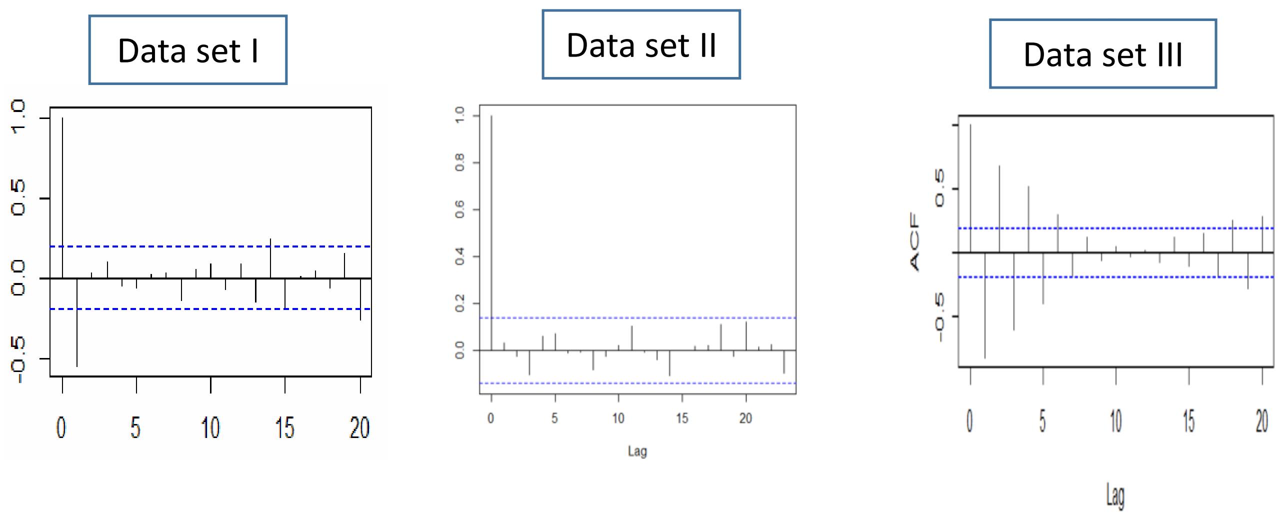
PSTATW 174/274, Fall 2023: Homework # 2.

Note: $\{Z_t\} \sim WN(0, \sigma_Z^2)$ denotes white noise.

1. Bellow, you are given the following graphs of autocorrelation functions for three separate data sets, each with n observations. The dotted lines in each graph correspond to 95% confidence intervals. Determine which of the above data sets exhibit statistically significant autocorrelations. Explain how you came to this conclusion.

A. I only; B. II only; C. III only; D. I, II and III; E. The answer is not given by (A), (B), (C), or (D).



2. For each of the two time series models, check stationarity and invertibility. Fully justify your answer.

(2.a) $X_t = \frac{4}{3}X_{t-1} - \frac{1}{3}X_{t-2} + Z_t$.
(2.b) $X_t = Z_t - \frac{5}{2}Z_{t-1} + Z_{t-2}$.

3. (3.a) For a MA(3) process with coefficients $\theta_1 = -1.5, \theta_2 = 0.75$, and $\theta_3 = -0.125$,
(i) write the mathematical equation for MA(3) model with these coefficients, and
(ii) calculate the autocorrelation function at lags 1, 2, 3, 4: $\rho(1), \rho(2), \rho(3)$ and $\rho(4)$.

(3.b) For an AR(1) process with coefficient $\phi_1 = -0.4$,
(i) write the mathematical equation for AR(1)) model with these coefficients, and
(ii) calculate the autocorrelation function at lags 1, 2, 3, 4: $\rho(1), \rho(2), \rho(3)$ and $\rho(4)$.

4. True/False, explain fully. For $X_t - 0.2X_{t-1} = Z_t$, $Z_t \sim WN(0, 1)$,
A. The mean of the process is equal to 0.2.
B. The variance of the process is equal to $(0.2)^2 = 0.04$.
C. The autocorrelation function $\rho_X(k)$ exhibits a slow decay across lags.
D. None of the above.

5. Let $X_t = Z_t + 2Z_{t-1} - 8Z_{t-2}$.
(i) Identify the model as the model as MA(q) or AR(p), specify q or p respectively.
(ii) Is the model stationary and invertible? Explain fully and show calculations where needed.
(*Hint: review 4 from homework 1!*)
(iii) Find $\rho_X(2)$. Use R to simulate 300 values of $\{X_t\}$ and use your simulated values to plot sample acf. Compare your sample estimate of $\rho_X(2)$ to its true value found by calculations. Redo this part using 10,000 simulated values of X_t.

The following problems are for students enrolled in PSTAT 274 ONLY

G1. Let $\{Z_t\} \sim WN(0,1)$ and $\{X_t\}$ be given by $X_t = Z_t + \theta Z_{t-2}$.
(a) Find the autocovariance and autocorrelation function for this process when $\theta = 0.8$.
(b) Compute the variance of the sample mean $(X_1 + X_2 + X_3 + X_4)/4$ when $\theta = 0.8$.
(c) Repeat (b) when $\theta = -0.8$ and compare your answer with the result obtained in (b).

G2. Let $X_t = Z_t + \theta Z_{t-1}$, $Z_t \sim WN(0, \sigma_Z^2)$ be a MA(1) process. Show that

$$\max_{-\infty<\theta<\infty} \rho_X(\theta) = 0.5 \ \textit{and} \ \min_{-\infty<\theta<\infty} \rho_X(\theta) = -0.5$$

G3. Let $X_t = Z_t + \theta Z_{t-1}$, $t = 1, 2, \ldots$, where $Z_t \sim IID(0, \sigma_Z^2)$. Show that X_t is both weakly and strictly stationary.
(*Hint: for the last part express the joint moment generating function* $E\exp(\sum_{i=1}^n \lambda_i X_i)$ *in terms of function* $m(\lambda) = E\exp(\lambda Z_i)$.)