Game Theory and Applications (博弈论及其应用)

Chapter 12: Repeated Games

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时间：2023年6月24日 8:00-10:00

地点：**本科生：仙I-206，研究生：仙I-207**

答疑时间：2023年6月22日- 6月23日下午3:00-5:00

答疑地点：国际学院A区 A-305

Recap on Previous Chapter

- Extensive game with imperfect information $G = \{N, H, P, I, \{u_i\}\}$
- Information set $I = \{I_1, I_2, \ldots I_N\}$
- Perfect recall and imperfect recall
- Mixed Strategies and behavioral strategies
- Subgame, subgame perfect Nash equilibrium
- Bayes consistency, consistency, sequential rational
- Sequential equivilibrum
- Signaling games

Repeated Games

- Many real interactions have an ongoing structure
  - Firms compete over and over time again
  - Chinese and American compete repeatedly for the future
  - OPEC cartel is also repeated over time
- In such situation, players should consider their long-term and short-term payoff simultaneously
- This yields behaviors which is different from one interaction (extensive and strategy games)
- Players' interaction in repeated game help us rationalize cooperation
  - Cooperate could not be sustained when players interact only once

Repeated Games

- Players play a stage game repeated over time
  - Stage game includes strategic and extensive game
- If there is a finial period: finitely repeated game
- If there is no definite end period: infinite repeated game
  - We could think of firm having infinite lives
  - Players do not know when the game will end

Implicit Cooperation

- The firms cooperate with fixing prices (explicit cooper.)
- Could firms cooperate without explicitly fixing prices?
- Some reward/punishment mechanisms are used to keep the firms cooperation
- Repeated interaction provides the opportunity to implement such mechanisms
  - A firm faces a trade-off between short- and long-term profits
- Repeated games is a model to study these questions

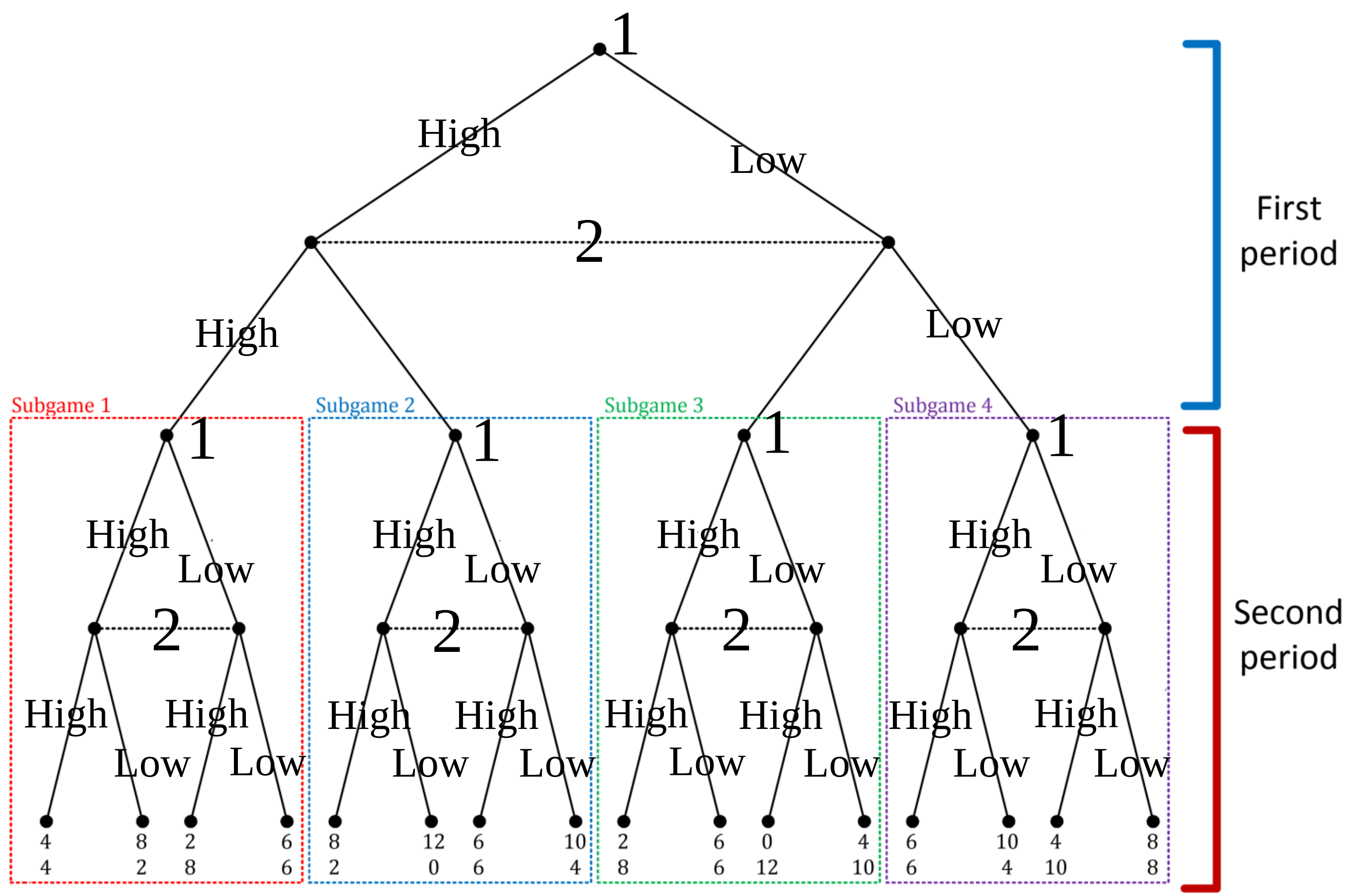
An example

		Firm 2	
		High	Low
Firm 1	High	2, 2	6, 0
	Low	0, 6	4, 4

- What happens if this game is played only once
- What do you think might happen if played repeatedly
  - Being caught cheating will yield punishment
  - Is cooperation always good?

An example (cont.)

For simplicity, let's see what happens for **twice**



An example (cont.)

We solve this twice-repeated game by backward induction

Second stage

- We first identify the proper subgames: there are four, as indicated in the figure, plus the game as a whole.
- We can then find the NE of each of these four subgames, separately.
- We will then be ready to insert the equilibrium payoffs from each of these subgames, constructing a reduced-form game.

First stage

- Using the reduced-form game we can then solve the first stage of the game

Subgame 1

Only one PSNE (High, High)

Firm 1 \ Firm 2	High	Low
High	4, 4	8, 2
Low	2, 8	6, 6

An example (cont.)

Subgame 2

Only one PSNE (High, High)

Firm 1 \ Firm 2	High	Low
High	8, 2	12, 0
Low	6, 6	10, 4

Subgame 3

Only one PSNE (High, High)

Firm 1 \ Firm 2	High	Low
High	2, 8	6, 6
Low	0, 12	4, 10

Subgame 4

Only one PSNE (High, High)

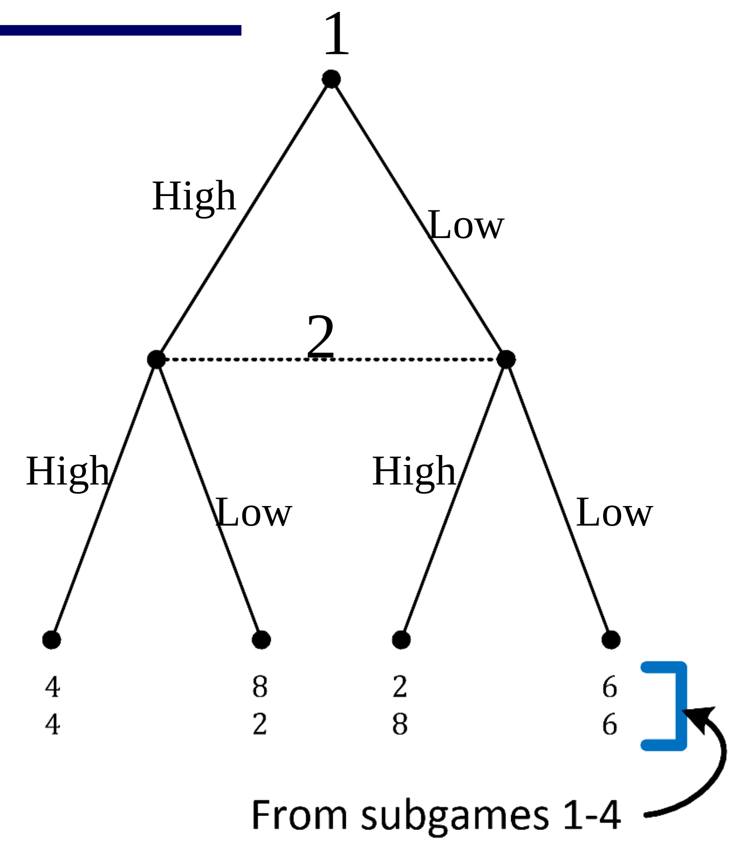
Firm 1 \ Firm 2	High	Low
High	6, 6	10, 4
Low	4, 10	8, 8

Prisoners' Dilemma

Inserting the payoffs from each sub-game, we construct the reduced-form game

Firm 2

Firm 1		High		Low	
High		4	4	8	2
Low		2	8	6	6



The unique SPNE:

Firm 1: (High, High) regardless of what happened in period 1

Firm 2: (High, High) regardless of what happened in period 2

Repeating game twice "was a failure!" to rationalize cooperation

Prisoners' Dilemma

This result provides us with some interesting insights:

- If the stage game has a unique NE, then there is a unique SPNE in the finitely-repeated game in which all players behave as in the stage-game equilibrium.
- Examples:
  - ✓ Prisoner's dilemma, Cournot competition ...

In finite repeated games, players know when the game will end

But... what if they don't?

Let us analyze the infinitely-repeated game

Repeated Games (cont.)

- Denote the discount factor (贴现因子) by $\delta \in (0,1)$
  - Control the short-term and long-term profits
- Today's \$1 payoff is more valuable than tomorrow's \$1
  - Represents how patient the players are
  - Think as probability with which the game will play next time
  - Think as the factor to calculate the values for different period
  - Guarantee the convergence of payoff

Payoffs

- If starting now, a player receives an infinite sequence of payoffs $u_1, u_2,\ u_3, \ldots$ i.e., the payoff is u_t for each stage
- Discount factor $\delta \in (0,1)$. Payoff is defined by

$$u_1 + \delta\, u_2 + \delta^2 u_3 + \delta^3 u_4 + \cdots$$

Example: Period payoffs are all 2

Firm 1 \ Firm 2	High	Low
High	2, 2	0, 3
Low	3, 0	1, 1

$$2 + 2\delta\ + 2\delta^2 + 2\delta^3 + \cdots = \frac{2}{1-\delta}$$

Repeated Prisoners' Dilemma

		Prisoner 2	
		Confess(c)	Don't confess(d)
Prisoner 1	Confess(c)	0 0	2 -1
	Don't confess(d)	-1 2	1 1

- Suppose two players are going to play the prisoner's dilemma game for $t = 1, 2, \ldots$
- The discount factor is $\delta \in (0,1)$
- Is cooperation is always good?

Repeated Prisoners' Dilemma

Assume Prisoner's Dilemma game proceeds infinite periods

- If one player is a "nice" guy, who plays "d" as long as you play "d" in all previous periods, then selects "c" for all future periods once you choose "c"
  He plays "nice" until you cheat him.

- If you consider the payoff by selecting "d" for first T periods, then choosing "c" in the $T+1$ period. Then the payoff from the strategy (d, d,..., d, c, c,c,…) is:

$$1 + \delta + \delta^2 + \cdots + \delta^{T-1} + 2\delta^T = \frac{1 + \delta^T - 2\delta^{T+1}}{1 - \delta}$$

Repeated Prisoners' Dilemma

- If your strategy is (d, ..., d, c, c,…), then

$$1+\delta+\delta^2+\cdots+\delta^{T-2}+2\delta^T+0=\frac{1+\delta^T-2\delta^{T+1}}{1-\delta}$$

- If your strategy is (d, ..., d, d, …), then

$$1+\delta+\delta^2+\cdots+\delta^{T-1}+\cdots=\frac{1}{1-\delta}$$

- By comparison, we have, if $\delta \le 1/2$,

$$\frac{1+\delta^T-2\delta^{T+1}}{1-\delta} \ge \frac{1}{1-\delta}$$

This looks like the noncooperation is going to occur, even if one player is willing to cooperate

Formal Definition of Repeated Game

Definition A repeated game $G^{\mathrm{T}}(\delta)$ is

- a stage game of finite length: $G = \{N, \{A_i\}, \{u_i\}\}$, which is usually independent of the calendar date.
- a terminal date $T = 1,2, \ldots,$ giving the number of interact times. The calendar date is given by $t = 1, 2, \ldots, T$.
- a discount factor, $0 \le \delta \le 1$, that represents both how patient the players are and how likely the game continues.

- If $a^t = (a_1^t, a_2^t, \ldots, a_N^t)$ is the strategy outcome that occurs in period t, the player i's payoff is

$$u_i(a^1) + \delta u_i(a^2) + \cdots + \delta^{T-1} u_i(a^T) = \sum_t \delta^{t-1} u_i(a^t)$$

History

- **Perfect information**: Players keep track of how players behave in previous periods; so as to choose strategies that reward or punish players for good or bad behavior.
- How to track what happens in repeated games?-**History**
- In prisoners' dilemma, all of the possible outcomes from the stage game are $\Sigma = \{(\mathrm{d}, \mathrm{d}), (\mathrm{d}, \mathrm{c}), (\mathrm{c}, \mathrm{d}), (\mathrm{c}, \mathrm{c})\}$
  - For two periods, $\Sigma^2 = \Sigma \times \Sigma$ is the set of all possible outcomes of two repetitions of the game
  - For three periods, $\Sigma^3 = \Sigma \times \Sigma \times \Sigma$ is the set of all possible outcomes of three repetitions of the game
  - and so on

History and SPNE

Definition Let Σ be the set of all strategy outcomes for one stage game, and let $\Sigma^{\mathrm{t}} = \Sigma \times \Sigma \times \cdots \times \Sigma$ denote all possible outcomes.

A **history** at time t is an element $h_t \in H_t = \Sigma^t$.

Definition A set of strategies is a **Subgame Perfect Nash Equilibrium (SPNE)** of a repeated game if, for any t-period history h_t, there is no subgame in which any player has a profitable deviation.

- No player can have a profitable deviation for any history, even if only one history actually occurs
- The players know the consequences of their actions

SPEN in Repeated Games

Proposition If one stage game has an Nash equilibrium $a^* = (a_1^*, \ldots, a_N^*)$, then the strategy

$$(a^*, a^*, \ldots, a^*)$$

is a subgame perfect Nash equilibrium (SPNE) of the repeated game, i.e., each player i plays a_i^* for every history.

Is this the only equilibrium of a repeated game?

SPNE of Finite Repeated Game

Theorem Consider a repeated game $G^T(\delta)$ with $T < +\infty$. Suppose that the stage game G has an unique pure strategy NE a^*. G^T has a unique SPNE with $a^t = a^*$ for each t.

Proof We use the backward induction.

For period T, we will have $a^T = a^*$ regards of history.

For period T-1, we also have $a^{T-1} = a^*$

…

By Induction, we have $a^t = a^*$ for $1 \leq t \leq T$

Infinite Repeated Prisoners' Dilemma

Prisoner 2

Prisoner 1

	Confess(c)	Don't confess(d)
Confess(c)	0 0	2 -1
Don't confess(d)	-1 2	1 1

- Consider the following strategies
  - If the history at time t is $\{(d, d), \ldots, (d, d)\}$, paly d
  - Else paly c

Is this a SPNE of the infinite repeated game?

Infinite Repeated Prisoners' Dilemma

- Suppose the history is $\{(d, d), (d, d), \ldots\}$, the payoff is

$$1 + \delta + \delta^2 + \cdots + \delta^t + \cdots$$

- Suppose the history is not $\{(d, d), (d, d), \ldots\}$

$$2 + 0\delta + 0\delta^2 + \cdots$$

- So it all comes down to whether it's better to cooperate than cheat in any periods,

$$\frac{1}{1-\delta} > = < 2$$

Infinite Repeated Prisoners' Dilemma

If $\delta > 1/2$, both players using the strategy

- if the history at time t is $\{(\mathrm{d}, \mathrm{d}), \ldots, (\mathrm{d}, \mathrm{d})\}$, paly d;
- for any other history at time t, play c.

is a Subgame Perfect Nash Equilibrium of the infinitely repeated prisoners' dilemma.

The Steps for Repeated Games

- Solve for all of the equilibria of the stage game (Competitive play)
- Find a strategy profile that gives all the players a higher payoff (Cooperative play)
- Enforce cooperation: If all players have previously cooperated, continue cooperating. If any player has previously defected, play competitively
- For sufficiently large discount factor δ, this will be an equilibrium of the repeated game

Folk Theorem

- Consider a N-player infinitely repeated game with a stage game equilibrium $a^* = (a_1^*, a_2^*, \dots, a_N^*)$ with payoffs $u^* = (u_1^*, u_2^*, \dots, u_N^*)$.

- Suppose there is another $\hat{a} = (\hat{a}_1, \hat{a}_2, \dots, \hat{a}_N)$ with payoffs $\hat{u} = (\hat{u}_1, \hat{u}_2, \dots, \hat{u}_N)$, where, for every player i,

$$\hat{u}_i > u_i^*$$

For some discount factor δ, there is a Subgame Perfect Nash Equilibrium in which the players use $\hat{a}$ in every period of the infinitely repeated game.

The Folk Theorem: Trigger Strategies (触发策略)

Consider the following trigger strategy for player i:

- If the history at t is $h_t = (\hat{a}, \hat{a}, \dots, \hat{a})$, play $\hat{a}_i$ in period t
- For any other history at time t, play a_i^* in period t

This is called a "trigger strategy" because it starts in "cooperative" mode, but after any defection by any player, it switches to "punishment" or "competitive" mode, and they play the stage game strategies forever.

Construct SPNE in Repeated Games

1. Solve all equilibria of the stage game (Competition)
2. Find an outcome (equilibrium/not) where all the players do at least as well as in the stage game (Cooperation)
3. Design trigger strategies that support cooperation and punish with competition
4. Compute the maximum discount factor so that cooperation is an equilibrium
5. The trigger strategies are an SPEN of the infinitely repeated game for some larger discount factor

Repeated Cournot Competition

- Two firms compete by choosing how much to produce

$$G = \{\{1,2\},\{q_1,q_2\},\{u_1,u_2\}\}$$

  - Price

$$p(q_1+q_2) = \max(0, a-(q_1+q_2))$$

  - Costs $(i = 1, 2)$

$$c_i(q_i) = 0$$

  - Payoffs $(i = 1, 2)$

$$u_i(q_1,q_2) = (\max\big(0, a-(q_1+q_2)\big))q_i$$

  - Condition $a > 0, q_1 \geq 0, q_2 \geq 0$

Step 1: Nash Equilibrium for One Stage

Best Response Correspondence is given by

$$B_i(q_{-i}) = \max(0, (a - q_{-i})/2)$$

The Nash equilibria is give by

$$q^* = (q_1^*, q_2^*) = \left(\frac{a}{3}, \frac{a}{3}\right)$$

The payoff is

$$u^* = (u_1^*, u_2^*) = \left(\frac{a^2}{9}, \frac{a^2}{9}\right)$$

What happens if two firms cooperate for their profits?

Maximal Payoff for Cooperation

Summing the firms' profits, we get

$$u_1 + u_2 = (a - q_1 - q_2)q_1 + (a - q_1 - q_2)q_2$$
$$= (a - q_1 - q_2)(q_1 + q_2)$$

Maximizing the above gives

$$q_1 + q_2 = a/2$$

The total payoff for cooperation: $a^2/4 = 2a^2/8$

The total payoff for completion: $2a^2/9$

Cooperation is potentially profitable

Step 2: Cooperation

Suppose the two firms are playing the Cournot game an infinite number of times, and they share a discount factor δ.

Let

$$\hat{q} = (\hat{q}_1, \hat{q}_2) = \left(\frac{a}{4}, \frac{a}{4}\right)$$

$$\hat{u} = (\hat{u}_1, \hat{u}_2) = \left(\frac{a^2}{8}, \frac{a^2}{8}\right)$$

In competitive model,

$$\hat{q} = (\hat{q}_1, \hat{q}_2) = \left(\frac{a}{3}, \frac{a}{3}\right)$$

$$\hat{u} = (\hat{u}_1, \hat{u}_2) = \left(\frac{a^2}{9}, \frac{a^2}{9}\right)$$

Step 3: Trigger Strategy

Consider the strategy:

- If the two firms have both used $\hat{q} = (a/4, a/4)$ in all previous periods, use $\hat{q}_j = a/4$ this period

- If either firm ever did anything besides $\hat{q}$, play the stage Cournot quantity $q_j^* = a/3$

Is this a subgame perfect Nash equilibrium of the infinitely repeated game?

Check the NE of Cooperative Strategy

To check whether $\hat{q} = (\hat{q}_1, \hat{q}_2) = (a/4, a/4)$ is a NE?

By symmetry, it is sufficient to check player 2. We solve

$$\max_{q_2}(a - \hat{q}_1 - q_2)q_2 = \max_{q_2}(a - a/4 - q_2)q_2$$

Maximizing the above gives

$$q_2' = \frac{3a}{8}, u_2'\left(\frac{a}{4}, \frac{3a}{8}\right) = \left(\frac{3a}{8}\right)^2$$

$\boldsymbol{\hat{q} = (\hat{q}_1, \hat{q}_2) = (a/4, a/4)}$ **is not a NE**

Step 4: Select Discounting Factor

For cooperating case, all players keep the cooperation model, and the payoff for player 2 is

$$\hat{u}_2(1 + \delta + \delta^2 + \cdots) = \frac{a^2}{8}\frac{1}{1-\delta}$$

For competitive case, deviating optimally in some period t after a history, and all players cooperated switches the game to competition. The pay off for player 2 is

$$u_2' + u_2^*(\delta + \delta^2 + \cdots) = \left(\frac{3a}{8}\right)^2 + \left(\frac{a}{3}\right)^2 \frac{\delta}{1-\delta}$$

Step 5: SPNE

- The cooperating is better than deviating if

$$\frac{a^2}{8}\frac{1}{1-\delta} \geq \left(\frac{3a}{8}\right)^2 + \left(\frac{a}{3}\right)^2 \frac{\delta}{1-\delta}$$

This implies $\delta \geq 9/17$.

If $\delta \geq 9/17$, then the strategy:

- If the two firms have both used $\hat{q} = (a/4, a/4)$ in all previous periods, use $\hat{q}_j = a/4$ this period

- If either firm ever did anything besides $\hat{q}$, play the stage Cournot quantity $q_j^* = a/3$

is a SPNE of the infinitely repeated game

How to construct NE in Repeated Games

Convex hull

- A set is said to be convex if it contains the line segments connecting each pair of its points
- The convex hull of set $S = \{x_1, \ldots, x_n\}$ is defined as

$$\mathrm{Conv}(S) = \left\{ \sum_i a_i x_i \mid a_i \in [0,1], \sum_i a_i = 1 \right\}$$

  - The set of all convex combinations of points in S
  - The (unique) minimal convex set containing S
  - The intersection of all convex sets containing S

Feasible Payoffs

- Consider stage game $G = \{N, \{A_i\}, \{u_i\}\}$ and infinitely repeated game $G^{\infty}(\delta)$.

- Let us introduce the set of feasible payoffs:

$$U = \mathrm{conv}\left\{ \begin{array}{ll} \mathrm{u} \in R^N: & \text{there exists } a = (a_1, \dots, a_N) \\ & \text{s.t.} \quad u = (u_1(a), \dots, u_N(a)) \end{array} \right\}$$

That is, U is the convex hull of all N-dimensional vectors that can be obtained by some (possibly mixed) strategy outcome.

Minmax Payoffs

- Minmax payoff of player i: the lowest payoff that player i's opponent can hold him to:

$$\underline{u}_i = \min_{a_{-i}} \left[\max_{a_i} u_i(a_i, a_{-i}) \right]$$

- The player can never receive less than this amount.
- Minmax strategy outcome against to i

$$a_{-i}^{i} = \arg\min_{a_{-i}} \left[\max_{\mathrm{a}} u_i(a_i, a_{-i}) \right]$$

- Let a_i^i denote the strategy of player i such that

$$u_i\left(a_i^i, a_{-i}^i\right) = \underline{u}_i$$

Notice that a_i may be a mixed strategy for each player i

Example

Player 2

Player 1

	L	R
U	-2, 2	1, -2
M	1, -2	-2, 2
D	0, 1	0, 1

How to find the minmax payoff for player 1

The payoffs of player 1 for different strategies are

‘U’ : $1 - 3q$

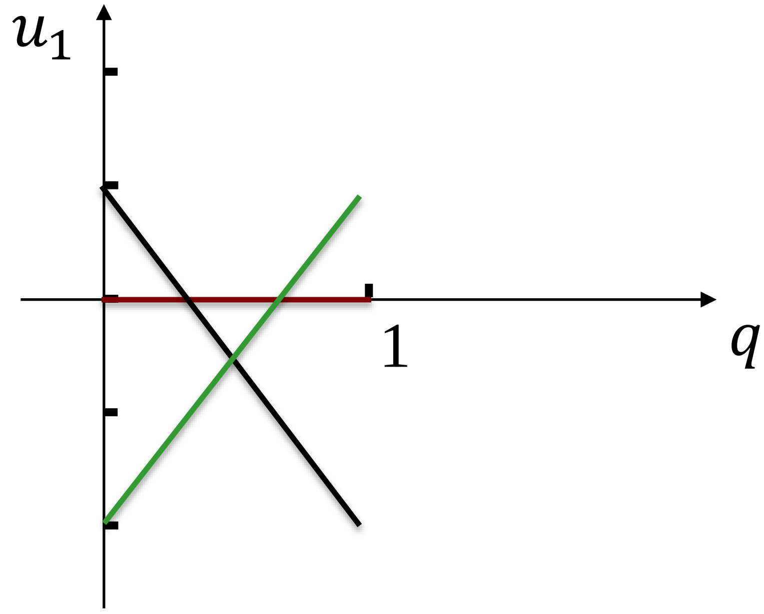
‘M’: $-2 + 3q$

‘D’: 0

Example (cont.)

We have

$$\underline{u}_1 = \min_{q\in[0,1]} [\max \{1 - 3q, -2 + 3q, 0\}]$$



Then, $\underline{u}_1 = 0$, and $a^1_{-1} = a^1_2$ is the mixed strategy with probability $q \in [\frac{1}{3}, \frac{2}{3}]$ over strategy 'L'

Example

Player 2

		L		R	
Player 1	q_1 U	-2	2	1	-2
	q_2 M	1	-2	-2	2
	D	0	1	0	1

How to find the minmax payoff for player 2

The payoffs of player 1 for different strategies are

$$\text{'L'}: 1 + q_1 - 3q_2$$

$$\text{'R'}: 1 + q_2 - 3q_1$$

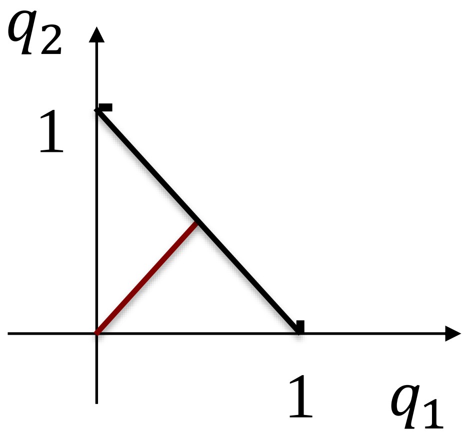
Example (cont.)

For player 2, we have

$$\underline{u}_2 = \min_{\substack{q_1, q_2 \in [0,1] \\ q_1 + q_2 \le 1}} [\max\{1 + q_1 - 3q_2, 1 + q_2 - 3q_1\}]$$

Case: if $1 + q_1 - 3q_2 \ge 1 + q_2 - 3q_1$, i.e., $q_1 \ge q_2$

$$\underline{u}_2 = \min_{\substack{q_1, q_2 \in [0,1] \\ q_1 + q_2 \le 1 \\ q_1 \ge q_2}} [1 + q_1 - 3q_2]$$



- $q_1 = 0, q_2 = 0, u_2 = 1$
- $q_1 = 1, q_2 = 0, u_2 = 2$
- $q_1 = q_2 = 1/2, u_2 = 0$

Example (cont.)

For player 2, we have

$$\underline{u}_2 = \min_{\substack{q_1, q_2 \in [0,1] \\ q_1 + q_2 \leq 1}} [\max\{1 + q_1 - 3q_2, 1 + q_2 - 3q_1\}]$$

Then, $\underline{u}_2 = 0$, and $a^2_{\underline{\;\;}1}$ is the mixed strategy with probability $q_1 = q_2 = 1/2$ over strategy 'U' and 'M'

Minmax Payoff Lower Bounds

Theorem Let $a' = (a'_1, a'_2, \dots, a'_n)$ be a (possibly mixed) Nash Equilibrium of game G and $u_i(a')$ be its payoff. Then

$$u_i(a') \geq \underline{u}_i$$

Proof. Write $a' = (a'_i, a'_{-i})$. If a' is a NE, then

$$u_i(a') = \max_{a_i \in A_i} u_i(a_i, a'_{-i})$$

On the other hand

$$\underline{u}_i = \min_{a_{-i}} \left[\max_{a_i} u_i(a_i, a_{-i}) \right] \leq \max_{a_i} u_i(a_i, a'_{-i}) = u_i(a')$$

Nash Folk Theorem

Definition A payoff vector $(u_1, u_2, \dots, u_N) \in R^N$ is strictly individually rational if $u_i > \underline{u}_i$ for all i

Nash Folk Theorem If $(u_1, u_2, \dots, u_N) \in U$ is strictly individually rational, then there exists some $\delta_0 < 1$ such that for all $\delta \geq \delta_0$, there is Nash equilibrium of $G^\infty(\delta)$ with payoff $(u_1, u_2, \dots, u_N)$

Any strictly individually rational payoff can be obtained as a Nash Equilibrium when players are patient enough

Proof of Nash Folk Theorem

Proof. Suppose $a = (a_1, a_2, \ldots, a_N)$ s.t. $u_i(a) = \mathrm{u}_i$

Let (a_i^i, a_{-i}^i) be the minmax strategy for player i with $\underline{u}_i$

Now we construct the following trigger strategy:

i) Each player plays a as long as no one strategy

ii) If player i deviates, then $a_{-i} = a_{-i}^i$

We next check if player i can gain by deviating from a_i

If player I does not deviate, then the payoff is

$$u_i + u_i\delta + \cdots + u_i\delta^{t-1} + u_i\delta^t + \cdots$$

Proof of Nash Folk Theorem

If player i deviates at step t, then the payoff at step t is

$$\bar{u}_i = \max_{a' \in A\_i} u(a', a_{-i}) > u_i$$

and the payoff for all step is

$$u_i + u_i\delta + \cdots + \bar{u}_i\delta^t + \underline{u}_i\delta^{t+1} + \cdots$$

To guarantee the cooperate, we have

$$\bar{u}_i\delta^t + \underline{u}_i\delta^{t+1} + \cdots < u_i\delta^t + u_i\delta^{t+1} + \cdots$$

$$\delta \geq \frac{\bar{u}_i - u_i}{\bar{u}_i - \underline{u}_i} \text{ finally setting } \delta_0 = \max_i \left\{\frac{\bar{u}_i - u_i}{\bar{u}_i - \underline{u}_i}\right\}$$

Problem with Nash Folk Theorem

- Nash Folk Theorem may be not a subgame perfect

		Player 2 L	R
Player 1	U	6, 6	0, -2
	D	7, 1	0, -2

- The unique NE in this game is (D,L).
- The minmax payoff are given by

$$\underline{u}_1 = \min_{q\in[0,1]} \max(6q, 7q) = 0 \quad \text{and} \quad a_{-1}^{2} = R$$

$$\underline{u}_2 = \min_{q\in[0,1]} \max(6q + 1 - q, -2) = 1$$

Problem with Nash Folk Theorem

		Player 2 L		Player 2 R	
Player 1	U	6	6	0	-2
Player 1	D	7	1	0	-2

- Nash Folk Theorem: the strategy
  - Play (U,L) as long as no one deviates
  - If Player 1 deviates, then player 2 select R
- While this will hurt player 1, it will hurt player 2 a lot.
- It is an threat, and it is not a SPNE

Nash Folk Theorem

Nash Folk Theorem If $(u_1, u_2, \ldots, u_N) \in U$ is strictly individually rational, then there exists some $\delta_0 < 1$ such that for all $\delta \geq \delta_0$, there is **Nash equilibrium** of $G^{\infty}(\delta)$ with payoff $(u_1, u_2, \ldots, u_N)$

Any strictly individually rational payoff can be obtained as a Nash Equilibrium when players are patient enough

Folk Theorem

- Consider a N-player infinitely repeated game with a stage game equilibrium $a^* = (a_1^*, a_2^*, \dots, a_N^*)$ with payoffs $u^* = (u_1^*, u_2^*, \dots, u_N^*)$.

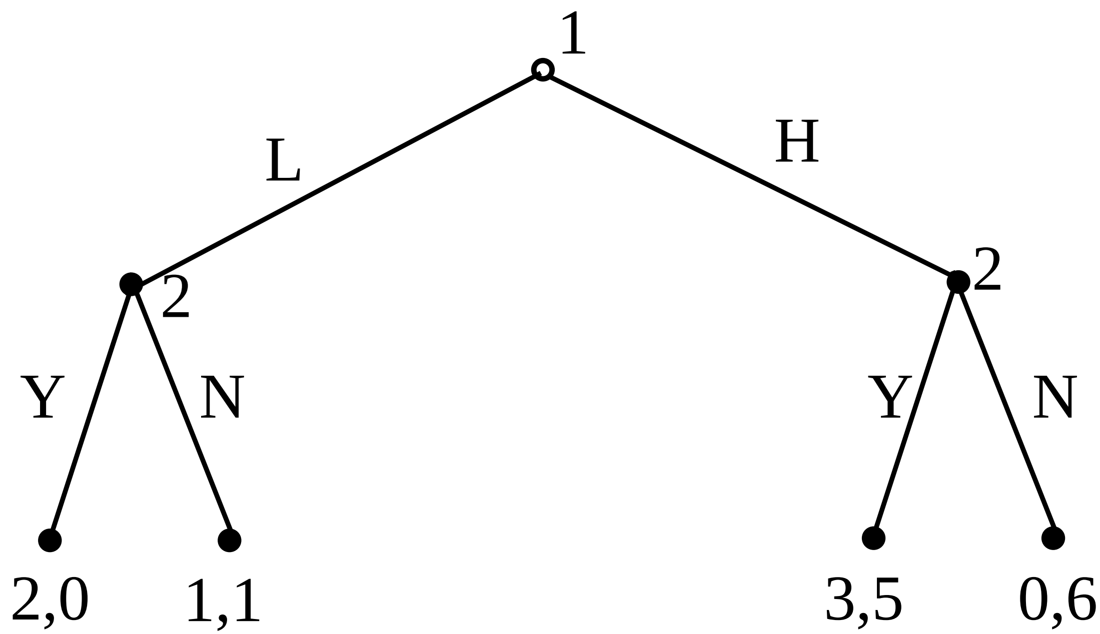
- Suppose there is another $\hat{a} = (\hat{a}_1, \hat{a}_2, \dots, \hat{a}_N)$ with payoffs $\hat{u} = (\hat{u}_1, \hat{u}_2, \dots, \hat{u}_N)$, where, for every player i,

$$\hat{u}_i > u_i^*$$

For some discount factor δ, there is a **Subgame Perfect Nash Equilibrium** in which the players use $\hat{a}$ in every period of the infinitely repeated game.

Home work 1

What happens for the following repeated extensive game



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Definition

A strategic game (normal form game) consists of

- A finite set N of players
- A non-empty strategy set A_i for each player $i \in N$
- A payoff function $u_i: A_1 \times A_2 \times \cdots \times A_N \to R$ for $i \in N$

$$G=\{N, \{A_i\}_{i=1}^N, \{u_i\}_{i=1}^N\}$$

- An outcome $a^* = (a_1^*, a_2^*, \ldots, a_N^*)$ is a **Nash equilibrium (NE)** if for each players i

$$u_i(a_i^*, a_{-i}^*) \geq u_i(a_i, a_{-i}^*) \text{ for all } a_i \in A_i.$$

How to Find Nash Equilibria

- One way of finding Nash equilibrium for continuous strategies A_i:

(1) Find the best response correspondence for each player

Best response correspondence

$$B_i(a_{-i}) = \{a_i \in \mathrm{argmax}_{a_i \in A_i}\, u_i(a_i, a_{-i})\}$$

(2) Find all Nash Equilibria $(a_1^*, a_2^*, \dots, a_N^*)$ such that

$$a_i^* \in B_i(a_{-i}^*) \quad \text{for each player}$$

Example

- Find all Nash equilibria

P2

		h	i	j	k	l	m
P1	a	7, 5	8, 6	2, 2	2, 3	6, 9	6, 5
	b	6, 5	9, 6	5, 8	6, 7	8, 8	7, 4
	c	9, 7	1, 1	7, 9	3, 2	9, 6	9, 2
	d	2, 14	10, 12	6, 5	6, 3	7, 2	9, 12
	e	8, 6	5, 9	3, 9	7, 5	13, 15	8, 9

Cournot Competition(古 偌 竞 争, 1838)

- Two firms compete by choosing how much to produce

$$G = \{\{1,2\},\{q_1,q_2\},\{u_1,u_2\}\}$$

  - Price

$$p(q_1+q_2) = \max(0, a - b(q_1+q_2))$$

  - Costs $(i = 1,2)$

$$c_i(q_i) = cq_i$$

  - Payoffs $(i = 1,2)$

$$u_i(q_1,q_2) = (\max\big(0, a - b(q_1+q_2)\big) - c)q_i$$

  - Condition $a > \mathrm{b}, c > 0, q_1 \geq 0, q_2 \geq 0$

Cournot: Best Response Correspondence

Best Response Correspondence is given by

$$B_i(q_{-i}) = \max(0, (a - c - bq_{-i})/2b)$$

Proof. We will prove for i=1 (similarly for i=2)

If $q_2 \geq (a-c)/b$, then $u_1(q_1, q_2) \leq 0$ for any $q_1 > 0$. $q_1 = 0$.

If $q_2 < (a-c)/b$, then

$$u_i(q_1, q_2) = (a - c - b(q_1 + q_2))q_i$$

$$\frac{\partial u_1(q_1, q_2)}{\partial q_1} = a - c - bq_2 - 2bq_1 = 0$$

$$q_1 = (a - c - bq_2)/2b$$

Cournot: Nash Equilibrium

The Nash equilibria is give by

$$\left\{\left(\frac{a-c}{3b},\frac{a-c}{3b}\right)\right\}$$

Proof. Assume that (q_1^*, q_2^*) is a Nash equilibrium.

1) Prove $q_1^* > 0$ and $q_2^* > 0$ by contradiction

2) (q_1^*, q_2^*) is such that $q_1^* > 0, q_2^* > 0$

$$q_1^* = B_1(q_2^*) = (a - c - bq_2^*)/2b$$

$$q_2^* = B_2(q_1^*) = (a - c - bq_1^*)/2b$$

Mixed Strategies

Strategy game

$$G = \{N, \{A_1, A_2, \dots, A_N\}, \{u_1, u_2, \dots, u_N\}\}$$

Pure strategy: each strategy in A_i

Mixed strategy: a probability over the set A_i of strategies

Pure strategy can be viewed as a special mixed strategy

Nash Theorem Every finite strategic game has a mixed strategy Nash equilibrium

How to calculate Mixed Nash Equilibria

Theorem If a mixed strategy is a best response, then each of the pure strategies (positive prob.) involved in the mixed strategy must be a best response. Particularly, each must yield the same expected payoff

		Player 2	
		L, π_2	$R, 1-\pi_2$
Player 1	U, π_1	1 2	0 4
	$D, 1-\pi_1$	0 5	3 2

Dominant Strategies and Nash Equilibrium

A pure strategy a_i **strictly dominates** a_i' if

$$u_i(a_i, a_{-i}) > u_i(a_i', a_{-i}) \text{ for all } a_{-i} \in A_{-i}$$

Theorem A **strictly** dominated strategy is never used with positive probability in a mixed strategy Nash equilibrium

How to find NE:

Step 1: eliminate all strictly dominated strategies

Step 2: Find all Nash Equilibria

Two-Player zero-sum game

Maxmin (最大化最小原则)

Maxmin (最小化最大原则)

		Player 2 L		Player 2 R	
Player 1	U	3	-3	-1	1
	D	-2	2	1	-1

The Minmax Theorem For two-player zero-sum finite game $G = \{\{1,2\}, \{A_1, A_2\}, u\}$, we have

$$\max_{p\in\Delta_1}\min_{q\in\Delta_2} pMq^\top = \min_{q\in\Delta_2}\max_{p\in\Delta_1} pMq^\top.$$

How to solve Nash Equilibrium? Player 1:

$$\max \quad v$$

$$3x_1 - 2x_2 \geq v,$$

$$-x_1 + x_2 \geq v$$

$$x_1 + x_2 = 1, x_1 > 0, x_2 > 0$$

Stackelberg game (主 从 博 弈)

Player 2

		Left	Right
Player 1	Up	1 1	3 0
	Down	0 0	2 1

Player 1: “leader” **Player 2 “follower”**

1) Leader selects a (possibly mixed) strategy x_1
2) Follower learns about x_1, selects the best response x_2
3) The payoff matrix is of size $d_1 \times d_2$

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Bayesian Games

A Bayesian game consists of

- A set of players N
- A set of strategies A_i for each player i
- A set of types Θ_i for each player i
  - Type set Θ_i includes all private information for player i
  - The types on payoff are adequate (Payoff types)
- Probability distribution $p = p(\theta_1, \dots, \theta_N)$ on $\times_{i=1..n} \Theta_i$
- A payoff function $u_i : \times_{i=1..N} A_i \times \times_{i=1..n} \Theta_i \rightarrow R$

$$u_i(a_1, \dots, a_N, \theta_1, \dots, \theta_N) \text{ for } a_i \in A_i \text{ and } \theta_i \in \Theta_i$$

$$\boldsymbol{G = \{N, \{A_i\}, \{\Theta_i\}, \{u_i\}, p\}}$$

Bayesian Games (cont.)

Definition The outcome $(a_1, a_2, \dots, a_N)$ is a **Bayesian Nash Equilibrium** if for each type θ_i, we have

$$U_i(a_i(\theta_i), a_{-i}) \geq U_i(a'_i(\theta_i), a_{-i}) \text{ for all } a'_i(\theta_i) \in A_i$$

How to find Bayesian Nash Equilibrium

1) Find the best response function for each player and type
2) Find Bayesian NE by $a_i(\theta_i) \in B_i(a_{-i}, \theta_i)$

Bank Runs

- Two players
- Strategies $A_1 = A_2 = \{W, N\}$
- Types $\Theta_1 = \{1\}; \Theta_2 = \{\mathrm{G}, \mathrm{B}\}$
- A probability distribution $p_1(\theta_2 = \mathrm{G}) = p$
- Payoffs

Player 2 (G, p)

Player 1	W	N
W	50, 50	100, 0
N	0, 100	150, 150

Player 2 (B, $1 - p$)

Player 1	W	N
W	50, 50	100, 0
N	0, 100	0, 0

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Extensive Game

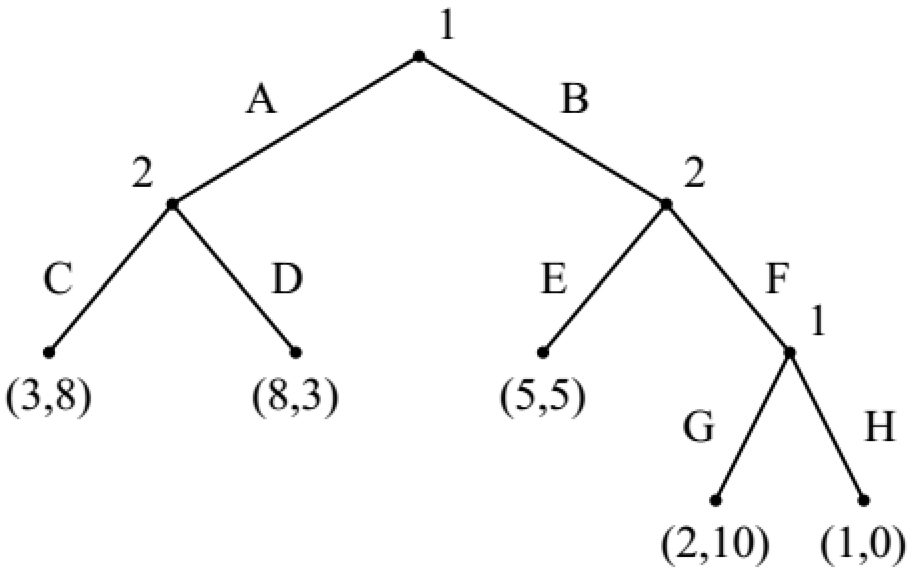
An extensive game with perfect information is defined by

- Players N is the set of N players
- Histories H is a set of sequence (finite or infinite)
- Player function
  - P assigns to each non-terminal history a player of N
  - $P(h)$ denotes the player who takes action after the history h
- Payoff function $u_i: Z \rightarrow R$

$$\boldsymbol{G} = \left\{\boldsymbol{N}, \boldsymbol{H}, \boldsymbol{P}, \{\boldsymbol{u_i}\}\right\}$$

Induced Strategic Game and NE

Every extensive game can be converted to a strategy game



	CE	CF	DE	DF
AG	3, 8	3, 8	8, 3	8, 3
AH	3, 8	3, 8	8, 3	8, 3
BG	5, 5	2, 10	5, 5	2, 10
BH	5, 5	1, 0	5, 5	1, 0

Subgame

Definition A subgame is a set of nodes, strategies and payoffs, following from a single node to the end of game.

Definition An outcome is $a = (a_1^*, a_2^*, \dots, a_N^*)$ is a subgame perfect (子博弈完美) if it is Nash Equilibrium in every subgame

- Subgame perfect is a Nash Equilibrium
- This definition rules out "non-credible threat"

Theorem Every extensive game with perfect information has a subgame perfect

Back Induction (后向归纳)

How to find subgame perfect Equilibria (SPE)

Back induction is the process of "pruning the game tree" described as follows:

- Step 1: start at each of the final subgame in the game, and solve for the player's equilibrium. Remove that subgame and replace it with payoff of the player's choice
- Step 2: Repeat step 1 until we arrive at the first node in the extensive game

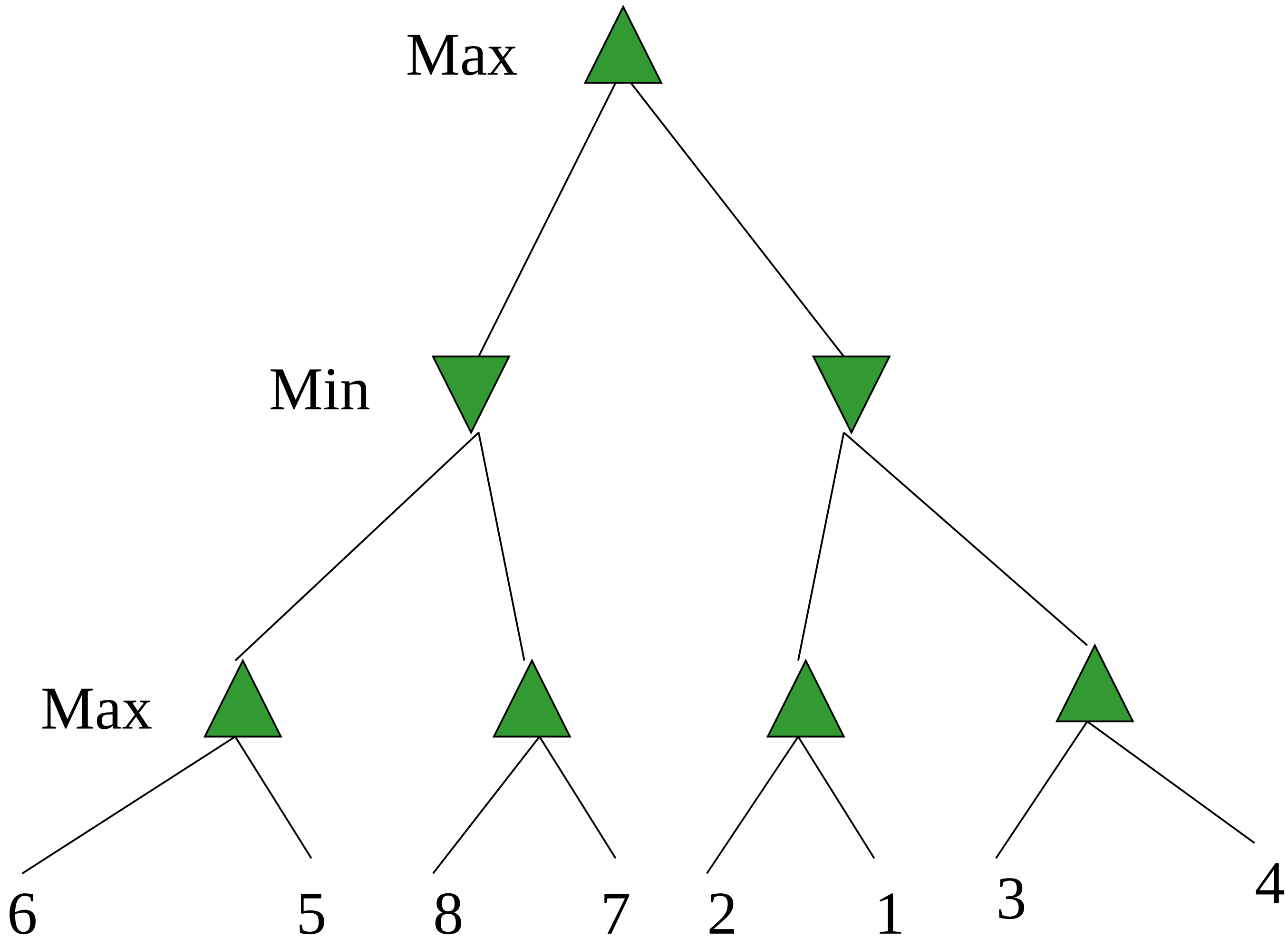
Theorem The set of strategy game constructed by backwards induction is equivalent to the set of SPE

Game Tree Search

- Two players made adversary actions alternately
- Game problem -> search problem
- Minimax algorithm
- Evaluation function
- Alpha-Beta Pruning

Example

What nodes can be pruned?



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Definition of Extensive Game with Imperfect Information

An extensive game with imperfect information is defined by $G = \{N, H, P, I, \{u_i\}\}$

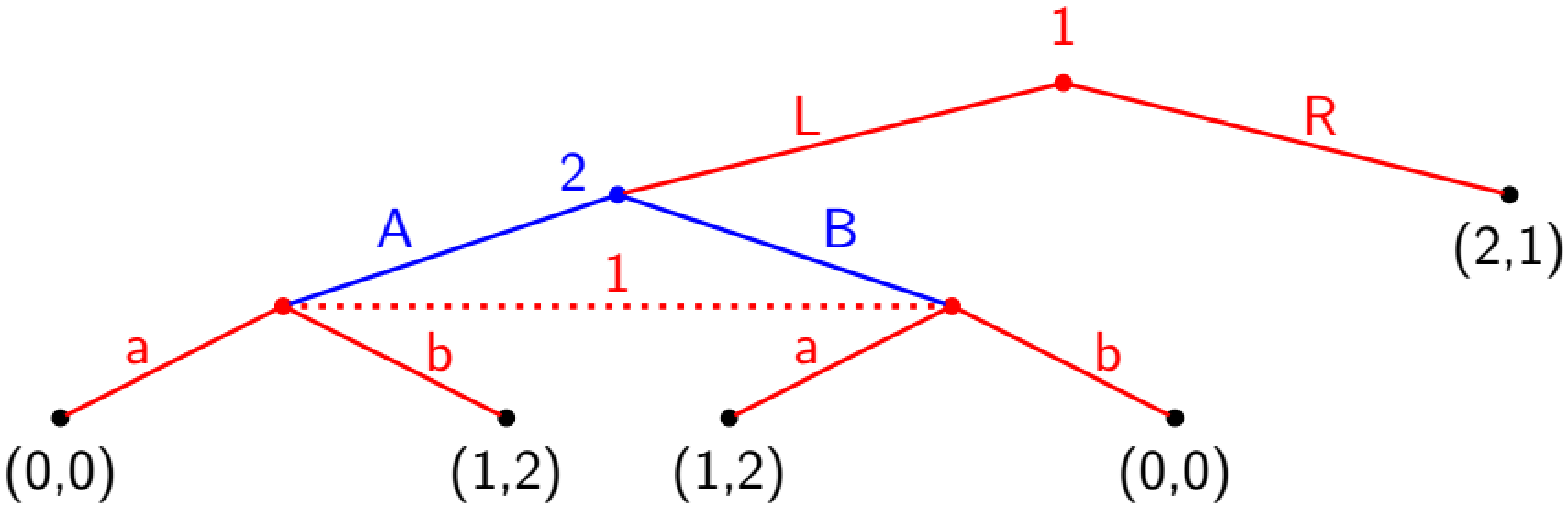
- Information set $I = \{I_1, I_2, \ldots I_N\}$ is the set of information partition of all players' strategy nodes, where the nodes in an information set are indistinguishable to player
  - $I_i = \{I_{i1}, \ldots, I_{ik_i}\}$ is the information partition of player i
  - $I_{i1} \cup \cdots \cup I_{ik_i}$ ={all nodes of player i}
  - $I_{ij} \cap I_{ik} = \emptyset$ for all $j \neq$ k
  - Action set $A(h) = A(h')$ for $h, h' \in I_{ij}$, denote by $A\left(I_{ij}\right)$
  - $P\left(I_{ij}\right)$ be the player who plays at information set I_{ij}
- An extensive game with perfect information is a special case where each I_{ij} contains only one node

Pure Strategies

- A pure strategy for player i selects an available action at each of i's information sets $I_{i1}, \ldots, I_{im}$
- All pure strategies for player i is

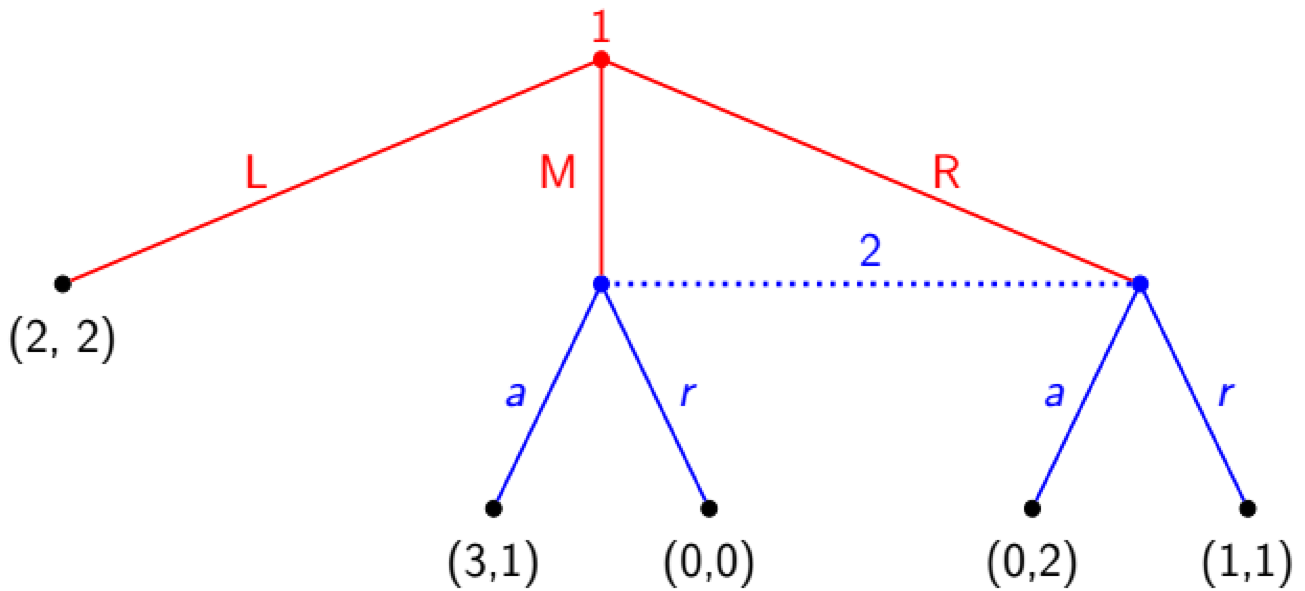
$$A(I_{i1}) \times A(I_{i2}) \times A(I_{im})$$

where $A(I_{ij})$ denotes the strategies available in I_{ij}



What's the pure strategies for players 1 and 2?

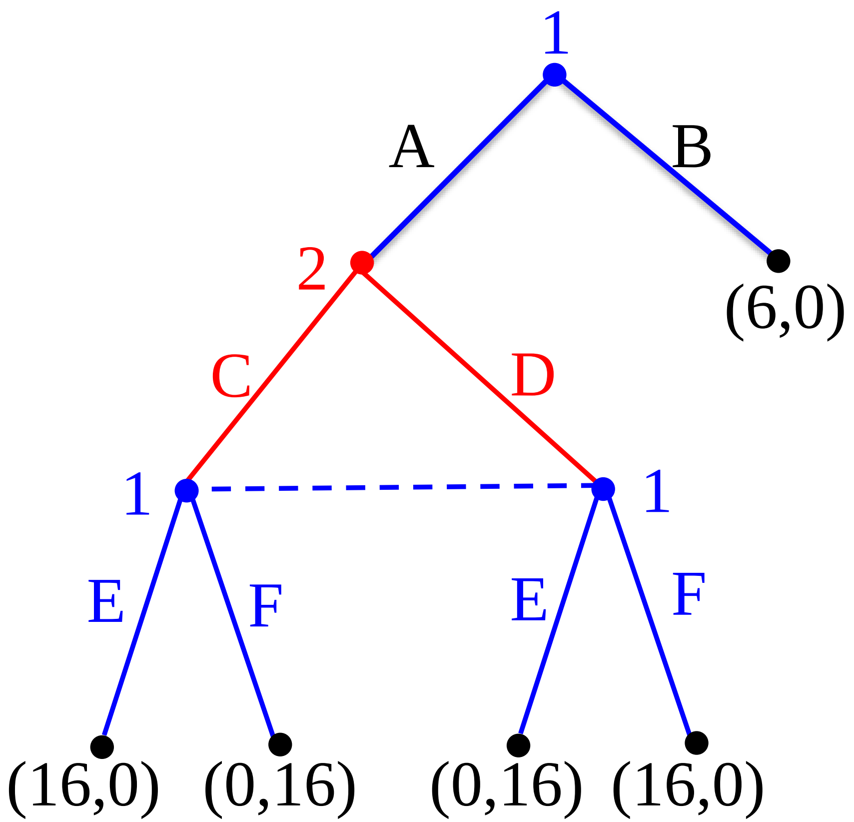
Normal-Form Representation of Extensive Imperf. Game



	a	r
L	2,2	2,2
M	3,1	0,0
R	0,2	1,1

- The pure and mixed strategy Nash Equilibrium remains?
- What's the difference from the extensive game with perfect information game?

Example



How to calculate the sequential equilibrium?

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Definition and Folk Theorem

- A repeated game $G^{\mathrm{T}}(\delta)$ consists of stage game G, terminal date T and discount factor δ

- Folk Theorem
  - An infinitely repeated game with a stage game equilibrium $a^* = (a_1^*, a_2^*, \dots, a_N^*)$ with payoffs $u^* = (u_1^*, u_2^*, \dots, u_N^*)$.
  - Suppose there is another $\hat{a} = (\hat{a}_1, \hat{a}_2, \dots, \hat{a}_N)$ with payoffs $\hat{u} = (\hat{u}_1, \hat{u}_2, \dots, \hat{u}_N)$, where, $\hat{u}_i \geq u_i^*$ for every player i
  - There is a Subgame Perfect Nash Equilibrium for some discount factor δ

Solving for Equilibria in Repeated Games

1. Solve all equilibria of the stage game (Competition)
2. Find an outcome (equilibrium/not) where all the players do at least as well as in the stage game (Cooperation)
3. Design trigger strategies that support cooperation and punish with competition
4. Compute the maximum discount factor so that cooperation is an equilibrium
5. The trigger strategies are an SPEN of the infinitely repeated game for some larger discount factor