Game Theory and Applications (博弈论及其应用)

# Chapter 3: Dominant Strategy Equilibrium and Rationality

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# Recap on Previous Chapter

- Mixed strategy game
  - $G$=$\{N, \{A_1, A_2, \dots, A_N\}, \{u_1, u_1, \dots, u_N\}\}$
  - $G = \left\{N, \{\Delta(A_1), \Delta(A_2), \dots, \Delta(A_N)\}, \{U_1, U_2, \dots, U_N\}\right\}$
- Mixed strategy Nash equilibrium
  - $p = (p_1 \dots p_N)$ is a MNE if and only if $p_i \in B_i(p_{-i})$
- Nash Theorem
  - Every finite strategic game has a mixed strategy Nash equilibrium

## Recap on Previous Chapter

- How to find mixed strategy Nash equilibria
  - If $p_i$ is a best response to $p_{-i}$ and $p_i(a_{ij}) > 0$ then $a_{ij}$ is a best response to $p_{-i}$

***Example*** A police and a chief are in a town. A bank and bar are required for protection in different direction. There are 2 and 1 million in bank and bar, respectively.

What’s the strategy for police?

# Dominant Strategy

- In most strategy games, one player's optimal choice depends on others' choice
- For some special cases, however, there is a optimal strategy independent of others' choice, e.g., dominant strategy

| | | Prisoner 2 | |
|---|---|---|---|
| | | Confess( c) | Don't confess(d) |
| Prisoner 1 | Confess(c) | -6 -6 | 0 -12 |
| | Don't confess(d) | -12 0 | -1 -1 |

Prisoner 2 will select c whatever Prisoner 1 how to choose

# Formal Definition

- $a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_N)$ outcome of strategy taken by all player other than $i$
- $A_{-i}$ denotes the set of all such outcomes

A pure strategy $a_i$ **strictly dominates** $a_i'$ if

$$u_i(a_i, a_{-i}) > u_i(a_i', a_{-i}) \text{ for all } a_{-i} \in A_{-i}$$

A pure strategy $a_i$ **weakly dominates** $a_i'$ if

$$u_i(a_i, a_{-i}) \geq u_i(a_i', a_{-i}) \text{ for all } a_{-i} \in A_{-i}$$

$$\text{while } u_i(a_i, a_{-i}) > u_i(a_i', a_{-i}) \text{ for some } a_{-i} \in A_{-i}$$

$a_i$ is **strictly dominant** if it strictly dominates all other strategies in $A_i$, and it is called **weak dominant** if it weakly dominates all other strategy in $A_i$

# Dominant Strategy Equilibrium

If every player has a (strictly and weakly) dominant strategy, then the corresponding outcome is a **(strictly and weakly) dominant strategy equilibrium**.

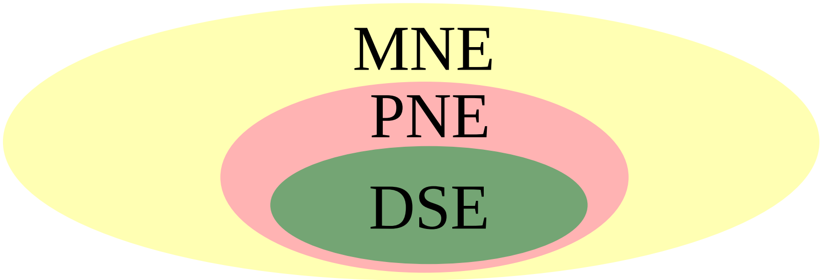
**Dominant strategy equilibrium is belong to NE.**

| Prisoner 1 \ Prisoner 2 | c | d |
|---|---|---|
| c | -6 -6 | 0 -12 |
| d | -12 0 | -1 -1 |

| Player 1 \ Player 2 | l | r |
|---|---|---|
| u | 3 3 | 3 0 |
| d | 0 3 | 3 3 |

It is very simple

It may not exist in many games



# Second Price Auction

$N$: players bid a building

$v_i \geq 0$: the true value for player $i$

$b_i \geq 0$: the bid price for  player $i$

$v_i - b_i$: the payoff for  player $i$

The rule of **second-price auction** is given as follow:

- Players make bids $b = (b_1, b_2, \ldots, b_N)$ simultaneously
- The higher player wins the building, yet pays the second highest bid price
- If there are more than one highest players, then randomly select one player and pay his own bid price

# Second Price Auction

**Theorem** In second price auction, the strategy $b_i = v_i$ is a weakly dominant strategy for each player $i$.

$(v_1, v_2, \dots, v_N)$ is a weakly dominant strategy equilibrium.

*Pf*. It suffice to show $u_i(v_i, b_{-i}) \geq u_i(b_i, b_{-i})$ for all $b_i, b_{-i}$

- If someone's bid $b_k \geq v_i$, then player $i$ has to pay $b_i > b_k \geq v_i$ by winning. Payoff is $v_i - \max_{k \neq i} b_k \leq 0$. It is optimal to select $b_i = v_i$.
- If each bid prize $b_k < v_i$, then payoff is $v_i - \max_{k \neq i} b_k > 0$, since the payoff is always the same when winning. It is optimal to select $b_i = v_i$.

# Second Price Auction

- Honesty strategy is the best strategy
- Many internet auctions can be regarded as variants of second price auction.
- How about the first price auction. Is it a dominant strategy to bid your true value?

# Dominated Strategies

A pure strategy $a_i$ **strictly dominates** $a_i'$ $\Rightarrow$ $a_i'$ is **strictly dominated** by $a_i$ if

$$u_i(a_i, a_{-i}) > u_i(a_i', a_{-i}) \text{ for all } a_{-i} \in A_{-i};$$

$a_i$ **weakly dominates** $a_i'$ $\Rightarrow$ $a_i'$ is **weakly dominated** by $a_i$ if

$$u_i(a_i, a_{-i}) \geq u_i(a_i', a_{-i}) \text{ for all } a_{-i} \in A_{-i}$$

$$\text{while } u_i(a_i, a_{-i}) > u_i(a_i', a_{-i}) \text{ for some } a_{-i} \in A_{-i}$$

# Iterated Elimination of Dominated Strategies

**How to find Dominant strategy equilibrium?**

**- Iterated Elimination of Dominated Strategies**

If every strategy eliminated is a strictly dominated strategy

- Iterated elimination of strictly dominated strategy

If at least one strategy eliminated is a weakly DS

- Iterated elimination of weakly dominated strategy

# Iterated Elimination and Pure Dominate Strategy

Player 2

| | | l | | m | | r | |
|---|---|---|---|---|---|---|---|
| Player 1 | u | 10 | 10 | 2 | 15 | 10 | 10 |
| | m | 15 | 2 | 5 | 5 | 5 | 5 |
| | d | 10 | 10 | 5 | 5 | 10 | 10 |

- For player 1, the strategy ‘u’ is weakly dominated by ‘d’
- For player 2, the strategy ‘l’ is weakly dominated by ‘r’

# Iterated Elimination and Pure DS (cont.)

Therefore, we have the game

| | | Player 2 | |
|---|---|---|---|
| | | m | r |
| Player 1 | m | 5, 5 | 5, 5 |
| | d | 5, 5 | 10, 10 |

- For player 1, the strategy ‘m’ is weakly dominated by ‘d’
- For player 2, the strategy ‘m’ is weakly dominated by ‘r’

By iterated elimination of weakly dominated strategy

(d,r) is a weakly dominant strategy Equilibrium

# Mixed Strategy and Dominant Strategy

- A strategy may be not dominated by other strategies, yet can be dominated by a mixed strategy

Player 2

| Player 1 | | l | | r | |
|---|---|---|---|---|---|
| | u | 1 | 1 | 1 | 0 |
| | m | 3 | 0 | 0 | 3 |
| | d | 0 | 1 | 4 | 0 |

- For player 1, no strategy dominates ‘u’
- The mixed strategy $p_1 = (0, 0.5, 0.5)$ dominates ‘u’

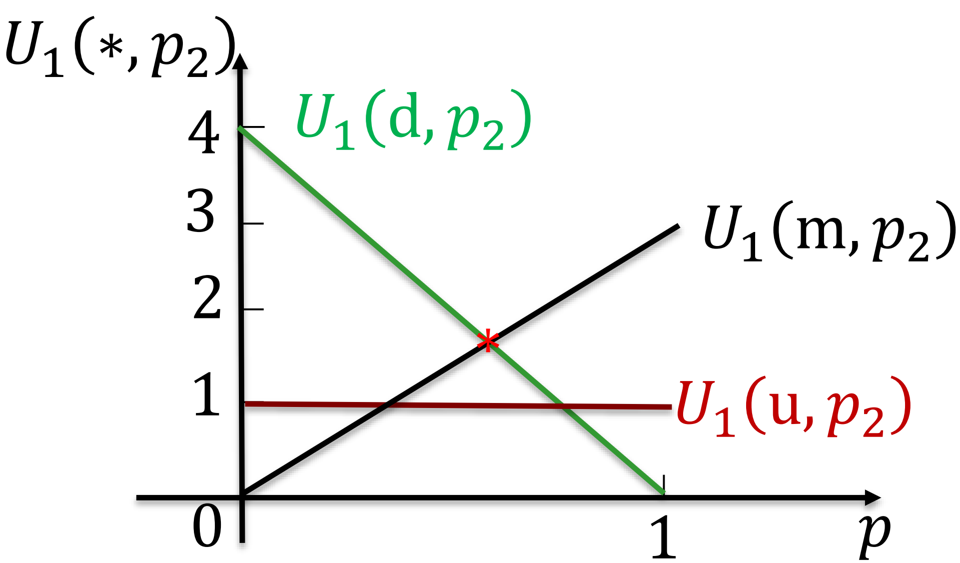
# Mixed Strategy and Dominant Strategy

Figure out the dominated strategy for expected payoff

Let $p_2 = (p, 1-p)$ be the mixed strategy for player 2

| Player 1 \ Player 2 | l | r |
|---|---|---|
| u | 1, 1 | 1, 0 |
| m | 3, 0 | 0, 3 |
| d | 0, 1 | 4, 0 |

$U_1(\mathrm{u}, p_2) = 1$
$U_1(\mathrm{m}, p_2) = 3p$
$U_1(\mathrm{d}, p_2) = 4(1-p)$




- The mixed strategy $p_1 = (0, 0.5, 0.5)$ dominates ‘u’
- ‘u’ is a never best strategy

# Mixed Strategy and Dominant Strategy

**Theorem** A **strictly dominated strategy** is never used with positive probability in a mixed strategy Nash equilibrium

Let $p = (p_1, \dots, p_N)$ be a mixed strategy NE.

For player $i$, $a_i, a_i' \in A_i$ s.t. $a_i$ is strictly dominated by $a_i'$,

$$\mathrm{U}_i(a_i, p_{-i}) < \mathrm{U}_i(a_i', p_{-i})$$

$$p_i(a_i) = 0$$

*Proof*. See board.

# Find Mixed Strategy Nash Equilibria

Step 1: eliminate all **strictly** dominated strategies (**mixed**)

Step 2: use our previous methods introduced in Chapter 2

| | | Prisoner 2 | |
|---|---|---|---|
| | | Confess( c) | Don't confess(d) |
| Prisoner 1 | Confess(c) | -6   -6 | 0   -12 |
| | Don't confess(d) | -12   0 | -1   -1 |

# Find Mixed Strategy Nash Equilibria

| | | Player 2 | | |
|---|---|---|---|---|
| | | l | m | r |
| Player 1 | ~~u~~ | ~~0~~ , ~~5~~ | ~~2~~ , ~~3~~ | ~~2~~ , ~~3~~ |
| | m | 2 , ~~3~~ | 1 , 5 | 3 , 2 |
| | d | 5 , ~~0~~ | 3 , 2 | 2 , 3 |

# Exercise on Class

Find all pure and mixed strategy NE

Player 2

| Player 1 | x | y | z | r |
|---|---|---|---|---|
| a | 0, 5 | 2, 3 | 2, 0 | 1, 3 |
| b | 2, 3 | 5, 5 | 2, 4 | 3, 2 |
| c | 5, 0 | 6, 2 | 2, 1 | 2, 3 |
| d | 4, 1 | 3, 1 | 2, 0 | 1, 1 |

# Belief

Given a strategy game $G = \{N, \{A_i\}, \{u_i\}\}$

- A mixed strategy outcome $p = (p_1, p_2, \dots p_N)$
- $p = (p_i, p_{-i})$
- $p_{-i}$ is called **a belief**

**A belief $\boldsymbol{p_{-i}}$ of player $\boldsymbol{i}$ is a probability over $\boldsymbol{A_{-i}}$**

A strategy $a_i \in A_i$ is a best response to belief $p_{-i}$ if

$$U_i(a_i, p_{-i}) \geq U_i(a, p_{-i}) \text{ for all } a \in A_i$$

# Rationality and NE

**A pure strategy $\boldsymbol{a_i \in A_i}$ is rational** if there is a belief $p_{-i}$ s.t. $a_i$ is a best response to belief $p_{-i}$

The relationship between NE and rationality:

**Theorem** Every pure strategy with positive probability in a mixed strategy Nash equilibrium is rational.

*Pf*. Assume $p = (p_1, p_2, \ldots, p_N)$ is a mixed strategy NE. Then, $p_{\mathrm{i}}$ is a best response to $p_{-\mathrm{i}}$, and every strategy with positive probability in $p_{\mathrm{i}}$ is also a best response to $p_{-\mathrm{i}}$.

# Rationality and Strictly Dominant Strategy

$a_i \in A_i$ is rational if $a_i$ is a best response to some belief $p_{-i}$

$$U_i(a_i, p_{-i}) \geq U_i(a, p_{-i}) \text{ for all } a \in A_i$$

A mixed strategy $p_i \in \Delta(A_i)$ strictly dominates $a_i \in A_i$

$$U_i(p_i, p_{-i}) > U_i(a_i, p_{-i}) \text{ for all } p_{-i} \in \Delta(A_{-i})$$

The relationship between rationality and strict domination

**Theorem** A strategy $a_i \in A_i$ is rational if and only if $a_i$ is not strictly dominated.

# Rationality and Strictly Dominant Strategy

**Theorem** A strategy $a_i \in A_i$ is rational if and only if $a_i$ is not strictly dominated.

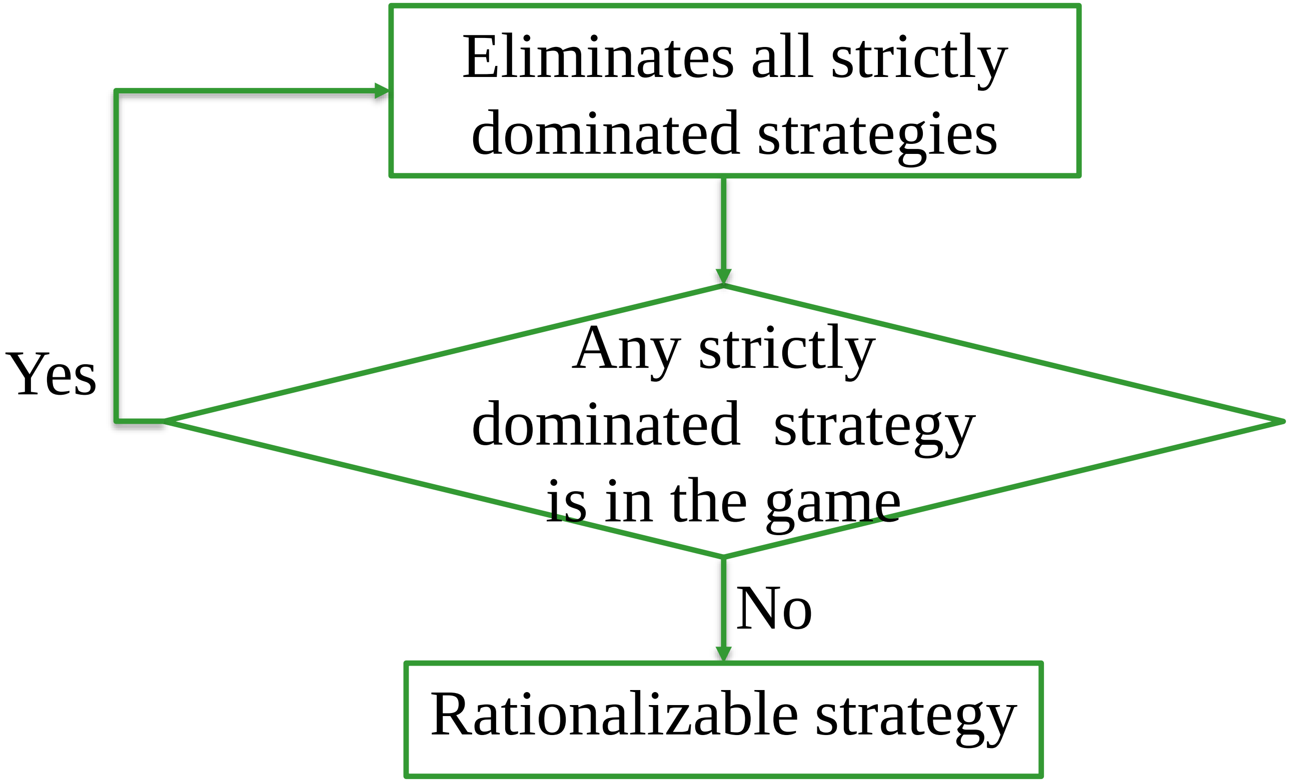
Proof. $\longrightarrow$ [By contradiction] If $a_i$ is rational, then there exists $p_{-i}$ such that $a_i$ is a best response to $p_{-i}$

$a_i'$ strictly dominates $a_i$: $U_i(a_i', p_{-i}) > U_i(a_i, p_{-i})$

$\leftarrow$ [By contradiction] if $a_i \in A_i$ is rational, then $\exists p_{-i}$, $\forall a' \in A_i$ such that $U_i(a', p_{-i}) \leq U_i(a_i, p_{-i})$

If $a_i \in A_i$ is not rational, then $\forall p_{-i}$, $\exists a' \in A_i$ such that $U_i(a', p_{-i}) > U_i(a_i, p_{-i})$; thus $a_i$ is strictly dominated

# Rationalizability




**Notice**

1) Eliminate all strictly DS and keep weakly DS

2) Eliminate all strictly DS by pure and mixed strategy

# An Example

Player 2

Player 1

| | l | | r | |
|---|---|---|---|---|
| u | 2 | 0 | -1 | 1 |
| m | ~~0~~ | ~~10~~ | ~~0~~ | ~~0~~ |
| d | -1 | -6 | 2 | 0 |

Player 1 is rational

Player 2 is rational and

knows that Player 1 is rational

Player 1 is rational, and knows that player 2 is rational

and knows that 2 knows that 1 is rational

# Beauty Contest (选 美 竞 赛 游 戏)

- There are $n$ players
- Each player selects a number $a_i \in [0,50]$
- The payoff for each player is $50 - \left(a_i - \frac{2}{3}\frac{\sum_i a_i}{n}\right)^2$

Given $a_{-i}$, the best strategy for player $i$ is

$$a_i^* = \frac{2}{3}\frac{\sum_{j,j\neq i} a_i}{n - 2/3}$$

$$a_i^* \in \left[0, \frac{2}{3}\frac{n-1}{n-2/3}50\right]$$

# Beauty Contest (cont)

- After round 1: $\left[0, \frac{2}{3}\frac{n-1}{n-2/3}50\right]$
- After round 2: $\left[0, \left(\frac{2}{3}\frac{n-1}{n-2/3}\right)^2 50\right]$

…

- After round k: $\left[0, \left(\frac{2}{3}\frac{n-1}{n-2/3}\right)^k 50\right]$

…

- Rational ={0}

# Symmetric Game

A game is symmetric if any player's payoff $u_i(a_i, a_j, a_{-i,j})$ can be converted into any other player's payoff $u_j(a_j, a_i, a_{-i,j})$ simply by re-arranging the player's "names"

**Theorem**. Any symmetric game has a symmetric NE, where each player uses the same strategy

| | | Prisoner 2 | |
|---|---|---|---|
| | | Confess( c) | Don't confess(d) |
| Prisoner 1 | Confess(c) | -6   -6 | 0   -12 |
| | Don't confess(d) | -12   0 | -1   -1 |

## Continuous Game

A game $G = \{N, \{A_i\}, \{u_i\}\}$ with complete information is **continuous** if each $A_i$ is non-empty and compact, and $u_i: A \to R$ are continuous.

- Many quantities are essentially continuous: If we're considering how many fish to catch in a season, where the measurement is in millions of tons

- Cournot game …

# Existence of Equilibria for Infinite Games

**(Nash)** Every finite game has a mixed strategy NE

**(Debreu, Glicksberg, Fan)** Consider a strategic form game $\{N, \{A_i\}, \{u_i\}\}$ such that for each player

- $A_i$ is compact and convex
- $u_i(a_i, a_{-i})$ is continuous in $a_{-i}$
- $u_i(a_i, a_{-i})$ is continuous and concave in $a_i$

There exists a pure strategy Nash equilibrium

# More Powerful Theorem

**(Glicksberg)** Consider a strategic form game $\{N, \{A_i\}, \{u_i\}\}$ such that for each player

- $A_i$ is compact and convex
- $u_i(a)$ is continuous in $a$

There exists a mixed strategy Nash equilibrium

For continuous pure strategy space, the space of mixed strategy has infinite dimension

# Summary

- Strictly/weakly Dominant Strategy
- Dominant Strategy Equilibrium
- Dominated strategy and Nash Equilibrium
- How to find NE
- Rational
- Rationalizability
- Continuous Game