Lab 3

PSTAT W 174/274

Model Identification

For each of the following time series process, establish the following:

`(a). State whether it is an MA(q), AR(p), or ARMA(p,q) and determine p and/or q.`

`(b). Express the model in terms of the back shift operator, B.`

`(c). Determine whether each process is causal and/or invertible.`

Causality [§4.2 from Week2 Slides]: An ARMA(p,q) process X_t is causal, or a causal function of Zt if:

$\phi(z) = 1 - \phi_1 z - ... - \phi_p z^p \neq 0$ for all $|z| \leq 1$.

Invertibility [§3.3 from Week2 Slides]: An ARMA(p,q) process X_t is invertible if:

$\theta(z) = 1 + \theta_1 z + ... + \theta_p z^p \neq 0$ for all $|z| \leq 1$.

1. $X_t = 0.7X_{t-1} - 0.1X_{t-2} + Z_t$

   (a). AR(2)

   (b). $(1 - 0.7B + 0.1B^2)X_t = Z_t$

   (c). The autoregressive models are by nature invertible. The AR polynomial for this process has the factorization $\phi(z) = 1 - 0.7z + 0.1z^2 = (1 - 0.5z)(1 - 0.2z)$ and is therefore zero at z = 2 and z = 5. Since these zeroes lie outside the unit circle, we conclude that X_t is a causal AR(2).

   If you prefer to find the roots of the AR(2) polynomial $\phi(z) = 1 - 0.7z + 0.1z^2$ using R, you may use the polyroot command:

   ```
   polyroot(c(1,-0.7,0.1))
   ```

   ```
   ## [1] 2+0i 5-0i
   ```

2. $X_t = 0.1Z_{t-2} - 0.7Z_{t-1} + Z_t$

   (a). MA(2)

   (b). $X_t = (1 - 0.7B + 0.1B^2)Z_t$

   (c). The moving average models are by nature casual. The MA polynomial is given by $\phi(z) = 1 - 0.7z + 0.1z^2$ with corresponding zero at z = 2 and z = 5, which is greater than 1 in absolute value so the process X_t is invertible.

   ```
   polyroot(c(1,-0.7,0.1))
   ```

   ```
   ## [1] 2+0i 5-0i
   ```

3. $X_t = 0.5X_{t-1} + Z_t + 0.4Z_{t-1}$

   (a). ARMA(1,1)

(b). $(1 - 0.5B)X_t = (1 + 0.4B)Z_t$

(c). The AR and MA polynomial are given by $\phi(z) = 1 - 0.5z$ and $\phi(z) = 1 + 0.4z$, respectively. The AR root is z $= 2$ so the model is causual; and the MA root is z $= -2.5$, which lies outside the unit circle, so the model is invertible.

To find roots using polyroot, we can use polyroot function:

```
polyroot(c(1,-0.5))
```

```
## [1] 2+0i
```

```
polyroot(c(1,0.4))
```

```
## [1] -2.5+0i
```

4. $X_t = 0.75X_{t-1} - 0.5625X_{t-2} + Z_t + 1.25Z_{t-1}$

   (a). ARMA(2,1)

   (b). $(1 - 0.75B + 0.5625B^2)X_t = (1 + 1.25B)Z_t$

   (c). $(1 - 0.75B + 0.5625B^2)X_t = (1 + 1.25B)Z_t$. The AR polynomial is $\phi(z) = 1 - 0.75z + 0.5625z^2$ has zeroes at $z = 2(1 \pm i\sqrt{3})/3$, which lie outside the unit circle. The process is therefore causal. On the other hand, the MA polynomial $\phi(z) = 1 + 1.25z$ has zero at z $= -0.8$, and hence the X_t is not invertible.

   We can also see directly that for MA part: $\theta = 1.25 > 1$, so that the absolute value of the root $|z| = |-1/\theta| < 1$ making the model noninvertible.

   In AR part: to calculate roots fast we can directly use polyroot. To check that the roots are outside the unit circle for complex roots:

```
polyroot(c(1,-0.75,0.5625))
```

```
## [1] 0.666667+1.154701i 0.666667-1.154701i
```

```
polyroot(c(1,1.25))
```

```
## [1] -0.8+0i
```

   $|z|^2 = (0.667)^2 + (1.154)^2 > 1$ because the second number $1.154 > 1$. Therefore, the roots are outside the unit circle.

5. Simulate 200 observations from each model and compare the sample ACF and PACF with the model ACF and PACF (white noise variance is 1).

   To calculate sample ACF, we use acf() function: acf(data)

   To calculate sample PACF, we use pacf() function: pacf(data)

   To calculate theoretical ACF, we use ARMAacf() function with pacf = FALSE: ARMAacf(model,pacf = FALSE)

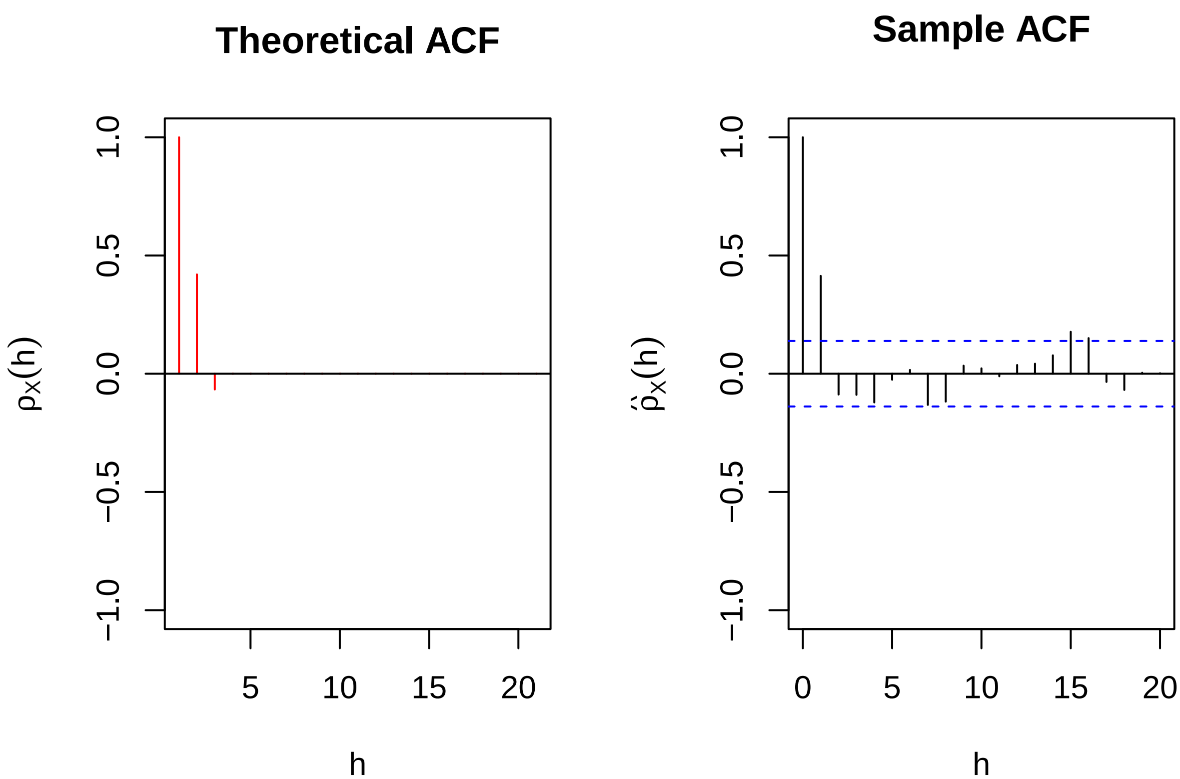
   To calculate theoretical ACF, we use ARMAacf() function with pacf = TRUE: ARMAacf(model,pacf = TRUE)

   Here are some examples:

   (a). MA(2)

```
# Simulate model
set.seed(2)
ma1 <- arima.sim(model = list(ma = c(0.7,-0.1),sd = 1),n = 200)
# Theoretical ACF
```

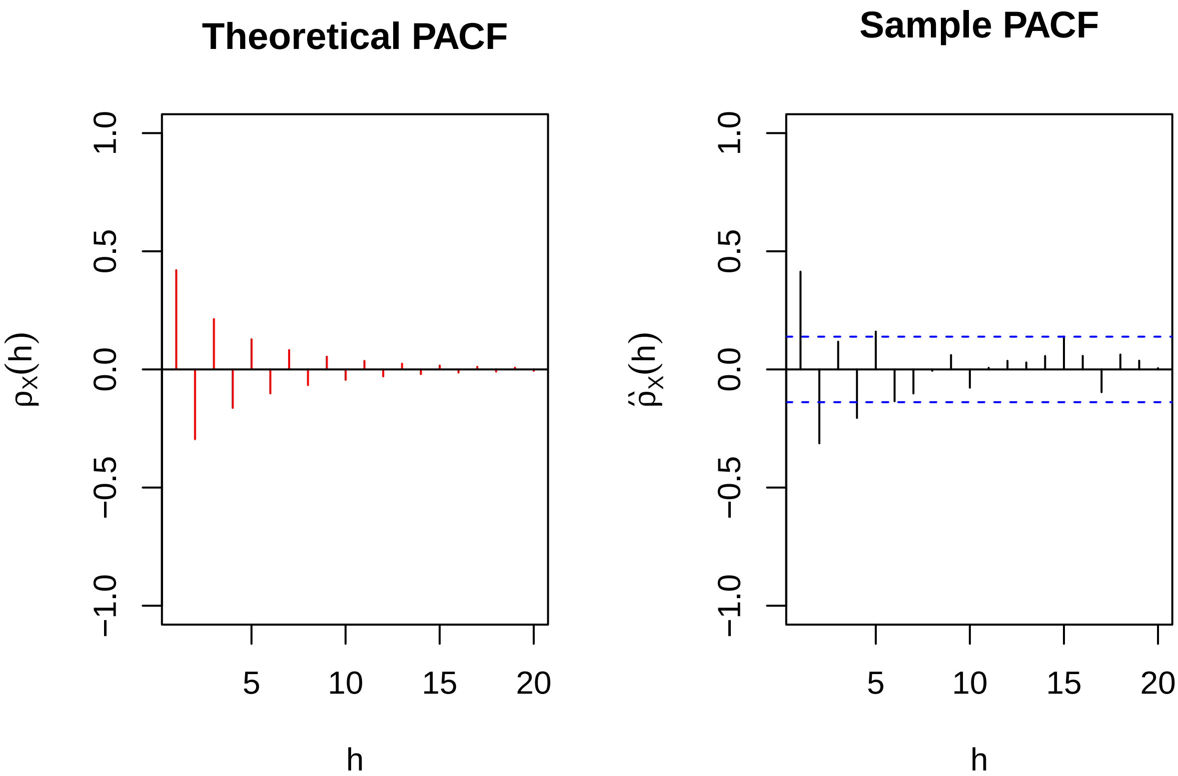
```r
theo_acf <- ARMAacf(ma = c(0.7,-0.1),lag.max = 20, pacf = FALSE) # Plot
op <- par(mfrow = c(1,2))
plot(theo_acf,type = "h",ylim = c(-1,1),
main = "Theoretical ACF",
col = "red",
ylab = expression(rho[X](h)), xlab = "h")
abline(h = 0) # Add horizontal line
# Sample ACF
acf(ma1,lag.max = 20,
main = "Sample ACF",
ylim = c(-1,1),
xlab = "h",
ylab = expression(hat(rho)[X](h)))
```




```r
par(op)
```

```r
# Theoretical PACF
theo_pacf <- ARMAacf(ma = c(0.7,-0.1),lag.max = 20, pacf = TRUE) # Plot
op <- par(mfrow = c(1,2))
plot(theo_pacf,type = "h",ylim = c(-1,1),
main = "Theoretical PACF",
col = "red",
ylab = expression(rho[X](h)), xlab = "h")
abline(h = 0) # Add horizontal line
# Sample PACF
pacf(ma1,lag.max = 20,
main = "Sample PACF",
ylim = c(-1,1),
xlab = "h",
```

```r
ylab = expression(hat(rho)[X](h)))
```

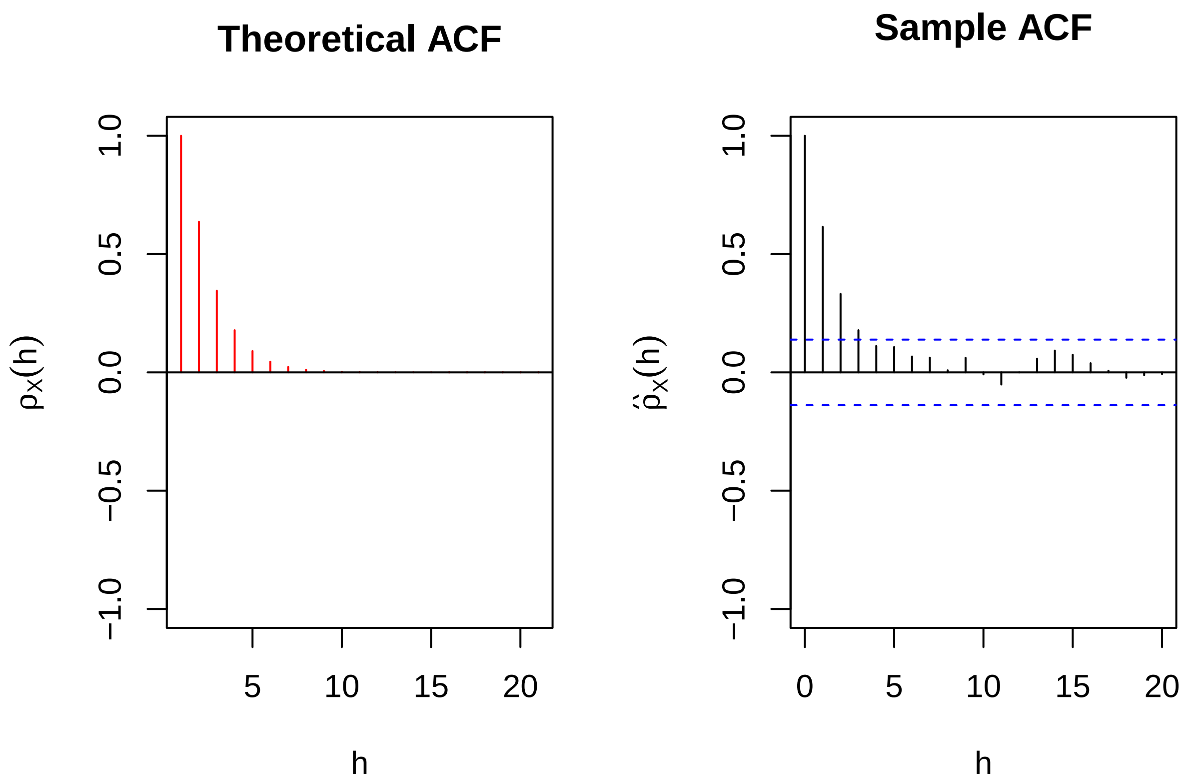



```r
par(op)
```

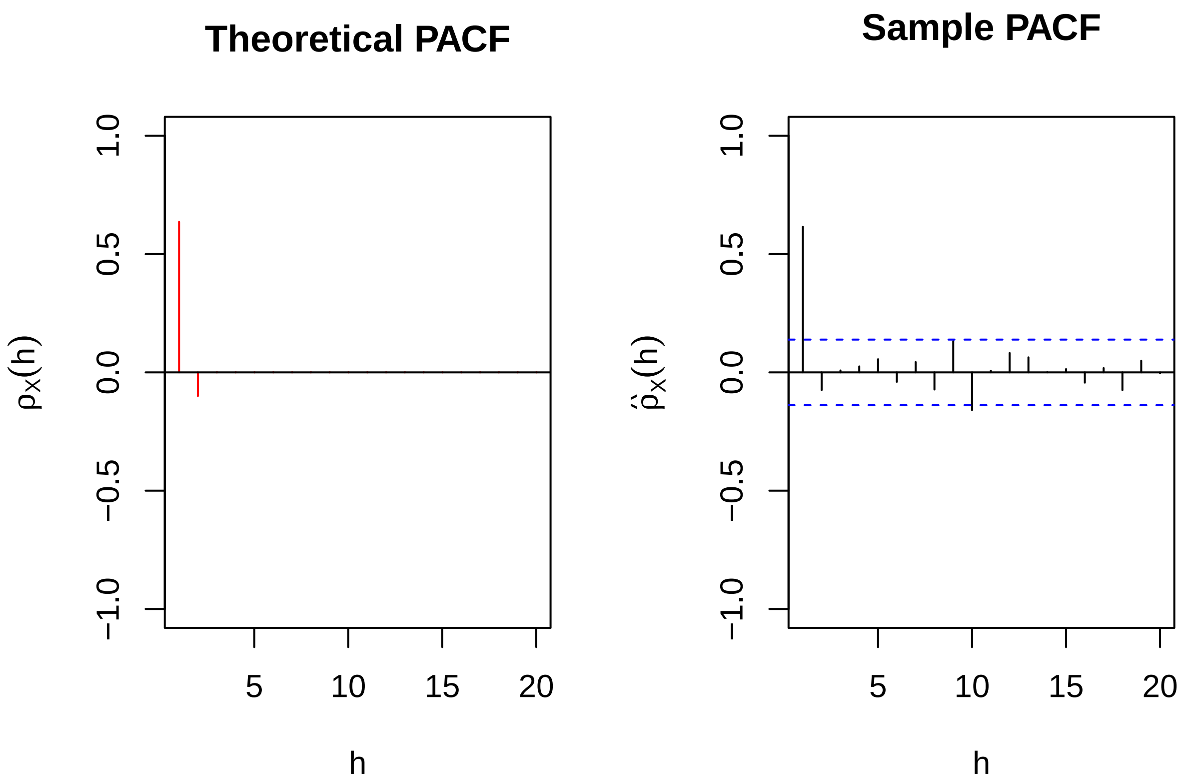
From the above plots, MA models are characterized by ACF=0 after lag q but PACFs do not become zeros. In the MA(2) model, theoretical ACFs cut off after lag k=2.

(b). AR(2)

```r
# Simulate model
set.seed(1)
ar2 <- arima.sim(model = list(ar = c(0.7,-0.1),sd = 1),n = 200)
# Theoretical ACF
theo_acf <- ARMAacf(ar = c(0.7,-0.1),lag.max = 20, pacf = FALSE)
op <- par(mfrow = c(1,2))
# Theoretical ACF
plot(theo_acf,type = "h",ylim = c(-1,1),
main = "Theoretical ACF",
col = "red",
ylab = expression(rho[X](h)), xlab = "h")
abline(h = 0) # Add horizontal line
# Sample ACF
acf(ar2,lag.max = 20,
main = "Sample ACF",
ylim = c(-1,1),
xlab = "h",
ylab = expression(hat(rho)[X](h)))
```




```
# Theoretical PACF
theo_pacf <- ARMAacf(ar = c(0.7,-0.1),lag.max = 20, pacf = TRUE)
op <- par(mfrow = c(1,2))
plot(theo_pacf,type = "h",ylim = c(-1,1),
main = "Theoretical PACF",
col = "red",
ylab = expression(rho[X](h)), xlab = "h")
abline(h = 0) # Add horizontal line
# Sample PACF
pacf(ar2,lag.max = 20,
main = "Sample PACF",
ylim = c(-1,1),
xlab = "h",
ylab = expression(hat(rho)[X](h)))
```

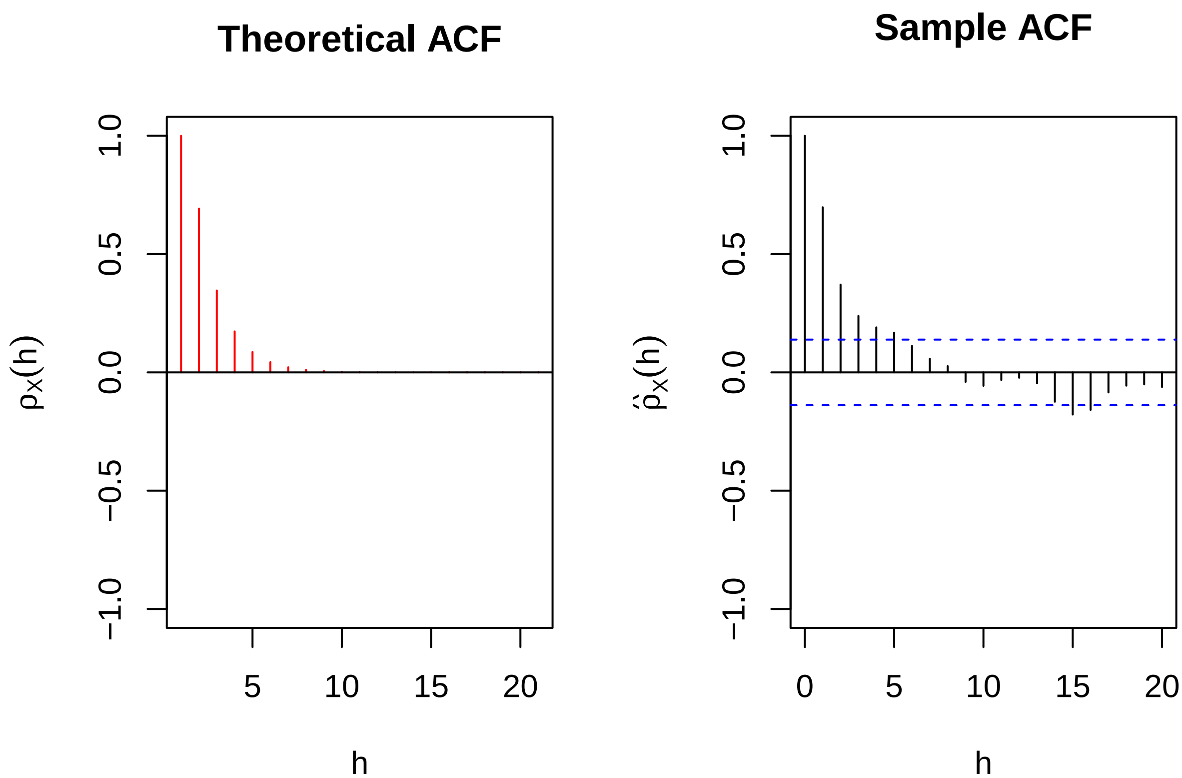



```
par(op)
```

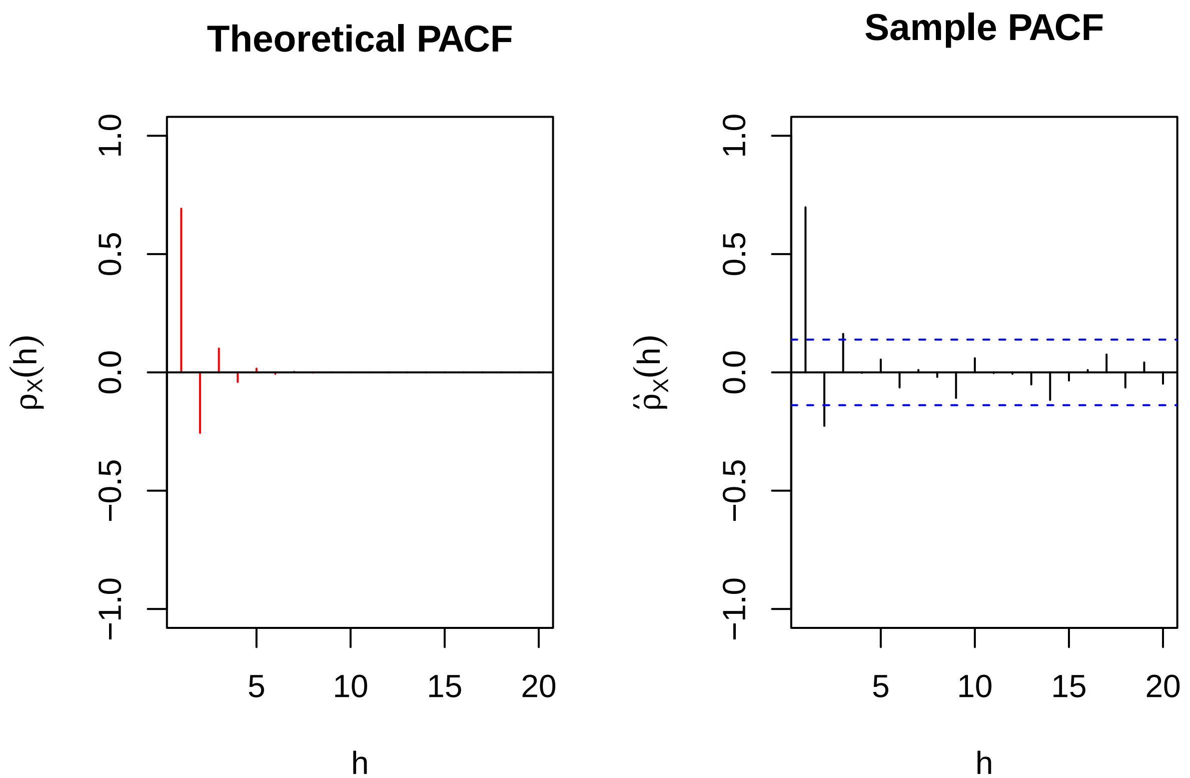
AR models are characterized by PACF=0 after lag p but ACFs do not become zeros. In the AR(2) model, theoretical PACFs cut off after lag k=2.

(c) ARMA(1,1)

```
# Simulate model
set.seed(3)
arma11 <- arima.sim(model = list(ar = c(0.5),ma = c(0.4),sd = 1),n = 200)
# Theoretical ACF
theo_acf <- ARMAacf(ar = 0.5, ma = 0.4,lag.max = 20, pacf = FALSE) # Plot
op <- par(mfrow = c(1,2))
plot(theo_acf,type = "h",ylim = c(-1,1),
main = "Theoretical ACF",
col = "red",
ylab = expression(rho[X](h)), xlab = "h")
abline(h = 0) # Add horizontal line
# Sample ACF
acf(arma11,lag.max = 20, main = "Sample ACF",
ylim = c(-1,1),
xlab = "h",
ylab = expression(hat(rho)[X](h)))
```



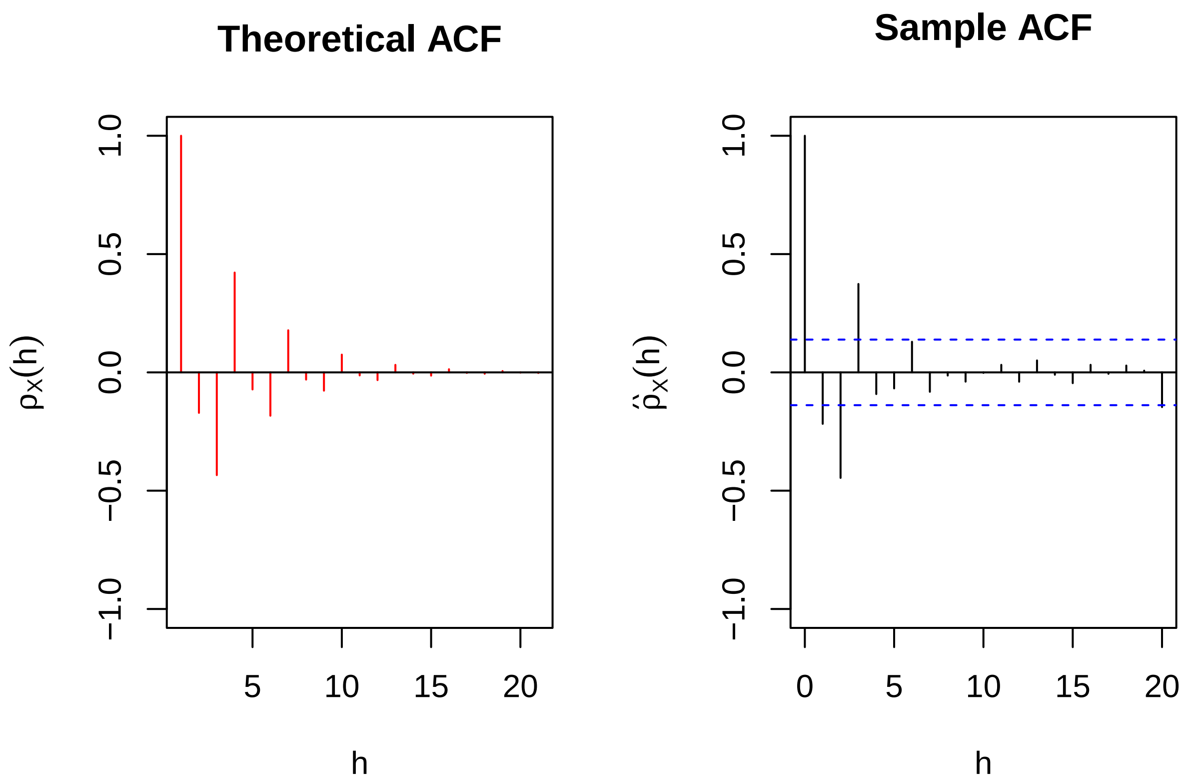

```r
# Theoretical PACF
theo_pacf <- ARMAacf(ar = 0.5, ma = 0.4,lag.max = 20, pacf = TRUE) # Plot
op <- par(mfrow = c(1,2))
plot(theo_pacf,type = "h",ylim = c(-1,1),
main = "Theoretical PACF",
col = "red",
ylab = expression(rho[X](h)), xlab = "h")
abline(h = 0) # Add horizontal line
# Sample PACF
pacf(arma11,lag.max = 20,
main = "Sample PACF",
ylim = c(-1,1),
xlab = "h",
ylab = expression(hat(rho)[X](h)))
```



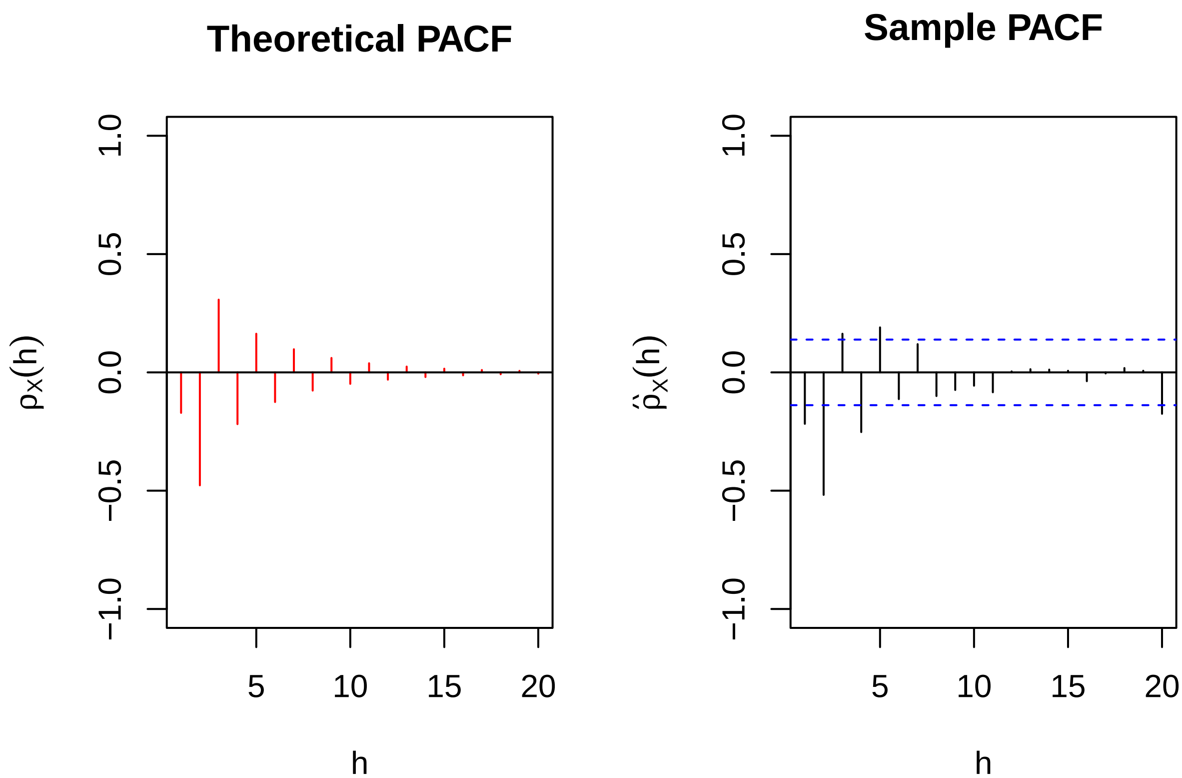

```
par(op)
```

(d) ARMA(2,1)

```
# Simulate model
set.seed(4)
arma21 <- arima.sim(model = list(ar = c(-0.75,-0.5625),ma = 1.25,sd = 1),n = 200)
# Theoretical ACF
theo_acf <- ARMAacf(ar = c(-0.75,-0.5625), ma = 1.25,lag.max = 20, pacf = FALSE) # Plot
op <- par(mfrow = c(1,2))
plot(theo_acf,type = "h",ylim = c(-1,1),
main = "Theoretical ACF",
col = "red",
ylab = expression(rho[X](h)), xlab = "h")
abline(h = 0) # Add horizontal line
# Sample ACF
acf(arma21,lag.max = 20, main = "Sample ACF",
ylim = c(-1,1),
xlab = "h",
ylab = expression(hat(rho)[X](h)))
```




```r
# Theoretical PACF
theo_pacf <- ARMAacf(ar = c(-0.75,-0.5625), ma = 1.25,lag.max = 20, pacf = TRUE) # Plot
op <- par(mfrow = c(1,2))
plot(theo_pacf,type = "h",ylim = c(-1,1),
main = "Theoretical PACF",
col = "red",
ylab = expression(rho[X](h)), xlab = "h")
abline(h = 0) # Add horizontal line
# Sample PACF
pacf(arma21,lag.max = 20,
main = "Sample PACF",
ylim = c(-1,1),
xlab = "h",
ylab = expression(hat(rho)[X](h)))
```




```
par(op)
```

ARMA is hard to identify the order p and q from the ACF and PACF plots.