Topological Sort

Zezhou Jing

Rev. September 24, 2016

Preliminaries

Definition (topological sort)**.** A *topological sort*, or *topological ordering* f of a directed graph G is a linear labeling of G's vertices such that

- The $f(v)$'s are the set $\{1, 2, 3, \ldots, n\}$;
- For any directed edge uv, u is labeled before v in the ordering:

$$(u, v) \in G \Longrightarrow f(u) < f(v)\,.$$

The topological sort is useful representing sequential tasks, such as work flow and completion towards a degree in terms of prerequisite courses.

Remark. A graph with some directed cycle violates the above properties and thus has no topological ordering.

Consequently, we have:

Theorem. *There exists a topological sort for a directed graph G if it has no directed cycle.*

A depth-first search algorithm computes the topological ordering of a directed graph G without directed cycle in $\mathcal{O}(m + n)$ time, where $|V| = m$ and $|E| = n$, the numbers of vertices and edges, respectively.