Introduction to Graphs

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Preliminaries

Proposition. *An undirected, connected graph $G(V, E)$ with n vertices but no parallel edges has minimum $n-1$ edges and maximum $\frac{n(n-1)}{2}$ edges.*

Thus in most cases the number of edges are of $\Omega(n)$ and $\mathcal{O}(n^2)$. A graph is considered *sparse* if the number of edges is close to $\mathcal{O}(n)$; it is considered *dense* if the number is close to $\mathcal{O}(n^2)$.

Representations

We may represent a graph $G(V, E)$ via a $n \times n$ *adjacency matrix* A, where n is the number of vertices. As for an undirected, connected graph without parallel edges, the only two possibilities are represented by 0 and 1, namely whether there is an edge between a pair of vertices. For example, entry $A_{ij} = 0$ denotes that there is no edge between vertices i and j. It is clear that this representation takes the space complexity $\theta(n^2)$.

We may also represent a graph via two *adjacency list*s. One array (or list) is used to store n vertices while the other is used to store m edges. The space complexity is $\theta(n + m)$.

Definition (vertex degree)**.** The *degree of a vertex v* of a graph G is the number of incident edges on it.

Proposition. *We have following properties regarding to the vertex degree.*

- *For each vertex v, its vertex degree is at least k, where k is the number of its crossing edges.*

$$deg\,(v) \geq k\,.$$

- *For any undirected graph, we have the sum of degrees of all vertices to be $2m$, where m is the number of edges.*

$$\sum_v deg\,(v) = 2m\,.$$

- *Recall the number of vertices is n, we have*

$$\sum_v deg\,(v) = 2m \geq nk\,,$$

and thus the relationship between the number of edges m, number of vertices n, and degrees of vertices k.

$$m \geq \frac{nk}{2}\,.$$