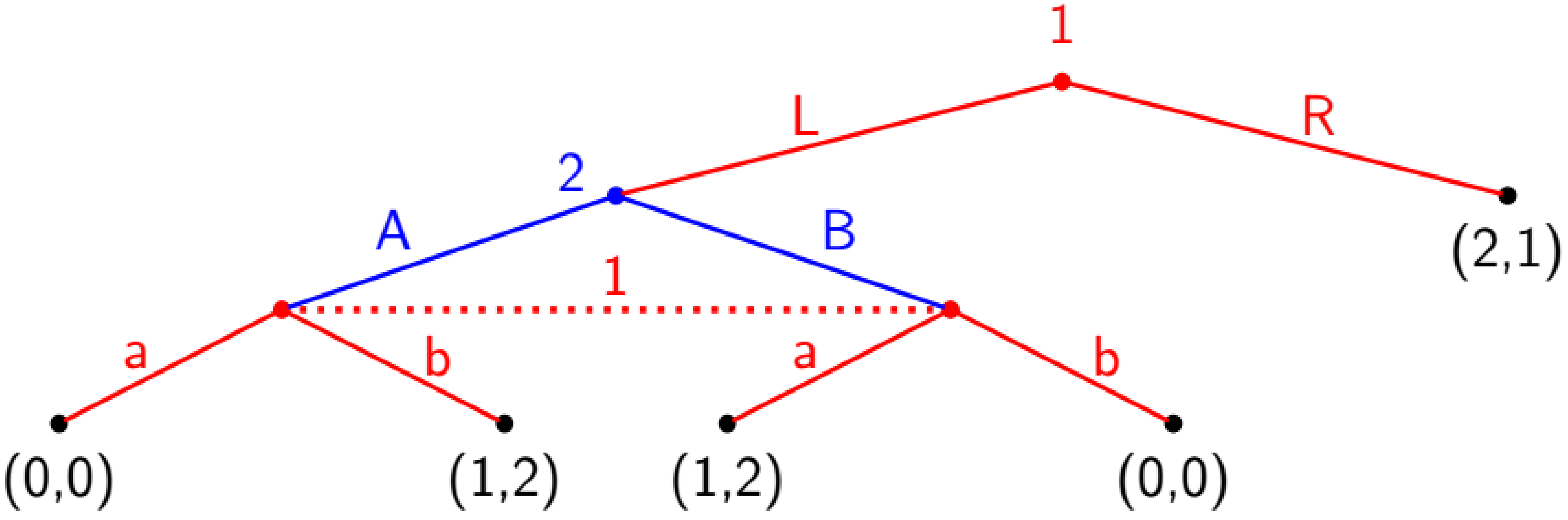
# Example

What mixed strategy is equivalent to behavioral strategy of prob. $p$ over A and $q$ over C



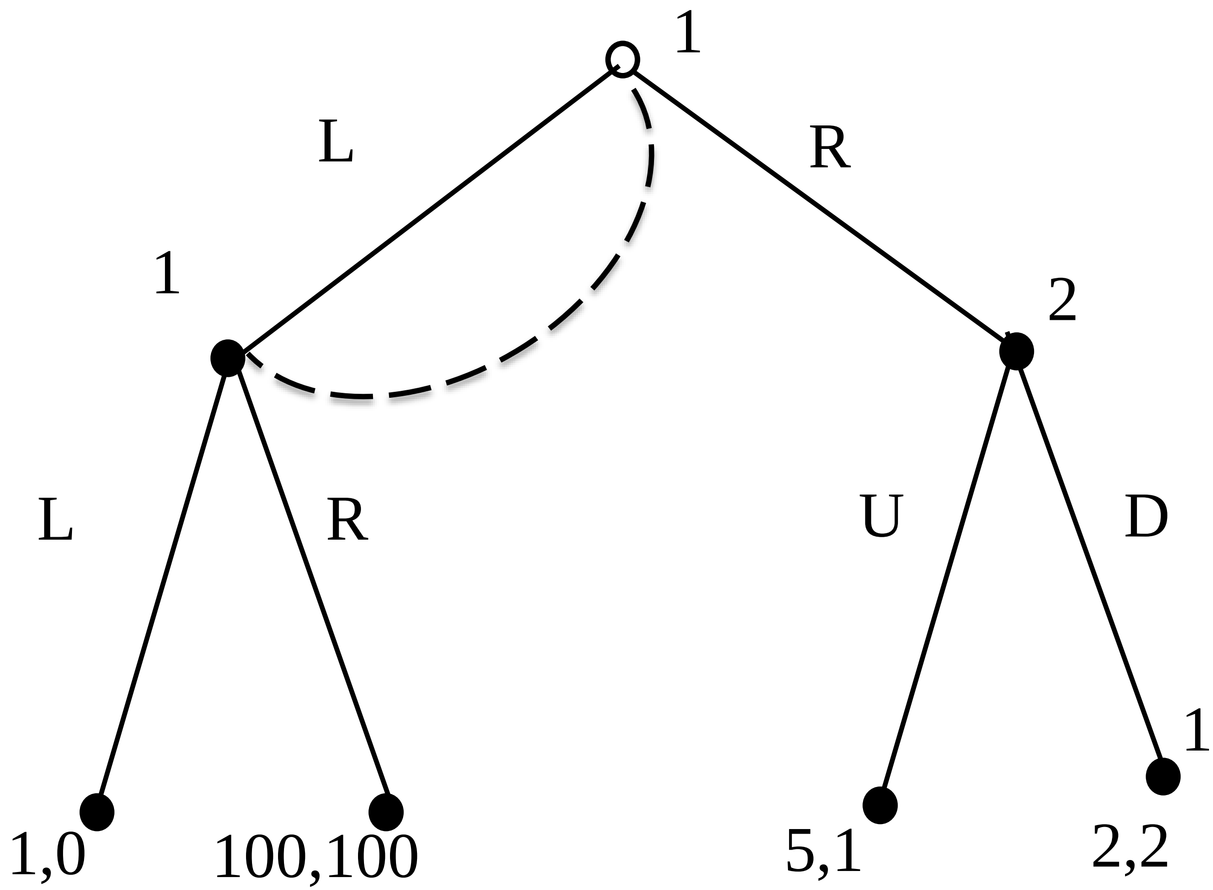

$$(p_{AC}, p_{AD}, p_{BC}, p_{BD})$$
$$= (pq, p(1-q), (1-p)q, (1-p)(1-q))$$

# Exercise



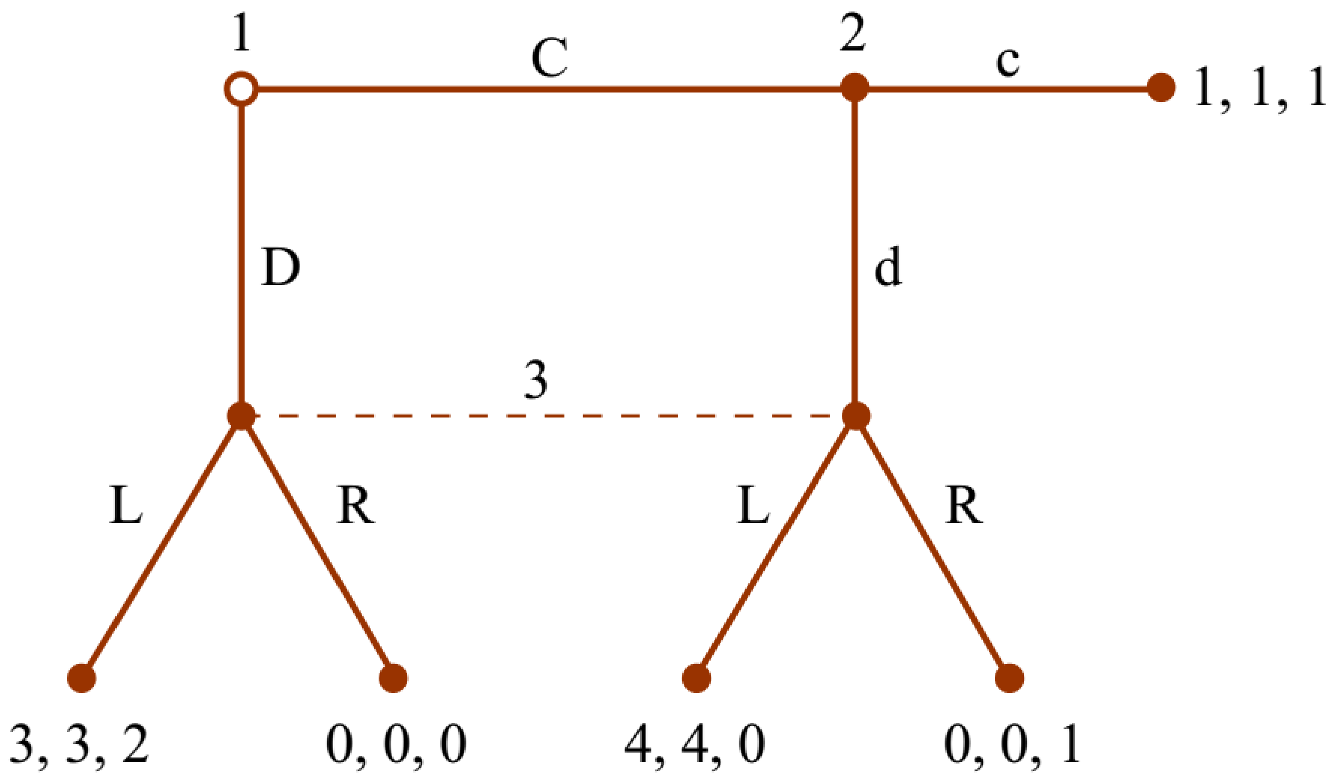

What are Nash Equilibria

# Exercise




What are Nash Equilibria

# Exercise




What are Nash Equilibria