Game Theory and Applications (博弈论及其应用)

Chapter 7: One Deviation, Back Induction and Bargaining

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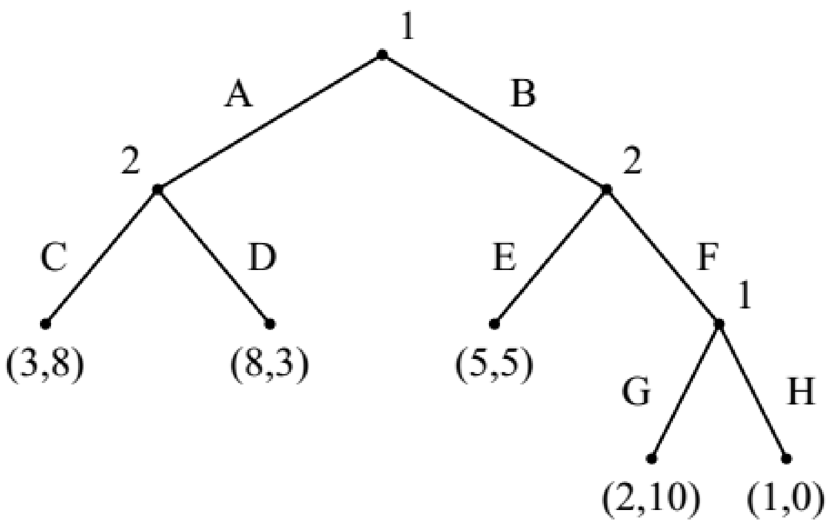
Recap on Previous Chapter

- The extensive game is an alternative representation that makes the temporal structure explicit
- Perfect information: game tree



Formalize $G = \{N, H, P, \{u_i\}\}$

Pure strategy (Mixed)

Nash Equilibrium

Subgame

Subgame Perfect

Motivation

- Existence:
  - Does every extensive game with perfect information have an SPE
  - If not, what conditions for the existence an SPE of extensive games with perfect information

- Computation:
  - If an SPE exists, how to compute it

Back Induction (后向归纳)

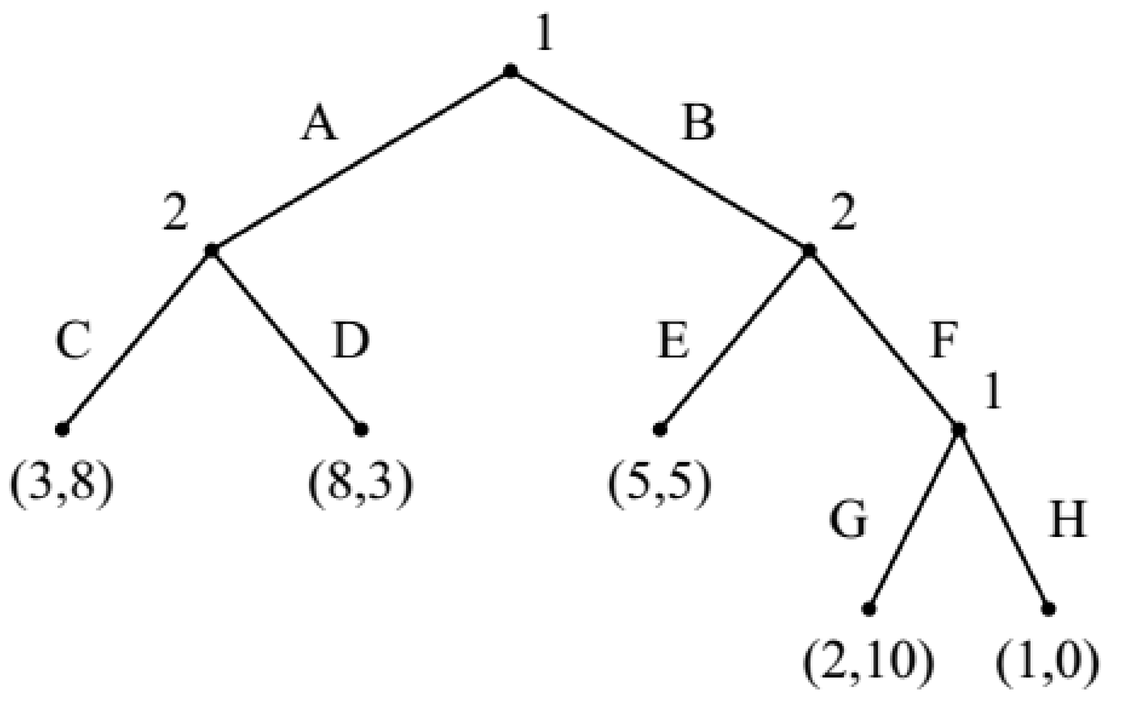
How to find subgame perfect Equilibria (SPE)

Back induction is the process of "pruning the game tree" described as follows:

- Step 1: start at each of the final subgame in the game, and solve for the player's equilibrium. Remove that subgame and replace it with payoff of the player's choice
- Step 2: Repeat step 1 until we arrive at the first node in the extensive game

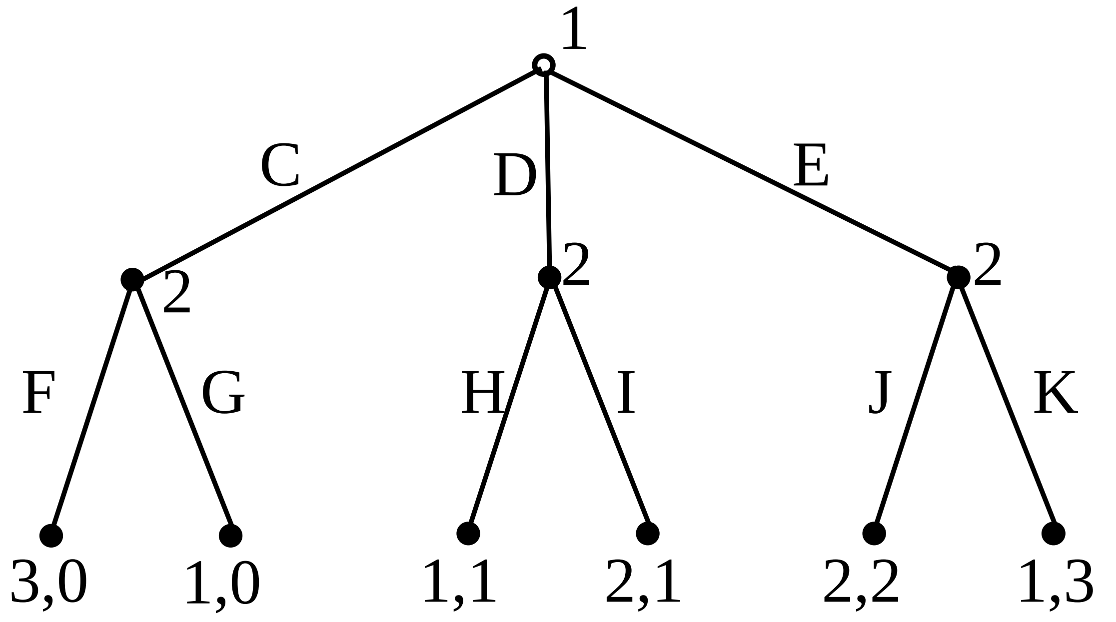
Theorem The set of strategy game constructed by backwards induction is equivalent to the set of SPE

Example



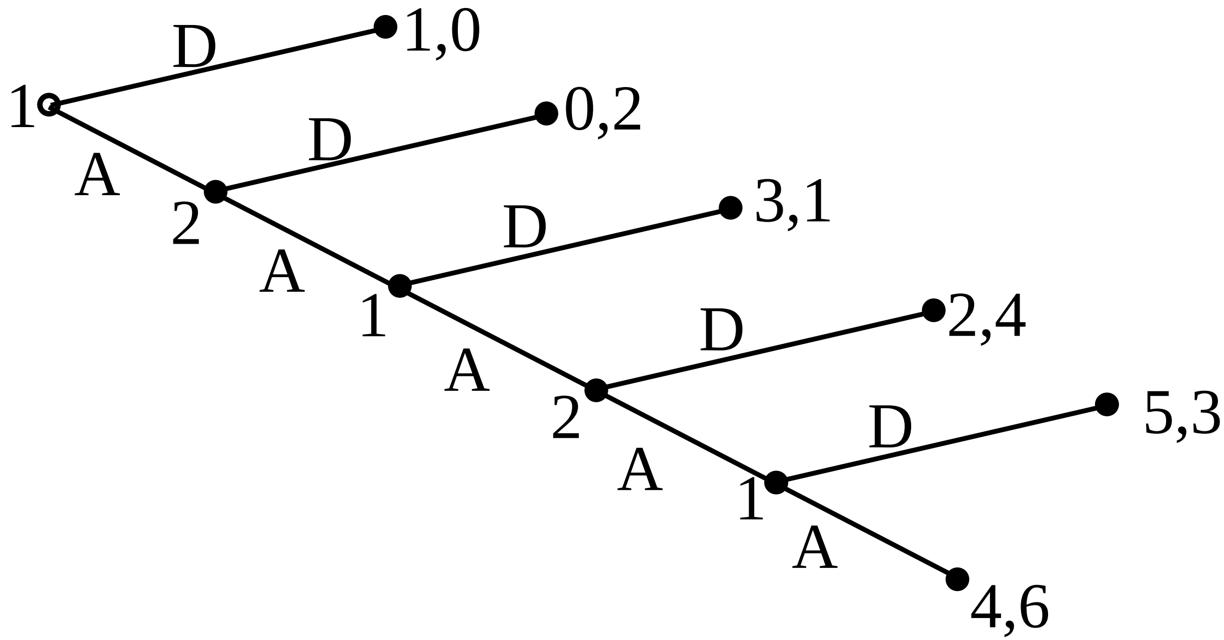
- Find a Sub-game perfect Equilibrium

Multiplicity of Subgame Perfect Equilibria



What happens for multiple optimal strategies?

Centipede Game（蜈蚣游戏）



What happens for centipede game?

Notations

Given game $G = \{N, H, P, \{u_i\}\}$

- define the initial history of $h \in H$ as

$$A(h) = \{a\colon (h, a) \in H\}$$

- define the length of G as

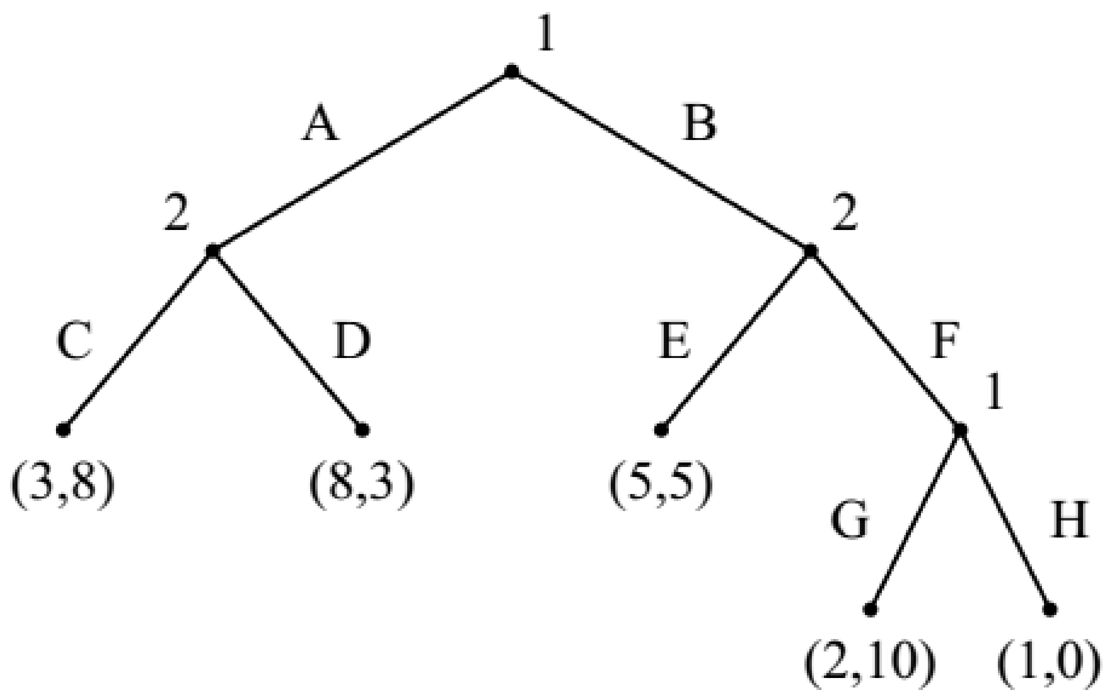
$$\ell(G) = \max_{h \in H}\{|h|\}$$

the length of the longest history in H

Given pure strategy s_i, and history h s.t. $P(h) = i$,

$$s_i(h) = a \text{ s.t. } a \in A(h) \text{ and } a \in s_i$$

Example



ℓ(G)=?

A(BF)=? A(A)=?

Given pure strategy $s_1 = (AG)$, s_1(BF)=?

Formal Definition of Subgame

Given $G = \{N, H, P, \{u_i\}\}$, the subgame of extensive game after the history h is

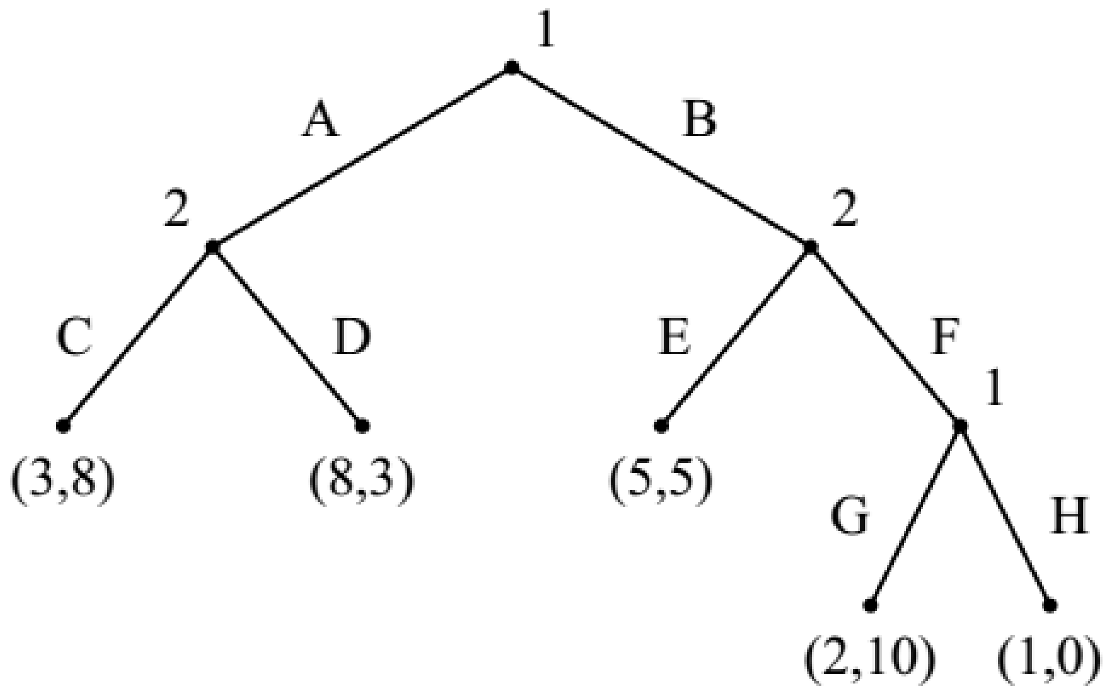
$$\boldsymbol{G(h)} = \{N, H|_h, P|_h, \{u_i|_h\}\}$$

- $H|_h$ is the set of sequence h' s.t. $(h, h') \in H$;
- $P|_h(h') = P(h, h')$ for every non-terminal his. $h' \in H|_h$;
- $u_i|_h(h') = u_i(h, h')$ for every terminal his. $h' \in H|_h$.

Given pure strategy s_i and history h

- $s_i|_h$ the strategy that s_i induces in subgame $G(h)$.
- $s_i|_h(h') = s_i(h, h')$ for every $h' \in H|_h$

Example



G(B)=$\{N, H|_{B}, P|_{B}, \{u_i|_{B}\}\}$

Subgame Perfect Equilibrium

Theorem For **finite** game $G = \{N, H, P, \{u_i\}\}$, $s^* = (s_1^*, s_2^*, \ldots, s_N^*)$ is a subgame perfect equilibrium (SPE) iff

$$\forall h \in H \setminus \mathrm{Z} \ \text{ s.t. } \ \mathrm{P}(h) = i$$

$$u_i|_h(s_i^*|_h, s_{-i}^*|_h) \geq u_i|_h(s_i, s_{-i}^*|_h)$$

for every s_i in $G(h)$.

In words: $s^*|_h$ is a NE in every $G(h)$

One Deviation Principle (单步偏离原则)

Theorem For finite game $G = \{N, H, P, \{u_i\}\}$, $s^* = (s_1^*, s_2^*, \ldots, s_N^*)$ is a subgame perfect equilibrium (SPE) iff

$$\forall h \in H \setminus Z \ \text{ s.t. } \ \mathrm{P}(h) = i$$

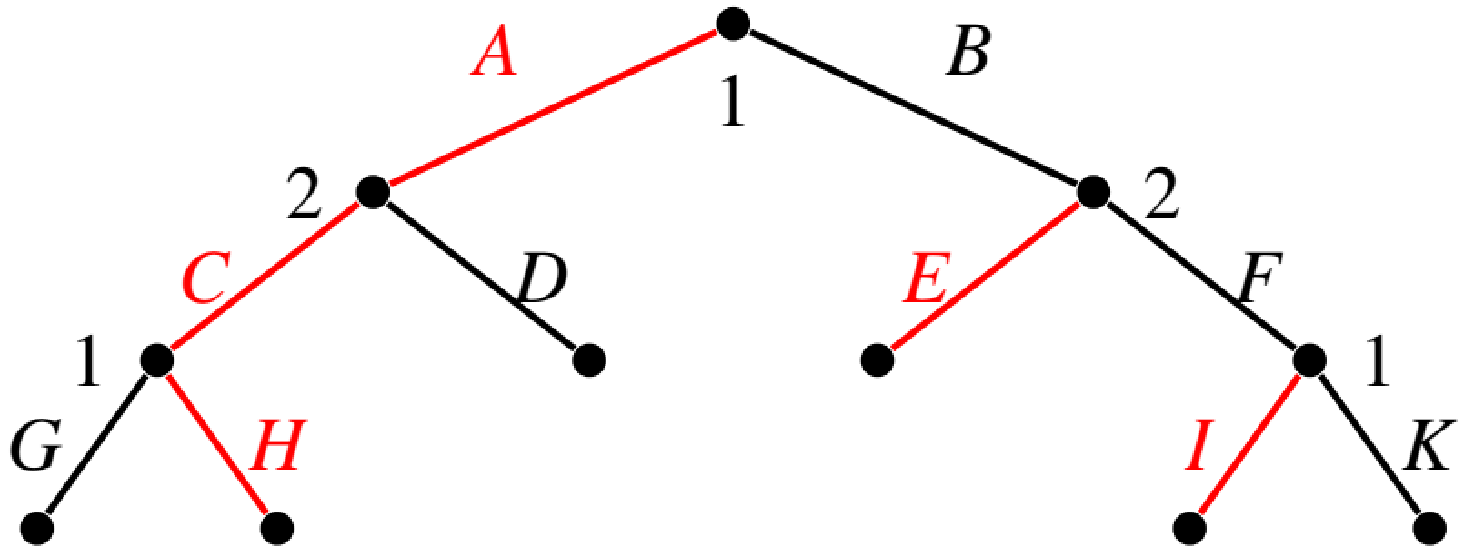
$$u_i|_h(s_i^*|_h, s_{-i}^*|_h) \geq u_i|_h(s_i, s_{-i}^*|_h)$$

for every s_i in $G(h)$ that differs from $s_i^*|_h$ only in $A(h)$.

- $s_i(\emptyset) \neq s_i^*|_h(\emptyset)$
- $s_i(h') = s_i^*|_h(h')$ for $(h, h') \in H$ and $h' \neq \emptyset$

One Deviation

Example: One Deviation Principle



Check whether (AHI, CE) is an SPE, it suffices to check

Player 1: Player 2

G in the subgame $G(AC)$ D in $G(A)$

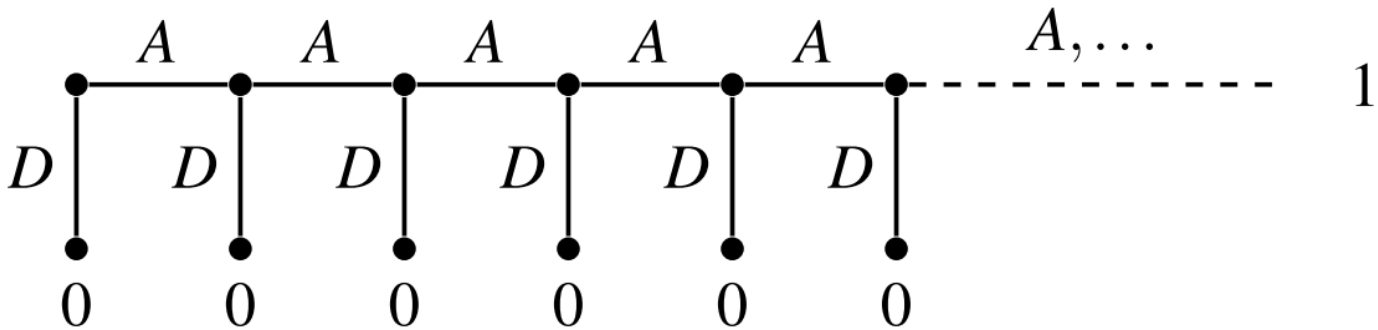
K in the subgame $G(BF)$ F in $G(B)$

BHI in G, and it is not necessary to check BGK, AHK, BHK ...

Infinite Games for One Deviation Property

One deviation does NOT hold for infinite-length game

For example



Strategy DDD… satisfies one-stage deviation property

AAA…is an SPE

Kuhn’s Theorem

Theorem Every **finite** extensive game with perfect information has a subgame perfect equilibrium.

- The SPE consists of pure strategies (not mixing);
- If all payoffs for each player are different, then SPE is unique;
- Proof is constructive and builds an SPE bottom-up (backward induction).
- Finite means ‘**finite length**’

Kuhn’s Theorem

Theorem Every **finite** extensive game with perfect information has a subgame perfect equilibrium.

Proof. Let $G = \{N, H, P, \{u_i\}\}$ be a finite extensive game. We proceed by induction on $\ell(\mathrm{G}(h))$ for h

- If $\ell(\mathrm{G}(h)) = 0$ (h is terminal history), $R(h) = h$;
- Now suppose $R(h)$ is defined for every $\ell(\mathrm{G}(h)) \leq k$, let h^* be a history s.t. $\ell(\mathrm{G}(h^*)) = k + 1$, and let $P(h^*) = i$.
  $R(h^*, a)$ is a SPE for every $a \in A(h^*)$ since $\ell(\mathrm{G}(h)) = k + 1$.

  Define

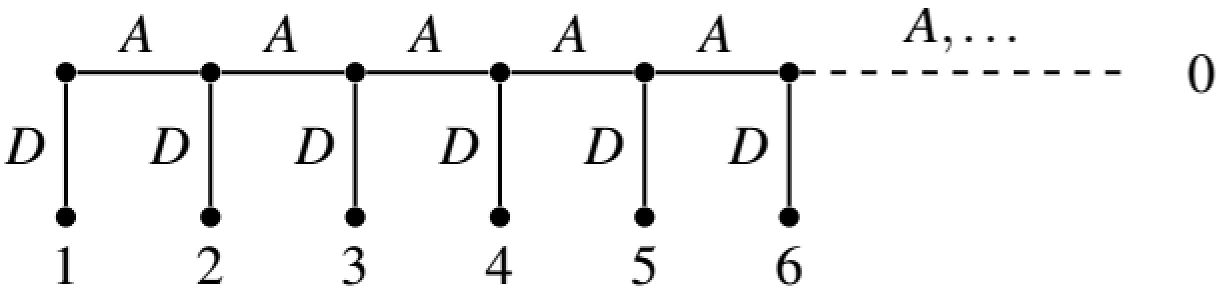
$$a^* = \max_{a \in A(h^*)} \{u_i(R(h^*, a))\}$$

  Define $R(h^*) = R(h^*, a^*)$. Based on one deviation principle, $R(h^*)$ is a SPE for $G(h^*)$. We complete the proof by $h^* = \emptyset$.

Infinite games

Kuhn's theorem does not holds for infinite-length games

Counter example (for one player)



$u_1(AAA\ldots) = 0$

$u_1(DDD\ldots) = 1$

$u_1(AAA\ldots D) = n + 1$ no SPE

Ultimatum Game (最后通牒博弈)

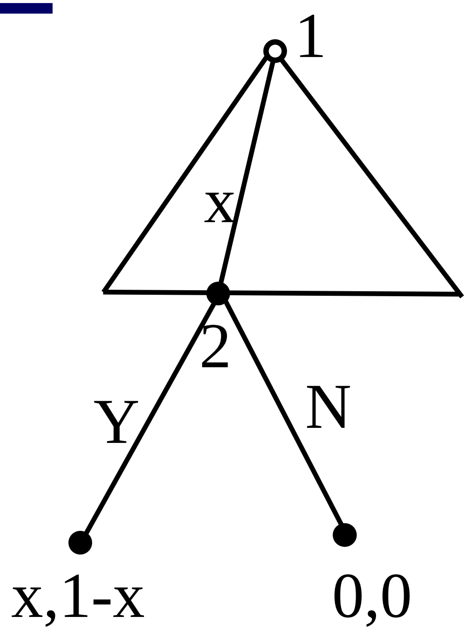
The ultimatum game

- Two players bargain over 1￥:
  - Player 1 offers player 2 some amount $1 - x \leq 1$
  - If player 2 accepts the outcome is: $(x, 1 - x)$ e.g. (0.7,0.3)
  - If player 2 rejects the outcome is: (0, 0)
- Each person cares about the amount of money received. Assume that x can be any scalar, not necessarily integral.
- Question: What is an SPE for this game?

Ultimatum Game

Back induction to find the SPE

- Player 1's optimal strategy
  - If $x < 1$, then accept
  - If $x = 1$, then accept or reject
- If player 2 accept for any $x \in [0,1]$
  - What is the optimal offer by player 1? $x = 1$
  - The SPE is (1,Y)
- If player 2 accept if and only if $x \in [0,1)$
  - What is the optimal offer by A? No solution



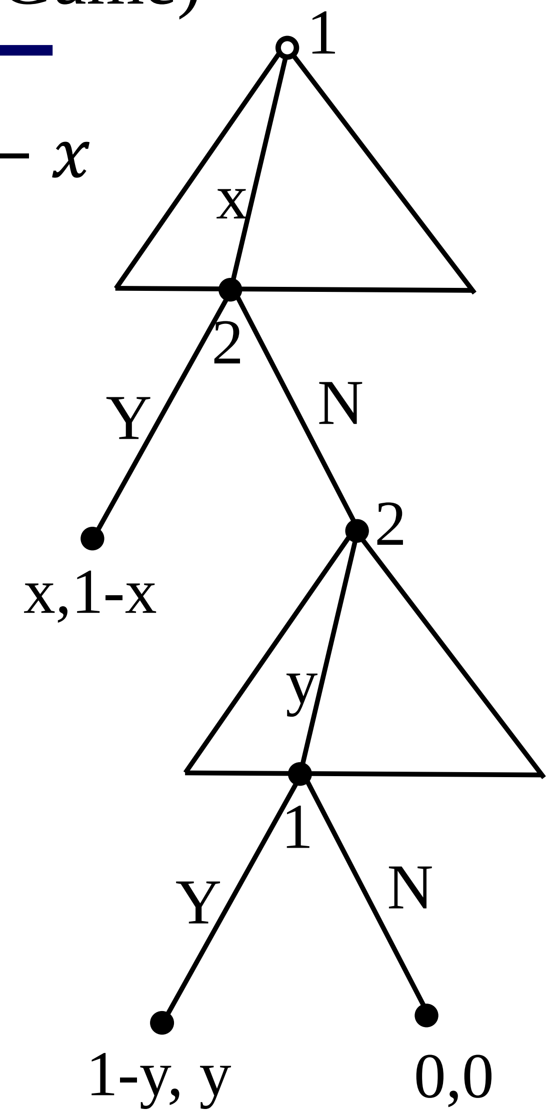
Unique SPE (1,Y)

Two-Period Bargaining Game (Ultimatum Game)

- Player 1 offers player 2 some amount $1 - x$
- Player 2 has two choices:
  - Accept: $(x, 1 - x)$
  - Reject: we flip the role and play again
    - This is the second period of the game

The second period is an ultimatum game:

- Player 2 offers player 1 some amount $1 - y$
- If player 1 accepts, the deal is done
- If player 1 rejects, none of them gets anything

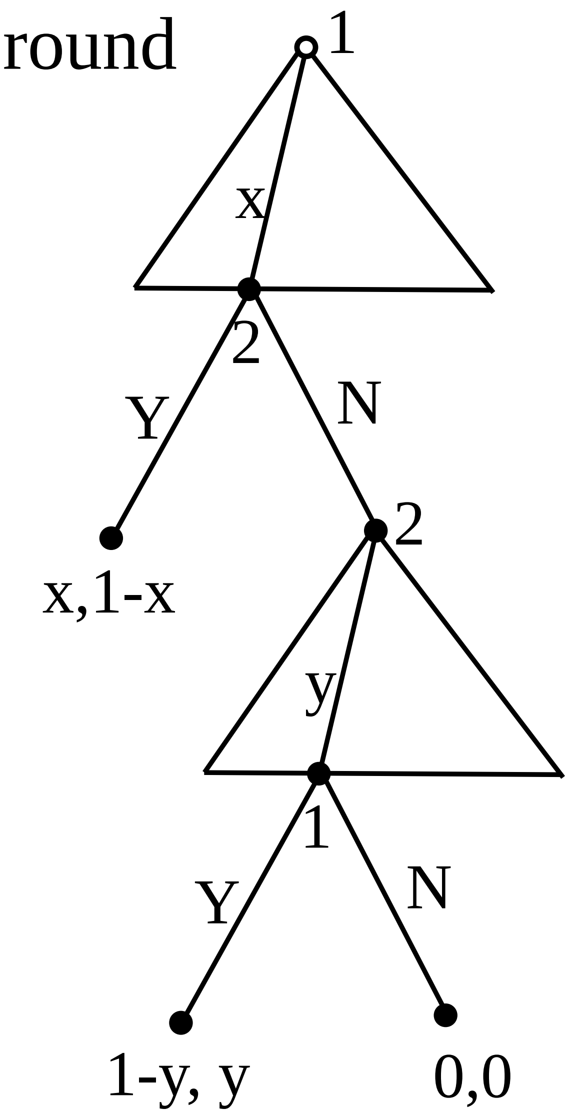


Discount Factor

- We add one important factor
  - In the first round, the pie is worth 1￥
  - If we end up in the second round, the pie is worth less
- Example:
  - If I give you 1￥ today, that's what you get
  - If I give you 1￥ in 1 year, we assume it's worth less, say $\delta < 1$
- **Discounting factor:**
  - From today perspective, 1￥ tomorrow is worth $\delta < 1$

Analysis for Two-Period Bargaining Game

It is clear that the decision to accept or reject partly depends on what you think in the second round



Backward induction:

- The unique SPE in the second period $(0, \delta)$
- What you should offer in the first period: palyer 1 offer $x = 1 - \delta$

Comparisons

	Player 1	Player 2
1-period	1	0
2-period	$1-\delta$	δ

- In the second round of the two-period game, player 2 gets the whole pie
- The pie in the second round, that player 2 gets, is worth less than 1￥

Three-Period Bargaining Game

The rules are the same as for the previous games, but now there are two possible flips:

- Period 1: player 1 offers first
- Period 2: if player 2 rejected in period 1, she gets to offer
- Period 3: if player 1 rejected in period 2, he gets to offer again

Discounting factor:

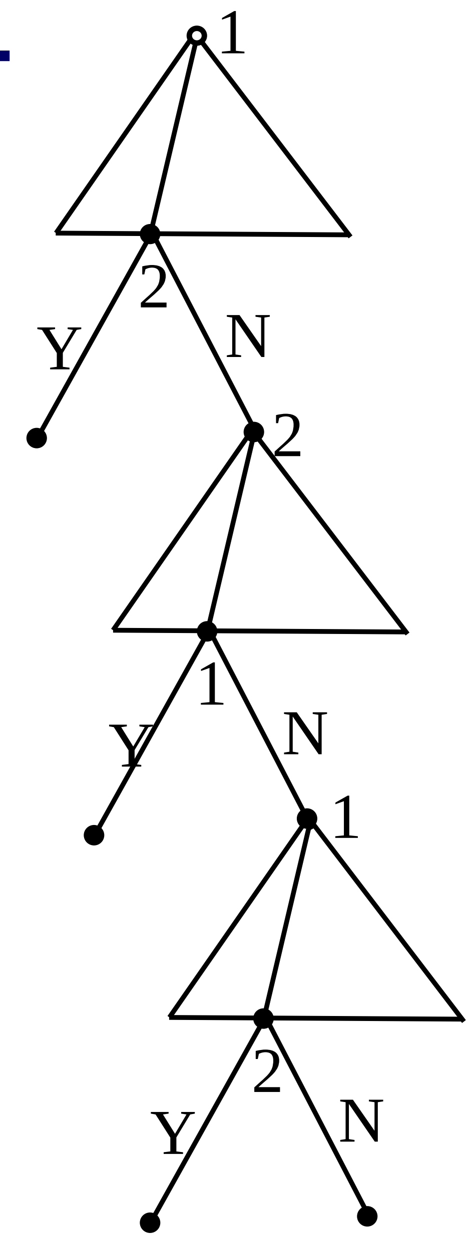
- the value of a pie in round three is discounted by δ
- the value of a pie in round three is discounted by δ^2

Analysis for Three-Period Bargaining Game

- In the third round
  - Unique SPE (δ^2, Y)
  - Payoffs $(\delta^2, 0)$

- In the second round
  - Unique SPE $(\delta - \delta^2, Y)$
  - payoffs $(\delta^2, \delta - \delta^2)$

- In the first round
  - Unique SPE $(1 - \delta + \delta^2, Y)$
  - payoffs $(1 - \delta + \delta^2, \delta - \delta^2)$



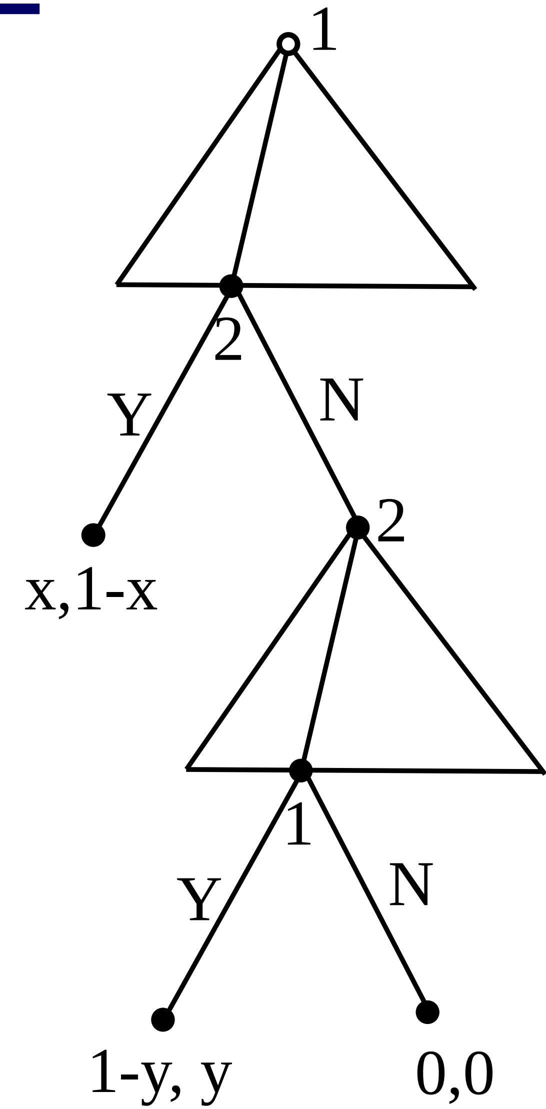
Result for Three-Period Bargaining Game

	Player 1	Player 2
1-period	1	0
2-period	$1-\delta$	δ
3-period	$1-\delta(1-\delta)$	$\delta(1-\delta)$

- In the second round of the two-period game, player 2 gets the whole pie
- The pie in the second round, that player 2 gets, is worth less than 1￥

Result for Four-Period Bargaining Game

	Player 1	Player 2
1-period	1	0
2-period	$1-\delta$	δ
3-period	$1-\delta(1-\delta)$	$\delta(1-\delta)$
4-period	?	?



Analysis for n-Period Bargaining Game

Geometric series

- payoff for player 1 on n $-$period bargaining

$$1-\delta+\delta^2-\delta^3+\cdots+(-\delta)^{n-1}=\frac{1-(-\delta)^n}{1+\delta}$$

- payoff for player 2 on n $-$period bargaining

$$1-\frac{1-(-\delta)^n}{1+\delta}=\frac{\delta-(-\delta)^n}{1+\delta}$$

Large Number of Period Bargaining Game

Let's look at the asymptotic behavior of this game, when there is an infinite number of stages

Player 1: $\frac{1-(-\delta)^n}{1+\delta} \rightarrow \frac{1}{1+\delta}$

Player 2: $\frac{\delta-(-\delta)^n}{1+\delta} \rightarrow \frac{\delta}{1+\delta}$

Let's imagine that the offers are made in rapid succession: this would imply that the discount factor we hinted at is almost negligible $\delta \rightarrow 1$

So, if we assume rapidly alternating offers, we end up with a 0.5-0.5 split!