Game Theory and Applications (博弈论及其应用)

Chapter 4: Two-Player Zero-Sum Game

南京大学

高 尉

Recap on Previous Chapter

- Dominant strategy and dominant strategy Equilibra

  $u_i(a_i, a_{-i}) > u_i(a_i', a_{-i})$ for all $a_{-i} \in A_{-i}$, $a_i' \in A_i$

- How to find mixed strategy Nash equilibrium for strictly dominated strategies

  A **strictly** dominated strategy is never used with positive probability in a mixed strategy Nash equilibrium

- Rationalization and iteration of strictly dominated strategies

Two-Player Zero-Sum Game

Definition A **two-player zero-sum game** is a strategy game $G = \{\{1,2\}, \{A_1, A_2\}, \{u_1, u_2\}\}$ such that

$$u_1(a_1, a_2) + u_2(a_1, a_2) = 0 \text{ for } a_1 \in A_1 \text{ and } a_2 \in A_2$$

One player wins while the other losses

Rock-Paper-Scissors

			Player 2	
		Rock	Paper	Scissors
	Rock	0 0	-1 1	1 -1
Player 1	Paper	1 -1	0 0	-1 1
	Scissors	-1 1	1 -1	0 0

Example

We consider a zero-sum game

Player 2

		L		M		R	
	U	1	-1	1	-1	8	-8
Player 1	M	5	-5	2	-2	4	-4
	D	7	-7	0	0	0	0

It is not necessary to keep track of both payoffs. We keep the first player payoff only by convention.

The abbreviation is

Player 2

		L	M	R
	U	1	1	8
Player 1	M	5	2	4
	D	7	0	0

Maxmin (最大化最小原则)

For this game, both player do not do too badly

Player 1 method

- Calculate minimization for each strategy, and maximize

$$\max_{a_1 \in A_1} \min_{a_2 \in A_2} u(a_1, a_2)$$

Player 2

		L	M	R
	U	1	1	8
Player 1	M	5	2	4
	D	7	0	0

Player 1 selects M

$$\mathbf{M \in \operatorname*{argmax}_{a_1 \in A_1} \min_{a_2 \in A_2} u(a_1, a_2)}$$

Maxmin

For this game, both player do not do too badly

Player 2 method:

- calculate minimization for each strategy and Maximize

$$\max_{a_2 \in A_2} \min_{a_1 \in A_1} u_2(a_1, a_2)$$

Player 2 selects M

Player 2

Player 1

	L	M	R
U	1	1	8
M	5	2	4
D	7	0	0

Maxmin (最小化最大原则)

Player 2 method:

$$\max_{a_2 \in A_2} \min_{a_1 \in A_1} u_2(a_1, a_2)$$

From $u_2(a_1, a_2) = -u(a_1, a_2)$, we have

$$\max_{a_2 \in A_2} \min_{a_1 \in A_1} u_2(a_1, a_2) = \max_{a_2 \in A_2} \min_{a_1 \in A_1} -u(a_1, a_2)$$

By $\max(-f(x)) = -\min(f(x))$ and $\max(-f(x)) = -\min(f(x))$

$$\max_{a_2 \in A_2} \min_{a_1 \in A_1} u_2(a_1, a_2) = -\min_{a_2 \in A_2} \max_{a_1 \in A_1} u(a_1, a_2)$$

Player 2 method:

$$\mathbf{\underset{a_2 \in A_2}{argmin} \max_{a_1 \in A_1} u(a_1, a_2)}$$

Minmax

For this game, both player do not do too badly

Player 2 method:

$$\min_{a_2 \in A_2} \max_{a_1 \in A_1} u(a_1, a_2)$$

Player 2 selects M

Player 2

Player 1

	L	M	R
U	1	1	8
M	5	2	4
D	7	0	0

Two-players zero-sum method

For this game, both player do not do too badly

Player 1 method

$$\max_{a_1 \in A_1} \min_{a_2 \in A_2} u(a_1, a_2)$$

Player 2 method

$$\min_{a_2 \in A_2} \max_{a_1 \in A_1} u(a_1, a_2)$$

Player 2

Player 1

	L	M	R
U	1	1	8
M	5	2	4
D	7	0	0

$$\max_{a_1 \in A_1} \min_{a_2 \in A_2} u(a_1, a_2) = \min_{a_2 \in A_2} \max_{a_1 \in A_1} u(a_1, a_2)$$

Another Example

Another example

Player 2

		L	M	R
	U	2	6	1
Player 1	M	3	1	4
	D	4	3	6

Player 1 method

$$\max_{a_1 \in A_1} \min_{a_2 \in A_2} u(a_1, a_2) = 3$$

Player 2 method

$$\min_{a_2 \in A_2} \max_{a_1 \in A_1} u(a_1, a_2) = 4$$

$$\min_{a_2 \in A_2} \max_{a_1 \in A_1} u(a_1, a_2) > \max_{a_1 \in A_1} \min_{a_2 \in A_2} u(a_1, a_2)$$

MinMax ≥ MaxMin

Lemma For two-player zero-sum finite game G, we have

$$\min_{a_2 \in A_2} \max_{a_1 \in A_1} u(a_1, a_2) \geq \max_{a_1 \in A_1} \min_{a_2 \in A_2} u(a_1, a_2)$$

Proof. For any function $F(x, y)$, we have

$$F(x, y) \geq \min_{y} F(x, y) \text{ for any } y$$

$$\max_{x} F(x, y) \geq \max_{x} \min_{y} F(x, y) \text{ for any } x$$

$$\min_{y} \max_{x} F(x, y) \geq \max_{x} \min_{y} F(x, y)$$

Two-Players Zero-Sum Nash Equilibrium

Theorem For two-player zero-sum finite game $G = \{\{1,2\}, \{A_1, A_2\}, u\}$, let player 1 select

$$a_1^* \in \operatorname*{argmax}_{a_1 \in A_1} \min_{a_2 \in A_2} u(a_1, a_2)\,,$$

and let player 2 select

$$a_2^* \in \operatorname*{argmin}_{a_2 \in A_2} \max_{a_1 \in A_1} u(a_1, a_2)\,.$$

The strategy outcome (a_1^*, a_2^*) is a Nash Equilibrium if and only if

$$\max_{a_1 \in A_1} \min_{a_2 \in A_2} u(a_1, a_2) = \min_{a_2 \in A_2} \max_{a_1 \in A_1} u(a_1, a_2)$$

Two-Players Zero-Sum Nash Equilibrium

Proof. If (a_1^*, a_2^*) is a NE, then

$$u_1(a_1^*, a_2^*) \geq u_1(a_1, a_2^*) \text{ for all } a_1 \in A_1$$

$$u_2(a_1^*, a_2^*) \geq u_2(a_1^*, a_2) \text{ for all } a_2 \in A_2$$

By using $u_1(\cdot,\cdot) = u(\cdot,\cdot), u_{2(\cdot,\cdot)} = -u(\cdot,\cdot)$, we have

$$(a_1^*, a_2^*) \text{ is a NE} \quad \text{iff} \quad u(a_1, a_2^*) \leq u(a_1^*, a_2^*) \leq u(a_1^*, a_2)$$

$$\text{iff} \;\; u(a_1, a_2^*) \leq u(a_1^*, a_2)$$

必要性：If (a_1^*, a_2^*) is a NE then we have

$$u(a_1, a_2^*) \leq u(a_1^*, a_2) \rightarrow \max_{a_1} u(a_1, a_2^*) \leq \min_{a_2} u(a_1^*, a_2)$$

$$a_2^* \in \operatorname{argmin}_{a_2 \in A_2} \max_{a_1 \in A_1} u(a_1, a_2) \; a_1^* \in \operatorname{argmax}_{a_1 \in A_1} \min_{a_2 \in A_2} u(a_1, a_2)$$

$$\min_{a_2 \in A_2} \max_{a_1 \in A_1} u(a_1, a_2) \leq \max_{a_1 \in A_1} \min_{a_2 \in A_2} u(a_1, a_2)$$

Two-Players Zero-Sum Nash Equilibrium

充分性：If $\min_{a_2 \in A_2} \max_{a_1 \in A_1} u(a_1, a_2) = \max_{a_1 \in A_1} \min_{a_2 \in A_2} u(a_1, a_2)$

then $u(a_1, a_2^*) \leq u(a_1^*, a_2)$.

$$u(a_1, a_2^*) \leq \max_{a_1} u(a_1, a_2^*) = \min_{a_2 \in A_2} \max_{a_1 \in A_1} u(a_1, a_2)$$

$$u(a_1^*, a_2) \geq \min_{\mathrm{a}_2} u(a_1^*, a_2) = \max_{a_1} \min_{a_2} u(a_1, a_2)$$

$$u(a_1, a_2^*) \leq u(a_1^*, a_2)$$

Find Nash Equilibrium

Player 2

		L	M	R
	U	1	1	8
Player 1	M	5	2	4
	D	7	0	0

(M, M) is a NE

Player 2

		L	M	R
	U	2	6	1
Player 1	M	3	1	4
	D	4	3	6

(D, L) is not a NE

Mixed strategy

Strategic game

$N = \{1,2\}$

$A_1 = \{a_1, a_2, \dots, a_m\}, \;\; A_2 = \{b_1, b_2, \dots b_n\}$

$u_1(a_i, b_j) = u(a_i, b_j) = u_{ij}$, $\mathrm{M} = (u_{ij})_{m \times n}$

Mixed strategy

$\mathrm{p} = (p_1, p_2, \dots, p_m) \in \Delta_1$ is a mixed strategy over A_1

$q = (q_1, q_2, \dots, q_n) \in \Delta_2$ is a mixed strategy over A_2

The expected payoff for player 1 on mixed outcome (p, q)

$$U(p,q) = \sum_{i,j} p_i q_j u(a_i, b_j) = \sum_{i,j} p_i q_j u_{ij} = pMq^\top$$

MinMax and MaxMin

Player 1's methods:

$$\max_{p\in\Delta_1}\min_{q\in\Delta_2} U_1(p,q) = \max_{p\in\Delta_1}\min_{q\in\Delta_2} pMq^\top$$

Player 2's methods:

$$\min_{q\in\Delta_2}\max_{p\in\Delta_1} U_2(p,q) = \min_{q\in\Delta_2}\max_{p\in\Delta_1} pMq^\top$$

Lemma We have

$$\max_{p\in\Delta_1}\min_{q\in\Delta_2} U(p,q) \leq \min_{q\in\Delta_2}\max_{p\in\Delta_1} U(p,q)$$

Nash Equilibrium

Theorem For two-player zero-sum finite game $G = \{\{1,2\}, \{A_1, A_2\}, u\}$, let player 1 select

$$p^* \in \underset{p \in \Delta_1}{\operatorname{argmax}} \min_{q \in \Delta_2} U(p, q),$$

and let player 2 select

$$q^* \in \underset{q \in \Delta_2}{\operatorname{argmin}} \max_{p \in \Delta_1} U(p, q).$$

The mixed strategy outcome (p^*, q^*) is a MNE if and only if

$$\max_{p \in \Delta_1} \min_{q \in \Delta_2} U(p, q) = \min_{q \in \Delta_2} \max_{p \in \Delta_1} U(p, q)$$

Proof excise.

General John von Neumann's minimax theorem

The Minmax Theorem Let $X \in \mathcal{R}^n$ and $Y \in \mathcal{R}^m$ be compact convex sets. If $f: X \times Y \to \mathcal{R}$ is a continuous function with

- $f(\cdot, y): X \to \mathcal{R}$ is concave for fixed y;
- $f(x,\cdot): Y \to \mathcal{R}$ is convex for fixed x,

then, we have

$$\max_{x \in X} \min_{y \in Y} f(x, y) = \min_{y \in Y} \max_{x \in X} f(x, y) .$$

John von Neumann’s Minimax Theorem (1928)

The Minmax Theorem For two-player zero-sum finite game $G = \{\{1,2\},\ \{A_1, A_2\}, u\}$, we have

$$\max_{p\in\Delta_1}\min_{q\in\Delta_2} pMq^\top = \min_{q\in\Delta_2}\max_{p\in\Delta_1} pMq^\top .$$

Corollary: Two-person finite zero-sum games have at least one mixed-strategy Nash-equilibrium: any pair of optimal strategies is a Nash equilibrium.

How to Solve???

Example

		Player 2	
		L	R
Player 1	U	3 -3	-1 1
	D	-2 2	1 -1

Player 1 mixture strat. (x_1, x_2)

$U_2(\mathrm{L}) = -3x_1 + 2x_2$

$U_2(\mathrm{R}) = x_1 - x_2$

The optimal solution for Player 2 is

$$\max(-3x_1 + 2x_2, x_1 - x_2)$$

The payoffs for Player 1 is

$$-\max(-3x_1 + 2x_2, x_1 - x_2) = \min(3x_1 - 2x_2, -x_1 + x_2)$$

The solution for Player 1 is

$$(x_1, x_2) \in \arg\max_{(x_1,x_2)} \min(3x_1 - 2x_2, -x_1 + x_2)$$

Example

		Player 2			
		L		R	
Player 1	U	3	-3	-1	1
	D	-2	2	1	-1

The solution for Player 1 is

$$(x_1, x_2) \in \arg \max_{(x_1,x_2)} \min(3x_1 - 2x_2, -x_1 + x_2)$$

This is equivalent to

$$\max \ v$$

$$3x_1 - 2x_2 \geq v$$

$$-x_1 + x_2 \geq v$$

Linear programming

$$x_1 + x_2 = 1, x_1 > 0, x_2 > 0$$

How to Solve?

Theorem The optimization problem of $\max_{p\in\Delta_1}\min_{q\in\Delta_2} pMq^\top$ is equivalent to

$$\max v$$

s.t.

$$pM \geq v\mathbf{1}$$
$$p = (p_1, \dots, p_m) \in \Delta_1$$
$$\mathbf{1} = (1, \dots, 1)^\top$$

Linear programming: can be solved in polynomial time

How to Solve?

Theorem The optimization problem of $\min_{q\in\Delta_2}\max_{p\in\Delta_1} pMq^\top$ is equivalent to

$$\min v$$

s.t.

$$Mq^\top \leq v\mathbf{1}$$
$$q = (q_1, \dots, q_n) \in \Delta_2$$
$$\mathbf{1} = (1, \dots, 1)^\top$$

Linear programming: can be solved in polynomial time

Example: solve NE

Player 2

Player 1

	r	x	y	z
a	1	-1	2	-2
b	3	-2	4	-2
c	-2	-4	-5	7
d	-5	2	6	3

Symmetric Game (2-player zero-sum)

Symmetric strategic game

$N = \{1,2\}$

$A_1 = \{a_1, a_2, \dots, a_n\}$, $A_2 = \{b_1, b_2, \dots b_n\}$

$u_1(a_i, b_j) = u_{ij}$, $\mathrm{M} = (u_{ij})_{n\times n}$, $\mathbf{M} = -\mathbf{M}^\top$

Theorem For a symmetric game, we have

$$\max_{p\in\Delta_1} \min_{q\in\Delta_2} pMq^\top = \min_{q\in\Delta_2} \max_{p\in\Delta_1} pMq^\top = 0$$

Symmetric Game (2-player zero-sum)

Theorem For a symmetric game, we have

$$\max_{p\in\Delta_1}\min_{q\in\Delta_2} pMq^\top = \min_{q\in\Delta_2}\max_{p\in\Delta_1} pMq^\top = 0$$

Proof. For any p, we have $pMp^\top = 0$ from

$$pMp^\top = (pMp^\top)^\top = -pMp^\top$$

$$\max_{p\in\Delta_1}\min_{q\in\Delta_2} pMq^\top \le \max_{p\in\Delta_1} pMp^\top = 0$$

$$\min_{q\in\Delta_2}\max_{p\in\Delta_1} pMq^\top \ge \min_{q\in\Delta_2} pMp^\top = 0$$

How to find Nash Equilibria

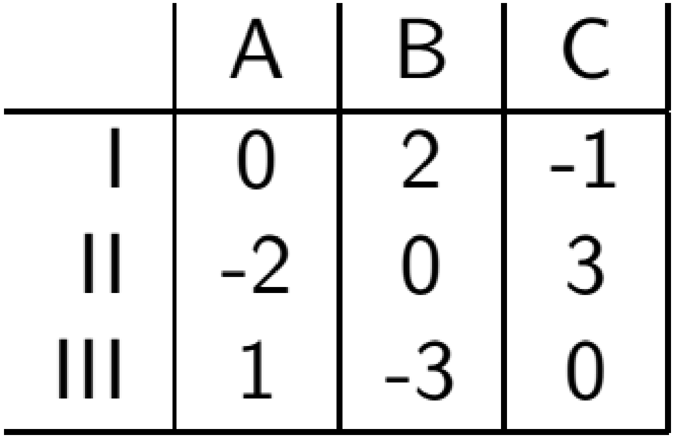
1) Calculate directly

- I) find the best response functions
- II) calculate Nash equilibria

2) Eliminate all dominated strategy

3) For two-player zero-sum player, linear programming

Exercise: solve NE



	A	B	C
I	0	2	-1
II	-2	0	3
III	1	-3	0

Excise: solve NE

Player 2

Player 1

	r	x	y	z
a	1	-2	6	-4
b	2	-7	2	4
c	-3	4	-4	-3
d	-8	3	-2	3

Excise: Proof of Nash Equilibrium

Theorem For two-player zero-sum finite game $G=\{\{1,2\},\{A_1,A_2\},u\}$, let player 1 select

$$p^* \in \operatorname*{argmax}_{p\in\Delta_1} \min_{q\in\Delta_2} U(p,q),$$

and let player 2 select

$$q^* \in \operatorname*{argmin}_{q\in\Delta_2} \max_{p\in\Delta_1} U(p,q).$$

The mixed strategy outcome (p^*, q^*) is a MNE if and only if

$$\max_{p\in\Delta_1} \min_{q\in\Delta_2} U(p,q) = \min_{q\in\Delta_2} \max_{p\in\Delta_1} U(p,q)$$

Excise: Proof of Minimax Theorem

The Minmax Theorem For two-player zero-sum finite game $G = \{\{1,2\},\ \{A_1, A_2\}, u\}$, we have

$$\max_{p\in\Delta_1} \min_{q\in\Delta_2} pMq^\top = \min_{q\in\Delta_2} \max_{p\in\Delta_1} pMq^\top .$$