Game Theory and Applications (博弈论及其应用)

Chapter 1：Strategy Game and Nash Equilibrium

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Recap

- Game theory: **study of mathematical models of conflict and cooperation between intelligent rational decision-makers**
  - Player
  - Strategy/Decision
  - Payoff
  - Information
  - Rationality
- **How to formalize games?**

Prisoners' Dilemma (Formally)

Prisoner 1 \ Prisoner 2	Confess(c)	Don't confess(d)
Confess(c)	-6 -6	0 -12
Don't confess(d)	-12 0	-1 -1

- Set of players $N = \{1, 2\}$
- Set of strategies $A_1 = A_2 = \{\text{c}, \text{d}\}$
- $u_1(c, c) = -6, u_1(c, d) = 0, u_1(d, d), u_1(d, c)$
- $u_2(c, c) = -6, u_2(c, d) = -12, u_2(d, d), u_2(d, c)$

Strategy Games (策略式博弈)

- How to model the game where each player select his strategy simultaneously with full information, without knowledge of strategy choices by the other players (non-cooperate)

- Strategy game = normal form game

A strategy game consists of

- A finite set N of players
- A non-empty strategy set A_i for each player $i \in N$
- A payoff function $u_i: A_1 \times A_2 \times \cdots \times A_N \rightarrow R$ for $i \in N$

Strategy Games (cont.)

A strategy game consists of

- A finite set N of players
- A non-empty strategy set A_i for each player $i \in N$
- A payoff function $u_i: A_1 \times A_2 \times \cdots \times A_N \to R$ for $i \in N$

- An outcome $a = (a_1, a_2, \ldots, a_N)$ is a collection of strategies, one for each player
- Outcome space:

$$A = A_1 \times A_2 \times \cdots \times A_N$$

- The payoff function can be replaced by a **preference relation(偏好关系)** $\succsim_i$ over A for each player i

Preference Relation

A preference relation (偏好关系) $\succsim$ over a set A satisfies

– complete

a $\succsim b$ or $b \succsim a$ for every a $\in A, b \in A$

– reflexive

a $\succsim a$ for every a $\in A$

– transitive

if a $\succsim b$ or $b \succsim c$, then a $\succsim c$ for every a$, b, c \in A$

We write

$a \succ b$ if $a \succsim b$ but not $b \succsim a$ for a $\in A, b \in A$

Prisoners' Dilemma (Formally)

Prisoner 2

Prisoner 1

	Confess(c)	Don't confess(d)
Confess(c)	-6 -6	0 -12
Don't confess(d)	-12 0	-1 -1

- Set of players $N = \{1, 2\}$
- Set of strategies $A_1 = A_2 = \{\mathrm{c}, \mathrm{d}\}$
- $(\mathrm{c}, \mathrm{c}) \gtrsim_1 (d, c)$, $(\mathrm{c}, \mathrm{d}) \gtrsim_1 (d, d)$, ...
- $(\mathrm{c}, \mathrm{c}) \gtrsim_2 (c, d)$, $(\mathrm{d}, \mathrm{d}) \gtrsim_2 (c, d)$, ...

Strategy Games (cont.)

A strategy game consists of

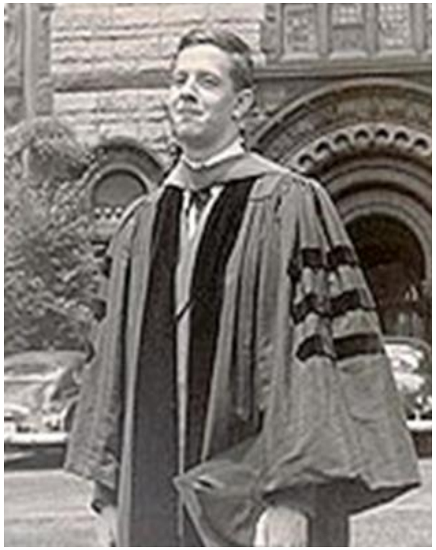
- A finite set N of players
- A non-empty strategy set A_i for each player $i \in N$
- A payoff function $u_i: A \to R$ for $i \in N$; or a preference relation $\succsim_i$ over A for $i \in N$

$$\boldsymbol{G=\{N, \{A_i\}_{i=1}^{N}, \{u_i\}_{i=1}^{N}\}}$$

$$\boldsymbol{G=\{N, \{A_i\}_{i=1}^{N}, \{\succsim_i\}_{i=1}^{N}\}}$$

John F. Nash

- 1928 Born in American
- 1950 PhD (Princeton)
  - Non-cooperative Games
  - 28 pages
- 1951 Teacher MIT

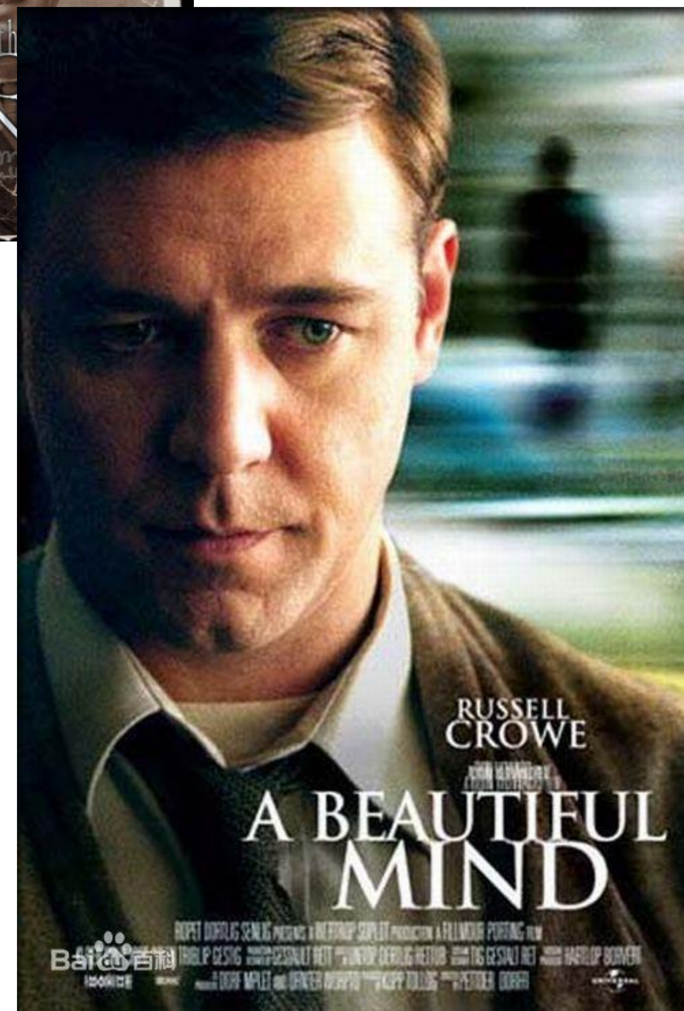


Louis De Broglie
3 pages

- 1957 Married

John F. Nash (cont.)

- 1959 Paranoia (精神分裂症)
- 1980s Health



- 1994 Nobel Prize
- 1999 Steel Prize
- 2015 Abel Prize
- 2015 Accident

Strategy Games (cont.)

A strategy game consists of

- A finite set N of players
- A non-empty strategy set A_i for each player $i \in N$
- A payoff function $u_i \colon A \to R$ for $i \in N$; or a preference relation $\succsim_i$ over A for $i \in N$

- An outcome $a = (a_1, a_2, \ldots, a_N)$ is a collection of strategies, one for each player
- Outcome space:

$$A = A_1 \times A_2 \times \cdots \times A_N$$

Notations

- For an outcome $a = (a_1, a_2, \dots, a_N)$, we define

$$\boldsymbol{a_{-i}} = (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_N)$$

the outcome of strategies taken by all players other than i

$$a = (a_i, a_{-i})$$

- For a game $G = (N, \{A_i\}, \{u_i\})$, we define $\boldsymbol{A_{-i}}$ the set of all such a_{-i}, i.e.,

$$\boldsymbol{A_{-i}} = A_1 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_N$$

Nash Equilibrium (NE)

Nash equilibrium is a strategy outcome (a collection of strategies, one for each player) such that each strategy is a best response (maximizes payoff) to all other strategies

An outcome $a^* = (a_1^*, a_2^*, \dots, a_N^*)$ is a **Nash equilibrium (NE)** if for each players i

$$u_i(a_i^*, a_{-i}^*) \geq u_i(a_i, a_{-i}^*) \text{ for all } a_i \in A_i.$$

Nash equilibrium is self-enforcing: no player has an incentive to alter his strategy unilaterally (单方面).

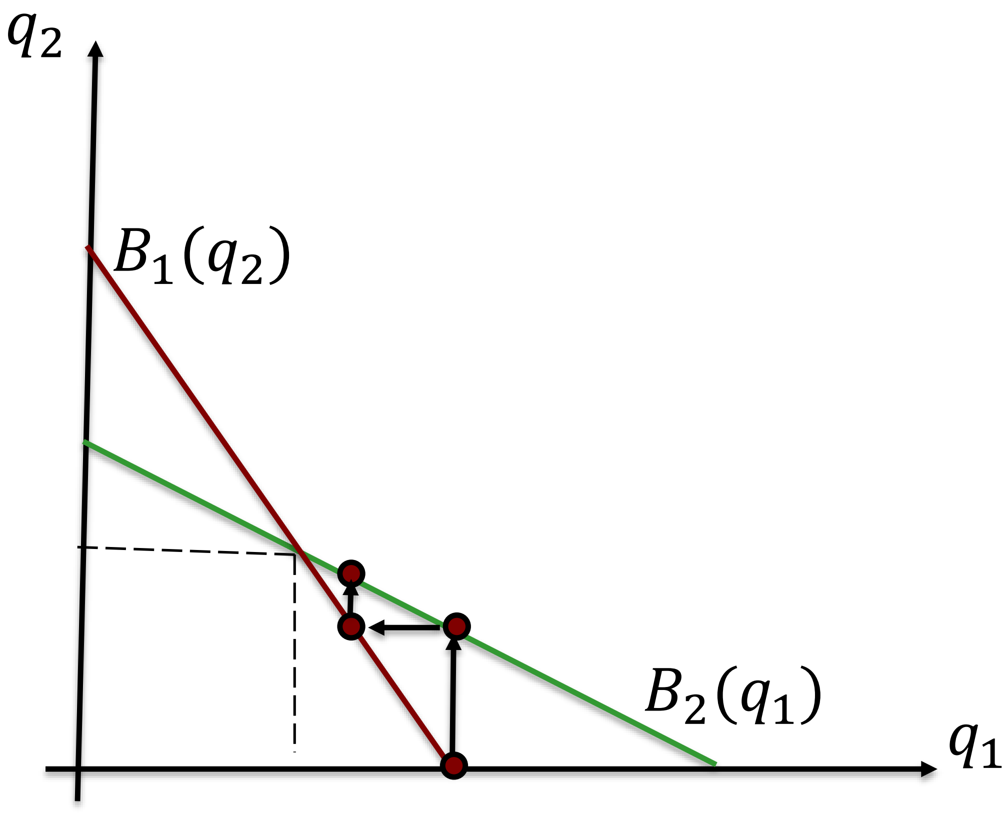
Nash Equilibrium

Player 2

Player 1		d		e		f	
	a	-6	-6	0	8	0	-2
	b	-12	0	1	3	1	1
	c	-10	2	4	0	1	4

Nash Equilibrium (cont.)

- Let (q_1^*, q_2^*) be an Nash equilibrium
  - Player i makes the best strategies q_i^* with respect to q_{-i}^*



Best Response Correspondence

- The best response correspondence of player i is given by

$$B_i(a_{-i}) = \{a_i \in A_i : u_i(a_i, a_{-i}) \geq u_i(b_i, a_{-i}) \text{ for all } b_i \in A_i\}$$

$B_i(a_{-i})$ may be a set or a singleton

Prisoner 1 \ Prisoner 2	Confess(c)	Don't confess(d)
Confess(c)	-6 -6	0 -12
Don't confess(d)	-12 0	-1 -1

$$B_1(c) = \{c\} \quad B_1(d) = \{c\}$$
$$B_2(c) = \{c\} \quad B_2(d) = \{c\}$$

How to Find Nash Equilibria

- One way of finding Nash equilibria for payoff matrix:

(1) Find the best response correspondence for each player

Best response correspondence gives the set of payoff maximizing strategies for each strategy profile of the other players

(2) Find where they intersect

Find all outcomes $(a_1^*, a_2^*, \dots, a_N^*)$ such that

$$a_i^* \in B_i(a_{-i}^*)$$

Nash Equilibrium: Example 1

Primitive hunting

Hunter 2

Hunter 1	Rabbit (r)	Bear (b)
Rabbit (r)	3, 3	3, 0
Bear (b)	0, 3	9, 9

Nash Equilibrium: Example 2

Prisoners' Dilemma

Prisoner 1 \ Prisoner 2	Confess(c)	Don't confess(d)
Confess(c)	-6 -6	0 -12
Don't confess(d)	-12 0	-1 -1

Nash Equilibrium: Example 3

Rock-Paper-Scissors

		Player 2 Rock	Player 2 Paper	Player 2 Scissors
Player 1	Rock	0, 0	-1, 1	1, -1
Player 1	Paper	1, -1	0, 0	-1, 1
Player 1	Scissors	-1, 1	1, -1	0, 0

An Exercise

- Find all Nash equilibria

P2

P1		h		i		j		k		l		m	
	a	7	5	8	6	2	2	2	3	6	9	6	5
	b	6	5	9	6	5	8	6	7	8	8	7	4
	c	9	7	1	1	7	9	3	2	9	6	9	2
	d	2	14	10	12	6	5	6	3	7	2	9	12
	e	8	6	5	9	3	9	7	5	13	15	8	9

3- Players Game

$$G = \{\{1,2,3\}, \{\{a,b,c\},\{x,y,z\},\{L,R\}\}, \{u_i\}_{i=1}^{3}\}$$

P3 chooses *L*

P2

		x			y			z		
P1	a	8	7	4	2	9	1	4	1	8
	b	4	6	5	7	2	6	1	3	7
	c	6	2	2	5	1	7	4	4	2

P3 chooses *R*

P2

		x			y			z		
P1	a	5	3	2	6	5	4	1	2	4
	b	8	6	2	2	8	10	5	2	6
	c	6	9	4	1	1	3	9	7	8

How to Find Nash equilibrium for continuous strategies

- One way of finding Nash equilibrium for continuous strategies A_i:

(1) Find the best response correspondence for each player

Best response correspondence

$$B_i(a_{-i}) = \{a_i \in \mathrm{argmax}_{a_i \in A_i}\, u_i(a_i, a_{-i})\}$$

(2) Find all Nash Equilibria $(a_1^*, a_2^*, \dots, a_N^*)$ such that

$$a_i^* \in B_i(a_{-i}^*) \text{ for each player}$$

Optimization

- Let $f: R \to R$ be a continuous and differential function. The optimization problem: $\max_{x \in [a,b]} f(x)$

This problems can be solved by following procedures:

- Find all critical points: $x \in [a, b], f'(x) = 0$
- Evaluate f at all critical point and boundaries a and b
- Find the highest f

Example

$$\max_{x \in [-2,5]} x^3 - 3x^2 - 9x + 6$$

Cournot Competition(古诺竞争)

- Two firms compete by choosing how much to produce

$$G = \{\{1,2\},\{q_1,q_2\},\{u_1,u_2\}\}$$

  - Price

$$p(q_1+q_2) = \max(0, a - b(q_1+q_2))$$

  - Costs $(i = 1,2)$

$$c_i(q_i) = cq_i$$

  - Payoffs $(i = 1,2)$

$$u_i(q_1,q_2) = (\max\big(0, a - b(q_1+q_2)\big) - c)q_i$$

  - Condition $a > \mathrm{b}, c > 0, q_1 \geq 0, q_2 \geq 0$

Cournot: Best Response Correspondence

Best Response Correspondence is given by

$$B_i(q_{-i}) = \max(0, (a - c - bq_{-i})/2b)$$

Proof. We will prove for i=1 (similarly for i=2)

If $q_2 \geq (a-c)/b$, then $u_1(q_1, q_2) \leq 0$ for any $q_1 > 0$. $q_1 = 0$.

If $q_2 < (a-c)/b$, then

$$u_i(q_1, q_2) = (a - c - b(q_1 + q_2))q_i$$

$$\frac{\partial u_1(q_1, q_2)}{\partial q_1} = a - c - bq_2 - 2bq_1 = 0$$

$$q_1 = (a - c - bq_2)/2b$$

Cournot: Nash Equilibrium

The Nash equilibria is give by

$$\left\{\left(\frac{a-c}{3b}, \frac{a-c}{3b}\right)\right\}$$

Proof. Assume that (q_1^*, q_2^*) is a Nash equilibrium.

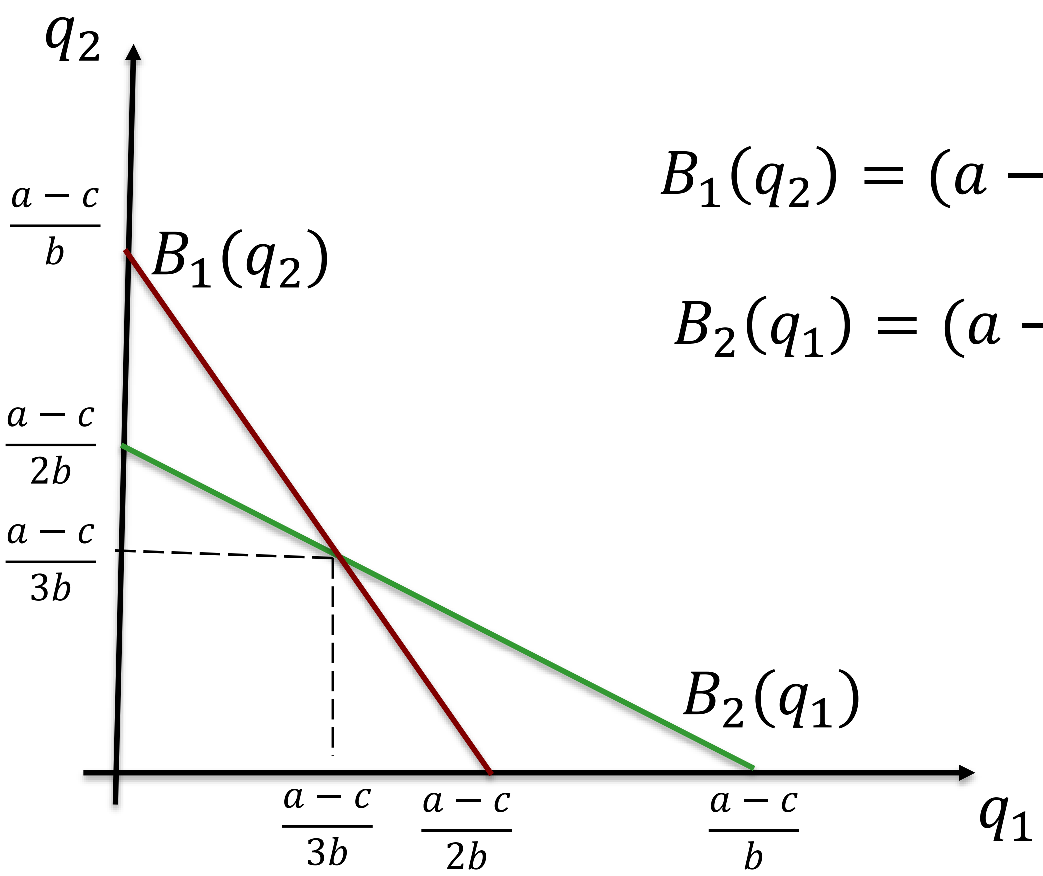
1) Prove $q_1^* > 0$ and $q_2^* > 0$ by contradiction

2) (q_1^*, q_2^*) is such that $q_1^* > 0, q_2^* > 0$

$$q_1^* = B_1(q_2^*) = (a - c - bq_2^*)/2b$$

$$q_2^* = B_2(q_1^*) = (a - c - bq_1^*)/2b$$

Cournot: Nash Equilibrium (cont.)



$$B_1(q_2) = (a - c - bq_2)/2b$$

$$B_2(q_1) = (a - c - bq_1)/2b$$

Cournot (Corporative)

- Two firms compete by choosing how much to produce

$$G = \{\{1, 2\}, \{q_1, q_2\}, \{u_1, u_2\}\}$$

  - Price

$$p(q_1, q_2) = a - q_1 - q_2$$

  - Costs $(i = 1, 2)$

$$c_i(q_i) = cq_i$$

  - Payoffs $(i = 1, 2)$

$$u_i(q_1, q_2) = (a - q_1 - q_2)q_i - cq_i$$

  - Condition $a > 0, c > 0, q_1 \geq 0, q_2 \geq 0$

Cournot (Corporative)

$$u_1(q_1, q_2) + u_2(q_1, q_2) = (a - c - q_1 - q_2)(q_1 + q_2)$$

$$u_1(q_1, q_2) + u_2(q_1, q_2) = (a - c - x)x,\ x = q_1 + q_2$$

$$x = (a - c)/2$$

Best Response Correspondence is given by

$$B_i(q_{-i}) = (a - c)/4$$

The Nash equilibria: $\left\{\left(\frac{a-c}{4}, \frac{a-c}{4}\right)\right\}$ Payoff: $\left\{\left(\frac{(a-c)^2}{8}, \frac{(a-c)^2}{8}\right)\right\}$

Cournot (Corporative vs Competition)

Best response correspondence (corporative) is given by

$$B_i(q_{-i}) = (a-c)/4$$

The Nash equilibria: $\left\{\left(\frac{a-c}{4}, \frac{a-c}{4}\right)\right\}$ Payoff: $\left\{\left(\frac{(a-c)^2}{8}, \frac{(a-c)^2}{8}\right)\right\}$

Best response correspondence (competition) is given by

$$B_i(q_{-i}) = (a-c-q_{-i})/2$$

The Nash equilibria: $\left\{\left(\frac{a-c}{3}, \frac{a-c}{3}\right)\right\}$ Payoff: $\left\{\left(\frac{(a-c)^2}{9}, \frac{(a-c)^2}{9}\right)\right\}$

The corporative payoffs are better than the competition cases

Election

- Several candidates vote for political office
- Each candidate chooses a policy position
- Each citizen, who has preferences over policy positions, votes for one of the candidates
- Candidate who obtains the most votes wins.

Strategic game:

- Players: candidates
- Set of actions of each candidate: set of possible positions
- Payoff is 1 for winner; is 0.5 for ties; and is 0 for loser

- Note: Citizens are not players in this game

Example

- Two candidates $N = \{1,2\}$
- Set of possible position: $b_1, b_2 \in [0,1]$
- Citizens are continuous, and are distributed uniformly on [0,1], and vote for the candidate with closet position.
- Payoff

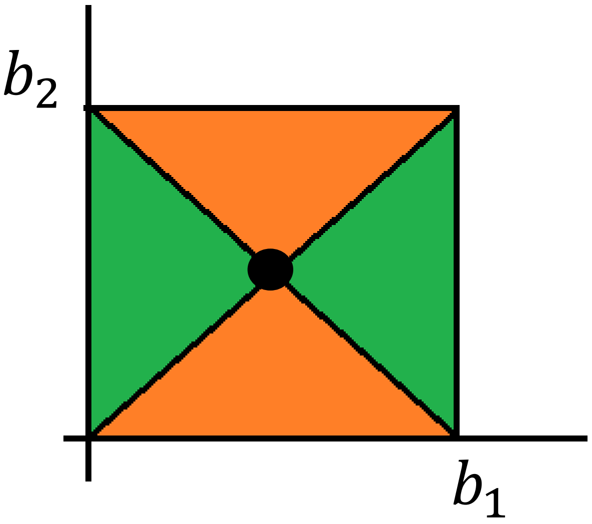
$$u_i(b_1, b_2) = \begin{cases} 1 & \text{if } i \text{ wins} \\ 0.5 & \text{if } i \text{ ties} \\ 0 & \text{if } i \text{ loses} \end{cases}$$

Best Response

The best response function $B_i(b_j)$ is give as follows:

- If $b_j < 1/2$, then $B_i(b_j) = \{b_i : b_j < b_i < 1 - b_j\}$
- If $b_j = 1/2$, then $B_i(b_j) = \{b_i : b_i = 1/2\}$
- If $b_j > 1/2$, then $B_i(b_j) = \{b_i : 1 - b_j < b_i < b_j\}$

The Nash Equilibrium (1/2,1/2)



An Exercise: n-Cournot Competition

- n firms compete by choosing how much to produce

$$G = \{\{1, \dots, n\}, \{q_1, \dots, q_n\}, \{u_1, \dots, u_n\}\}$$

  - Price

$$p(q_1 + \cdots + q_n) = a - b(q_1 + \cdots + q_n)$$

  - Costs $(i = 1, \dots, n)$

$$c_i(q_i) = cq_i$$

  - Payoffs $(i = 1, \dots, n)$

$$u_i(q_1, \dots, q_n) = (a - b(q_1 + \cdots + q_n) - c)q_i$$

  - Condition $a > c > 0, b > 0\ q_i \geq 0$

Homework

Best Response Correspondence is given by

$$B_i(q_{-i}) = \max\left(0, \frac{a - c - b\sum_{k=1,k\neq i}^{n} q_k}{2b}\right)$$

Proof. Solve

$$\max_{q_i\geq 0} u_i(q_1, \dots, q_n) = \max_{q_i\geq 0} (a - b(q_1 + \cdots + q_n) - c)q_i$$

$$\frac{\partial u_i(q_1, \dots, q_n)}{\partial q_i} = a - c - b\sum_{k=1,k\neq i}^{n} q_k - 2bq_i = 0$$

$$q_i = \frac{\left(a - c - b\sum_{k=1,k\neq i}^{n} q_k\right)}{2b}$$

Summary

- Formulations of Game
- Nash Equilibrium
- How to find Nash Equilibrium for payoff matrices
- How to find Nash Equilibrium for continuous strategies