PSTATW 174/274 Fall 2023: Homework 6

1. The sunspot numbers $\{X_t, t = 1, 2, \dots, 100\}$ have sample autocovariances $\hat{\gamma}(0) = 1382.2, \hat{\gamma}(1) = 1114.4, \hat{\gamma}(2) = 591.73$, and $\hat{\gamma}(3) = 96.216$. Let $Y_t = X_t - 46.93$, $t = 1, 2, \dots, 100$, be the mean-corrected sunspot series.

(a) Use the Durbin-Levinson algorithm to compute the sample partial autcorrelations $\hat{\phi}_{11}, \hat{\phi}_{22}$ and $\hat{\phi}_{33}$ for Y_t. Is the value of $\hat{\phi}_{33}$ compatible with the hypothesis that the data are generated by an AR(2) process? (Use significance level 0.05).

(b) Find the Yule-Walker estimates of ϕ_1, ϕ_2, σ^2 in the model: $Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + Z_t$, $\{Z_t\} \sim WN(0, \sigma^2)$.

2. In modeling the weekly sales of a certain commodity over the past few months, the time series model $X_t - \phi_1 X_{t-1} = Z_t + \theta_1 Z_{t-1}$ was thought to be appropriate. Suppose the model was fitted and the autocorrelations of the residuals were:

k	1	2	3	4	5	6	7	8
$\hat{\rho}_{\hat{W}}(k)$	-.04	-.50	.03	-.01	.01	.02	.03	-.01
st. dev $\hat{\rho}_{\hat{W}}(k)$	.08	.10	.11	.11	.11	.11	.11	.11

Is the assumed model really appropriate? If not, how would you modify the model? Explain.

Hint: Check slides 8 - 9 of Lecture 11 for the 95% confidence intervals for autocorrelation function of the fitted residuals. You might also find slide 26 of Week 2 and slide 15 of Lecture 11 useful.

The following problem is for students enrolled in PSTATW 274 ONLY

G1. Two hundred observations of a time series, $X_1, \dots, X_{200}$, gave the following sample statistics: sample mean: $\bar{x}_{200} = 3.82$; sample variance: $\hat{\gamma}(0) = 1.15$; sample ACF: $\hat{\rho}(1) = .427$, $\hat{\rho}(2) = .475$, $\hat{\rho}(3) = .169$.

(a) Based on these sample statistics, is it reasonable to suppose that $\{X_t - \mu\}$ is a white noise?

(b) Assuming that $\{X_t - \mu\}$ can be modeled as the AR(2) process, $X_t - \mu - \phi_1(X_{t-1} - \mu) - \phi_2(X_{t-2} - \mu) = Z_t$, where $Z_t \sim \text{IID}(0, \sigma_Z^2)$, find estimates of μ, ϕ_1, ϕ_2, and σ_Z^2.

(c) Assuming that the data was generated from an AR(2) model, derive estimates of the PACF for lags $h \geq 1$.

...

Some useful formulas.

- A 95%, the confidence interval for the theoretical acf $\rho_{\hat{W}}(h)$ of the fitted residuals $\hat{W}$ is $\hat{\rho}_{\hat{W}}(h) \pm 1.96 \times sd(\hat{\rho}_{\hat{W}}(h))$ where $sd(\hat{\rho}_{\hat{W}}(h)) \approx 1/\sqrt{n}$ (cp. slide 11 of week 6)
- For MA(2), $\rho(1) = \frac{\theta_1(1+\theta_2)}{1+\theta_1^2+\theta_2^2}$ and $\rho(2) = \frac{\theta_2}{1+\theta_1^2+\theta_2^2}$ (cp. slide 26 of week 2)
- Yule-Walker estimates for model parameters of AR(p) process are (slide 67 of week 4)
  $\underline{\hat{\phi}}_p = \hat{R}_p^{-1} \underline{\hat{\rho}}_p$, $\hat{\sigma}_Z^2 = \hat{\gamma}(0)\{1 - \underline{\hat{\rho}}_p' \hat{R}_p^{-1} \underline{\hat{\rho}}_p\}$
- Durbin-Levinson Algorithm to calculate sample PACF (slide 70 of week 4; §§10.5, 11.1.1 of lecture notes)

$$\hat{\phi}_{hh} = \frac{\hat{\rho}(h) - \sum_{j=1}^{h-1} \hat{\phi}_{h-1,j}\hat{\rho}(h-j)}{1 - \sum_{j=1}^{h-1} \hat{\phi}_{h-1,j}\hat{\rho}(j)}, \; \hat{\phi}_{h,j} = \hat{\phi}_{h-1,j} - \hat{\phi}_{hh}\hat{\phi}_{h-1,h-j}, \; j = 1, \dots, h-1.$$

In particular, Example 10.5.1 calculates:

$\hat{\phi}_{11} = \hat{\rho}(1)$;

$\hat{\phi}_{22} = \frac{\hat{\rho}(2) - \hat{\phi}_{11}\hat{\rho}(1)}{1 - \hat{\phi}_{11}\hat{\rho}(1)}$; $\hat{\phi}_{21} = \hat{\phi}_{11} - \hat{\phi}_{22}\hat{\phi}_{11}$

$\hat{\phi}_{33} = \frac{\hat{\rho}(3) - (\hat{\phi}_{21}\hat{\rho}(2) + \hat{\phi}_{22}\hat{\rho}(1))}{1 - (\hat{\phi}_{21}\hat{\rho}(1) + \hat{\phi}_{22}\hat{\rho}(2))}$.

For AR(p), $\hat{\sigma}_Z^2 = \hat{v}_p$ with $\hat{v}_h = \hat{v}_{h-1}[1 - \hat{\phi}_{hh}^2]$ and $\hat{v}_0 = \hat{\gamma}(0)$

- For AR(p) process, large sample size n, and $h > p$, $\hat{\phi}_{hh} \sim \mathcal{N}(0, 1/n)$ (§10.4; slides 68-69 of Week 4)