Game Theory and Applications (博弈论及其应用)

Chapter 10: Extensive Game with Imperfect Information-II

南京大学

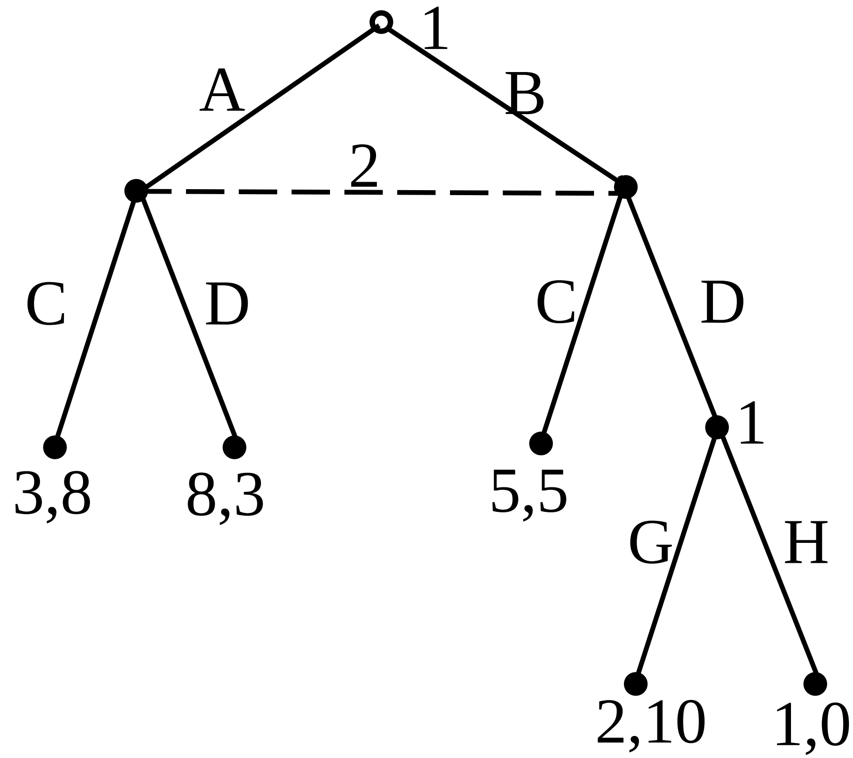
高 尉

Recap on Previous Course

- Extensive game with imperfect information $G = \{N, H, P, I, \{u_i\}\}$
- Information set $I = \{I_1, I_2, \dots I_N\}$
- Pure strategies $A(I_{i1}) \times A(I_{i2}) \times A(I_{im})$
- How to solve NE
- Perfect recall and imperfect recall
- Mixed Strategies and behavioral strategies, **equivalence**

Definition of Mixed and Behavioral Strategies

Mixed Strategies: A mixed strategy of player i in an extensive game is a probability over the set of player i's pure strategy

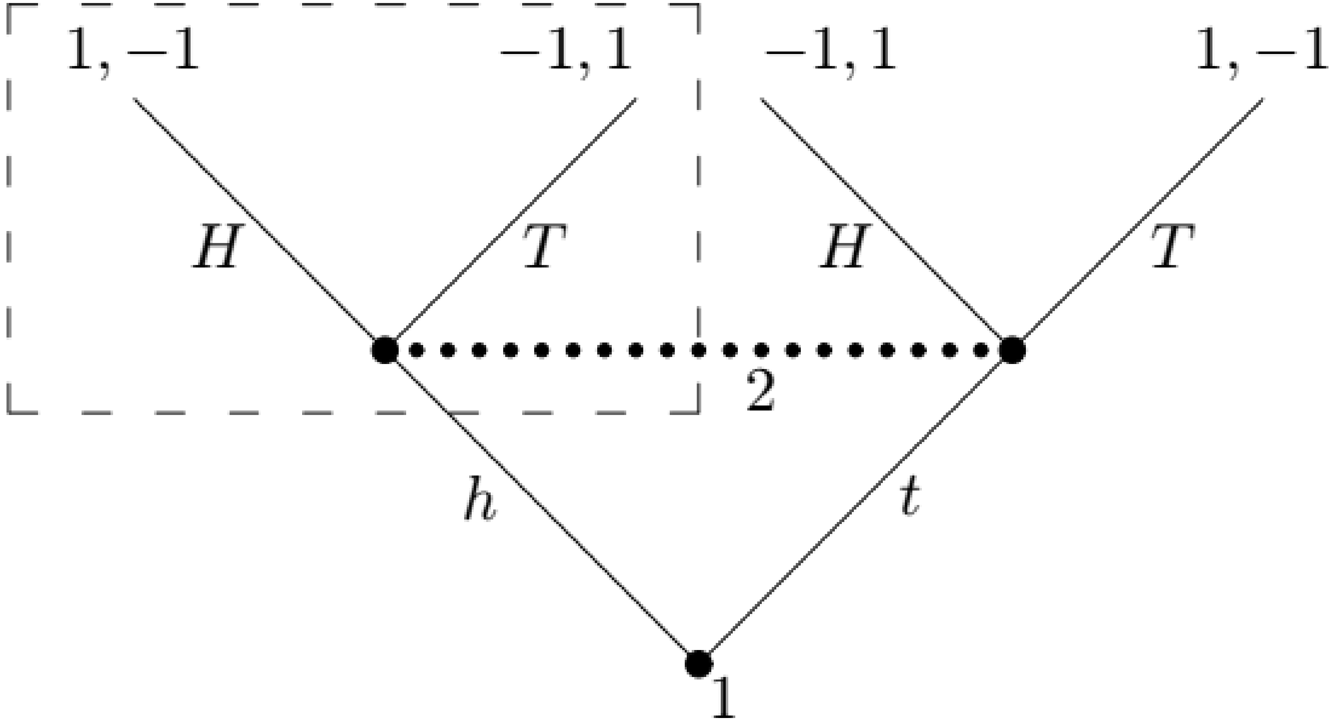


Behavioral strategies:

A behavior strategy of player i is a collection $\beta_{ik}(I_{ik})_{I_{ik}\in I_i}$ of independent probability measure, where $\beta_{ik}(I_{ik})$ is a probability measure over $A(I_{ik})$

Extensive Imperfect Subgame

Definition A subgame of an extensive imperfect game G is some node in the tree G and all the nodes that follow it, with the properties that any information set of G is either completely in or outside the subgame



Subgame Perfect Nash Equilibrium

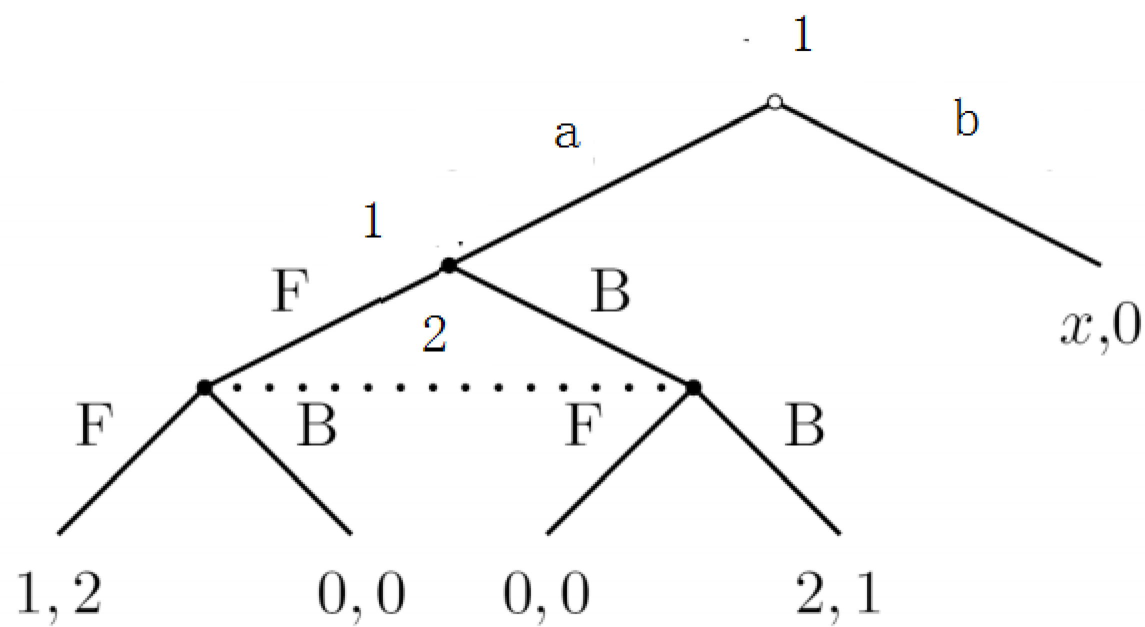
Definition A subgame perfect Nash equilibrium of an extensive form game G with perfect recall is a outcome of behavior strategies $(\beta_1, \beta_2, \dots, \beta_N)$ such that it is a Nash Equilibrium for every subgame

Theorem Every finite extensive game with perfect recall has at least one subgame perfect Nash Equilibrium

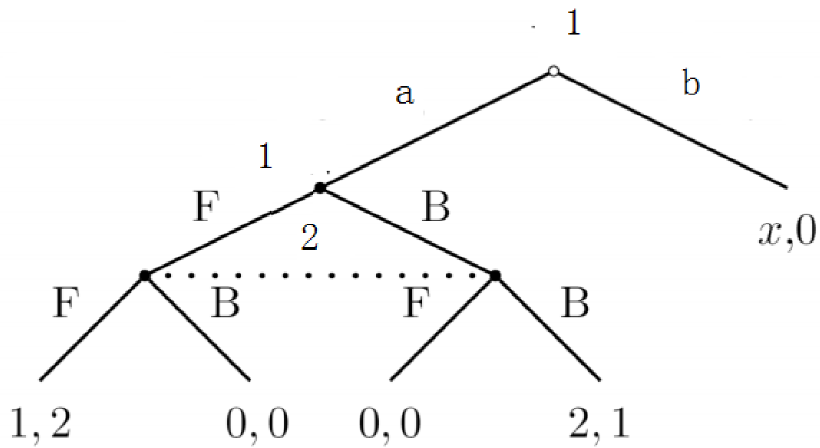
How to find SPNE

Backwards Induction

Example



Example



For NE (F,F)
If $x \geq 1$, player 1 select b

For NE (B,B)
If $x \geq 2$, player 1 select b

2

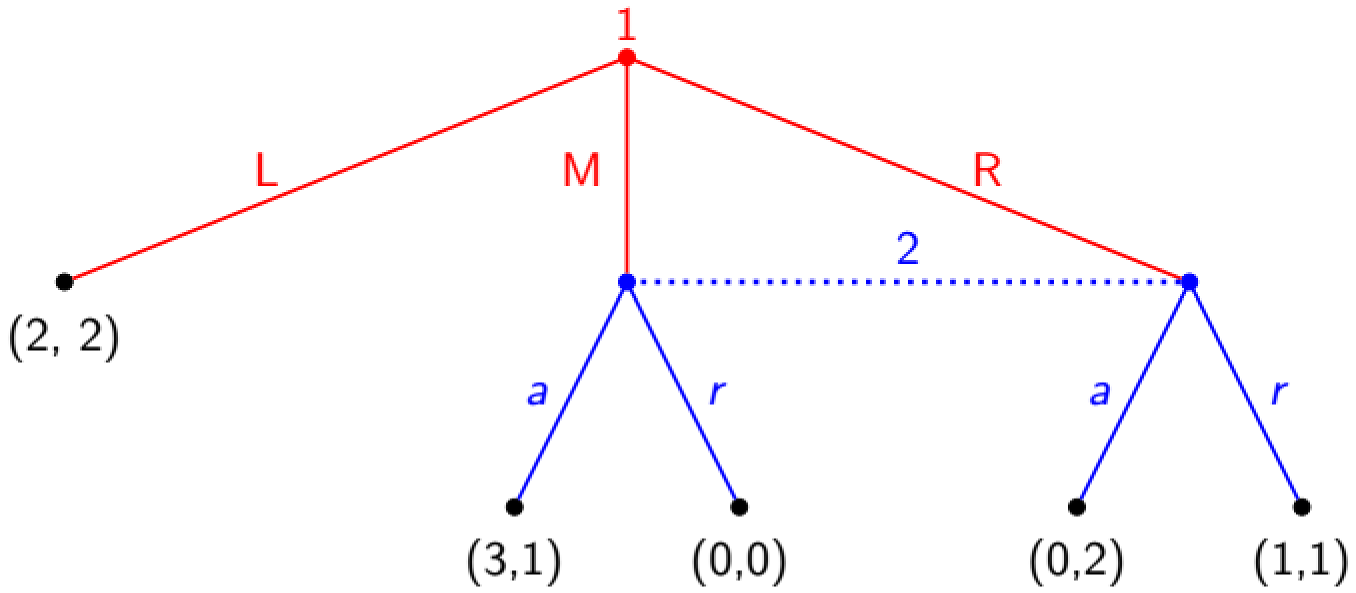
		F	B
1	F	1 2	0 0
	B	0 0	2 1

If $x < 1$, SPNE: (aF,F), (aB,B)

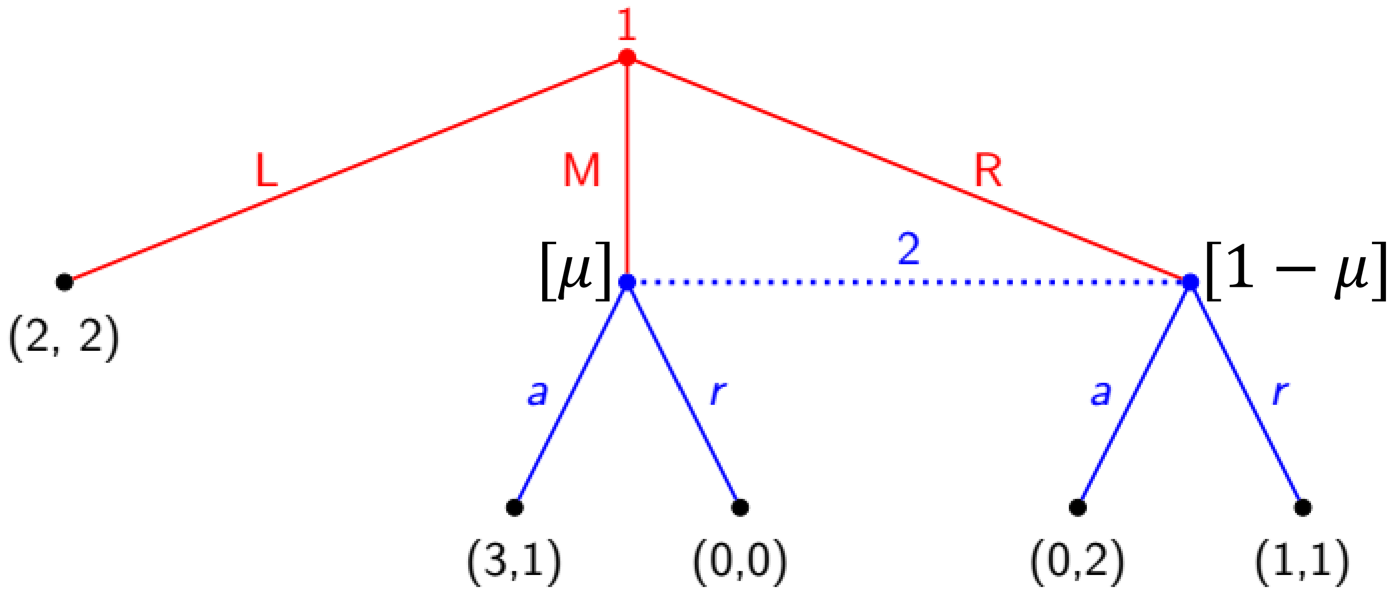
If $1 \leq x < 2$,
SPNE: (bF,F), (aB,B)

If $x \geq 2$,
SPNE: (bF,F), (bB,B)

An Example

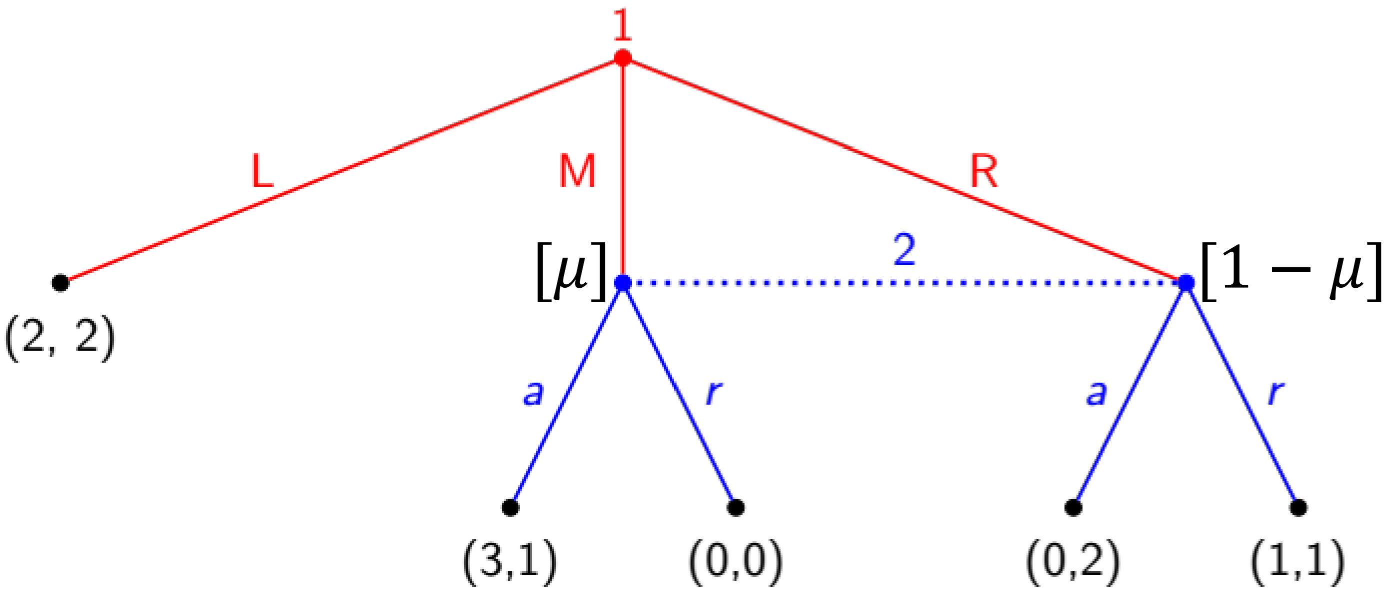


Belief



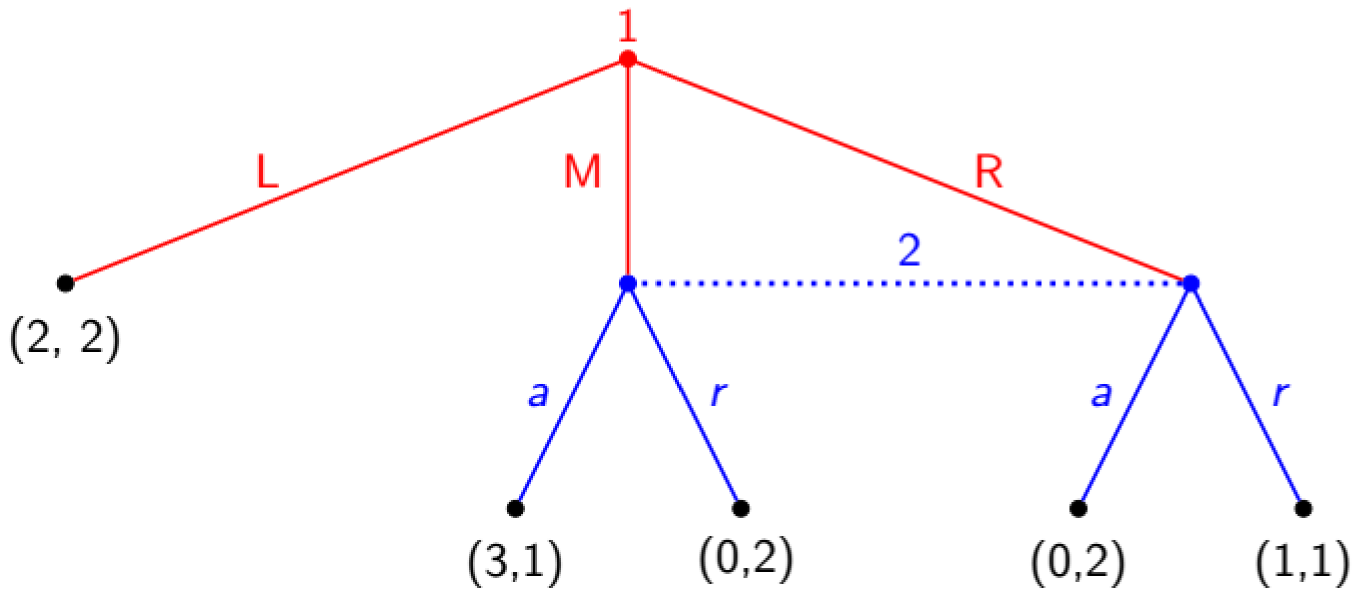
- A belief μ is a function that assigns to every information set a probability measure on the set of histories in the information set
- The probability is 1 for the information set of size 1

Behavior Strategy



- A behavior strategy β a collection of independent probability measure over the actions after information set

Beliefs and Optimal Behavior Strategies

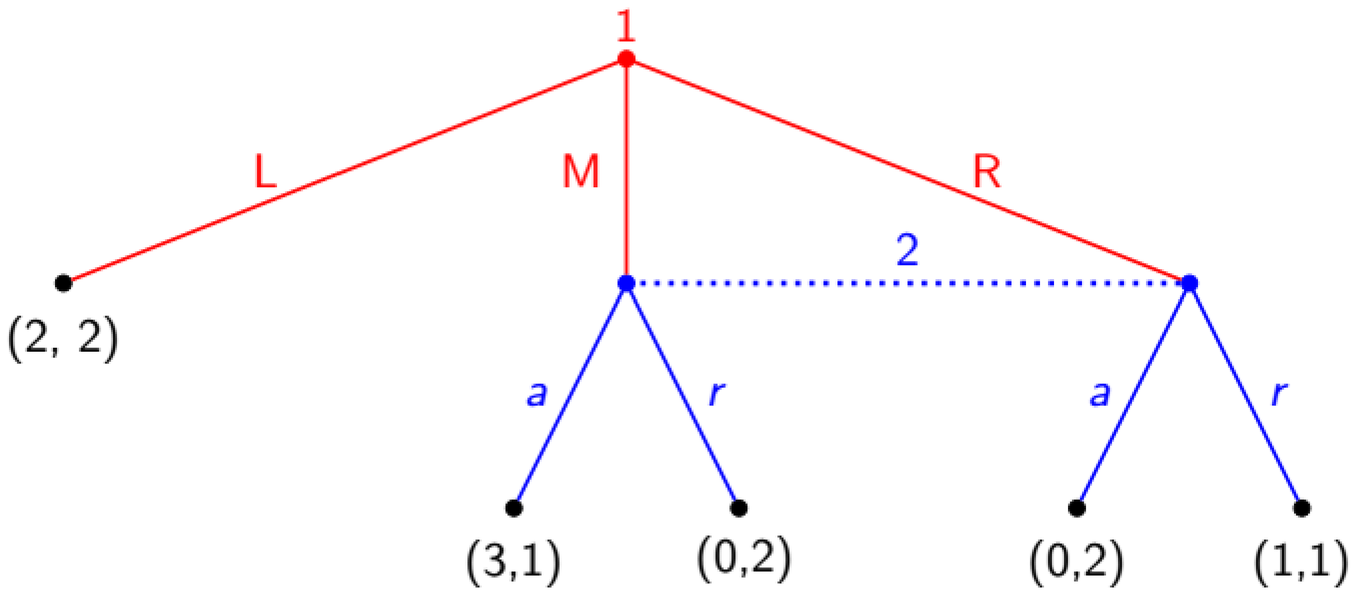


- Beliefs affect optimal strategies: For 2, a is the best strategies iff 2 assigns a belief $\mu(M) \leq 1/2$
- Strategies affect reasonable beliefs: If 1 assigns to action (L,M,R) prob. (0.1,0.3,0.6), then Bayes rule requires the belief (1/3,2/3) of 2
- What are reasonable beliefs if 1 select L with prob. 1

Two Requirements to Beliefs

Bayes consistency: beliefs are determined by Bayes' law in information sets of positive probability; otherwise, beliefs are allowed to be arbitrary for 0 probability.

Consistency: beliefs are determined as a limit of case



1: (L,M,R) with probability $(1-\epsilon, 3\epsilon/4, \epsilon/4)$.

2: belief is well-defined for $\epsilon > 0$, as well as $\epsilon = 0$

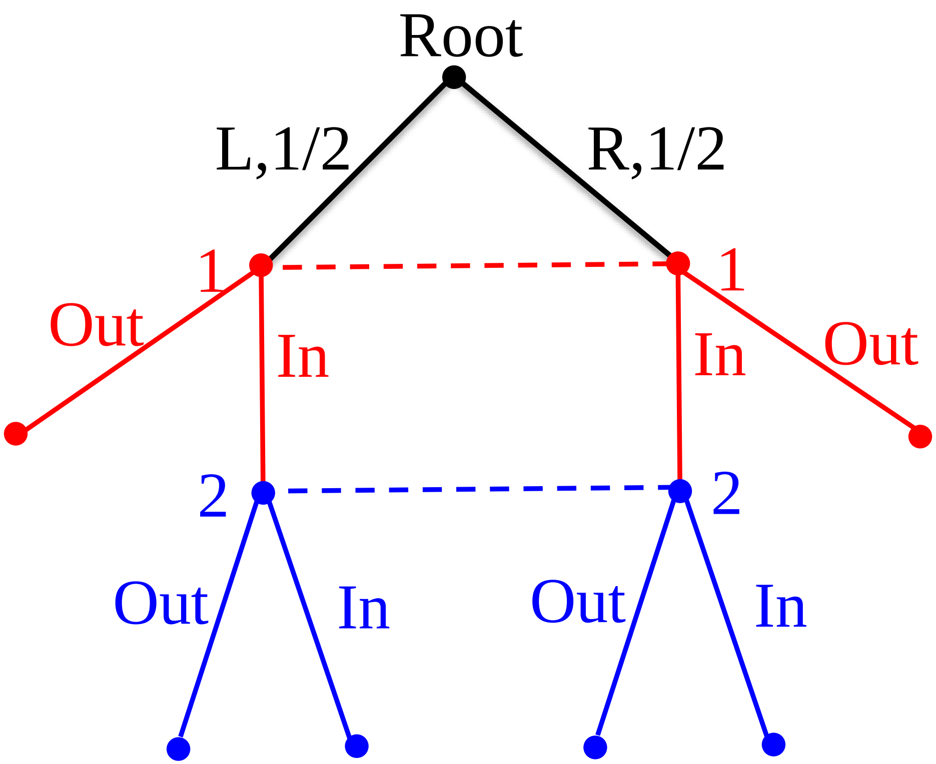
Assessment (评 估)

- An assessment is a pair (β, μ)
  - β is an outcome of behavioral strategies
  - μ is a belief system
- Assessment (β, μ) is:
  - **Bayesian consistent** if beliefs in information sets reached with positive probability are determined by Bayes' law:

$$\mu_{\mathrm{h,a}}(\mathrm{h,a}) = \beta_{h,a}(h,a) / \sum_a \beta_{h,a}(h,a)$$

    for every information set.
  - **Consistent** if there is a sequence of Bayesian consistent and $(\beta^n, \mu^n) \to (\beta, \mu)$ as $n \to \infty$
- (β, μ) is consistent → (β, μ) Bayesian consistent

Example

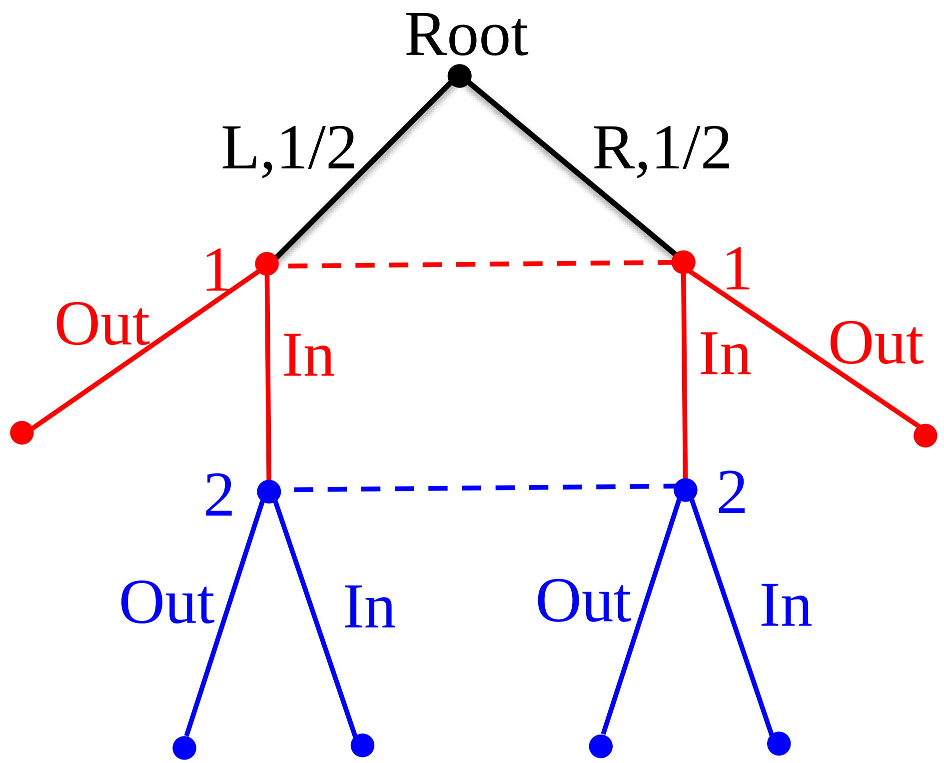


- The payoffs are omitted since they are irrelevant
- Find all Bayesian consistent assessments
- Find all consistent assessments

Bayesian consistency

An assessment (β, μ) by a 4-tuple $(\beta_1, \beta_2, \mu_1, \mu_2) \in [0,1]^4$

- β_1 is the probability that 1 chooses In
- β_2 is the probability that 2 chooses In
- μ_1 is the belief assigns to the left node in 1's info set
- μ_2 is the belief assigns to the left node in 2's info set



Bayesian consistency

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Two cases:

- i) If $\beta_1 \in (0,1]$, 2's information set is reached with positive probability. Bayes' Law dictates that $\mu_1 = \mu_2 = 1/2$.

  $$(\beta_1, \beta_2, \mu_1, \mu_2) = (0,1] \times [0,1] \times \{1/2\} \times \{1/2\}$$

  are Bayesian consistent

- ii) If $\beta_1 = 0$, then 2's information set is reached with zero probability and $\mu_2 \in [0,1]$

  $$(\beta_1, \beta_2, \mu_1, \mu_2) = \{0\} \times [0,1] \times \{1/2\} \times [0,1]$$

  are Bayesian consistent

Consistency

- Every complete outcome of behavioral strategies leads to $\mu_1 = \mu_2 = 1/2$.
- 2's information set, both nodes are reached with equal probability.
- Conclusion:

$$(\beta_1, \beta_2, \mu_1, \mu_2) = [0,1] \times [0,1] \times \{1/2\} \times \{1/2\}$$

are consistent

Expected Payoffs in Information Sets

Fix assessment (β, μ) and information set I_{ij} of player i. We consider the expected payoff of player i on I_{ij} as

- Given I_{ij}, the belief μ assigns probability over I_{ij} with $\mu(h)$ for $h \in I_{ij}$
- For $h \in I_{ij}$, let $P(e|h, \beta)$ the probability from h to e under the behavioral strategy β, and the payoff is $u_i(e)$

The expected payoff for player i in the information I_{ij} w.r.t. (β, μ), is

$$u_i\left(\beta_i, \beta_{-i} \middle| I_{ij}, \mu\right) = \sum_{h \in I_{ij}} \mu(h) \left(\sum_e P(e|h, \beta) u_i(e)\right)$$

Sequential Rational

Assessment (β, μ) is **sequentially rational** if for each information set I_{ij}, player i makes a best response w.r.t. belief μ, that is,

$$u_i\left(\beta_i, \beta_{-i} \middle| I_{ij}, \mu\right) \geq u_i\left(\beta_i', \beta_{-i} \middle| I_{ij}, \mu\right)$$

for all other behavior strategies β_i' of player i

- Consistency: beliefs have to make sense w.r.t strategies, without requirements on strategies
- Sequential rationality: strategies have to make sense w.r.t. beliefs, without requirements on beliefs

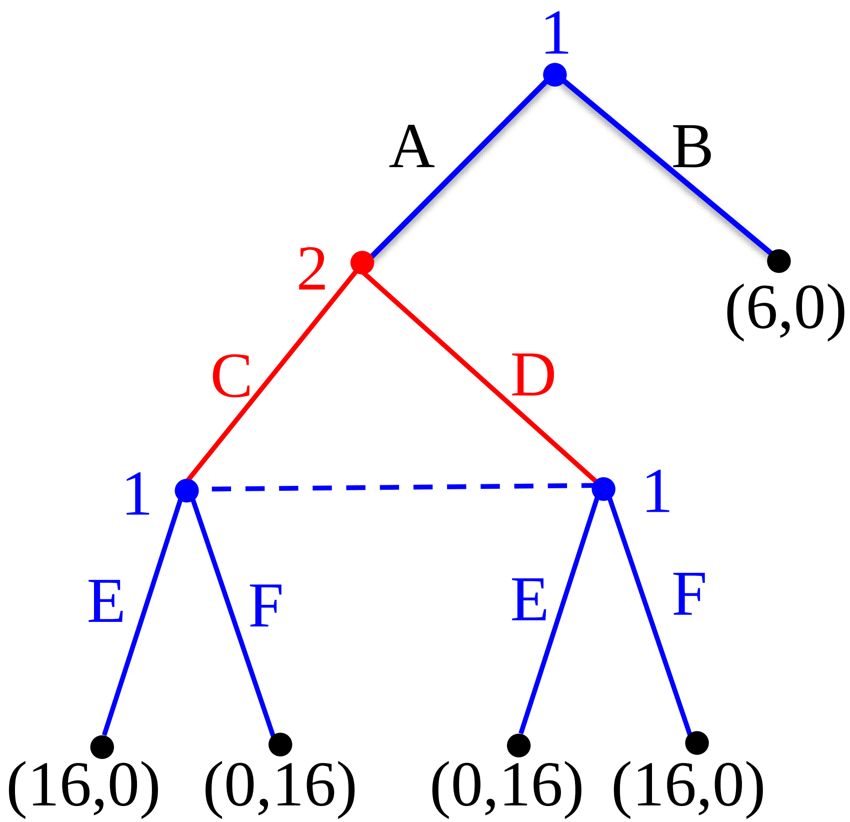
Sequential Equilibrium

An assessment (β, μ) is a **sequential equilibrium** if it is both consistent and sequentially rational.

Theorem

- a) Each finite extensive form game with perfect recall has a sequential equilibrium.
- b) If assessment (β, μ) is a sequential equilibrium, then β is a subgame perfect equilibrium.

Example



How to calculate the sequential equilibrium?

Example (Consistency)

Behavioral strategies $\beta = (\beta_1, \beta_2) = (p, r; q)$, where

- p: probability that 1 chooses A;
- q: probability that 2 chooses C;
- r: probability that 1 chooses E;

Belief μ can be summarized by one probability α

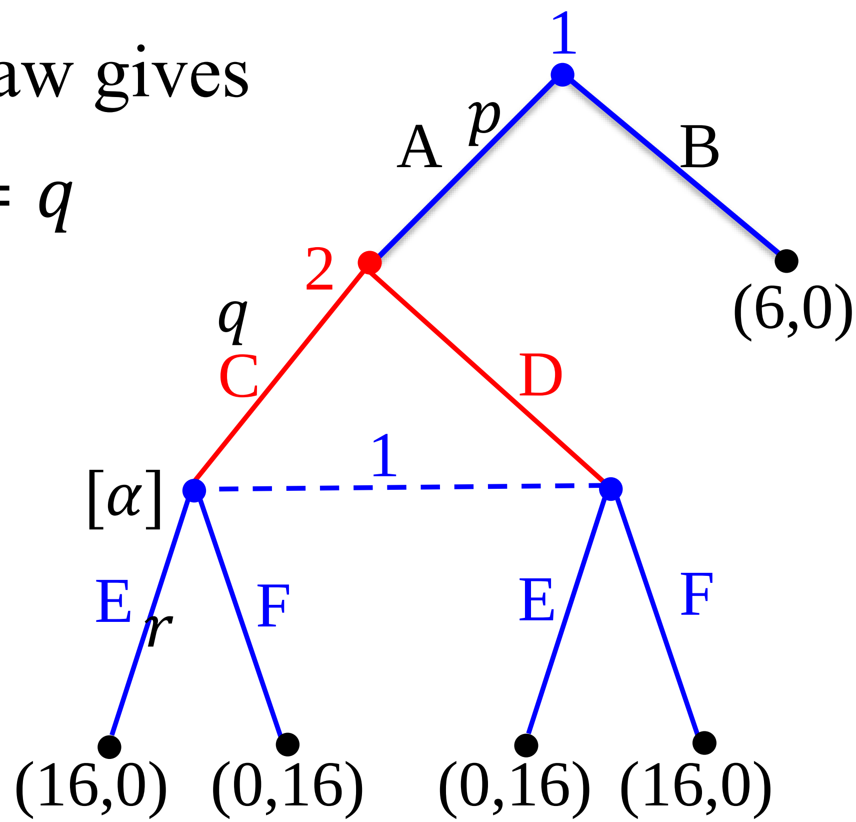
- α: probability assigns to history AC in inform. set {AC,AD}

Example (Consistency)

If $p, q, r \in (0,1)$, then Bayes' law gives

$$\alpha = \frac{pq}{pq + p(1-q)} = q$$

For each consistent (β, μ)
we have $\alpha = q$



Example (Rationality)

- If $q = 0$, then $\alpha = 0$ and $r = 0$ is player 1's unique best reply in the final info set. But if $r = 0$, then $q = 0$ is not a best reply in 2's info set. Contradiction.
- If $q = 1$, then $\alpha = 1$ and $r = 1$ is player 1's unique best reply in the final info set. But if $r = 1$, then $q = 1$ is not a best reply in 2's info set. Contradiction.
- If $q \in (0,1)$
  - rationality of 2 dictates that both C and D must be optimal and equal, i.e., $16(1-r) = 16r$, this gives $r = 1/2$
  - In info set (AC,AD), the expected payoff of player 1 is $\alpha 16r + (1-\alpha)16(1-r) = 16 - 16\alpha + 16r(2\alpha - 1)$
    - $r = 1$ if $\alpha > 1/2$; $r = 0$ if $\alpha < 1/2$; and $r \in (0,1)$ if $\alpha = 1/2$

$r = 1/2$ if and only if $\alpha = 1/2$. Finally $p = 1$

Signaling games (信号传递博弈)

The most interesting class of games that are solved used the sequential Equilibrium concept are signaling games

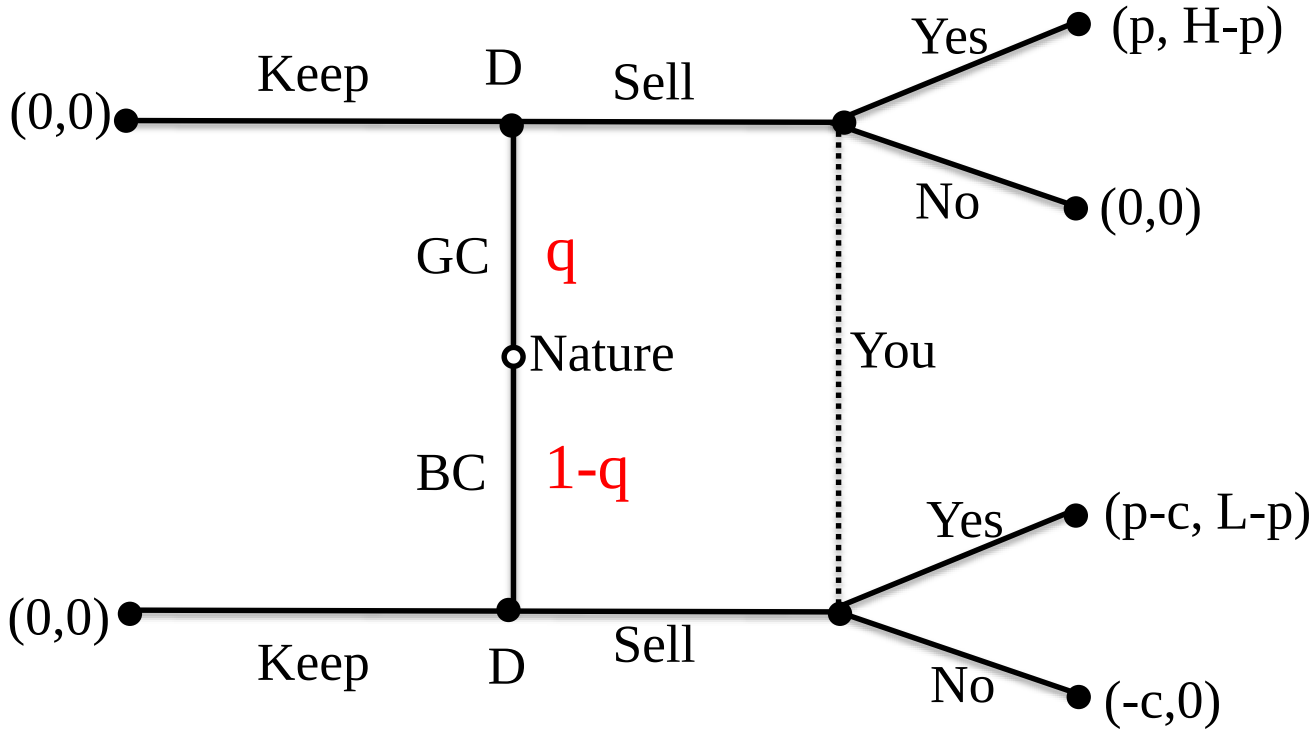
Michael Spence, 2001 Nobel Memorial Prize in economics: job-market signaling model

- A prospective employer can hire an applicant.
- The applicant has high or low ability, but the employer doesn't know which
- Applicant can give a signal about ability, e.g., education

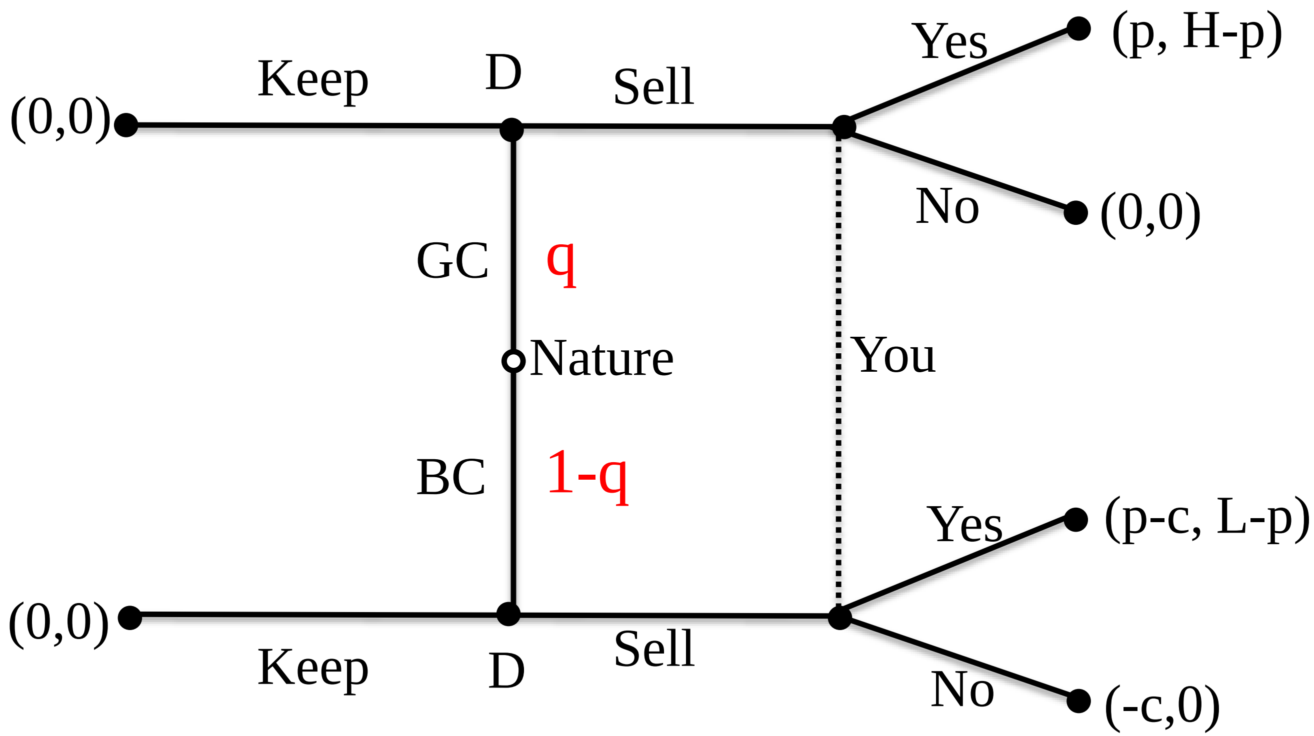
Signaling Games: Used-Car Market

- You want to buy a used-car which may be either good or bad
- A good car is worth H and a bad one L dollars
- You cannot be told a good car or a bad one but believe a proportion q of cars are good
- The car you are interested in has a price p
- The dealer knows quality but you don't
- The bad car needs additional costs c to make it look like good
- The dealer decides whether to put a given car on sale or keep
- You decide whether to buy or not
- Assume $H > p > L$

Signaling Games: Used-Car Market



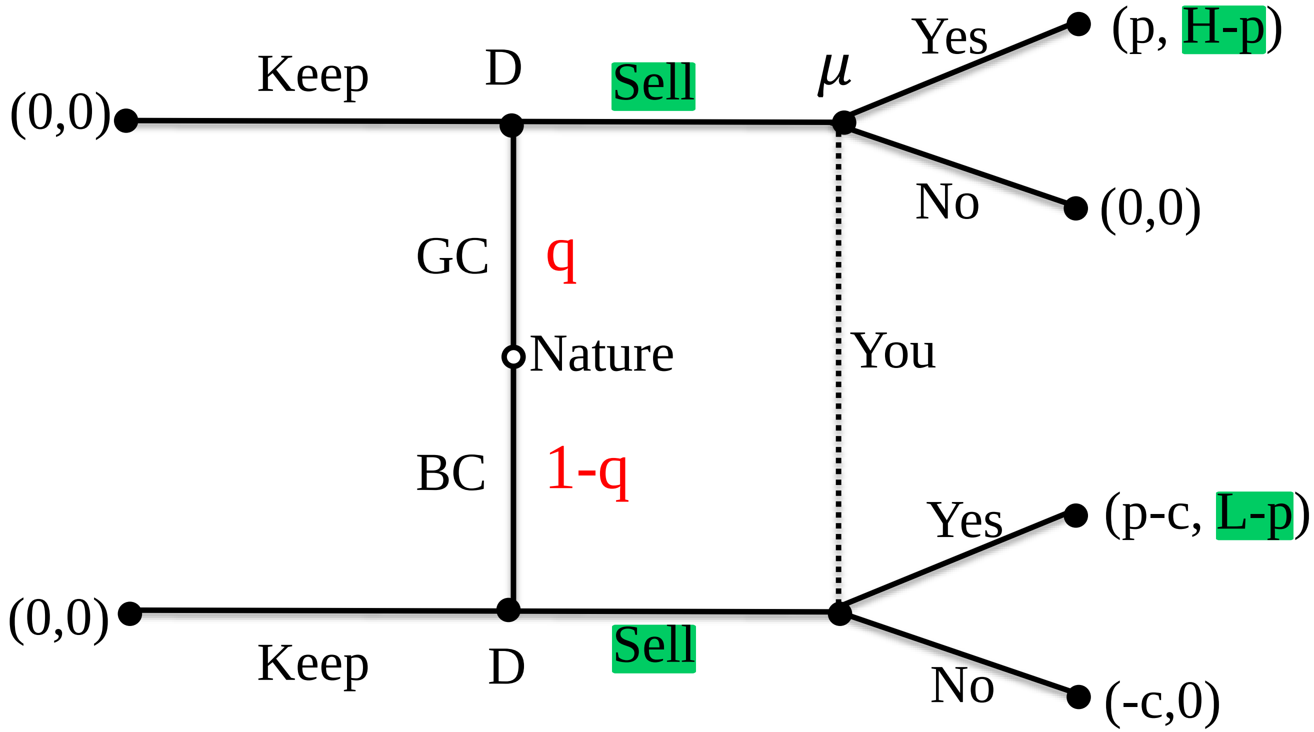
Signaling Games: Used-Car Market



We require to consider two cases:

1) GC and BC dealer paly the same strategy
2) GC and BC dealer paly different strategy

Both Sell



Both strategies: Sell

Belief:

$$\mu = \frac{q}{1 \times q + 1 \times \ (1 - q)} = q$$

Pooling Strategy: Both Sell

- If Y buys a car with your prior beliefs q your expected payoff is
  $$V = q \times (H - p) + (1 - q) \times (L - p) \geq 0$$
- What does sequential rationality of seller imply?
- You must be buying and it must be the case that $p \geq c$

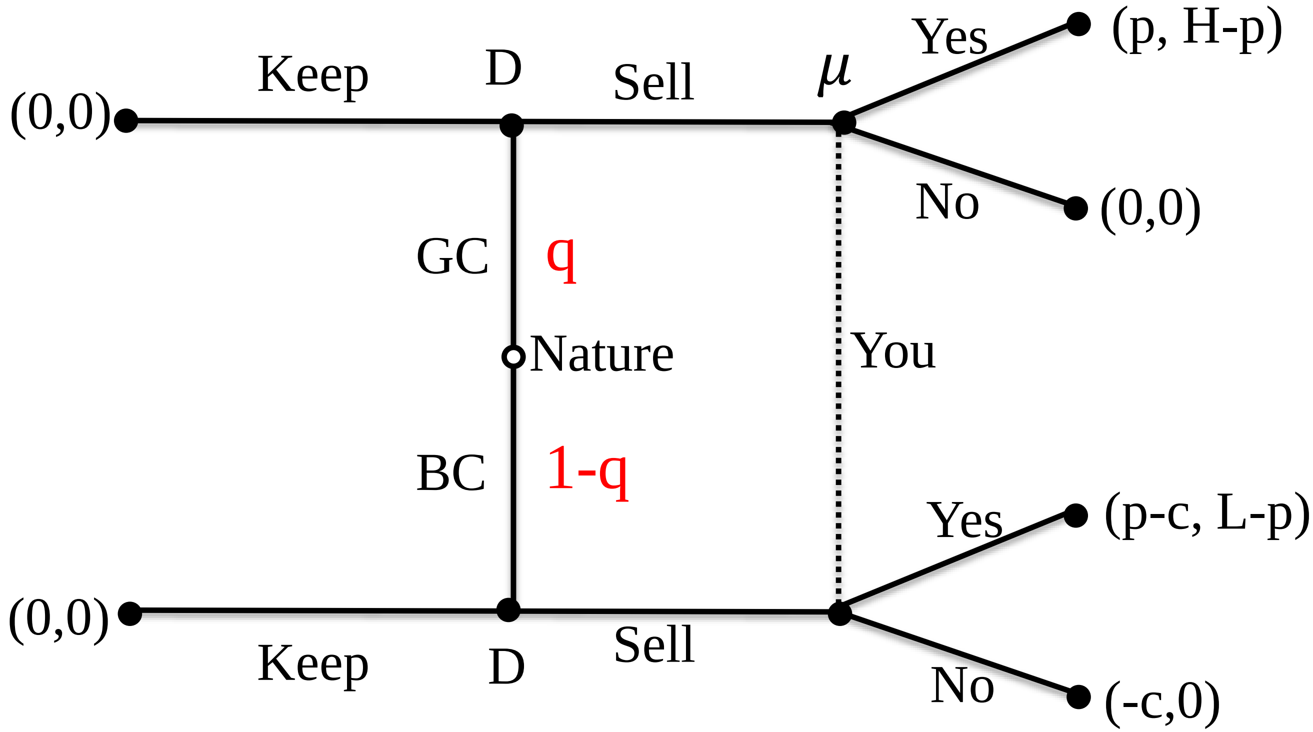
Equilibrium I

If $p \geq c$ and $V \geq 0$ the following is a PNE

Behavioral Strategy Profile: (GC: Sell, BC: Sell), (Y: Yes)

Belief System: $\mu = q$

Pooling Strategy: Both keep



Pooling Equilibria: Both Keep

You must be saying No

- Otherwise Good car dealer would offer

Under what conditions would Ysay No?

$$\mu \times (H - p) + (1 - \mu) \times (L - p) \leq 0$$

So we can set $\mu = 0$

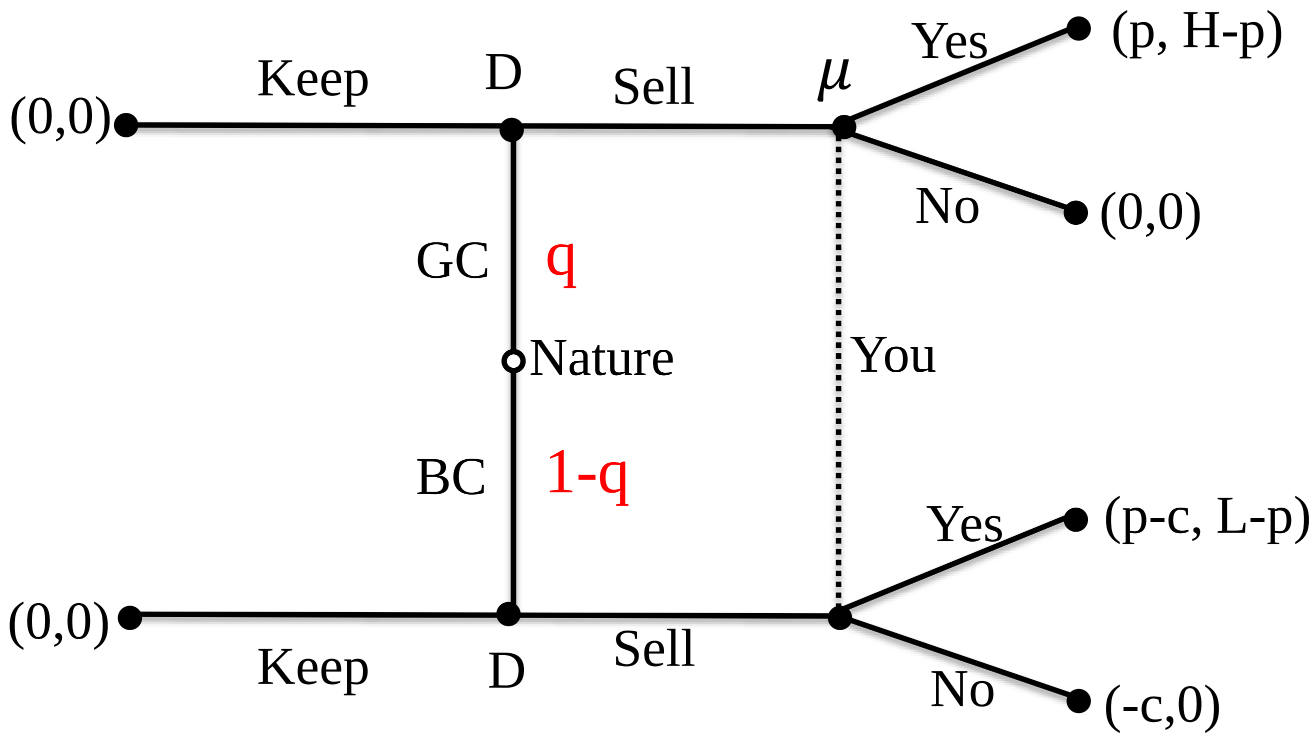
The following is a PNE

Behavioral Strategy Profile: (Good: Keep, Bad: Keep),(You: No)

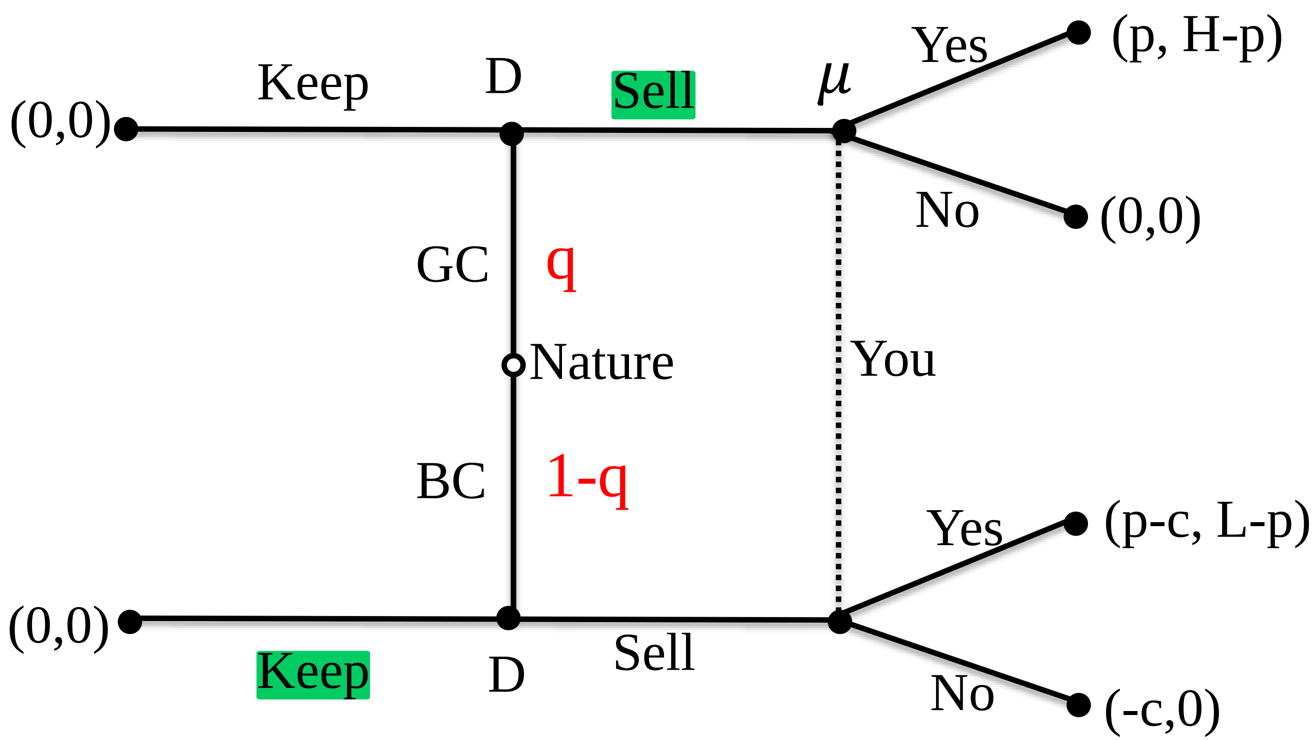
Belief System: $\mu = 0$

Market failure: a few bad car can ruin a market

Pooling Strategy: Good: Sell and Bad: Keep



Belief



Dealer strategy: Sell if good; Keep if bad

What is your consistent belief if you observe the dealer sell a car?

$$\mu = \frac{P(\text{GC and Sell})}{P(\text{Sell})} = \frac{q \times 1}{q \times 1 + 0 \times (1 - q)} = 1$$

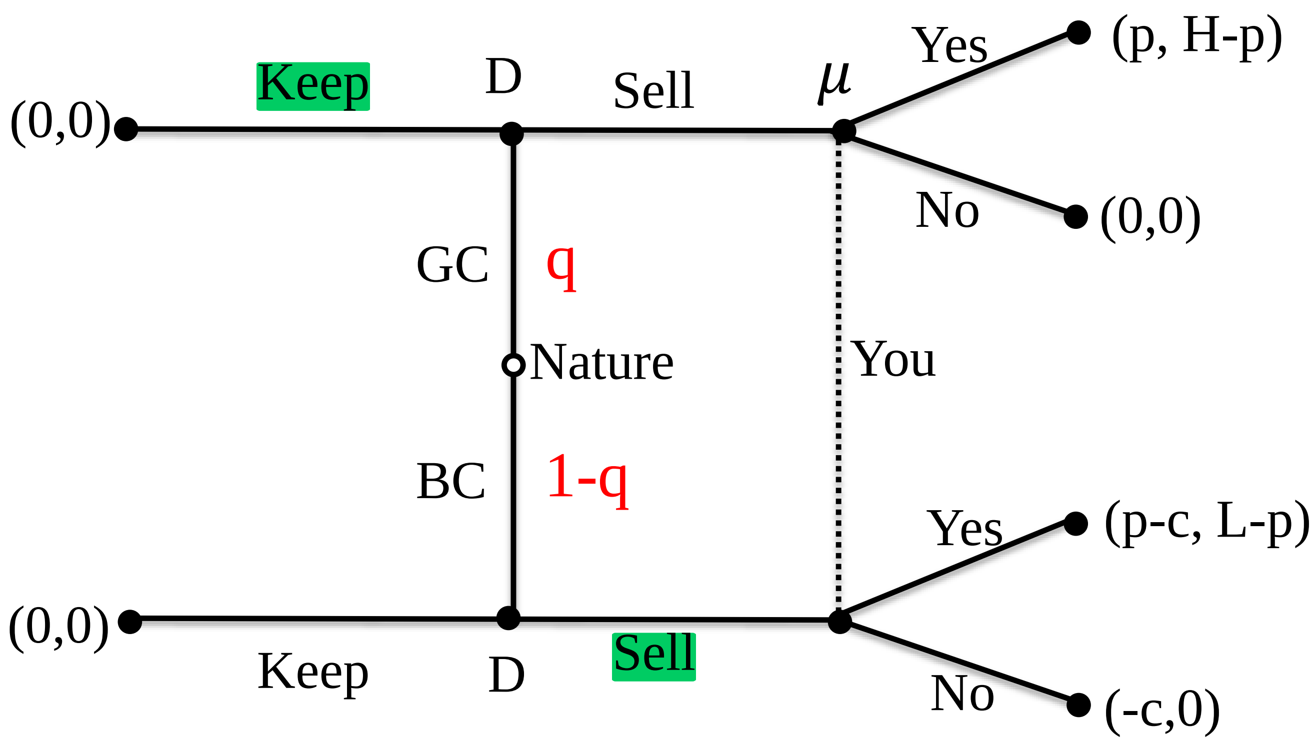
Separating Equilibria - Good: Sell and Bad: Keep

- What about your beliefs?

$$\mu = 1$$

- What does you sequential rationality imply?
  - You say Yes
- Is Good car dealer’s sequential rationality satisfied?
  - Yes
- Is Bad car dealer’s sequential rationality satisfied?
  - Yes if $p \leq c$
- If $p \leq c$ the following is a SNE
  Behavioral Strategy Profile: (Good: Offer, Bad: Hold), (You: Yes)
  Belief System: $\mu = 1$

Pooling Strategy - Good: Keep and Bad: Sell



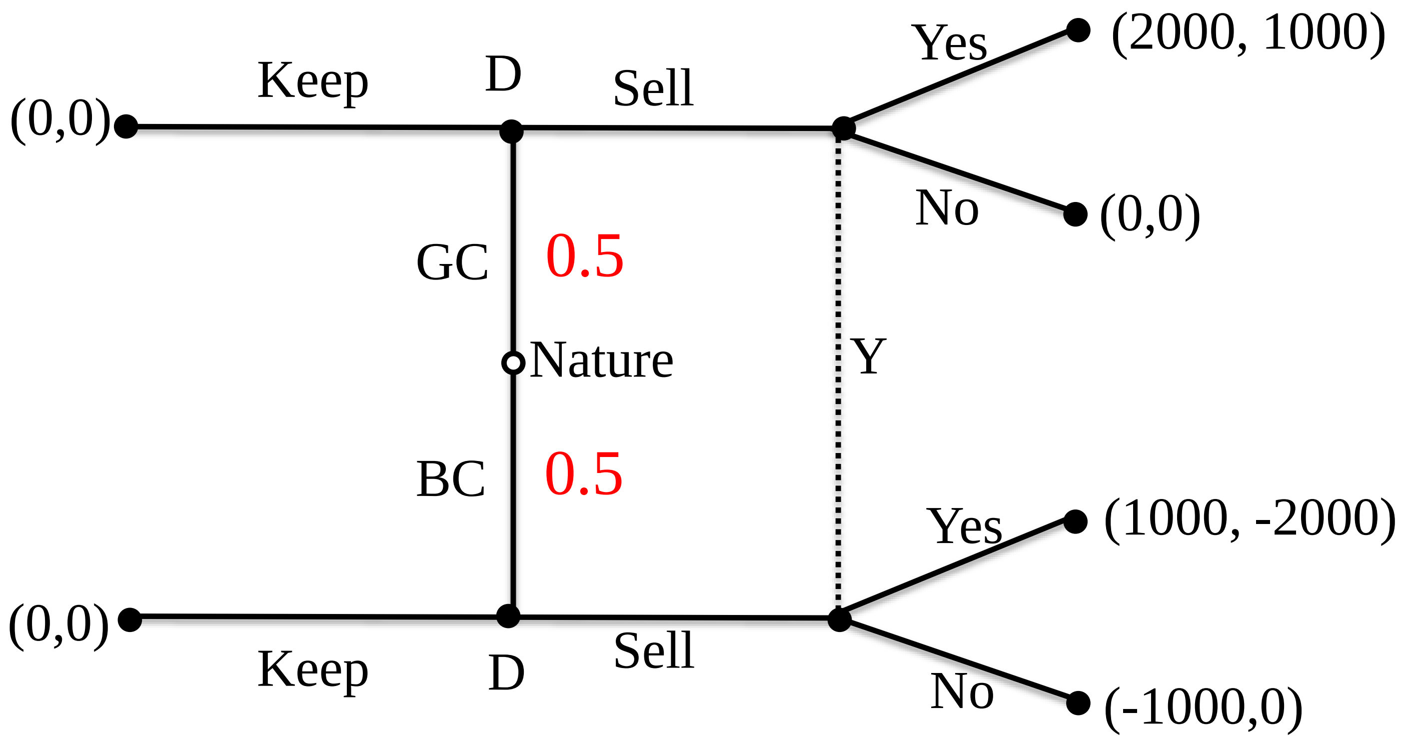
Separating Equilibria - Good: Keep and Bad: Sell

- What does Bayes Law imply about your beliefs?

$$\mu = 0$$

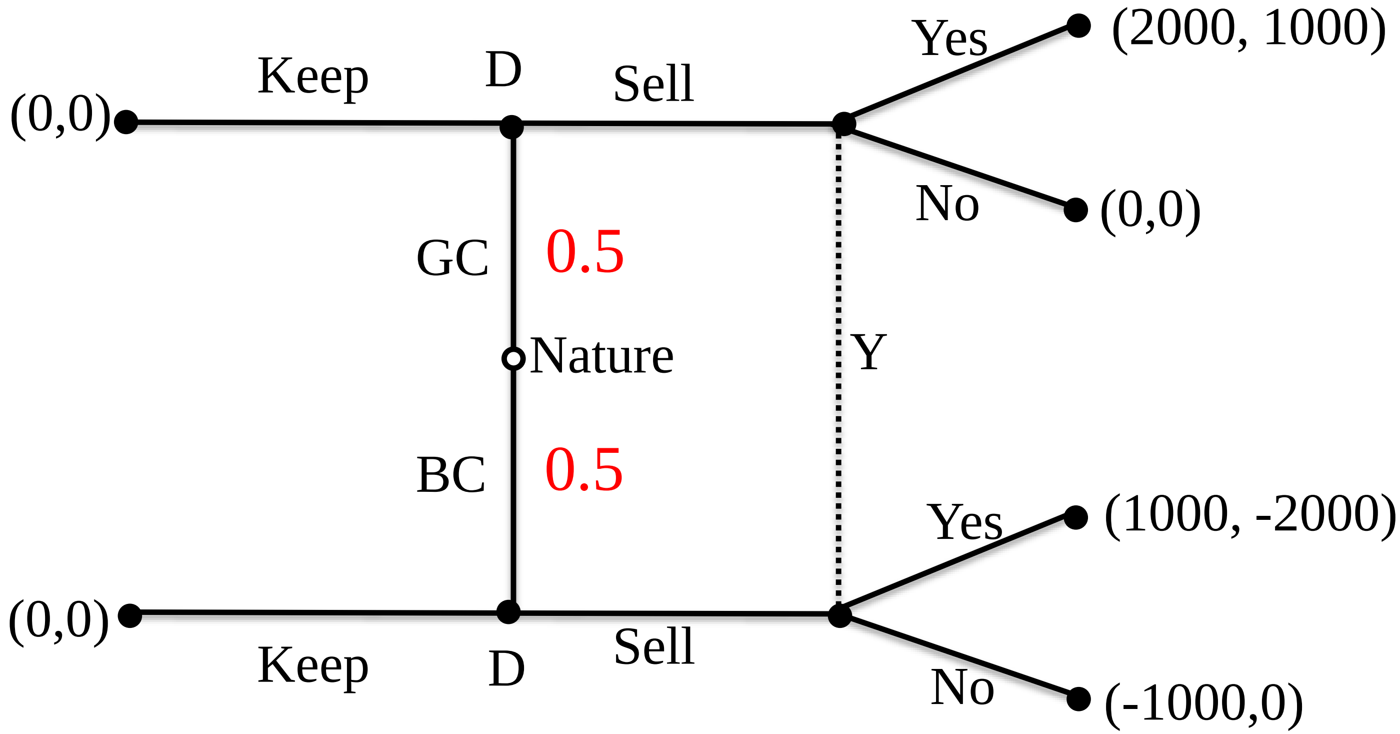
- What does you sequential rationality imply?
  - You say No
- Is Good car dealer's sequential rationality satisfied?
  - Yes
- Is Bad car dealer's sequential rationality satisfied?
  - No
- There is no SNE in which Good dealer Holds and Bad dealer Offers

Example=homework



How to find all sequential equivalence?

Behavior Strategy



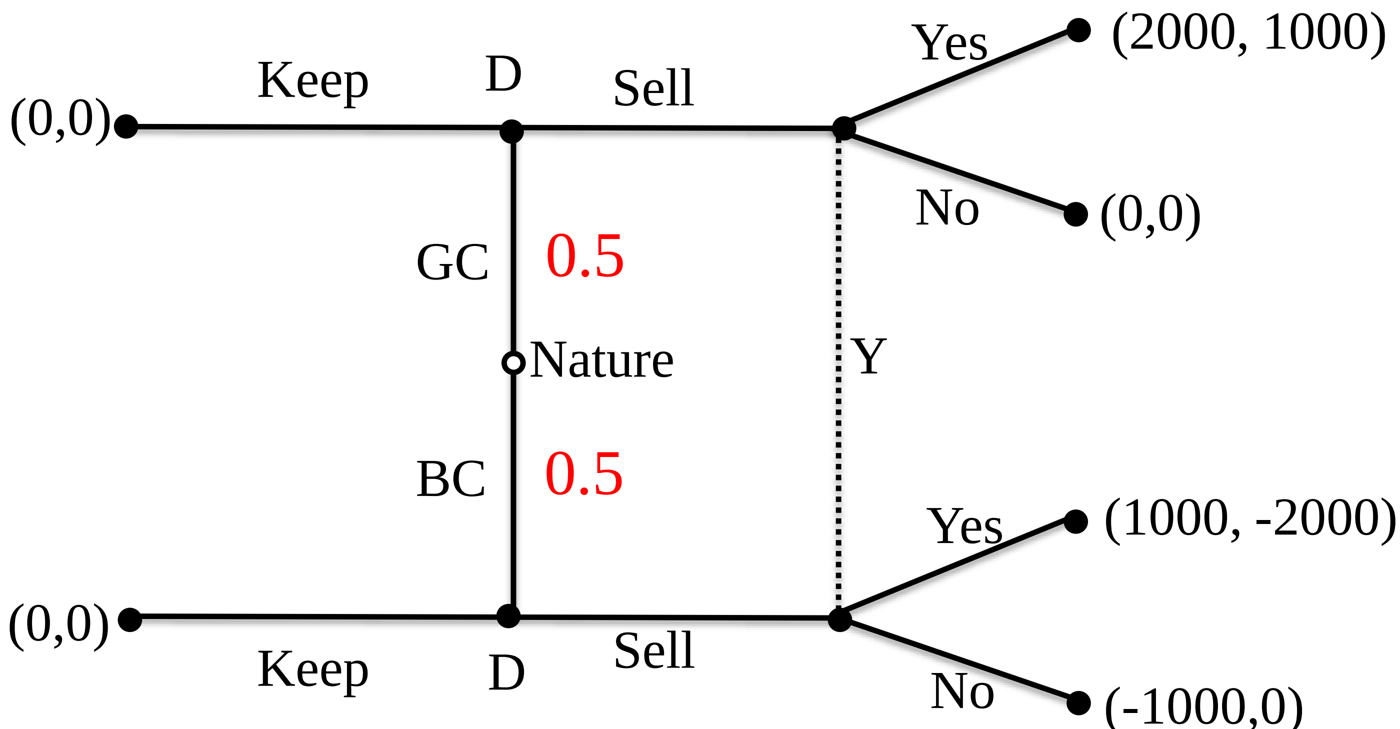
Behavior strategy: Yes Prob. $x = 0$

Dealer strategy: GC: keep or sell BC: keep $\rightarrow x = 0$ is not best resp.

Behavior strategy: Yes Prob. $x = 1$

No NE

Behavior Strategy



Behavior strategy: Yes Prob. $x \in (0,1)$

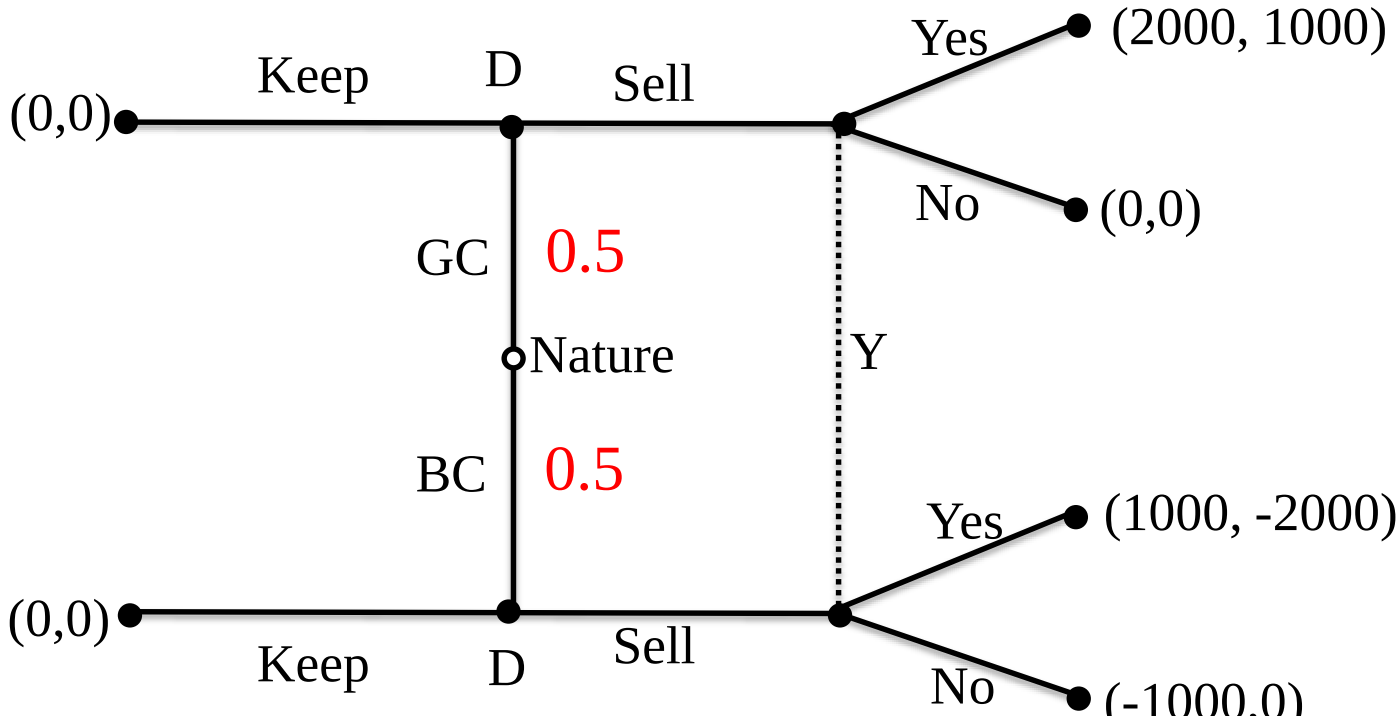
Behavior strategy: BC – sell Prob. y

Belief: GC – sell Prob. μ

You must be indifferent between Yes and No

$1000\mu - (1-\mu)2000 = 0$ implies μ=2/3

Behavior Strategy



You must be indifferent between Yes and No

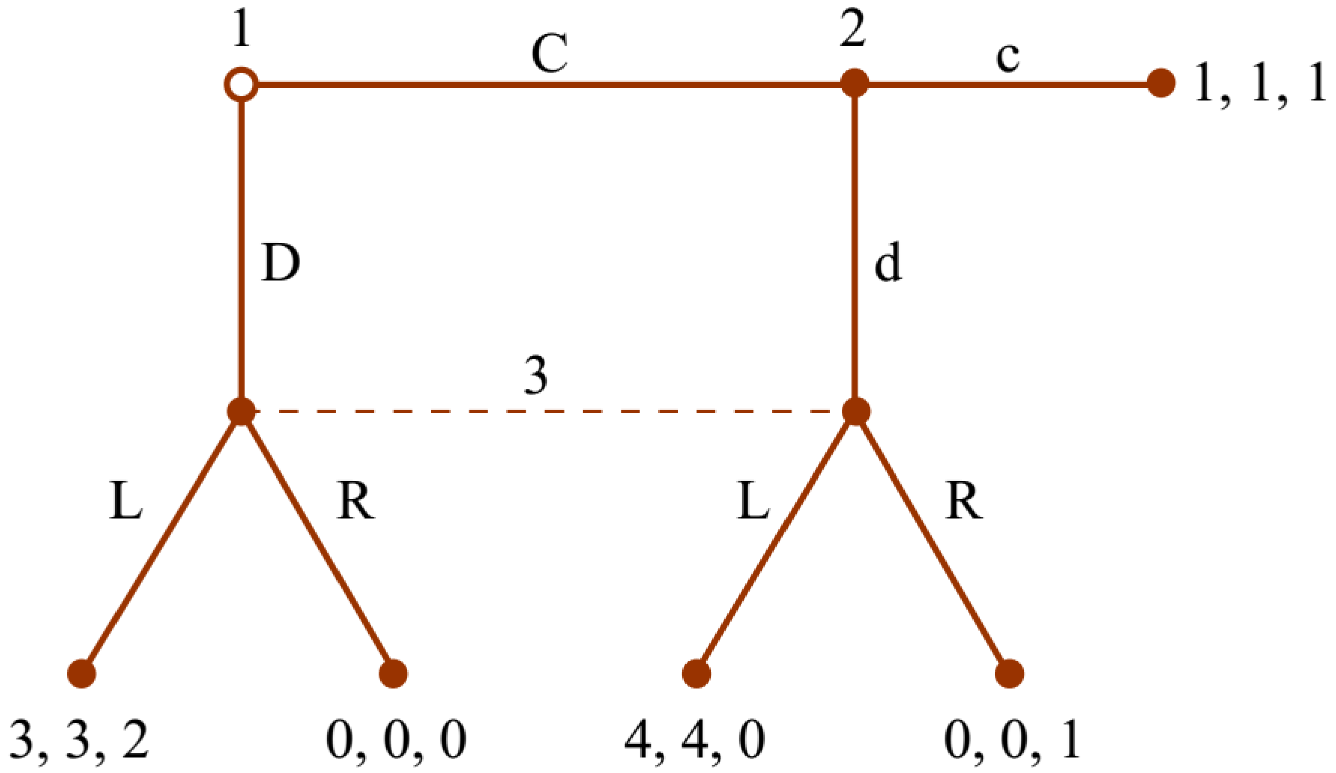
$$1000\mu - (1-\mu)2000 = 0 \text{ implies } \mu = 2/3$$

$$\frac{0.5}{0.5+0.5y} = \frac{2}{3} \text{ implies } y = 0.5$$

Bad car dealers must be indifferent between Keep and Sell

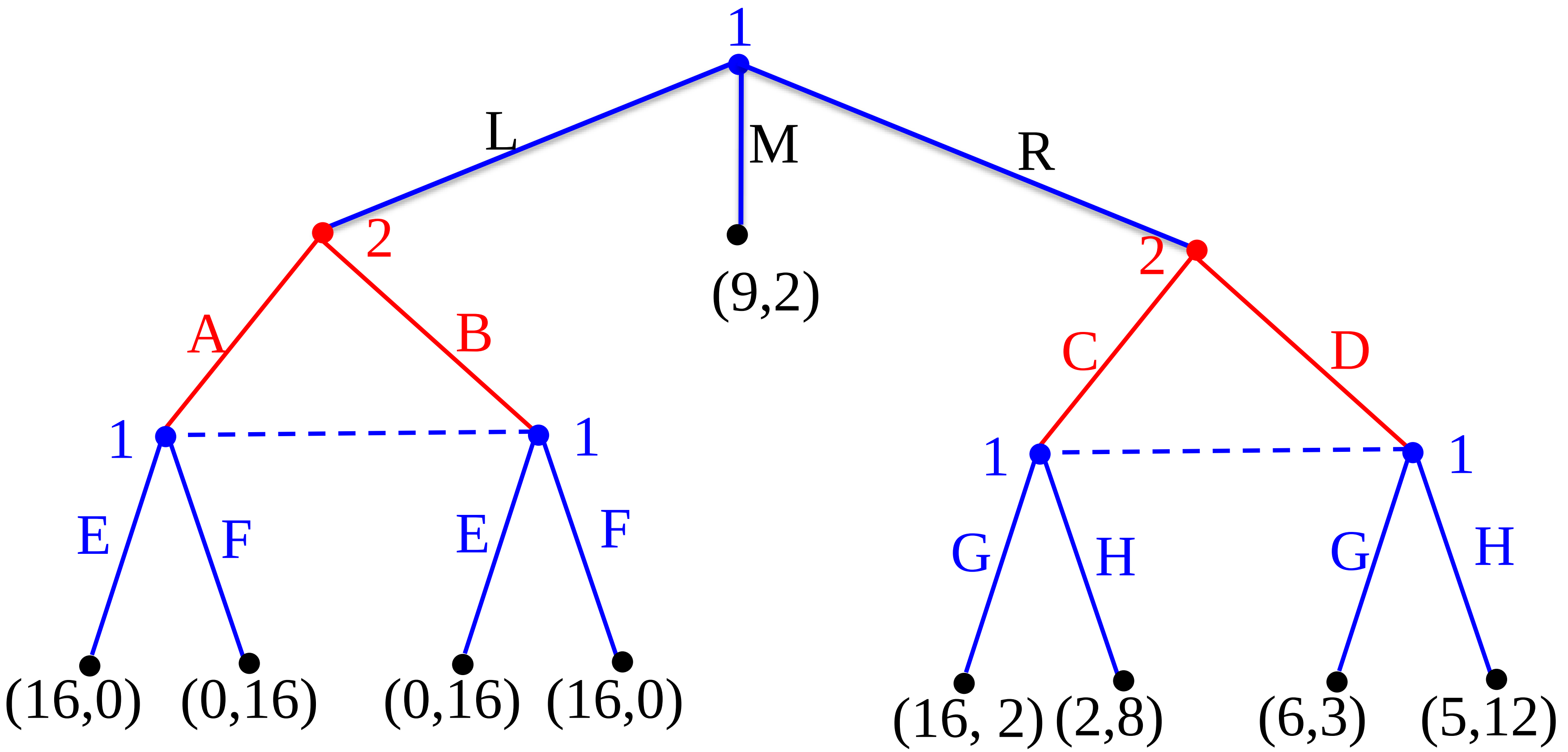
$$0 = 1000x - 1000(1-x) \text{ implies } x = 0.5$$

Homework



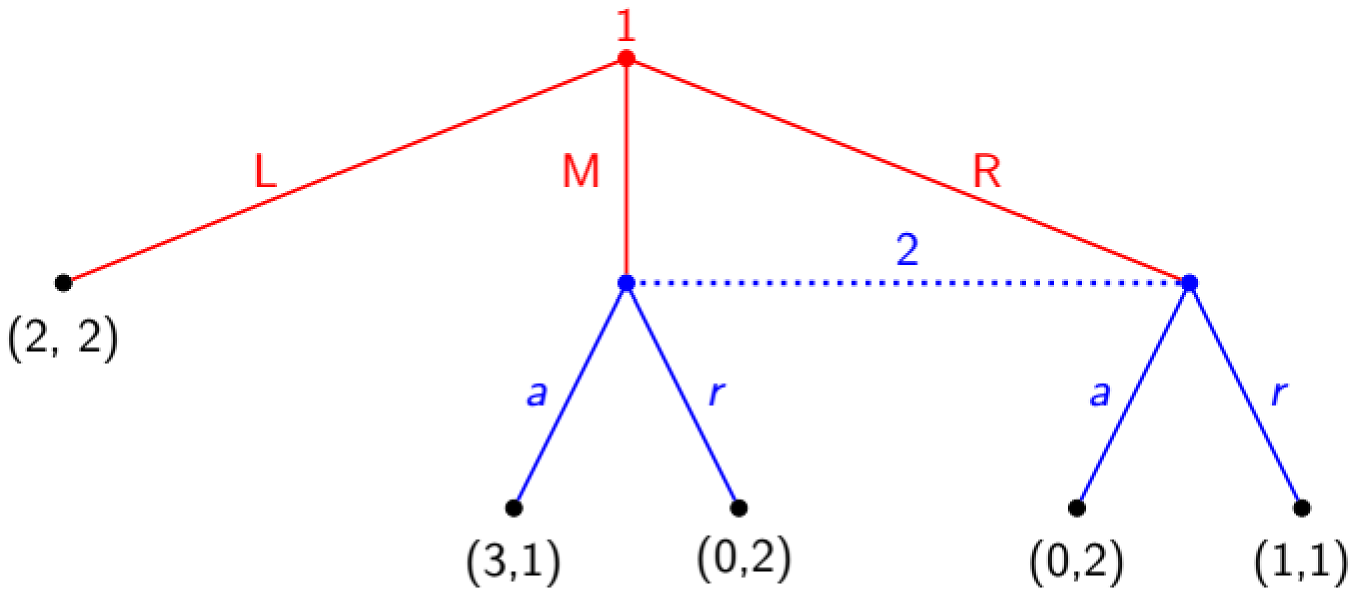
What are Nash Equilibria

Homework



- How many SPNE for this game?

Homework



Find a sequential equilibrium?